PRAMANA — journal of physics

Vol. 69, No. 1
July 2007
pp. 49–76

The Raychaudhuri equations: A brief review

SAYAN KAR[1] and SOUMITRA SENGUPTA[2]
[1]Department of Physics and Centre for Theoretical Studies, Indian Institute of Technology, Kharagpur 721 302, India
[2]Department of Theoretical Physics, Indian Association for the Cultivation of Science, Jadavpur, Kolkata 700 032, India
E-mail: sayan@phy.iitkgp.ernet.in; sayan@cts.iitkgp.ernet.in; tpssg@mahendra.iacs.res.in

Abstract. We present a brief review on the Raychaudhuri equations. Beginning with a summary of the essential features of the original article by Raychaudhuri and subsequent work of numerous authors, we move on to a discussion of the equations in the context of alternate non-Riemannian spacetimes as well as other theories of gravity, with a special mention on the equations in spacetimes with torsion (Einstein–Cartan–Sciama–Kibble theory). Finally, we give an overview of some recent applications of these equations in general relativity, quantum field theory, string theory and the theory of relativisitic membranes. We conclude with a summary and provide our own perspectives on directions of future research.

Keywords. Raychaudhuri equations; general relativity; alternate theories of gravity.

PACS Nos 04.20.-q; 04.50.+h; 11.25.-w

1. The beginnings

About half a century ago, general relativity (GR) was young (just forty years old!), and even the understanding of the simplest solution, the Schwarzschild, was incomplete. Cosmology was virtually in its infancy, despite the fact that the Friedmann–Lemaitre–Robertson–Walker (FLRW) solutions had been around for quite a while. The question about the then-known exact solutions of GR, which worried the serious relativist quite a bit, concerned their singular nature. Both the Schwarzschild and the cosmological solutions were singular. It is well-known that the creator of GR, Einstein himself, was quite worried about the appearance of singularities in his theory. Was there a way out? Was it correct to believe in a theory which had singular solutions? Were singularities inevitable in GR?

It was during these days in the early 1950s, Raychaudhuri began examining some of these questions in GR. One of his early works during this era involved the construction of a non-static solution of the Einstein equations for a cluster of radially moving particles in an otherwise empty space [1]. A year before, he had also written an article related to condensations in an expanding Universe [2] where he dealt with cosmological perturbations (in a sense, this article deals with what is today known as structure formation). Subsequent to these papers, in 1955, appeared *Relativistic Cosmology I* [3], which contains the derivation of the now-famous Raychaudhuri equation.

Fifty years hence, the Raychaudhuri equations have been discussed and analysed in a variety of contexts. Their rise to prominence was largely due to their use (through the notion of geodesic focusing) in the proofs of the seminal Hawking–Penrose singularity theorems of GR. Today, the importance of this set of equations, as well as their applicability in diverse scenarios, is a well-known fact.

This article is a brief review on these equations. We shall deal with some selected aspects in greater detail. We, of course, would like to emphasize that there are many topics which we leave untouched or, barely touched. We hope to do justice to these in a later, and more extensive article.

The overall plan of this article is as follows. In the remainder of this section, we shall recall the basic ideas and results in the 1955 paper and the singularity theorems. The next section introduces (with illustrative examples) the kinematical quantities (expansion, rotation and shear) which govern the characteristics of geodesic flows and also outlines the derivation and consequences of the equations. Section 3 considers the equations in alternative Riemannian and non-Riemannian theories of gravity ($R+\beta R^2$ theory and the Einstein–Cartan–Sciama–Kibble (ECSK) theory, in particular). In §4, we give a glimpse of the diverse uses of these equations in contexts within, as well as outside the realm of GR. Finally, we present a summary and provide our perspectives on possible future work.

1.1 *The original 1955 paper*

The derivation of the Raychaudhuri equation, presented in the 1955 article, is somewhat different from the way it is presented in standard textbooks today. It must however be mentioned, that in a subsequent paper in 1957 [4], Raychaudhuri presented further results which bear a similarity with the modern approach to the derivation. Let us now briefly summarise the main points of the original derivation of Raychaudhuri.

(a) Raychaudhuri's motivation behind this article is almost entirely restricted to cosmology. He assumes that the Universe is represented by a time-dependent geometry but does *not* assume homogeneity or isotropy at the outset. In fact, one of his aims is to see whether non-zero rotation (spin), anisotropy (shear) and/or a cosmological constant can succeed in avoiding the initial singularity.

(b) The entire analysis is carried out in the comoving frame (in the context of cosmological line elements) – the frame in which the observer is at rest in the fluid.

(c) The quantity R_4^4 (spacetime coordinates in the 1955 paper are labeled as x^1, x^2, x^3, x^4 with the fourth one being time), is evaluated in two ways – once using the Einstein equations (with a cosmological constant Λ) and, again, using the geometric definition of R_4^4 in terms of the metric and its derivatives. In the second way of writing this quantity, Raychaudhuri introduces the definitions of shear and rotation.

(d) Finally, equating the two ways of writing R^4_4 the equation for the evolution of the expansion rate is obtained. Note that the definition of expansion given in this paper refers to the special case of a cosmological metric.
(e) Apart from obtaining the equation, the article also arrives at the focusing theorem (though it is not mentioned with this name) and some additional results (the last section).

In the same year (1955), Heckmann and Schucking [5], while dealing with Newtonian cosmology arrived at a set of equations, one of which is the Raychaudhuri equation (in the Newtonian case). Prompted by this work, Raychaudhuri re-derived his equations in a somewhat different way in an article where he also showed that Heckmann and Schucking's work for the Newtonian case could be generalised without any problems to the fully relativistic scenario. It must also be noted that Komar [6], a year after Raychaudhuri's article appeared, obtained conclusions similar to what is presented in Raychaudhuri's article. Raychaudhuri pointed this out in a letter published in 1957 [7].

Subsequently, in 1961, Jordan *et al* wrote an extensive article on the relativistic mechanics of continuous media where the derivation of the evolution equations of shear and rotation seem to appear for the first time [8]. Furthermore, for null geodesic flows, the kinematical quantities: expansion, rotation and shear (related to the so-called optical scalars) and the corresponding Raychaudhuri equations, were first introduced by Sachs [9].

The Raychaudhuri equation is sometimes referred to as the Landau–Raychaudhuri equation. It may be worthwhile to point out precisely, the work of Landau, in relation to this equation. Landau's contribution appears in his treatise *The Classical Theory of Fields* [10] and is also discussed in detail in [6,11]. Working in the synchronous (comoving) reference frame, Landau defines a quantity $\chi_{\alpha\beta} = \partial\gamma_{\alpha\beta}/\partial t$, where $\gamma_{\alpha\beta}$ is the 3-metric. Subsequently, using the fact that $\chi^\alpha_\alpha = (\partial/\partial t)\gamma$, where γ is the determinant of the 3-metric, he writes down an expression for R^0_0 and then, an inequality $(\partial/\partial t)\chi^\alpha_\alpha + \frac{1}{6}\left(\chi^\alpha_\alpha\right)^2 \geq 0$. While deriving the inequality, Landau implicitly assumes the strong energy condition (though it is not mentioned with this name). Then, using it, he is able to show that γ must necessarily go to zero within a finite time. However, he mentions quite clearly that this does not imply the existence of a physical singularity in the sense of curvature. Though Landau's work captures the essence of focusing, he does not explicitly mention geodesic focusing. Moreover, he does not introduce shear and rotation or write down the complete equation for the expansion.

Even though it was mentioned in [11] and [8], Raychaudhuri's contribution found its true recognition only after the seminal work of Hawking and Penrose which appeared a decade later. It was at that time, along with the proofs of the singularity theorems, the term Raychaudhuri equations came into existence in the physics literature.

1.2 *The singularity theorems*

It must be mentioned that Raychaudhuri did point out the connection of his equations to the existence of singularities in his 1955 article. However, more general results (based on global techniques in Lorentzian spacetimes) appeared in the form of singularity theorems following Penrose's work [12] and, then, Hawking's contributions [13,14]. The crucial element of the singularity theorems is that the existence of singularities is proved by using a minimal set of assumptions (loosely speaking, these are: Lorentz signature metrics and causality, the generic condition on the Riemann tensor components, the existence of trapped surfaces and energy conditions on matter). In fact, a precise definition of 'what is a singularity?' first appeared in the works of Hawking and Penrose. The notion of geodesic incompleteness and its relation to singularities (not necessarily curvature singularities) was also born in their work. One should also realise that the focusing of geodesics arrived at by Raychaudhuri and discussed in much detail in later articles by other authors could be completely benign (irrespective of any actual singularity being present in the manifold). Thus, a singularity would always imply focusing of geodesics but focusing alone cannot imply a singularity (also pointed out by Landau [10]). We refrain from discussing the singularity theorems any further here – excellent discussion on global aspects in gravitation as well as the Hawking–Penrose theorems are available in [15–17].

2. The geometry and physics of the equations

Let us now review the basic ingredients and the derivation of the equations. First, of course, we need to know – what do these equations deal with? In a sentence, one may say that they are concerned about the kinematics of flows. Flows are generated by a vector field – they are the integral curves of the given vector field. These curves may be geodesic or non-geodesic, though the former is more useful in the context of gravity. Thus, a flow is a congruence of such curves – each curve may be time-like or null or, in the Euclidean case, have tangent vectors with a positive definite norm. One does not, in the context of these equations, ask, how the flow is generated. In other words, we are more interested, in deriving the kinematic characteristics of such flows. The evolution equations (along the flow) of the quantitites that characterise the flow in a given background spacetime, are the Raychaudhuri equations. Historically speaking, it is the equation for one of the quantitites (the expansion), which is termed as the Raychaudhuri equation. However, in this article, we will refer to the full set of equations as Raychaudhuri equations.

2.1 *Expansion, rotation, shear*

What quantities characterise a flow? If λ denotes the parameter labeling points on the curves in the flow, then, in order to characterise the flow, we must have different functions of λ. In other words, the gradient of the velocity field being a second rank tensor is split into three parts: the symmetric traceless part, the antisymmetric part and the trace. These define for us the shear, rotation and the expansion of the flow. Specifically,

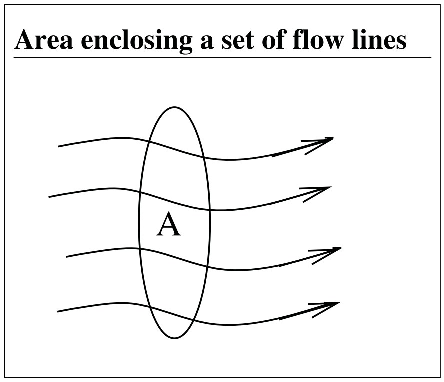

Figure 1. The cross-sectional area enclosing a congruence of geodesics.

$$\nabla_b v_a = \sigma_{ab} + \omega_{ab} + \frac{1}{n-1} h_{ab}\Theta, \tag{1}$$

where the symmetric, traceless part, the shear, is defined as $\sigma_{ab} = \frac{1}{2}(\nabla_b v_a + \nabla_a v_b) - (1/(n-1))h_{ab}\Theta$, the trace, expansion, is $\Theta = \nabla_a v^a$ and the antisymmetric rotation is given as, $\omega_{ab} = \frac{1}{2}(\nabla_b v_a - \nabla_a v_b)$. n is the dimension of spacetime, and $h_{ab} = g_{ab} \pm v_a v_b$ is the projection tensor (the plus sign is for time-like curves whereas the minus one is for space-like ones). Also, correspondingly, $v_a v^a = \mp 1$. Later, we shall discuss the case of null geodesic congruences briefly.

The geometric meaning of these quantities is shown through figures 1 and 2. The expansion, rotation and shear are related to the geometry of the cross-sectional area (enclosing a fixed number of geodesics) orthogonal to the flow lines (figure 1). As one moves from one point to another, along the flow, the shape of this area changes. It still includes the same set of geodesics in the bundle but may be isotropically smaller (or larger), sheared or twisted. The analogy with elastic deformations or fluid flow is, usually, a good visual aid for understanding the change in the geometry of this area. A recent, nice discussion is available in [18]. In refs [19,20] these quantities are explained in quite some detail.

2.2 *Examples*

It is useful to illustrate these quantities with a set of examples. We first choose to work with Schwarzschild spacetime. Our examples here will involve (i) rotation-free time-like geodesic flows and (ii) time-like flows with all three kinematical quantities non-zero. We focus on examples with non-zero shear and rotation because these are not usually available in standard texts on GR. Our choice of examples are primarily based on the problems suggested in a recent monograph by Poisson [18]. In a third example (iii) we discuss briefly a time-like geodesic flow in the FLRW Universe. Finally, in (iv) we briefly deal with geodesic flows in wormhole spacetimes.

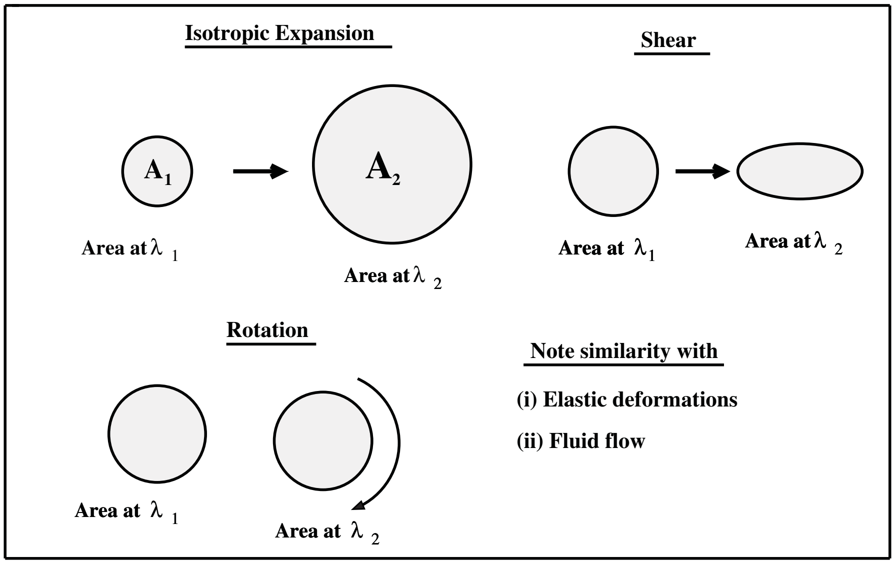

Figure 2. Illustrating expansion, rotation and shear.

(i) From the Frobenius theorem [18], we know that hypersurface orthogonal vector fields must necessarily have zero rotation though the shear can have non-zero components. In Schwarzschild spacetime, we construct a congruence which has the above properties (i.e. it is irrotational but has non-zero shear and expansion). Consider the vector field

$$u^a \partial_a = \frac{1}{1-(2M/r)} \partial_t \pm \sqrt{\frac{2M}{r}} \partial_r . \tag{2}$$

The geodesics corresponding to the above vector field are marginally bound (i.e. $u_t = -1$) and the upper and lower signs refer to outgoing and incoming geodesics. It is easy to show that the above vector field can be written as $u_a = \partial_a \phi$, where $\phi(x^a)$ = constant would represent the hypersurface with respect to which u^a is orthogonal. In fact, ϕ is the same as the new time T (the proper time as measured by a freely falling observer starting from rest at infinity and moving radially inward) used in the Painleve-Gullstrand representation [18] of the Schwarzschild line element.

It is easy to calculate the expansion, which turns out to be

$$\Theta = \pm \frac{3}{2} \sqrt{\frac{2M}{r^3}} . \tag{3}$$

Notice that the expansion is positive for outgoing and negative for incoming geodesics. We can also find the non-zero shear tensor components which are given as

$$\sigma_{tt} = \mp\frac{2M}{r^2}\sqrt{\frac{2M}{r}}; \quad \sigma_{rr} = \mp\frac{\sqrt{2M/r^3}}{(1-(2M/r))^2}, \tag{4}$$

$$\sigma_{tr} = \frac{2M/r^2}{1-(2M/r)}; \quad \sigma_{\theta\theta} = \pm\sqrt{\frac{Mr}{2}} = \frac{1}{\sin^2\theta}\sigma_{\phi\phi}. \tag{5}$$

One can then check that the Raychaudhuri equations (given in the next sub-section) hold with the above expressions for the shear and expansion.
(ii) We now move on to an example where all three kinematical quantities are non-zero.

Consider the following vector field in Schwarzschild spacetime:

$$u^a\partial_a \equiv \frac{1}{\sqrt{1-(3M/r)}}\left(\partial_t + \sqrt{\frac{M}{r^3}}\partial_\theta\right), \tag{6}$$

where M is the usual mass. It can be verified that the geodesics corresponding to this vector field are time-like and they are circular (r is a constant). We intend to calculate the expansion, rotation and shear of the above vector field. The expansion is given as

$$\Theta = \cot\theta\sqrt{\frac{M/r^3}{1-(3M/r)}}. \tag{7}$$

Notice that the expansion is positive in the northern hemisphere and negative in the southern hemisphere.

Following the definition, one can show that the rotation tensor for this vector field is given as

$$\omega_{tr} = \frac{M}{4r^2}\frac{1-(6M/r)}{(1-(3M/r))^{3/2}} = \sqrt{\frac{M}{r^3}}\omega_{r\theta} \tag{8}$$

and the shear tensor is

$$\begin{aligned}\sigma_{tt} &= \frac{M}{r^3}\sigma_{\theta\theta} = -\frac{M}{2r^3\sin^2\theta}\frac{(1-(2M/r))}{(1-(3M/r))}\sigma_{\phi\phi} \\ &= -\sqrt{\frac{M}{r^3}}\sigma_{t\theta} = \frac{M}{r}\frac{(1-(2M/r))^2}{(1-(3M/r))}\sigma_{rr} \\ &= -\frac{1}{3}\cot\theta\sqrt{\frac{M^3}{r^5}}\frac{(1-(2M/r))}{(1-(3M/r))^{3/2}} \\ \sigma_{tr} &= \sqrt{\frac{M}{r^3}}\sigma_{r\theta} = -\frac{3M}{4r^2}\frac{(1-(2M/r))}{(1-(3M/r))^{3/2}}.\end{aligned} \tag{9}$$

One can further verify that here too, the Raychaudhuri equations (given below) are satisfied for the above quantities. One can also evaluate $\sigma^2 - \omega^2$ and show that it is positive for $r > 3M$. The expansion is defined for this domain of r ($>3M$) and can diverge to negative infinity (focusing) at $\theta = \pi$ (south pole).

(iii) As a third example, we now quickly discuss the expansion, rotation and shear with respect to the vector field $u^a\partial_a = \partial_t$ in the standard cosmological line element (FLRW). The shear and rotation are identically zero. The expansion is given as

$$\Theta = 3\frac{\dot{a}}{a} = \frac{1}{\sqrt{a^6}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt{a^6}\right). \tag{10}$$

Note that a^6 is the volume of the expanding 3-space. Hence the term 'volume' expansion is also used in the literature. Raychaudhuri, in his original article, defined the expansion in this way. However, his treatment did include non-zero shear and rotation because he did not assume, to start with, maximally symmetric metrics on spatial slices representing R^3, S^3 or H^3. It may be mentioned here that the equation for the expansion reduces to the equation for $(\ddot{a}/a) = (4\pi G/3)(\rho + 3p)$.

(iv) Our final example concerns the case of geodesic flows in a traversable wormhole [21]. It is known that traversable wormholes require energy-condition-violating matter. These are non-singular spacetimes where the spatial slices resemble two asymptotically flat regions connected by a throat. Thus a geodesic congruence passing through the throat from one asymptotic region to the other would necessarily tend to focus first, not quite reach a focal point, and then defocus. For a typical wormhole, the line element is given as

$$\mathrm{d}s^2 = -\chi^2(l)\mathrm{d}t^2 + \mathrm{d}l^2 + r^2(l)\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right), \tag{11}$$

where, for a wormhole, $\chi(l)$ is non-zero and finite for all l and $r(l = 0) = b_0$ with $r(l \to \pm\infty) \sim l$. It is easy to check that the expansion is proportional to r'/r (the prime denoting a derivative with respect to l). Thus, the expansion never becomes negative infinity and thus there is no focusing. One can work out the expansion for a typical example using $r(l) = \sqrt{b_0^2 + l^2}$ (the so-called Ellis wormhole) [21].

2.3 *The equations and the focusing theorem*

We now turn towards writing down the evolution equations for the expansion, shear and rotation along the flow representing a time-like geodesic congruence. A fact worth mentioning here is that, these evolution equations (and their generalisations) are essentially geometric statements and are independent of any reference to the Einstein field equations.

The modern (textbook) way to derive these equations (see [15]) is as follows. Consider the quantity $v^c\nabla_c B_{ab}$ (where $B_{ab} = \nabla_b v_a$). Evaluate this as an identity and then split it into its trace, antisymmetric and symmetric traceless parts. The equations that emerge are the ones given below (for $n \equiv$ the dimension of spacetime $= 4$).

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\lambda} + \frac{1}{3}\Theta^2 + \sigma^2 - \omega^2 = -R_{ab}v^a v^b, \tag{12}$$

$$\frac{\mathrm{d}\sigma_{ab}}{\mathrm{d}\lambda} = -\frac{2}{3}\Theta\sigma_{ab} - \sigma_{ac}\sigma^c_b - \omega_{ac}\omega^c{}_b + \frac{1}{3}h_{ab}\left(\sigma^2 - \omega^2\right) + C_{cbad}v^c v^d + \frac{1}{2}\tilde{R}_{ab}, \tag{13}$$

$$\frac{\mathrm{d}\omega_{ab}}{\mathrm{d}\lambda} = -\frac{2}{3}\Theta\omega_{ab} - 2\sigma^c{}_{[b}\omega_{a]c}, \tag{14}$$

where $\sigma^2 = \sigma_{ab}\sigma^{ab}$, $\omega^2 = \omega_{ab}\omega^{ab}$, C_{cbad} is the Weyl tensor and the quantity $\tilde{R}_{ab} = h_{ac}h_{bd}R^{cd} - \frac{1}{3}h_{ab}h_{cd}R^{cd}$.

There are a few points to note here. Firstly, one must realise that these are *not* equations but, essentially, identitites. Hence, in some references [22–24] we find the usage Raychaudhuri identity or Codazzi–Raychaudhuri identity (in the context of surface congruences to be discussed later) which is, indeed, rigorously correct. The identities, however become equations once we use the Einstein equations or any other geometric property (e.g. Einstein space, or vacuum, etc.) as an extra input. However, we shall continue to use the term equations in this article.

Furthermore, the equations are coupled, nonlinear and first order. The equation for the expansion is of central interest (in the context of the singularity theorems) and it is rather straightforward to analyse. In mathematical parlance, it (the equation for the expansion) is known as a Riccati equation. Such equations can be transformed into a second-order linear form (more precisely a Hill-type equation or a harmonic oscillator equation with a time-varying frequency) [25,26]. Redefining $\Theta = 3(F'/F)$ one gets

$$\frac{\mathrm{d}^2 F}{\mathrm{d}\,\lambda^2} + \frac{1}{3}(R_{ab}v^a v^b + \sigma^2 - \omega^2)F = 0. \tag{15}$$

The analysis of the expansion equation can be done using the above form. One notes that the expansion Θ is nothing but the rate of change of the cross-sectional area orthogonal to the bundle of geodesics. Therefore, the expansion approaching negative infinity implies a convergence of the bundle, whereas a value of positive infinity would imply a complete divergence. What are the conditions for convergence? Firstly, for convergence we must have an initially negative expansion. Finally, with F' negative we must end up at a zero of F (at a finite λ), in order to have a negatively infinite expansion [25,26]. Thus, the criterion for the existence of zeros in F at finite values of the affine parameter is what is required for convergence. Using the well-known Sturm comparison theorems in the theory of differential equations one can show that convergence occurs if

$$R_{ab}v^a v^b + \sigma^2 - \omega^2 \geq 0. \tag{16}$$

Thus, rotation defies convergence, while shear assists it. The equation for the evolution of the rotation ω_{ab}, has a trivial solution $\omega_{ab} = 0$. The criterion for convergence then becomes particularly simple for such hypersurface orthogonal congruences (zero rotation): $R_{ab}v^a v^b \geq 0$. This leads to geodesic focusing.

If we make use of the Einstein field equations and rewrite the Ricci tensor in terms of the energy–momentum tensor $R_{ab} = T_{ab} - \frac{1}{2}g_{ab}T$ then the so-called timelike convergence condition becomes a condition on matter stress energy. This, given as, $\left(T_{ab} - \frac{1}{2}g_{ab}T\right) v^a v^b \geq 0$ is known as the strong energy condition (SEC). For a diagonal T_{ab} (with $T_{00} = \rho, T_{aa} = p_a$) we must have $\rho + p_a \geq 0, \rho + \sum_a p_a \geq 0$ if the SEC is to be obeyed. In other words, geodesic focusing encodes the simple statement that if matter is attractive, geodesics must be eventually drawn towards each other. This seemingly trivial statement is proved via the focusing theorem.

In the late seventies, Tipler [25,26] realised that the assumption of the SEC imposed to prove focusing and hence the existence of singularities could be further weakened. Among other results, he was able to show how, in the proof of the Hawking–Penrose theorem one might replace SEC by the weak energy condition (WEC: $T_{ab}v^a v^b \geq 0 \; \forall$ non-space-like v^a). Tipler also introduced in his article, for the first time, the notion of an averaged energy condition (the averaged strong and weak energy conditions (ASEC and AWEC)) which are global in nature. For instance, the AWEC is obtained by integrating the WEC along a non-space-like geodesic and gives a number ($\int_{\lambda_1}^{\lambda_2} T_{ab}v^a v^b \mathrm{d}\lambda \geq 0$). The question of whether a violation of the energy conditions could lead to a non-singular solution was also addressed by him. One should mention here that a few years before Tipler's work, Bekenstein [27] and Murphy [28] had proposed certain non-singular spacetimes. These models turned out to be singular, following the above-mentioned generalisations of the singularity theorems. Tipler also quotes, in his paper, the well-known Epstein–Glaser–Yaffe theorem in quantum field theory where it is said that there can exist a quantum state with respect to which the expectation value of the stress–energy tensor can be negative [29]. However, he does mention that the existence of one state with respect to which the $\langle T_{00}\rangle$ is less than zero cannot really lead to the prevention of singularities. Of late, however, energy-condition violations have been discussed in great detail in the context of wormholes [21], dark energy [30], braneworld models [31] and semi-classical gravity (quantum field theory in curved spacetimes) [32]. The experimentally observed Casimir effect (the feeble attraction between parallel, conducting capacitor plates) [33] has been cited as an example of the existence of negative energy density (though, truly speaking this effect is concerned with negative pressures as opposed to actual negative energy densities).

The quest for further weakening the criteria for geodesic focusing and thereby making the singularity theorems stronger was continued later through the work of Borde [34] and Roman [35]. Further work on issues related to geodesic focusing, singularities, energy conditions and causality violations have been carried out in [36–40].

2.4 *Null geodesic congruences*

The Raychaudhuri equations for null geodesic congruences were first derived by Sachs [9] in 1961. Let us briefly recall the salient features of these equations. It must be mentioned, however, that these equations are not very different (in structure as well as consequences thereof) from the equations for time-like congruences.

The central issue in the case for null geodesic congruences is the construction of the transverse parts of the deviation vector and the spacetime metric. Assuming an affine parametrisation in the sense $\mathrm{d}x^a = k^a \mathrm{d}\lambda$ with $k^a k_a = 0$ and $k^a \xi_a = 0$ (with ξ^a being the deviation vector) we realise that we are in trouble because of the above two orthogonality relations. Naively writing $h_{ab} = g_{ab} + k_a k_b$ will not work here ($k^a h_{ab} \neq 0$). The transverse metric is thus constructed by introducing an auxiliary null vector N^a with $k_a N^a = -1$. (the choice of -1 is by convention, the essence is that the quantity must be non-zero). If we choose $k_a = -\partial_a u$ ($u = t - x$) then we can have $N_a = -\frac{1}{2}\partial_a v$ and hence $h_{ab} = g_{ab} + k_a N_b + k_b N_a$. This satisfies $k^a h_{ab} = 0$ and $N^a h_{ab} = 0$. Note that h_{ab} now is entirely two-dimensional. Keeping this transverse metric in mind we can proceed in the same way as for the time-like case by constructing $\hat{B}_{ab} = \nabla_b k_a$. We quote below the equation for the expansion:

$$\frac{\mathrm{d}\hat{\Theta}}{\mathrm{d}\lambda} + \frac{1}{2}\hat{\Theta}^2 + \hat{\sigma}^2 - \hat{\omega}^2 = -R_{ab}k^a k^b, \tag{17}$$

where the hatted quantitites are the expansion, rotation and shear for the null geodesic congruence. The focusing theorem for null geodesic congruences follows in the same way as for time-like congruences, with the null convergence condition $R_{ab}k^a k^b \geq 0$ being the requirement. Using Einstein equations one can obtain the so-called null energy condition $T_{ab}k^a k^b \geq 0$. Similar to the case for time-like congruences, we have corresponding equations for the evolution of shear and rotation for null geodesic congruences too. These are available in [15,18].

2.5 *The acceleration term and non-affine parametrisations*

The discussions above were exclusively for geodesic congruences. In the non-geodesic (i.e. time-like or null congruences) case, it is obvious that there will be differences. We state below, how the equation for the expansion changes via the addition of the so-called acceleration term. The equation is now given as

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\lambda} + \frac{1}{3}\Theta^2 + \sigma^2 - \omega^2 - \nabla_a(v^b \nabla_b v^a) = -R_{ab}v^a v^b, \tag{18}$$

where the fifth term on the LHS is the acceleration term. Notice that this term is zero for geodesic congruences (zero acceleration). The average distance between the world lines in the flow is changed due to the divergence of the acceleration. The non-geodesic character of the flow also affects the equation for the shear.

For a non-affine parametrisation of null geodesic congruences, with $k^a \nabla_a k^b = \kappa k^b$ (κ, a constant, defined through the above equation) the definition of the expansion changes: $\Theta = \nabla_a k^a - \kappa$ and the Raychaudhuri equation for the expansion takes the form

$$\frac{\mathrm{d}\hat{\Theta}}{\mathrm{d}\lambda} - \kappa\Theta + \frac{1}{2}\hat{\Theta}^2 + \hat{\sigma}^2 - \hat{\omega}^2 = -R_{ab}k^a k^b. \tag{19}$$

The new feature here is the presence of the linear (in Θ) term. The conclusions on geodesic focusing however do not change, except for differences in the values of the expansion.

2.6 *A theorem for non-rotating singularity-free Universes: Raychaudhuri's later papers*

Over the entire period of more than forty years, Raychaudhuri did not work much on the equations which bear his name today. Interestingly, he came back to have a look at them once again in the late nineties with a series of papers [41,42]. In these articles, he constructed a theorem on non-rotating singularity-free Universes. The main inspiration behind these papers was the singularity-free cosmological solutions due to Senovilla [43], which created a lot of interest and curiosity among relativists in the 1990s. In fact, the work of Raychaudhuri sets out to show that these solutions may not quite be physically relevant, though surely, mathematically correct. Let us now briefly recall the theorem of Raychaudhuri.

The basic premise of [41,42] is the use of spacetime averages of quantitites, defined as

$$\langle \chi \rangle = \left[\frac{\int_{-x_0}^{x_0} \int_{-x_1}^{x_1} \int_{-x_2}^{x_2} \int_{-x_3}^{x_3} \chi \sqrt{|g|} \mathrm{d}^4 x}{\int_{-x_0}^{x_0} \int_{-x_1}^{x_1} \int_{-x_2}^{x_2} \int_{-x_3}^{x_3} \sqrt{|g|} \mathrm{d}^4 x} \right]_{\lim x_{0,1,2,3} \to \infty} . \tag{20}$$

Raychaudhuri shows that for a singularity-free non-rotating Universe, open in all directions, the spacetime average of all stress–energy invariants, including the energy density vanishes. The proof is worked out using the spacetime averages of the scalars that appear in the Raychaudhuri equation for the expansion. Following the statement of the theorem, Raychaudhuri claimed that an observationally consistent Universe cannot have a zero average density and hence one must necessarily give up the hope of having singularity-free solutions. Subsequent to Raychaudhuri's work, Saa and Senovilla [41] wrote a couple of comments which were mainly concerned with the converse of Raychaudhuri's theorem and the question whether spacetime averages could really be a property which can distinguish between singular and non-singular models. In fact, one must note that the theorem does not say that if the spacetime averages are zero the spacetime must be non-singular. Therefore, the theorem cannot really be used to distinguish between singular and non-singular cosmological models. Furthermore, Senovilla, in his comment, made a conjecture that spatial and not spacetime averages could be a distinguishing property between singular and non-singular models. More precisely, his claim was that for non-singular, non-rotating, globally hyperbolic and everywhere expanding models where SEC holds, the spatial averages of all stress–energy invariants must vanish. Therefore, if such spatial averages do not vanish then the model must be singular. More details and a recent proof of the conjecture is available in the article by Senovilla in this volume [44]. Despite the conflict between whether spatial or spacetime averages was the crucial distinguishing factor between singular and non-singular models, it goes without saying that Raychaudhuri's idea about averages being a deciding factor will surely be remembered as a lasting contribution apart from his coveted equations.

3. The equations in different geometries and different theories of gravity

The Raychaudhuri equations discussed in the previous sections do not change as long as we respect the Riemannian (pseudo-Riemannian) metric structure of space or spacetime. However, in an alternative theory of gravity, the Einstein field equations are of course different and hence the relation between the energy–momentum tensor and other geometric quantities do change. This results in a modification of the consequences that arise while analysing these equations. On the other hand, if the usual Riemannian structure of spacetime changes, such as in the case of spaces with torsion (Einstein–Cartan–Sciama–Kibble theory), then the Raychaudhuri equations are surely different. We shall give an example of both these scenarios in the following two subsections. The equations in spacetimes with torsion is discussed in relatively greater detail because, this is one scenario where we actually notice a generalisation of the usual Raychaudhuri equations.

3.1 *Metric theories with symmetric connections*

As mentioned above, for theories with a symmetric connection, the Raychaudhuri equations do not change, though the RHS of the expansion equation when written using matter stress–energy can be very different from what it is in GR. This change can surely affect geodesic focusing. It must be noted here that any change in the conclusions about geodesic focusing in this case is inherently due to the new solutions (spacetime geometries) of the modified Einstein equations for a given stress–energy. The above-mentioned aspects have been analysed in the context of the Brans–Dicke and Hoyle–Narlikar theories [45–47] as well as the $R+\beta R^2$ theory [48], on which we focus our attention below.

The study of the Raychaudhuri equations in the presence of higher curvature terms in the action has been a subject of interest for a long time [49]. The equation for the expansion, in a spacetime background solution of $R+\beta R^2$ gravity in the presence of matter ($\rho \propto a(t)^{-n}$) has been analysed in [50]. The strong energy condition (SEC) is examined and it is shown that the condition for a Big-Bang singularity changes in such a scenario. Recall that in GR, the singularity theorem indicates the inevitable presence of singularities when the SEC (as well as some other conditions) are satisfied. However, at the microscopic scale, in the evolution of the expansion of a geodesic congruence, quantum effects are expected to show up – which, in turn, may completely change classical predictions. One such quantum effect leads to the appearance of higher curvature terms, namely, quadratic gravity [48]. Even though it might seem that the analysis in [50] for quadratic gravity is classical, it is, in a broader sense, a semi-classical analysis. Classical solutions of quadratic gravity have been studied to explore the nature of singularities and it has been shown that the Big-Bang singularity may be avoided [51]. In [50], quadratic gravity, i.e. $R+\beta R^2$ gravity, was considered as a backreaction effect on pure Einstein's gravity. It was found that the SEC for $R+\beta R^2$ gravity is different from that of Einstein gravity. We briefly summarise the results of [50] below.

In GR, the SEC follows from the use of Einstein's equations through the relation

$$R_{ab}v^a v^b = 8\pi G\left[T_{ab} - \frac{1}{2}T g_{ab}\right] v^a v^b \geq 0 \tag{21}$$

for all time-like v^a. This, as mentioned before, leads to focusing.

The author in [50] is primarily interested in the cosmological singularity. To analyse the effects of the quadratic terms one only needs to find the new expression for the Ricci tensor R_{ab} in terms of the energy–momentum tensor T_{ab}. Therefore, we require the modified field equations for quadratic gravity. It is known that this equation is

$$-G_{ab} + 16\pi G\beta\left(\frac{1}{2}R^2 g_{ab} - 2RR_{ab} - 2R_{;n}^{;n} g_{ab} + 2R_{;a;b}\right) = -8\pi G T_{ab}. \tag{22}$$

In a perturbative analysis, we are primarily interested in the first-order contribution from βR^2 to the Raychaudhuri equation. After some algebra one arrives at

$$R_{ab}v^a v^b = \left[\tilde{G}\left(T_{ab} - \frac{1}{2}T g_{ab}\right) + 2\beta\tilde{G}^2 \times\left(\frac{1}{2}T^2 g_{ab}\tilde{G} - 2TT_{ab}\tilde{G} - T_{;n}^{;n} g_{ab} - 2T_{;a;b}\right)\right] v^a v^b + O(\beta^2), \tag{23}$$

where $\tilde{G} = 8\pi G$.

Comparing the above equation with that in Einstein gravity, we find that an effective energy–momentum tensor appears in the RHS of the above equation. Assuming a matter-dominated Universe with a characteristic dependence on the scale factor (i.e., $\rho = \rho_0/a^n$) and a local conservation of T_{ab} leads to $p = [(n-3)/3](\rho_0/a^n)$. Thus, finally, we have

$$\begin{aligned} R_{ab}v^a v^b = &-\frac{n-2}{2}\tilde{G}\rho_n \\ &+\beta\tilde{G}^2\left[3n(n-1)(n-4)\tilde{G}\rho_n^2 + 6n^2(n-4)ka^{-2}\rho_n\right] \\ &+(A^2-1)\left[-\frac{n}{3}\tilde{G}\rho_n + \beta\tilde{G}^2\left[2n^2(n-4)\tilde{G}\rho_n^2 \right.\right. \\ &\left.\left.+4n(n+2)(n-4)ka^{-2}\rho_n\right]\right], \end{aligned} \tag{24}$$

where A is a constant taking values in $(1,\infty)$, $v^a = At^a + \sqrt{A^2-1}x^a$ and $\{t^a, x^a, y^a, z^a\}$ are eigenvectors of T_b^a. The above expression is the modified term which appears in the RHS of the Raychaudhuri equation for the expansion. It is thereafter analysed in the context of different values of n and the possibilities of avoiding the Big–Bang singularity or its inevitable occurrence are pointed out for different cases.

We repeat once again that for any modified theory of gravity (e.g. Brans–Dicke, Einstein–Gauss–Bonnet in higher dimensions, induced gravity, low energy effective stringy gravity etc.) as long as the Riemannian structure of spacetime is respected, changes in conclusions related to geodesic focusing can arise only through the modified field equations and its use in the convergence condition.

3.2 *The Raychaudhuri equation in spacetimes with torsion*

The symmetric nature of the affine connection is one of the underlying assumptions of Riemannian geometry. This fact is also assumed while constructing GR. An antisymmetric connection may originate from the presence of spin matter fields in spacetime leading to a transition from the $\mathbf{V}_4$ to $\mathbf{U}_4$ manifolds [52]. This asymmetric part is known as torsion. Generalisation of the theory of gravity in such a spacetime with torsion was proposed by Einstein–Cartan–Sciama–Kibble (ECSK) [53]. Absence of any experimental signature of torsion however is the primary criticism of such models although there has been a renewed theoretical interest in the context of superstring theories [54] where spacetime torsion appears in the form of a massless string mode. The third rank field strength of the massless second rank antisymmetric tensor field of string theory (known as the Kalb–Ramond field) is identified with spacetime torsion in the low energy limit of the theory. Torsion also appears naturally in a theory of gravity where twistors are used [55] as well as in the supergravity scenario where torsion, curvature and matter fields are treated in an analogous way [56]. It has been shown in several articles [57] that in string inspired models such a background with torsion results in a departure from the experimentally predicted values of the well-known phenomena like gravitational lensing, perihelion precession of planetary orbits, gravitational redshift, rotation of the plane of polarization of the distant galactic radiowaves etc. All the above arguments, and several more, compel us to include torsion in any comprehensive theory of gravity. In the context of geodesic congruences it has further been shown that the kinematical quantities – shear, rotation, acceleration, expansion and their evolution equations are modified by the presence of torsion. This naturally leads to a generalisation of the Raychaudhuri equations in the presence of torsion leading to a more general understanding of the phenomenon of geodesic focusing. Here we give a brief review of the work presented in [58] where the role of torsion in modifying the Raychaudhuri equations and some of its implications have been discussed.

The torsion tensor ${T_{ab}}^c$ which is the antisymmetric part of the affine connection Γ^c_{ab}, is given by

$$ {T_{ab}}^c = \frac{1}{2}\left(\Gamma^c_{ab} - \Gamma^c_{ba}\right) \equiv \Gamma^c_{[ab]}\,, \tag{25} $$

where $a, b, c = 0, \ldots, 3$.

In GR, ${T_{ab}}^c$ is postulated to be zero.

From the torsion tensor one constructs the contortion tensor as

$$ {K_{ab}}^c = -{T_{ab}}^c - {T^c}_{ab} + {T_b}^c{}_a = -{K_a}^c{}_b. \tag{26} $$

This leads to a general expression for the affine connection given as

$$\Gamma^c_{ab} = \{^{\,c}_{ab}\} - K_{ab}{}^c\,, \tag{27}$$

where $\{^{\,c}_{ab}\}$ is the symmetric part of the connection (Christoffel symbols). With this modified connection the commutator of the covariant derivatives of a scalar field ϕ is

$$\tilde{\nabla}_{[a}\tilde{\nabla}_{b]}\phi = -T_{ab}{}^c\tilde{\nabla}_c\phi; \tag{28}$$

which is zero in the absence of torsion.

Similarly for a vector v^a the commutator of the derivative gives

$$\tilde{\nabla}_{[a}\tilde{\nabla}_{b]}v^c = R_{abd}{}^c v^d - 2T_{ab}{}^d\tilde{\nabla}_d v^c, \tag{29}$$

where the Riemann tensor is defined as

$$R_{abc}{}^d = \partial_a\Gamma^d_{bc} - \partial_b\Gamma^d_{ac} + \Gamma^d_{ae}\Gamma^e_{bc} - \Gamma^d_{be}\Gamma^e_{ac}. \tag{30}$$

The contribution of torsion, to the Riemann tensor, is explicitly given through the following expression:

$$R_{abc}{}^d = R_{abc}{}^d(\{\}) - \nabla_a K_{bc}{}^d + \nabla_b K_{ac}{}^d + K_{ae}{}^d K_{bc}{}^e - K_{be}{}^d K_{ac}{}^e, \tag{31}$$

where $R_{abc}{}^d(\{\})$ is the tensor generated by the Christoffel symbols. The symbols $\tilde{\nabla}$ and ∇ have been used to indicate the covariant derivative with and without torsion respectively.

From eq. (31), the expressions for the Ricci tensor and the curvature scalar are

$$R_{ab} = R_{ab}(\{\}) - 2\nabla_a T_c + \nabla_b K_{ac}{}^b + K_{ae}{}^b K_{bc}{}^e - 2T_e K_{ac}{}^e \tag{32}$$

and

$$R = R(\{\}) - 4\nabla_a T^a + K_{ceb}K^{bce} - 4T_a T^a \tag{33}$$

where

$$T_a = T_{ab}{}^b. \tag{34}$$

T_a can be either time-like, space-like or light-like.

3.2.1 *Contributions of torsion to shear, expansion, vorticity and acceleration.* Beginning with the behaviour of fluids and moving on to the initial singularity problem in cosmological models, the Raychaudhuri equations have been shown to be the key equation to explore the role of torsion in such diverse phenomena [59–62]. In [63] an inflationary model of the Universe in the context of ECSK theory was considered. References [64] and [65] addressed the crucial role of Raychaudhuri equation in the context of a gauge invariant formalism for cosmological perturbations in theories with torsion. We now explain the kinematical quantities mentioned before, in a scenario with torsion.

Torsion in a spacetime modifies the definition of kinematical quantities. The covariant derivative of the four-velocity U_a [16] can be decomposed as

$$\tilde{\nabla}_a U_b = \tilde{\sigma}_{ab} + \frac{1}{3} h_{ab} \tilde{\Theta} + \tilde{\omega}_{ab} - U_a \tilde{a}_b, \tag{35}$$

where $h_{ab} = g_{ab} + U_a U_b$ and

$$\tilde{\Theta} = \tilde{\nabla}_a U^a = \Theta - 2T^c U_c, \tag{36}$$

$$\tilde{\sigma}_{ab} = h_a^c h_b^d \tilde{\nabla}_{(c} U_{d)} = \sigma_{ab} + 2h_a^c h_b^d K_{(cd)}{}^e U_e, \tag{37}$$

$$\tilde{\omega}_{ab} = h_a^c h_b^d \tilde{\nabla}_{[c} U_{d]} = \omega_{ab} + 2h_a^c h_b^d K_{[cd]}{}^e U_e, \tag{38}$$

and the acceleration

$$\tilde{a}_c = U^a \tilde{\nabla}_a U_c = a_c + U^a K_{ac}{}^d U_d. \tag{39}$$

The quantities without the tilde are the values of the corresponding expressions in a spacetime without torsion.

3.2.2 *The Raychaudhuri equation*. Using the identity for the four-velocity U_a ($U_a U^a = -1$),

$$U^b \tilde{\nabla}_c \tilde{\nabla}_b U_a = \tilde{\nabla}_c (U^b \tilde{\nabla}_b U_a) - \tilde{\nabla}_c U^b \tilde{\nabla}_b U_a \tag{40}$$

and from eq. (30)

$$U^b \tilde{\nabla}_c \tilde{\nabla}_b U_a = U^b \tilde{\nabla}_b \tilde{\nabla}_c U_a + R_{cba}{}^d U_d U^b - 2U^b T_{ab}{}^c \tilde{\nabla}_d U_c \tag{41}$$

we find the equation

$$\begin{aligned}
&\frac{1}{3} h_{ca} \tilde{\Theta} + \tilde{\sigma}_{ca} + \tilde{\omega}_{ca} - U_c \tilde{a}_a \\
&\quad = \tilde{\nabla}_c \tilde{a}_a - \Bigg(\frac{1}{9} h_{ca} \tilde{\Theta} + \frac{2}{3} \tilde{\Theta} \tilde{\sigma}_{ca} + \frac{2}{3} \tilde{\Theta} \tilde{\omega}_{ca} + 2\tilde{\sigma}_c^b \tilde{\omega}_{ba} \\
&\qquad\qquad + \tilde{\sigma}_c^b \tilde{\sigma}_{ba} + \tilde{\omega}_c^b \tilde{\omega}_{ba} - \frac{1}{3} U_c \tilde{\Theta} \tilde{a}_a - U_c \tilde{a}^b \tilde{\sigma}_{ba} - U_c \tilde{a}^b \tilde{\omega}_{ba} \Bigg) \\
&\qquad\qquad - R_{cba}{}^d U_d U^d - 2U^b T_{ab}{}^c \left(\frac{1}{3} h_{dc} \tilde{\Theta} + \tilde{\sigma}_{dc} + \tilde{\omega}_{dc} - U_d \tilde{a}_c \right).
\end{aligned} \tag{42}$$

Contracting the indices in eq. (42), one obtains a general expression for the Raychaudhuri equation for the expansion, in the presence of torsion as

$$\begin{aligned}
\dot{\tilde{\Theta}} &= \tilde{\nabla}_c \tilde{a}^c - \frac{1}{3} \Theta^2 - \tilde{\sigma}^{ab} \tilde{\sigma}_{ab} + \tilde{\omega}^{ab} \tilde{\omega}_{ab} - R_{ab} U^a U^b \\
&\quad - 2U^b T_{ab}{}^d \left(\frac{1}{3} h_d^a \tilde{\Theta} + \tilde{\sigma}_d^a + \tilde{\omega}_d^a - U_d \tilde{a}^a \right).
\end{aligned} \tag{43}$$

This is the most general form of Raychaudhuri equation for the expansion in the presence of torsion. Simpler versions of this equation have been discussed in

[61,62,66,67]. It is obvious that there would be corresponding equations for shear and rotation too, which we do not mention here. Interestingly, it can be shown that if we have torsion as a phantom field (a scalar field with a negative kinetic energy term) through the dual of the third rank tensor field strength of a string-inspired Kalb–Ramond field, one can get a positive contribution in the RHS of the above equation, which, in turn, may be useful in eliminating singularities.

4. The equations in diverse contexts

4.1 *General relativity and relativistic astrophysics*

We have already discussed one of the main uses of the Raychaudhuri equations – the geodesic focusing theorem. Though it is an entirely geometric result, its use is mostly confined to the domain of GR. It is also quite obvious that the null version of these equations are useful in the study of gravitational lensing. The focal point in this case is nothing but the intersection of trajectories representing light rays and is known as the caustic of the bundle of trajectories. The optical scalars (Sachs scalars) are the quantitites of interest and their evaluation enables us to understand the nature of the null geodesic flow (light ray bundles). Is there anything else one can say apart from focusing, which will be relevant for lensing. Some of these issues have been addressed in [68]. For example, it is possible to rewrite the Raychaudhuri equation for null geodesics in a form involving the angular diameter distance d_A:

$$\frac{d_A''}{d_A} = -\frac{1}{2}R_{00} - \frac{1}{2}\sigma^2, \tag{44}$$

where $(\mathrm{d}/\mathrm{d}\lambda)\ln d_A = \frac{1}{2}\Theta$. Thus, for the case of zero shear one can determine d_A entirely in terms of geometry. In [68] the authors have asked the question 'how do voids affect light propagation?' and made use of the above relation to provide an answer.

A second example within relativistic astrophysics is a study of crack formation [69,70] using this equation. This is an interesting application, quite unique – but not pursued much later. The basic question here is – when will a spherical object develop cracks? Appearance of total radial forces of different signs in different regions in a perturbed configuration leads to the occurrence of such cracking, which, in turn leads to local anisotropy of fluid/emission of incoherent radiation. Using the Einstein and Raychaudhuri equations it is possible to express the net radial force R (after perturbation) in terms of $\mathrm{d}\Theta/\mathrm{d}\lambda$, p_i, ρ, g_{ij}. In this way a criterion for cracking can be obtained.

In 1995, Jacobson [71] in a rather unique paper demonstrated how one might view the Einstein field equation as a thermodynamic equation of state. In this calculation, Jacobson starts out by arguing that energy flux across a causal horizon is some kind of heat flow and the entropy of the system beyond is proportional to the area of the horizon. The heat flux δQ is related to the stress–energy tensor whereas the area variation is related to the expansion of a bundle of null geodesics. He then makes use of the Raychaudhuri equation in its linearised form (ignoring the Θ^2 term) to write down the expansion as an integral over the $R_{ij}k^ik^j$. Thereafter, with the input of the entropy–area relation he arrives at the Einstein equation as an equation of state. It is important to note here that Jacobson uses the geometric content of the Raychaudhuri equation and views it as more fundamental than the Einstein equation, which, in his approach is a derived relation. Extensions of Jacobson's work in the setting of non-equilibrium thermodynamics have appeared recently [72].

Scattered around the literature, are innumerable instances where the equations have been used in the context of GR and astrophysics. Prominent among them is its use in black hole physics – in studying the properties of black holes and in deriving the laws of black hole mechanics (see [18] and references therein). Apart from black holes, we mention, *en passant*, a few other situations which we thought were interesting: (i) use in a fluid-flow description of density irregularities in cosmology [73], (ii) quantum gravitational optics and an effective Raychaudhuri equation [74], (iii) magnetic tension and gravitational collapse [75], (iv) the effective Einstein and Raychaudhuri equations derived from higher-dimensional warped braneworld models [76].

4.2 *The Capovilla–Guven equations for relativistic membranes*

An obvious question, which was never asked till the work of Capovilla and Guven [77] appeared in the scene is: what happens if we consider a congruence of extremal surfaces instead of geodesics (extremal curves)? Are there similar Raychaudhuri equations?

To address this issue, we must first find out how to generalise the notions of expansion, rotation and shear for the case of a family of surfaces. Unlike a curve, a surface is parametrised by more than one parameter. Thus, it is natural to imagine an expansion, a rotation and a shear along each of these independent directions. Introducing a separate label for the surface coordinates, we find that we now have Θ^a, Σ^a_{ij} and Ω^a_{ij} (here $i, j, ...,$ denote the indices representing the normals to the surface). The relation between the i, j indices and the spacetime indices (say μ, ν) is given through the embedding of the surface. Let us now make things more concrete and explicit.

Define a D-dimensional surface in a N-dimensional background through an embedding $x^\mu = x^\mu(\xi^a)$. E^μ_a constitute the tangent vector basis chosen such that $g_{\mu\nu}E^{\mu a}E^{\nu b} = \eta_{ab}$ (a, b run from 1 to D). $n^{\mu i}$ are the normals, with $g_{\mu\nu}n^{\mu i}n^{\nu j} = \delta^{ij}$ (i, j run from 1 to $N - D$). Also $g_{\mu\nu}n^{\mu i}E^{\nu a} = 0$. K^{abi} are the $N - D$ extrinsic curvatures (one along each normal direction). The embedded surface is minimal provided $\mathrm{Tr}(K_i) = 0$.

With the above definitions, one can follow the derivation for the case of geodesic curves and obtain the equation for the generalised expansion (assuming zero values for Σ_{ij} and Ω_{ij}):

$$\nabla^a\Theta_a + \frac{1}{N-D}\Theta_a\Theta^a + \left(M^2\right)^i_i = 0, \tag{45}$$

where

$$\left(M^2\right)^i_i = K^{abi}K_{abi} + R_{\mu\nu\rho\alpha}E^{\mu a}E^\rho_a n^{\nu i}n^\alpha_i. \tag{46}$$

Note that the above equation is a partial differential equation – the reason being that we require more than one parameter to describe a surface. Structurally, the equation is similar to the original Raychaudhuri equation for the expansion – one can easily note this by comparing the two. However, there are crucial differences – one of which involves the appearance of extrinsic curvature terms. It is possible to rewrite this equation in a second order form by choosing $\Theta_a = \nabla_a F$. We obtain

$$\nabla_a \nabla^a F + \left(M^2\right)^i_{\ i} F = 0 \tag{47}$$

which resembles a variable-mass wave equation on a D-dimensional surface with a non-trivial induced line element. For special cases, one can work out focusing criteria, although the notion of focusing will be largely different for surface congruences. For the case of string world-sheets, it is possible to make use of the conformal character of any two-dimensional line element and rewrite the equation as

$$\frac{\partial^2 F}{\partial \sigma^2} - \frac{\partial^2 F}{\partial \tau^2} + \Omega^2(\tau, \sigma) \left(M^2\right)^i_{\ i} F = 0, \tag{48}$$

where τ, σ are the coordinates on the string world sheet and the induced metric is $\mathrm{d}s^2 = \Omega^2(-\mathrm{d}\tau^2 + \mathrm{d}\sigma^2)$.

It can be shown [78] that the criterion for focusing is given as (where ${}^2R$ is the Ricci scalar of the worldsheet):

$$-{}^2R + R_{\mu\nu} E^{\mu a} E^{\nu}_a > 0 \tag{49}$$

which is a requirement for the function F to have zeros. However, there are many questions related to focusing of surfaces which remain un-answered. Some of these have been recently addressed in [79]. Earlier references on examples of solutions of the Raychaudhuri equations for surface congruences and related issues are available in [23,24,80].

It must be mentioned here that the above-generalised equation for the expansion vector field is a subset of a full set of equations which include those for the generalised shear and rotation too. Moreover, the analysis is entirely for extremal membranes of Nambu–Goto type. Here too, if the action changes, we find that the equations change – an example is available in [81].

4.3 *Quantum field theory*

In recent times, the kinematic quantitites (expansion, shear and rotation), as well as the Raychaudhuri equations, have appeared, quite unexpectedly, in the context of quantum field theory. We outline below, some of these scenarios briefly.

4.3.1 *The Langevin–Raychaudhuri equation.* A couple of years ago, Borgman and Ford investigated gravitational effects of quantum stress tensor fluctuations [82,83]. They showed that these fluctuations produce fluctuations in the focusing of a bundle of geodesics. An explicit calculation using the Raychaudhuri equation, treated as a Langevin equation (ignoring the Θ^2 term by a smallness assumption converts the Raychaudhuri equation to the form $\mathrm{d}\Theta/\mathrm{d}\lambda = f(\lambda)$, a Langevin-type equation) was performed to estimate angular blurring and luminosity fluctuations of the images of distant sources. The stress-tensor flucututations were obtained assuming the case of a massless, minimally coupled scalar field in a flat background in a thermal state. Scalar field fluctuations drove the Ricci tensor fluctuations (via the semi-classical Einstein equations), which, in turn led to fluctuations in the expansion Θ. These authors also made some numerical estimates for the quantity $\Delta L/L$ (the fractional luminosity fluctuation) and pointed out possible astrophysical situations (gamma-ray bursts, for example), where an observable value of this quantity might exist. However, it is true that their work has many assumptions (ignoring shear, rotation as well as the Θ^2 term) and much further analysis is required to have a clearer picture of the possibility of actually seeing such effects in the real Universe. Nevertheless, the idea, on the whole, is novel and interesting in its own right and provides us with another application of the Raychaudhuri equation in a very different scenario.

4.3.2 *RG flows in theory (coupling) space.* Till now we have been looking at geodesic flows in spacetime. However, one might also contemplate geodesic flows in fictitious spaces where a metric is defined. Such spaces may correspond to certain physical scenarios. We highlight one example here in order to show how ubiquitous the notions of expansion, rotation, shear and the Raychaudhuri equation are.

For a given Lagrangian field theory, the set of couplings can form a space which is known as theory/coupling space. A curve in such a space will therefore represent a flow of couplings. In such a space, we can define a distance function using two-point functions integrated over physical spacetime. Such a definition of metric goes back to the well-known Cramer–Rao metric in probability theory and has also been highlighted in the context of field theory by Zamolodchikov as well as O'Connor–Stephens (see [84] and references therein).

Once we have a metric, we can use it to define derivatives of a vector field. The vector field of interest here is the so-called β-function vector field, which generates the RG flow. The covariant derivative of this vector field when split into the usual trace, anti-symmetric and symmetric traceless parts define the usual expansion, rotation and shear. However, we also know that the β-function vector field must satisfy the conformal Killing condition (this is the geometric form of the RG equation). These facts together imply the result that the focal point of the congruence generated by the β-function vector field must necessarily be a fixed point (i.e. all β_a ('a' labels the coupling space coordinates) vanish there) [84].

This line of thought is useful in obtaining generic results in field theory – the understanding of shear and rotation in the context of RG flows might provide a better geometric understanding of RG flows.

4.3.3 *Holography, c-function and the Raychaudhuri equation for the expansion.* The holographic principle has recently played a crucial role in our understanding of quantum aspects of gravity. The principle states that the information of gravity degrees of freedom in a D-dimensional volume is encoded in a quantum field theory defined on the $(D-1)$-dimensional boundary of this volume. As an immediate realisation of this, it is shown that the renormalisation group flow equation for the β-function of a four-dimensional quantum gauge field theory defined on the boundary of a five-dimensional volume can be described by geodesic congruences in a scalar-coupled five-dimensional gravitational theory. Such a gauge-gravity duality was proposed in a more general framework through the Maldacena conjecture and can be elegantly described through the renormalisation group equation with the bulk coordinate (or the holographic coordinate) as the renormalisation group parameter. It is shown that if the central charge or the c-function in a quantum field theory evolves monotonically under RG then the holographic principle indicates that in the corresponding dual gravity in five dimensions, the picture is realised through the Raychaudhuri equation governing the monotonic flow of the expansion parameter Θ for the geodesic congruences in the gravity sector [85–87]. The central charge of the boundary theory, or the c-function, is a measure of the degrees of freedom of the theory. As a consequence it is also a measure of the entropy of the black hole in the dual gravity sector which is related to the area of the horizon through the Bekenstein–Hawking formula. The effective central charge is a function of the couplings of the theory, which monotonically decreases as one flows to lower energies through the RG equation. The fixed point described by the boundary conformal field theory corresponds to the extrema of this function. It turns out that the null geodesic congruences can be used as a probe to decode the holographically encoded informations.

Consider a D-dimensional spacetime with a negative cosmological constant where the metric g_{ab} is foliated by appropriate choices of constant time surfaces. If we now focus on a $(D-2)$-dimensional spacetime surface M at a fixed time t, then one may construct a null vector field m^a on the light sheets (consisting of spacetime points scanned by null geodesic congruences) which is orthogonal to M such that $m^a n_a = -1$, where n_a are tangents to the geodesic. The metric on M is given by

$$h_{ab} = g_{ab} + n_{(a} m_{b)}. \tag{50}$$

Defining,

$$B_{ab} = D_b n_a \tag{51}$$

such that $B_{ab} = B_{ba}$ (Frobenius theorem ensures the symmetry property) and

$$\Theta = \mathrm{Tr}\, B \tag{52}$$

it can be shown from the Raychaudhuri equation for the expansion that the geodesics converge since,

$$\Theta \leq 0. \tag{53}$$

The question that arises now is – how is the information on the light sheet encoded in M? Assuming that the spacetime admits a time-like Killing vector field, we may visualise the time flow of M to constitute the $(D-2)+1 = (D-1)$-dimensional boundary of the D-dimensional bulk. The c-function of the corresponding dual gauge theory sits on the boundary. The RG flow to lower energy scales is then associated to the motion along the converging congruences of null geodesics described by Raychaudhuri equation for the expansion. In this sense, the null geodesics act as a probe for the dual theory at a lower energy scale. Pictorially, it may be described as follows. The lower the energy scale, the RG takes us into the gauge theory, deeper inside the bulk we move along null geodesic congruences following Raychaudhuri equation for the expansion. The caustic point where the geodesics meet correspond to the fixed point of the RG flow. This establishes a remarkable significance of the Raychaudhuri equation for the expansion in describing the holographic principle.

As described previously the c-function tells us about the degree of freedom of the system, which, in turn, is related to black hole entropy through the area law. Therefore, the c-function is naturally related to the area function $A(r)$ as

$$c(r) = \frac{A(r)}{4}. \tag{54}$$

In the context of the recently developed attractor mechanism [88] in describing black holes in a scalar coupled gravitational theory, one uses this flow of the c-function to decode some interesting properties of both supersymmetric and non-supersymmetric attractors. The Raychaudhuri equation for the expansion once again plays a crucial role. It is shown that in any spherically symmetric scalar coupled static asymptotically flat solution, $c(r)$ decreases monotonically as one moves radially inward from infinity. It is further shown that the minimum value of $c(r)$ corresponds to the entropy of the horizon [89]. In order to establish the c-theorem in this context once again the Raychaudhuri equation for the expansion is used. Taking a congruence of null geodesics, the expansion is given by

$$\Theta = \frac{\mathrm{d}\ln A}{\mathrm{d}\lambda}, \tag{55}$$

(where λ is the affine parameter). Using the null energy condition,

$$T_{ab}k^a k^b \geq 0 \tag{56}$$

and the Raychaudhuri equation, we get

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\lambda} \leq 0. \tag{57}$$

The c-theorem can thus be proved on very general grounds.

In summary, these rather novel applications of the equations in the context of quantum field theory seems to assert once again the generality and the ubiquitous nature of the equations.

5. Summary and scope

We have reviewed, in this article, the Raychaudhuri equations and its applications in diverse contexts. It must be admitted that since its inception, the equations have used extensively to enhance our understanding of various situations within, as well as outside the scope of GR. The primary reason behind its wide-ranging applicability is (as emphasised several times in this article) the fact that the equations encode geometric statements about flows. Since flows appear in many different contexts in physics, it goes without saying, that the equations will be useful in furthering our understanding in different ways.

Among the applications within classical GR, the utility of the equations in understanding the singularity problem, is surely the most prominent one. In astrophysics, we have quoted the use of the equations in the context of lensing, cracking of self-gravitating objects. At a more fundamental level, Jacobson's work exploits the equation and, in some sense, makes use of its geometric nature to arrive at a new way of looking at the Einstein equations.

On the other hand, in a non-Riemannian spacetime, we have seen how the equations change – in particular, in spacetimes with torsion. Torsion appears in disguise in string theory and is therefore not totally irrelevant today, even though observations might have ruled out the Einstein–Cartan–Sciama–Kibble theory long ago.

Furthermore, what has come as a pleasant surprise in recent times, is the utility of these equations (and the quantities that appear in the equations: expansion, shear and rotation) in the context of quantum field theory. In addition, the generalisation of these equations to surface congruences is also an interesting development, despite the fact that not much has been done on such generalised Raychaudhuri (Capovilla–Guven) equations.

Despite the large body of work on or based on the Raychaudhuri equations, there still remain many unanswered questions. In conclusion, we list a few of them. The choice is surely biased by our own perspective.

(a) Normally, in fluid mechanics, the independent variable is the velocity field. The Raychaudhuri equations are for the gradient of the velocity field – they are one derivative higher. We mentioned before that they are essentially identities and become equations when we choose a geometric property (e.g. vacuum, Einstein space etc.) or use Einstein's field equations of GR. Is it possible to set up and solve the initial value problem for this coupled system of equations such that we know the characteristics of a geodesic flow in a given spacetime geometry once we specify the initial conditions on expansion, shear and rotation? Similar analysis is done while deriving the focusing theorem but can we do it by analysing the full system of equations analytically/numerically. Following this approach one might be able to reconstruct the gradient of the velocity field and also obtain the velocity field itself.

(b) In the context of RG flows, is it possible to understand the role of shear and rotation and use it to improve our geometric analysis of such flows?

(c) For surface congruences, can we obtain equations for families of null surfaces? Or, for surfaces defined by the extremality of actions other than Nambu–Goto (rigidity corrections, Willmore functionals etc.)?

(d) In quantum mechanics, we talk about the flow of probability. Imagining this as a fluid, can we obtain expansion, rotation and shear of such flows and then write down conclusions based on the corresponding Raychaudhuri equations?

(e) Electrodynamics, is, after all, the physics of the electric and magnetic vector fields. Suppose we construct the gradient of the electric/magnetic fields, split it up into expansion, rotation and shear and write down Raychaudhuri equations for the so-called Faraday lines of force! What does such an analysis tell us?

To sum up, we would like to conjecture that wherever there are vector fields describing a physical/geometrical quantity, there must be corresponding Raychaudhuri equations. We believe that there are many avatars of the same equations, of which, till today, we are aware of only a few.

Acknowledgements

The authors thank the editors (N Dadhich, P S Joshi and Probir Roy) of this special volume, for giving them the opportunity to write this review. They also specially thank Probir Roy and Anirvan Dasgupta for their comments and a careful reading of the manuscript. This work was partially supported by funds from a DST, Government of India project (Grant No. SR/S2/HEP-10/2005).

References

[1] A Raychaudhuri, *Phys. Rev.* **89**, 417 (1953)
[2] A Raychaudhuri, *Phys. Rev.* **86**, 90 (1952)
[3] A Raychaudhuri, *Phys. Rev.* **98**, 1123 (1955)
[4] A Raychaudhuri, *Z. Astrophysik* **43**, 161 (1957)
[5] O Heckmann and E Schucking, *Z. Astrophysik* **38**, 95 (1955)
[6] A Komar, *Phys. Rev.* **104**, 544 (1956)
[7] A Raychaudhuri, *Phys. Rev.* **106**, 172 (1957)
[8] J Ehlers, *Gen. Relativ. Gravit.* **25**, 1225 (1993); English translation of original German article by P Jordan, J Ehlers, W Kundt and R K Sachs, *Proceedings of the Mathematical–Natural Sciences Section of the Mainz Academy of Sciences and Literature*, Nr. 11, 792 (1961)
[9] R K Sachs, *Proc. R. Soc. (London)* **A264**, 309 (1961); *ibid* **A270**, 103 (1962)
[10] L Landau and E M Lifshitz, *Classical theory of fields* (Pergamon Press, Oxford, UK, 1975)
[11] E M Lifshitz and I M Khalatnikov, *Adv. Phys.* **12**, 185 (1963)
[12] R Penrose, *Phys. Rev. Lett.* **14**, 57 (1965)
[13] S W Hawking, *Phys. Rev. Lett.* **15**, 689 (1965)
[14] S W Hawking, *Phys. Rev. Lett.* **17**, 444 (1966)
[15] R M Wald, *General relativity* (University of Chicago Press, Chicago, USA, 1984)
[16] S W Hawking and G F R Ellis, *The large scale structure of spacetime* (Cambridge University Press, Cambridge, UK, 1973)
[17] P S Joshi, *Global aspects in gravitation and cosmology* (Oxford University Press, Oxford, UK, 1997)
[18] E Poisson, *A relativist's toolkit: the mathematics of black hole mechanics* (Cambridge University Press, Cambridge, UK, 2004)
[19] G F R Ellis, in *General relativity and cosmology*, International School of Physics, Enrico Fermi–Course XLVII (Academic Press, New York, 1971)
[20] I Ciufolini and J A Wheeler, *Gravitation and inertia* (Princeton University Press, Princeton, USA, 1995)
[21] M Visser, *Lorentzian wormholes from Einstein to Hawking* (AIP Press, USA, 1995)
[22] T Frankel, *Gravitational curvature* (W H Freeman, USA, 1979)
[23] E Zafiris, *J. Geom. Phys.* **28**, 271 (1998)
[24] B Carter, *Contemp. Math.* **203**, 207 (1997)
[25] F J Tipler, *Phys. Rev.* **D17**, 2521 (1978)
[26] F J Tipler, *J. Diff. Equ.* **30**, 165 (1978)
[27] J D Bekenstein, *Phys. Rev.* **D11**, 2072 (1975)
[28] G L Murphy, *Phys. Rev.* **D8**, 4231 (1973)
[29] H Epstein, V Glaser and A Jaffe, *Nuovo Cimento* **36**, 1016 (1965)

[30] R R Caldwell, M Kamionkowski and N N Weinberg, *Phys. Rev. Lett.* **91**, 071301 (2003)
S M Carroll, M Hoffman and M Trodden, *Phys. Rev.* **D68**, 023509 (2003)
P Frampton, *Phys. Lett.* **B555**, 139 (2003)
V Sahni and Y Shtanov, *J. Cosmol. Astropart. Phys.* **0311**, 014 (2003)
[31] L Randall and R Sundrum, *Phys. Rev. Lett.* **83**, 4690 (1999)
C Csaki, *TASI Lectures on Extra Dimensions and Branes*, hep-ph/0404096
[32] N D Birrell and P C W Davies, *Quantum fields in curved space* (Cambridge University Press, Cambridge, UK, 1982)
[33] H B G Casimir, On the attraction between two perfectly conducting plates, *Proc. Kon. Nederl. Akad. Wetenschap* **51**, 793 (1948)
[34] A Borde, *Class. Quant. Grav.* **4**, 343 (1987)
[35] T Roman, *Phys. Rev.* **D37**, 546 (1988)
[36] F J Tipler, *Ann. Phys.* **108**, 1 (1977); *ibid. Phys. Rev. Lett.* **37**, 879 (1976)
[37] C Chicone and P Ehrlich, *Manuscripta Math.* **31**, 297 (1980)
[38] T Roman and P G Bergmann, *Phys. Rev.* **D28**, 1265 (1983)
[39] T Roman, *Phys. Rev.* **D33**, 3526 (1986)
[40] M Visser, *Phys. Rev.* **D47**, 2395 (1993)
[41] A K Raychaudhuri, *Phys. Rev. Lett.* **80**, 654 (1998)
A Saa, *Phys. Rev. Lett.* **81**, 5031 (1998)
J M M Senovilla, *Phys. Rev. Lett.* **81**, 5032 (1998)
A K Raychaudhuri, *Phys. Rev. Lett.* **81**, 5033 (1998)
[42] A K Raychaudhuri, *Mod. Phys. Lett.* **A15**, 391 (2000)
[43] J M M Senovilla, *Phys. Rev. Lett.* **64**, 2219 (1990)
E Ruiz and J M M Senovilla, *Phys. Rev.* **D45**, 1995 (1992)
[44] J M M Senovilla, *Pramana – J. Phys.* **69(1)**, 31 (2007)
[45] A K Raychaudhuri and S Banerji, *Zeit. fur. Astrophysik* **58**, 187 (1964)
[46] S Banerji, *Phys. Rev.* **D9**, 877 (1974)
[47] A K Raychaudhuri, *Prog. Theor. Phys.* **53**, 1360 (1975)
[48] J H Kung, *Phys. Rev.* **D52**, 6922 (1995)
[49] V Muller and H J Schmidt, *Gen. Relativ. Gravit.* **17**, 769 (1985)
D Page, *Phys. Rev.* **D36**, 1607 (1987)
J D Barrow and A Ottewill, *J. Phys.* **A16**, 2757 (1983)
[50] J H Kung, *Phys. Rev.* **D53**, 3017 (1996)
[51] B Whitt, *Phys. Lett.* **B145**, 176 (1984)
J C Alonso, F Barbero, J Julve and A Tiemblo, *Class. Quant. Grav.* **11**, 865 (1994)
M Mijic *et al*, *Phys. Rev.* **D34**, 2934 (1986)
[52] J A Schouten, *Ricci calculus* (Springer-Verlag, Berlin, 1954)
[53] F W Hehl, P von der Heyde, G D Kerlick and J M Nester, *Rev. Mod. Phys.* **48**, 393 (1976)
[54] M B Green, J H Schwarz and E Witten, *Superstring theory* (Cambridge University Press, Cambridge, UK, 1987)
R Hammond, *Il Nuovo Cimento* **B109**, 319 (1994); *Gen. Relativ. Gravit.* **28**, 419 (1986)
V De Sabbata, *Ann. der Physik* **7**, 419 (1991)
V de Sabbata, *Torsion, string tension and quantum gravity*, in 'Erice 1992', *Proceedings, String Quantum Gravity and Physics at the Planck Energy Scale*, p. 528
Y Murase, *Prog. Theor. Phys.* **89**, 1331 (1993)
[55] P S Howe, A Opfermann and G Papadopoulos, *Twistor spaces for QKT manifold*, hep-th/9710072

P S Howe and G Papadopoulos, *Phys. Lett.* **B379**, 80 (1996)
V de Sabbata and C Sivaram, *Il Nuovo Cimento* **A109**, 377 (1996)
[56] G Papadopoulos and P K Townsend, *Nucl. Phys.* **B444**, 245 (1995)
C M Hull, G Papadopoulos and P K Townsend, *Phys. Lett.* **B316**, 291 (1993)
B D B Figueiredo, I Damiao Soares and J Tiomno, *Class. Quant. Grav.* **9**, 1593 (1992)
L C Garcia de Andrade, *Mod. Phys. Lett.* **A12**, 2005 (1997)
O Chandia, *Phys. Rev.* **D55**, 7580 (1997)
P S Letelier, *Class. Quant. Grav.* **12**, 471 (1995); **12**, 2221 (1995)
R W Kühne, *Mod. Phys. Lett.* **A12**, 2473 (1997)
D K Ross, *Int. J. Theor. Phys.* **28**, 1333 (1989)
[57] S Kar, S SenGupta and S Sur, *Phys. Rev.* **D67**, 044005 (2003)
S Kar, P Majumdar, S SenGupta and S Sur, *Class. Quant. Grav.* **19**, 677 (2002)
S SenGupta and S Sur, *Phys. Lett.* **B521**, 350 (2001)
S Kar, P Majumdar, S SenGupta and A Sinha, *Eur. Phys. J.* **C23**, 357 (2002)
P Majumdar and S SenGupta, *Class. Quant. Grav.* **16**, L89 (1999)
S SenGupta and S Sur, *Europhys. Lett.* **65**, 601 (2004)
[58] S Capozziello, G Lambiase and C Stornaiolo, *Ann. der Phys.* **10**, 713 (2001), gr-qc/0101038
[59] R de Ritis, M Lavorgna and C Stornaiolo, *Phys. Lett.* **A95**, 425 (1983)
[60] A Trautman, *Nature (Phys. Sci.)* **242**, 7 (1973)
[61] J Stewart and P Hajicek, *Nature (Phys. Sci.)* **244**, 96 (1973)
[62] G Esposito, *Fortschritte der Physik* **40**, 1 (1992)
[63] M Demianski, R de Ritis, G Platania, P Scudellaro and C Stornaiolo, *Phys. Rev.* **D35**, 1181 (1987)
[64] G F R Ellis and M Bruni, *Phys. Rev.* **D40**, 1804 (1989)
G F R Ellis, J Hwang and M Bruni, *Phys. Rev.* **D40**, 1819 (1989)
G F R Ellis, M Bruni and J Hwang, *Phys. Rev.* **D42**, 1035 (1990)
[65] D Palle, *Nuovo Cimento* **B114**, 853 (1999)
[66] J Tafel, *Phys. Lett.* **A45**, 341 (1973)
[67] A K Raychaudhuri, *Theoretical cosmology* (Clarendon Press, Oxford, UK, 1979)
[68] N Sugiura, K Nakao, D Ida, N Sakai and H Ishihara, *Prog. Theor. Phys.* **103**, 73 (2000)
[69] L Herrera, *Phys. Lett.* **A165**, 206 (1992)
[70] A di Prisco, E Fuenmayor, L Herrera and V Varela, *Phys. Lett.* **A195**, 23 (1994)
[71] T Jacobson, *Phys. Rev. Lett.* **75**, 1260 (1995)
[72] C Eling, R Guedens and T Jacobson, *Phys. Rev. Lett.* **96**, 121301 (2006)
[73] D Lyth and M Mukherjee, *Phys. Rev.* **D38**, 485 (1988)
[74] N Ahmadi and M Nouri-Zonoz, *Quantum gravitational optics: Effective Raychaudhuri equation*, gr-qc/0605009
[75] C Tsagas, *Magnetic tension and gravitational collapse*, gr-qc/0311085
[76] R Maartens, *Phys. Rev.* **D62**, 084023 (2000)
[77] R Capovilla and J Guven, *Phys. Rev.* **D52**, 1072 (1995)
S Kar, *Phys. Rev.* **D52**, 2036 (1995)
S Kar, *Phys. Rev.* **D53**, 2071 (1996)
[78] S Kar, *Phys. Rev.* **D55**, 7921 (1997)
[79] S Kar, Focusing of branes in warped backgrounds, *Ind. J. Phys.* (to appear in the special issue commemorating A K Raychaudhuri) (2006)
[80] S Kar, *Phys. Rev.* **D54**, 6408 (1996)

[81] S Kar, Generalised Raychaudhuri equations for strings and membranes, *Proceedings of IAGRG–XVIII* (Matscience, Madras, India, Feb. 15–17, 1996) Matscience Report no. 117
[82] J Borgman and L H Ford, *Phys. Rev.* **D70**, 064032 (2004)
[83] J Borgman, *Fluctuations of the expansion: The Langevin–Raychaudhuri equation*, Ph.D. Thesis, Tufts University, USA (2004)
[84] S Kar, *Phys. Rev.* **D64**, 105017 (2001)
[85] V Balasubramanian, E Gimon and D Minic, *J. High Energy Phys.* **05**, 014 (2000)
[86] V Sahakian, *Phys. Rev.* **D62**, 126011 (2000)
[87] E Alvarez and C Gomez, *Nucl. Phys.* **B541**, 441 (1999); *ibid. Holography and the c-theorem*, hep-th/0009203
[88] M Cvetic and A A Tseytlin, *Phys. Rev.* **D53**, 5619 (1996)
A Stromonger, *Phys. Lett.* **B383** (1996)
S Ferrara and R Kallosh, *Phys. Rev.* **D54**, 1525 (1996)
S Ferrara, G W Gibbons and R Kallosh, *Nucl. Phys.* **B500**, 75 (1997)
[89] K Goldstein, R P Jena, G Mandal and S P Trivedi, *J. High Energy Phys.* **02**, 053 (2006)