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A singularity theorem based on spatial averages*

J M M SENOVILLA†
Física Teórica, Universidad del País Vasco, Apartado 644, 48080 Bilbao, Spain

Abstract. Inspired by Raychaudhuri's work, and using the equation named after him as a basic ingredient, a new singularity theorem is proved. Open non-rotating Universes, expanding everywhere with a non-vanishing spatial average of the matter variables, show severe geodesic incompleteness in the past. Another way of stating the result is that, under the same conditions, any singularity-free model must have a vanishing spatial average of the energy density (and other physical variables). This is very satisfactory and provides a clear decisive difference between singular and non-singular cosmologies.

Keywords. Raychaudhuri equation; singularity-free cosmologies; singularity theorems.

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1. Introduction

In this paper I would like to present a result which confirms – at least partially – a 10-year-old conjecture: reasonable non-rotating cosmological models can be non-singular only if they are open and have vanishing spatial averages of the matter and other physical variables. This conjecture arose as a result of the interactions and discussions between Prof. A K Raychaudhuri (AKR from now on) and myself concerning the question of feasibility of singularity-free cosmological solutions

*The first instance of a non-singular cosmological solution, compatible with the energy and causality conditions and the Raychaudhuri equation, was given by Senovilla. This generated an intense interest in AKR on the nature and behaviour of such solutions. His initial proposal of the vanishing of the space-time average of the energy density (and other physical and kinematical quantities) as a condition for their existence got refined through many discussions with Dadhich and Senovilla. It evolved finally to the realization that the vanishing of the spatial average of the energy density is what distinguishes all these non-singular solutions, as stated in this theorem proved by Senovilla. In our opinion, this theorem, to which both Raychaudhuri and Senovilla have contributed, provides a succinct and complete characterization of the nature of all non-singular cosmological models – Editors.
†In memory of Amal Kumar Raychaudhuri (1924–2005).

(see [1,2]). This is a subject to which, as is well-known, AKR made fundamental pioneering contributions [3,4], and I influenced in a much more modest and lateral manner [5–7].

The next section contains a more or less detailed historical review of the antecedents, birth, and hazardous life of the conjecture. However, any reader, not interested in this historical appraisal, who only wishes to see and learn about the results and the new theorem – which are interesting on their own – may go directly to §§3 and 4. The former contains a brief derivation of the celebrated Raychaudhuri equation and some discussion on the definition of spatial averages; the latter presents the new theorem and its implications. A brief discussion and an Appendix have been placed at the end of the paper.

2. History of a conjecture

This is divided into three parts: the appearance of the first non-singular solution and subsequent developments are summarized in §2.1; the importance of the fresh new ideas brought in by AKR, and the formulation of the conjecture, are analyzed in §2.2; finally, the reaction by AKR about the conjecture, the sometimes chaotic discussions in publications-comments-replies, and some recent developments on the subject are reviewed in §2.3.

2.1 *Anadi Vishva*

In 1990, I published [5] which is considered to be the first 'interesting' singularity-free cosmological model. It is a spatially inhomogeneous solution of Einstein's field equations for a perfect fluid with a realistic equation of state $p = \varrho/3$. This solution came as a big surprise: it was widely believed that a model such as the one in [5] should be singular due to the powerful singularity theorems developed by Penrose, Hawking and others [8–10]. Therefore, the solution caused some impact (see e.g. [11]), and some discussions within the relativity community [12,13].

Shortly afterwards, the geodesic completeness of this solution was explicitly proven in [14], along with a list of its main properties. The solution turns out to be *cylindrically symmetric*, globally hyperbolic and everywhere expanding for half the history of the model. It satisfies the stronger energy requirements (the dominant and strict strong energy conditions) and is singularity-free (see the Appendix for a short summary). A detailed analysis of how the model fits in with the general conclusions of the singularity theorems was also performed in [14], and the solution was proven to be in full accordance with the theorems: in all versions of the theorems at least one of their hypotheses was not satisfied, usually the so-called boundary or initial condition (see [7]).

Almost simultaneously, another paper [15] extended the particular model in [5] to a general family of perfect-fluid solutions with an Abelian G_2-symmetry acting on spatial surfaces. This family was obtained under the assumption of separation of variables. It contains a diversity of models, with different properties, in particular a 2-parameter subfamily of geodesically complete, singularity-free, solutions – which include the original solution in [5] – as well as other relevant solutions such as the one in [16] (see also [17]). There were also solutions with time-like singularities, and solutions with no singularities in the matter variables but with time-like singularities in the Weyl conformal part of the curvature [15]. All such behaviour was somehow unexpected. As a matter of fact, one could prove that, within the subfamily of solutions in [15] with a barotropic equation of state and no initial singularity, the singularity-free solutions lie precisely at the boundary separating the solutions with only time-like Weyl singularities from the solutions with time-like singularities in both Weyl and matter variables. This was not very encouraging, since one could suspect the instability and the zero measure of the non-singular models.

Several generalizations of these solutions were later found by allowing the fluid to have heat flow [18] and scalar fields [19]. It was claimed in [19,20] that a particular 'inhomogeneization' procedure (by requiring the separation of variables) of the Robertson–Walker open cosmological models would lead to the non-singular models of [5,15]. This was later pursued in a review published by Dadhich in [21], where he intended to prove the uniqueness of those non-singular models.

Up to this point, all known non-singular solutions were (i) *diagonal*, that is to say, with the existence of a global coordinate chart adapted to the perfect fluid (so-called 'co-moving') such that the metric takes a diagonal form, (ii) *cylindrically symmetric*, and (iii) *separable* in comoving coordinates. The last feature meant that the metric components could be written as the product of a function of the separable time coordinate times a function of the separable radial coordinate. The first example of a non-diagonal non-singular perfect fluid model was presented in [22], though the solution had been previously published in quite a different context in [23]. This was a solution for a cylindrically symmetric stiff fluid (the equation of state is $p = \varrho$) or equivalently, for a massless scalar field. It was also separable but could be almost immediately generalized to a family of non-separable, non-diagonal, non-singular stiff fluid solutions in [24]. Other non-singular solutions followed (see e.g. [7,25,26]).

It was then shown in [6] that the general family in [15] as well as the whole class of Robertson–Walker cosmologies belong to a single unified wider class of cylindrically symmetric [26a], separable, diagonal (non-necessarily perfect) fluid solutions. Those depended on one arbitrary function of time – essentially the scale factor – and four free parameters selecting the openness or closeness of the models, the anisotropy of the fluid pressures, or the anisotropy and spatial inhomogeneity of the models. The physical properties of this general class were analyzed in detail [6,7], in particular the deceleration parameter, leading to natural inflationary models (without violating the strong energy condition), and the generalized Hubble law. The possibility of constructing realistic cosmological models by 'adiabatically' changing the parameters in order to start with a singularity-free model which at later times becomes a Robertson–Walker model was also considered in [6] and in §7.7 in [7].

Some interesting lines of research appeared in print in 1997–8. First, by keeping the cylindrical symmetry, a new diagonal but non-separable family of stiff fluid singularity-free solutions was presented in [27]. The family contained the same static limit as the solution in [22], thereby suggesting that they both form part of a more general class, perhaps of non-zero measure, of non-singular cylindrically symmetric stiff fluid models. Second, the role of shear in expanding perfect fluid models was analyzed in [28] proving that non-singular models with an Abelian spatial G_2-symmetry should be spatially inhomogeneous. This result was much improved and proven in a more general context in [29], showing in particular that the symmetry assumption was superfluous. And third, by giving up cylindrical symmetry, a family of non-singular (non-perfect) fluid solutions with spherical symmetry was presented in [30]. These models depend on one arbitrary function of time. Once again, they can avoid the singularity theorems due to the failure of the boundary/initial condition: there are no closed trapped surfaces. It was also shown in §7.8 of [7] that these models cannot represent a finite star, since this would require a place where the radial pressure vanished, which is impossible for appropriate selections of the arbitrary function of time. This is a property shared by all models mentioned so far in this subsection.

2.2 *Raychaudhuri comes into play: The conjecture*

In December 1995, I attended the International Conference on Gravitation and Cosmology (ICGC-95), held in Pune (India), where I had the chance to meet Prof. Raychaudhuri for the first time. I was impressed by his personality and accessibility, specially for a man of his age and reputation. But more importantly, I was deeply influenced by his remarks in a brief conversations that we – AKR, Naresh Dadhich and myself – had at that time. If I remember well, AKR mentioned averages in these informal conversations, but just by the way. This came at a critical time: in a short talk at the workshop on 'Classical General Relativity' (see [31]). I presented the above-mentioned combined Robertson–Walker plus non-singular general family [6] and its properties. I thought that the paper, which had already been accepted, would open the door for realistic models.

Even though AKR meant spacetime averages, I immediately realized the relevance of his idea, especially to discriminate between singular and non-singular cosmological models, but using purely spatial averages. As remarked at the end of the previous subsection, all known non-singular models were 'cosmological' in the sense that they could not describe a finite star surrounded by a surface of vanishing pressure. However, it can certainly happen that (say) the energy density falls off too quickly at large distances (this certainly occurred in all known singularity-free solutions). Thus one may raise the issue whether or not this will better describe the actual Universe or rather a weakly-localized object such as a very large galaxy. Of course, a good way to distinguish between these two possibilities is to use the spatial average of the energy density. Thus, I was inspired by AKR's remarks and believed that this was the right answer to the existence of non-singular models such as the one in [5]. I incorporated this view to the review [7] (see p. 821).

I met AKR for the second time in Pune again, on the occasion of the 15th International Conference on General Relativity and Gravitation (GR15), held in December 1997. Either at GR15 or in an informal seminar (I cannot exactly recall), I attended a talk where he discussed some of the non-singular models and made some comments about the importance of the averages of the physical quantities such as the energy density, the pressure, or the expansion of the fluid. In 1998, AKR proved [1] that, under some reasonable assumptions, open non-rotating non-singular models must have vanishing spacetime averages of the matter and kinematical variables. Later, it was shown [2,32] explicitly that, in the open Robertson–Walker models (with an initial singularity), the same spacetime averages vanish too. Actually, this holds true for most open spatially homogeneous models as well. Since this property is shared by all models, it cannot be used to decide between singularity-free and singular spacetimes.

In my comment, I stressed the following fact. I came to understand, after listening to AKR, that pure spatial averages (at a given instant of time) vanish in the known non-singular solutions, while they are non-vanishing in open Robertson–Walker models. This, together with a well-known singularity theorem for expanding globally hyperbolic models (Theorem 1 below), enabled the formulation of the following conjecture [2]:

> In every singularity-free, non-rotating, expanding, globally hyperbolic model satisfying the strong energy condition, spatial averages of the matter variables vanish.

This will be made precise and proven in §4 below.

2.3 *History of the conjecture*

My comment [2] was only submitted after electronic correspondence with AKR [32a]. It seems that he was initially skeptical about the use of purely spatial – and not spacetime – averages, for he also replied in [33] to our comments. Nevertheless, in an e-mail dated 9 September 1998, he mentioned that a letter proving the vanishing of spatial averages 'following his earlier method' had already been submitted for publication. This private announcement was followed, shortly thereafter, by (i) another paper by Dadhich and AKR [34] where they proved the existence of oscillatory non-singular models within the non-perfect fluid spherically symmetric family of [30] mentioned above, (ii) general theorems providing sufficient conditions for the geodesic completeness of general cylindrically symmetric spacetimes [35,36] and (iii) some work [37] showing the relevance that the singularity-free solutions might have in the fashionable String Cosmology (see also [38] and references therein).

The letter that AKR mentioned was published in [39]. However, in it global hyperbolicity was assumed without being mentioned (due to the assumption of the existence of global coordinates associated to a hypersurface-orthogonal time-like eigenvector field of the Ricci tensor). Moreover, the openness of some local coordinates was taken for granted. Further, the statement that the spatial average of the divergence of the acceleration associated to the time-like eigenvector field vanishes was not clearly proved. More importantly, the blow-up of the kinematical quantities of this eigenvector field was incorrectly related to the blow up of some Ricci scalar invariants [39a]. All in all, the result in [39], involving spatial averages, was not completely proved.

This led to some interesting works by others [40,41], where the existence of a wide class of singularity-free (geodesically complete) cylindrically symmetric stiff fluid cosmologies was explicitly demonstrated, and many solutions were actually exhibited. In these papers, the family of regular cylindrically symmetric stiff fluids was proven to be very abundant, allowing for arbitrary functions. Furthermore, strong support for the conjecture was also provided in the second of these papers [41]: the vanishing of the energy density (and pressure) of the fluid at spatial infinity on every Cauchy hypersurface was demonstrated to be a necessary requirement if the spacetime was to be geodesically complete. This was quite encouraging, and constituted the first serious advance towards the proof of the conjecture.

AKR may not have been aware of these important developments and results. He put out a preprint [42] to which Fernández-Jambrina [43] found a counter-example. A revised version was published in [44] which was also not fully free of the shortcomings, as neatly pointed out in [45]. AKR acknowledged these deficiencies in [46], yet he believed that some of his results were still valid.

Reference [46] was AKR's last published paper. In my opinion, after having identified the clue to non-singular models (i.e. averages), which led us all to the right track, he tried to prove more ambitious and challenging results which were perhaps beyond the techniques he was using. This, of course, does not in any way diminishes his fundamental contribution to the field of singularities in Cosmology, a subject in which, probably, the most important ideas came from his insight and deep intuition. It is in this sense that the spirit of the theorem to be proven in §4 should be credited to him. I can only hope that he would have welcomed the new results.

3. Spatial averages and the Raychaudhuri equation

3.1 *The Raychaudhuri equation*

As is known, the first result predicting singularities under reasonable physical conditions was published in 1955 – exactly the year of Einstein's demise – by AKR [3]. In this remarkable paper, he presented what is considered to be the first singularity theorem, and included a version (the full equation appeared soon after in [47], see also [4]) of the equation named after him which is the basis of later developments and of all the singularity theorems [7–10]. The Raychaudhuri equation can be easily derived from the general Ricci identity:

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) u^\alpha = R^\alpha_{\rho\mu\nu} u^\rho .$$

Contracting α with μ here, then with u^ν, one gets

$$u^\nu \nabla_\mu \nabla_\nu u^\mu - u^\nu \nabla_\nu \nabla_\mu u^\mu = R_{\rho\nu} u^\rho u^\nu,$$

where $R_{\mu\nu}$ is the Ricci tensor. Reorganizing by parts the first summand on the left-hand side, one derives

$$u^\nu \nabla_\nu \nabla_\mu u^\mu + \nabla_\mu u_\nu \nabla^\nu u^\mu - \nabla_\mu (u^\nu \nabla_\nu u^\mu) + R_{\rho\nu} u^\rho u^\nu = 0 \tag{1}$$

which is the Raychaudhuri equation. AKR's important contribution was to understand and explicitly show the fundamental physical implications of this simple geometrical relation.

Let us analyze some of these implications. Observe that in the case that u^μ defines a (affinely parametrized) geodesic vector field, then $u^\nu \nabla_\nu u^\mu = 0$ and the third term vanishes. The second term can then be rewritten by splitting

$$\nabla_\mu u_\nu = S_{\mu\nu} + A_{\mu\nu}$$

into its symmetric $S_{\mu\nu}$ and antisymmetric $A_{\mu\nu}$ parts, so that

$$\nabla_\mu u_\nu \nabla^\nu u^\mu = S_{\mu\nu} S^{\mu\nu} - A_{\mu\nu} A^{\mu\nu} \, .$$

Now the point is to realize two things. (i) If u^μ is time-like (and normalized) or null, then both $S_{\mu\nu} S^{\mu\nu}$ and $A_{\mu\nu} A^{\mu\nu}$ are non-negative, (ii) u_μ is also proportional to a gradient (therefore defining orthogonal hypersurfaces) if and only if $A_{\mu\nu} = 0$. In summary, for hypersurface–orthogonal geodesic time-like or null vector fields u^μ, one has

$$u^\nu \nabla_\nu \nabla_\mu u^\mu = -S_{\mu\nu} S^{\mu\nu} - R_{\rho\nu} u^\rho u^\nu ,$$

so that the sign of the derivative of the divergence or expansion $\theta \equiv \nabla_\mu u^\mu$ along the geodesic congruence is governed by the sign of $R_{\rho\nu} u^\rho u^\nu$. If the latter is non-negative, then the former is non-positive. In particular, if the expansion is negative at some point and $R_{\rho\nu} u^\rho u^\nu \geq 0$ then one can prove, by introducing a scale factor L such that $u^\mu \nabla_\mu (\log L) \propto \theta$ and noting that $S^\mu_\mu = \theta$, that necessarily the divergence will reach an infinite negative value in finite affine parameter (unless all the quantities are zero everywhere).

If there are physical particles moving along these geodesics, then clearly a physical singularity is obtained, since the mean volume decreases and the density of particles will be unbounded (see Theorem 5.1 in [7], p. 787). This was the situation treated by AKR for the case of irrotational dust. In general, no singularity is predicted, though, and one only gets a typical caustic along the flow lines of the congruence defined by u^μ. This generic property is usually called the focusing effect on causal geodesics. For this to take place, of course, one needs the condition

$$R_{\rho\nu} u^\rho u^\nu \geq 0 \tag{2}$$

which is a geometric condition and independent of the particular theory. However, in General Relativity, one can relate the Ricci tensor to the energy–momentum tensor $T_{\mu\nu}$ via Einstein's field equations ($8\pi G = c = 1$)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = T_{\mu\nu} \, . \tag{3}$$

Here R is the scalar curvature and Λ the cosmological constant. Therefore, condition (2) can be rewritten in terms of physical quantities. This is why sometimes (2), when valid for all time-like u^μ, is called the time-like convergence condition, also the strong energy condition in the case where $\Lambda = 0$ [10]. One should bear in mind, however, that this is a condition on the Ricci tensor (a geometrical object) and therefore will not always hold (see the discussion in §6.2 in [7]).

The focusing effect on causal geodesics predicted by the Raychaudhuri equation was the fundamental ingredient needed to derive the powerful singularity theorems. Nevertheless, as remarked above, this focusing does not lead to singularities on its own in general. As a trivial example, observe that flat spacetime satisfies condition (2) trivially, but there are no singularities: the focusing effect simply leads to focal points or caustics of the geodesic congruences. This is why it took some time to understand the necessity of combining the focusing effect with the theory of existence of geodesics maximizing the interval (which necessarily cannot have focal points or caustics) in order to prove results on geodesic incompleteness, which is a sufficient condition in the accepted definition of a singularity. With the imaginative and fruitful ideas put forward by Penrose in the 1960s, later followed by Hawking, this led to the celebrated singularity theorems (see [7–10]).

As a simple but powerful example, which we shall later need to prove the new theorem, let us present the following standard singularity theorem (Theorem 9.5.1 in [48], Theorem 5.2 in [7]).

Theorem 1. *If there is a Cauchy hypersurface* Σ *such that the time-like geodesic congruence emanating orthogonal to* Σ *has an initial expansion* $\theta|_\Sigma \geq b > 0$ *and condition* (2) *holds along the congruence, then all time-like geodesics are past incomplete.*

The idea of the proof is simple [7,10,48]: since Σ is a Cauchy hypersurface, the spacetime is globally hyperbolic so that one knows that there is a maximal time-like curve from Σ to any point. From standard results any such maximal curve must be a time-like geodesic orthogonal to Σ without any point focal to Σ between Σ and the point. But the Raychaudhuri equation (1) implies that these focal points should exist in the past at a proper time less than or equal to a fixed value (given by $3/\theta|_\Sigma \leq 3/b$). As every causal curve crosses the Cauchy hypersurface Σ, no time-like geodesic can have length greater than $3/b$ back to the past.

3.2 *Spatial averages*

Let Σ be any space-like hypersurface in the spacetime and let $\boldsymbol{\eta}_\Sigma$ be the canonical volume element 3-form on Σ. The average $\langle f \rangle_S$ of any scalar f on a finite portion S of Σ is defined as

$$\langle f \rangle_S \equiv \frac{\displaystyle\int_S f\boldsymbol{\eta}_\Sigma}{\displaystyle\int_S \boldsymbol{\eta}_\Sigma} = [\mathrm{Vol}(S)]^{-1} \int_S f\boldsymbol{\eta}_\Sigma,$$

where $\mathrm{Vol}(S)$ is the volume of $S \subseteq \Sigma$. The spatial average on the whole Σ is defined as (the limit of) the previous expression when S approaches the entire Σ

$$\langle f \rangle_\Sigma \equiv \lim_{S\to\Sigma} \frac{\displaystyle\int_S f\boldsymbol{\eta}_\Sigma}{\displaystyle\int_S \boldsymbol{\eta}_\Sigma} \equiv [\mathrm{Vol}(\Sigma)]^{-1} \int_\Sigma f\boldsymbol{\eta}_\Sigma. \tag{5}$$

Obvious properties of these averages are

1. (linearity) For any $S \subseteq \Sigma$, any functions f, g and any constants a, b:

$$\langle af + bg \rangle_S = a \langle f \rangle_S + b \langle g \rangle_S .$$

2. For any $S \subseteq \Sigma$, $\langle f \rangle_S \leq \langle |f| \rangle_S$.
3. If $S \subseteq \Sigma$ such that its closure is compact (so that its volume is finite, $\mathrm{Vol}(S) < \infty$), then for any $f \geq b \geq 0$ on S, $\langle f \rangle_S \geq b$ and the equality holds only if f is constant, $f = b$, almost everywhere on S. In particular, if $f \geq 0$ on such an S, then $\langle f \rangle_S \geq 0$ and the equality holds only if f vanishes almost everywhere on S.
4. If $S \subseteq \Sigma$ does not have a finite volume (so that it cannot be of compact closure), then for any $f \geq 0$ on S, $\langle f \rangle_S \geq 0$ and the equality requires necessarily that $f \to 0$ when 'approaching the boundary' (i.e., when going to infinity). Conversely, if $f > 0$, f is bounded on S and f is bounded from below by a positive constant at most along a set of directions of zero measure, then $\langle f \rangle_S = 0$.
5. If $|f|$ is bounded on S ($|f| \leq M$), then $\left\langle f^2 \right\rangle_S \leq M \langle |f| \rangle_S$. In particular, $\langle |f| \rangle_S = 0$ implies that $\left\langle f^2 \right\rangle_S = 0$.
6. Similar results hold, of course, for negative and non-positive functions (just use $-f$).

4. The theorem

Let us consider any space-like hypersurface Σ in the spacetime, and let u^μ be its unit normal vector field (ergo time-like). The projector $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ defines the canonical first fundamental form of Σ. A classical result relates the intrinsic Riemannian structure of $(\Sigma, h_{\mu\nu})$ with the Lorentzian one of the spacetime, and in particular their respective curvatures. The main result is the Gauss equation, which reads (e.g. [10,48])

$$R_{\alpha\beta\gamma\delta} h^\alpha_\lambda h^\beta_\mu h^\gamma_\nu h^\delta_\tau = \bar{R}_{\lambda\mu\nu\tau} + K_{\lambda\nu} K_{\mu\tau} - K_{\lambda\tau} K_{\mu\nu},$$

where $K_{\mu\nu} = K_{\nu\mu} \equiv h^\beta_\mu h^\gamma_\nu \nabla_\beta u_\gamma$ is the second fundamental form of Σ, and $\bar{R}_{\lambda\mu\nu\tau}$ is the intrinsic curvature tensor of $(\Sigma, h_{\mu\nu})$. Observe that $u^\mu K_{\mu\nu} = 0$ and $u^\mu \bar{R}_{\lambda\mu\nu\tau} = 0$. Contracting all indices here one derives the standard result (e.g. §10.2 in [48])

$$K^2 = K_{\mu\nu} K^{\mu\nu} + 2 R_{\mu\nu} u^\mu u^\nu + R - \bar{R},$$

where $\bar{R}$ is the scalar curvature of Σ and $K \equiv K^\mu_\mu$. Using the Einstein's field equations (3) this can be rewritten in terms of the energy–momentum tensor as

$$K^2 = K_{\mu\nu} K^{\mu\nu} + 2 T_{\mu\nu} u^\mu u^\nu + 2\Lambda - \bar{R}. \tag{5}$$

Recall that $T_{\mu\nu} u^\mu u^\nu$ is the energy density of the matter content relative to the observer u^μ, and thus it is always non-negative. Note, also, that for any extension of u^μ outside Σ as a hypersurface-orthogonal unit time-like vector field (still called

u^μ), their previously defined kinematical quantities $\theta = \nabla_\mu u^\mu$ and $S_{\mu\nu} = \nabla_{(\mu} u_{\nu)}$ are simply

$$\theta|_\Sigma = K, \quad \left(S_{\mu\nu} + a_{(\mu} u_{\nu)}\right)\big|_\Sigma = K_{\mu\nu},$$

where $a^\mu = u^\nu \nabla_\nu u^\mu$ is the acceleration vector field of the extended u^μ.

Combining this with Theorem 1 one immediately deduces a very strong result concerning the average of the energy density:

Proposition 1. *Assume that* (1) *there is a non-compact Cauchy hypersurface* Σ *such that the time-like geodesic congruence emanating orthogonal to* Σ *is expanding and condition* (2) *holds along the congruence,* (2) *the spatial scalar curvature is non-positive on average on* Σ*:* $\left\langle \bar{R} \right\rangle_\Sigma \leq 0$*,* (3) *the cosmological constant is non-negative* $\Lambda \geq 0$ *and* (4) *the spacetime is past time-like geodesically complete. Then,*

$$\left\langle T_{\mu\nu} u^\mu u^\nu \right\rangle_\Sigma = \left\langle K_{\mu\nu} K^{\mu\nu} \right\rangle_\Sigma = \left\langle \bar{R} \right\rangle_\Sigma = \Lambda = 0\,. \tag{6}$$

Remarks

- The first assumption requires spacetime to be globally hyperbolic (so that it is causally well-behaved). Also, the time-like convergence condition has to hold along the geodesic congruence orthogonal to one of the Cauchy hypersurfaces. Furthermore, the Universe is assumed to be non-closed (non-compactness assumption) and expanding everywhere at a given instant of time as described by the hypersurface Σ. All this is standard (Theorem 1).
- The second assumption demands that the space of the Universe (at the expanding instant) be non-positively curved on average. Observe that this still allows for an everywhere positively curved Σ (see the example in the Appendix). This is in accordance with our present knowledge of the Universe and with all indirect observations and measures, e.g. the recent data [49] from WMAP.
- The third assumption is also in accordance with all theoretical and observational data [49]. Observe that the traditional case with $\Lambda = 0$ is included. Notice, however, that the time-like convergence condition (2) is also assumed. This may impose very strict restrictions on the matter variables if $\Lambda > 0$.
- The second and third assumptions could be replaced by milder ones such as $\left\langle \bar{R} \right\rangle_\Sigma \leq 2\Lambda$, allowing for all signs in both Λ and $\left\langle \bar{R} \right\rangle_\Sigma$. The conclusion concerning the vanishing of the averaged energy density would be unaltered, as well as the next one in (6), but the last two equalities in (6) should be replaced by $\left\langle \bar{R} \right\rangle_\Sigma = 2\Lambda$.
- Probably, the fourth condition may be relaxed substantially, as one just needs that the set of past-incomplete geodesics be not too big. There are some technical difficulties, however, to find a precise formulation of the mildest acceptable condition.
- The conclusion forces the second and third assumptions to hold in the extreme cases and, much more importantly, it implies that the energy density of the matter on Σ has a vanishing spatial average. This conclusion was the main goal in this paper.

Proof. From Theorem 1 and the fourth hypothesis it follows that $0 < \theta|_\Sigma = K$ cannot be bounded from below by a positive constant. Furthermore, the existence of a complete maximal time-like curve (which must be a geodesic without focal points) from the Cauchy hypersurface Σ to any point to the past implies that, actually, K can be bounded from below away from zero only along a set of directions of zero measure. Point 4 of the list of properties for averages implies that $\langle\theta\rangle_\Sigma = \langle K\rangle_\Sigma = 0$, and point 5 in the same list provides then $\langle K^2\rangle_\Sigma = 0$. Taking averages on formula (5) and using point 1 in that list one arrives at

$$\langle K_{\mu\nu}K^{\mu\nu}\rangle_\Sigma + 2\langle T_{\mu\nu}u^\mu u^\nu\rangle_\Sigma + 2\Lambda - \langle\bar{R}\rangle_\Sigma = 0.$$

Then, given that all the summands here are non-negative, the result follows.

This result can be made much stronger by using, once again, the Raychaudhuri equation. To that end, we need a lemma first.

Lemma 1. *If the energy–momentum tensor satisfies the dominant energy condition and $\langle T_{\mu\nu}u^\mu u^\nu\rangle_\Sigma = 0$ for some unit time-like vector field u^μ, then all the components of $T_{\mu\nu}$ in any orthonormal basis $\{e^\mu_\alpha\}$ have vanishing average on Σ*

$$\langle T_{\mu\nu}e^\mu_\alpha e^\nu_\beta\rangle_\Sigma = 0, \quad \forall\alpha,\beta = 0,1,2,3. \tag{7}$$

Proof. The dominant energy condition implies that [10,50], in any orthonormal basis $\{e^\mu_\alpha\}$ (where e^μ_0 is the time-like leg)

$$T_{\mu\nu}e^\mu_0 e^\nu_0 \geq \left|T_{\mu\nu}e^\mu_\alpha e^\nu_\beta\right|, \quad \forall\alpha,\beta = 0,1,2,3,$$

so that, by taking any orthonormal basis with $u^\mu = e^\mu_0$, points 2 and 6 in the list of properties of spatial averages lead to eq. (7) in those bases. As any other orthonormal basis is obtained from the selected one by means of a Lorentz transformation – so that the components of $T_{\mu\nu}$ in the new basis are linear combinations, with bounded coefficients, of the original ones – the result follows.

The combination of this lemma with Proposition 1 leads to the following result.

Proposition 2. *Under the same assumptions as in Proposition 1, if $T_{\mu\nu}$ satisfies the dominant energy condition then not only the averages shown in eqs* (6) *and* (7) *vanish, but furthermore*

$$\langle u^\mu\nabla_\mu\theta - \nabla_\mu a^\mu\rangle_\Sigma = \langle v^\mu\nabla_\mu\vartheta\rangle_\Sigma = \langle R_{\mu\nu}e^\mu_\alpha e^\nu_\beta\rangle_\Sigma = 0, \tag{8}$$

where v^μ is the unit time-like geodesic vector field orthogonal to Σ, $\vartheta = \nabla_\mu v^\mu$ its expansion, u^μ is any hypersurface-orthogonal unit time-like vector field orthogonal to Σ, and $a^\mu = u^\nu\nabla_\nu u^\mu$ its acceleration.

Remarks

- The hypersurface-orthogonal vector field u^μ may represent the mean motion of the matter content of the Universe. Observe that, thereby, an acceleration of the cosmological fluid is permitted. This is important, since such an acceleration is related to the existence of pressure gradients, and these forces oppose gravitational attraction.

- The expression $u^\mu\nabla_\mu\theta$ can be seen as a 'time derivative' of the expansion θ: a derivative on the transversal direction to Σ. In particular, $v^\mu\nabla_\mu\vartheta$ is the time derivative, with respect to proper time, of the expansion for the geodesic congruence orthogonal to Σ. This proper time derivative of ϑ does have a vanishing average on Σ. Notice, however, that the generic time derivative of the expansion θ does not have a vanishing average in general: this is governed by the average of the divergence of the acceleration.
- Evidently, $a^\mu u_\mu = 0$ so that a^μ is space-like and tangent to Σ on Σ. Furthermore, one can write

$$\nabla_\mu a^\mu = (h^{\mu\nu} - u^\mu u^\nu)\nabla_\mu a_\nu = h^{\mu\nu}\nabla_\mu a_\nu + a_\mu a^\mu.$$

Letting $\vec{a}$ represent the spatial vector field $a^\mu|_\Sigma$ on Σ, this implies

$$\nabla_\mu a^\mu|_\Sigma = \mathrm{div}_\Sigma\vec{a} + \vec{a}\cdot\vec{a},$$

where div_Σ stands for the three-dimensional divergence within $(\Sigma, h_{\mu\nu})$ and $\cdot$ is its internal positive-definite scalar product; hence $\vec{a}\cdot\vec{a} \geq 0$ and this vanishes only if $\vec{a} = \vec{0}$. Note that the average $\langle \mathrm{div}_\Sigma\vec{a}\rangle_\Sigma$ will vanish for any reasonable behaviour of $a^\mu|_\Sigma$, because the integral in the numerator leads via Gauss theorem to a boundary (surface) integral 'at infinity', which will always be either finite or with a lower-order divergence than $\mathrm{vol}(\Sigma)$. Therefore, by taking averages of the previous expression one deduces

$$\langle\nabla_\mu a^\mu\rangle_\Sigma = \langle\vec{a}\cdot\vec{a}\rangle_\Sigma = \langle a_\nu a^\nu\rangle_\Sigma \geq 0.$$

In other words, the first conclusion in (8) can be rewritten in a more interesting way as $\langle u^\mu\nabla_\mu\theta - a_\nu a^\nu\rangle_\Sigma = 0$, or equivalently

$$\langle u^\mu\nabla_\mu\theta\rangle_\Sigma = \langle a_\nu a^\nu\rangle_\Sigma \geq 0.$$

Observe that, if these averages do not vanish, this implies in particular that there must be regions on Σ where $u^\mu\nabla_\mu\theta$ is positive.

Proof. From Proposition 1 it follows that necessarily $\Lambda = 0$. Hence in any orthonormal basis $\{e^\mu_\alpha\}$ eq. (3) implies

$$R_{\mu\nu}e^\mu_\alpha e^\nu_\beta = T_{\mu\nu}e^\mu_\alpha e^\nu_\beta - \frac{1}{2}T^\rho_\rho\eta_{\alpha\beta},$$

where $(\eta_{\alpha\beta}) = \mathrm{diag}(-1,1,1,1)$. By taking averages on Σ here and using Lemma 1, one deduces

$$\left\langle R_{\mu\nu}e^\mu_\alpha e^\nu_\beta\right\rangle_\Sigma = 0, \quad \forall\alpha,\beta = 0,1,2,3$$

which are the last expressions in (8).

The Raychaudhuri equations (1) for u^μ and v^μ are respectively

$$u^\nu\nabla_\nu\theta + \nabla_\mu u_\nu\nabla^\nu u^\mu - \nabla_\mu a^\mu + R_{\mu\nu}u^\mu u^\nu = 0, \tag{9}$$

$$v^\nu\nabla_\nu\vartheta + \nabla_\mu v_\nu\nabla^\nu v^\mu + R_{\mu\nu}v^\mu v^\nu = 0. \tag{10}$$

Obviously $u^\mu|_\Sigma = v^\mu|_\Sigma$ and it is elementary to check that $\theta|_\Sigma = \vartheta|_\Sigma$ and

$$\nabla_\mu u_\nu \nabla^\nu u^\mu|_\Sigma = \nabla_\mu v_\nu \nabla^\nu v^\mu|_\Sigma = K_{\mu\nu}K^{\mu\nu}.$$

From (9) and (10) one thus gets

$$(u^\nu \nabla_\nu \theta - \nabla_\mu a^\mu)|_\Sigma = v^\nu \nabla_\nu \vartheta|_\Sigma = -K_{\mu\nu}K^{\mu\nu} - R_{\mu\nu}u^\mu u^\nu|_\Sigma \leq 0$$

so that $u^\nu \nabla_\nu \theta - \nabla_\mu a^\mu$ is everywhere non-positive on Σ. Taking averages on Σ, using the second equation in (6) and the previous result on the averages of the Ricci tensor components one finally gets

$$\langle u^\nu \nabla_\nu \theta - \nabla_\mu a^\mu \rangle|_\Sigma = \langle v^\nu \nabla_\nu \vartheta \rangle|_\Sigma = 0.$$

That ends the proof.

The main theorem in this paper is now an immediate corollary of the previous propositions.

Theorem 2. *Assume that* (1) *there is a non-compact Cauchy hypersurface* Σ *such that the time-like geodesic congruence emanating orthogonal to* Σ *is expanding and the time-like convergence condition* (2) *holds along the congruence,* (2) *the spatial scalar curvature is non-positive on average on* Σ*:* $\langle \bar{R} \rangle_\Sigma \leq 0$, (3) *the cosmological constant is non-negative* $\Lambda \geq 0$, (4) *the energy–momentum tensor satisfies the dominant energy condition.*

If any single one of the following spatial averages

$$\Lambda,\ \langle \theta \rangle_\Sigma\,,\ \langle \vartheta \rangle_\Sigma\,,\ \langle \theta^2 \rangle_\Sigma\,,\ \langle \vartheta^2 \rangle_\Sigma\,,\ \langle K_{\mu\nu}K^{\mu\nu} \rangle_\Sigma\,,\ \langle \bar{R} \rangle_\Sigma\,,\ \langle v^\mu \nabla_\mu \vartheta \rangle_\Sigma\,,$$
$$\langle u^\mu \nabla_\mu \theta - \nabla_\mu a^\mu \rangle_\Sigma\,,\ \langle T_{\mu\nu} e^\mu_\alpha e^\nu_\beta \rangle_\Sigma\,,\ \ \langle R_{\mu\nu} e^\mu_\alpha e^\nu_\beta \rangle_\Sigma \quad \forall \alpha, \beta = 0,1,2,3$$

does not vanish, then the spacetime is past time-like geodesically incomplete.

As before, notice that the first average in the second row can be replaced by $\langle u^\mu \nabla_\mu \theta - a_\mu a^\mu \rangle_\Sigma$. Therefore, another sufficient condition is that $u^\mu \nabla_\mu \theta$ is non-positive everywhere but its average be non-zero.

5. Conclusions

Observe the following implication of the theorem. Under the stated hypotheses, a non-vanishing average of any component of the energy–momentum tensor (or of the Ricci tensor) – such as the energy density, the different pressures, the heat flux, etc. – leads to the existence of past singularities. It is quite remarkable that one does not need to assume any specific type of matter content (such as a perfect fluid, scalar field, ...); only the physically compelling and well-established dominant energy condition is required. The theorem is valid for 'open' models, as the Cauchy hypersurface is required to be non-compact. This is, however, no real restriction, because for closed models there are stronger results [7–10,48]. As a matter of fact, closed expanding non-singular models require the violation of the strong energy condition [7]. There is no hope of physically acceptable closed non-singular models satisfying the time-like convergence condition (2) (without this condition, there are some examples [7,26]).

Let me stress that the conclusion in Theorem 2 is quite strong: it tells us that the incompleteness is to the past. Besides, I believe that one can in fact prove a stronger theorem such that the geodesic time-like incompleteness is universal to the past.

The main implication of the above results is this: a clear, decisive, difference between singular and regular (globally hyperbolic) expanding cosmological models is that the latter must have a vanishing spatial average of the matter variables. Somehow, one could then say that the regular models are not cosmological, if we believe that the Universe is described by a more or less not too inhomogeneous distribution of matter. This is, on the whole, a very satisfactory result.

Appendix: The singularity-free model of [5]

In cylindrical coordinates $\{t, \rho, \varphi, z\}$ the line-element reads

$$\begin{aligned} \mathrm{d}s^2 = {} & \cosh^4(at)\cosh^2(3a\rho)(-\mathrm{d}t^2 + \mathrm{d}\rho^2) \\ & + \frac{1}{9a^2}\cosh^4(at)\cosh^{-2/3}(3a\rho)\sinh^2(3a\rho)\mathrm{d}\varphi^2 \\ & + \cosh^{-2}(at)\cosh^{-2/3}(3a\rho)\mathrm{d}z^2, \end{aligned} \tag{11}$$

where $a > 0$ is a constant. This is a cylindrically symmetric (the axis is defined by $\rho \to 0$) solution of the Einstein's field equations (3) (with $\Lambda = 0$ for simplicity) for an energy–momentum tensor describing a perfect fluid: $T_{\mu\nu} = \varrho u_\mu u_\nu + p\,(g_{\mu\nu} + u_\mu u_\nu)$. Here ϱ is the energy density of the fluid given by

$$\varrho = 15a^2\cosh^{-4}(at)\cosh^{-4}(3a\rho),$$

and

$$u_\mu = \left(-\cosh^2(at)\cosh(3a\rho), 0, 0, 0\right)$$

defines the unit velocity vector field of the fluid. Observe that u^μ is not geodesic (except at the axis), the acceleration field being

$$a_\mu = (0, 3a\tanh(3a\rho), 0, 0).$$

The fluid has a realistic barotropic equation of state relating its isotropic pressure p to ϱ by

$$p = \frac{1}{3}\varrho.$$

This is the canonical equation of state for radiation-dominated matter and is usually assumed to hold at early stages of the Universe. Note that the energy density and the pressure are regular everywhere, and one can in fact prove that the spacetime (11) is completely free of singularities and geodesically complete [14]. For complete discussions on this spacetime, see [14] and §7.6 in [7]. One can nevertheless see that the focusing effect on geodesics takes place fully in this spacetime. This does not lead to any problem with the existence of maximal geodesics between any pair of chronologically related points (see the discussion in [7], pp. 829–830).

Spacetime (11) satisfies the strongest causality condition: it is globally hyperbolic, any t =const. slice is a Cauchy hypersurface. All typical energy conditions, such as the dominant or the (strictly) strong ones (implying in particular condition (2) with the strict inequality) also hold everywhere. The fluid expansion is given by

$$\theta = \nabla_\mu u^\mu = 3a \frac{\sinh(at)}{\cosh^3(at)\cosh(3a\rho)}. \tag{12}$$

Thus this Universe is contracting for half of its history ($t < 0$) and expanding for the second half ($t > 0$), having a rebound at $t = 0$ which is driven by the spatial gradient of pressure, or equivalently, by the acceleration a_μ. Observe that the entire Universe is expanding (that is, $\theta > 0$) everywhere if $t > 0$. Note that this is one of the assumptions in Propositions 1, 2 and Theorem 2. It is however obvious that, for any Cauchy hypersurface Σ_T given by $t = T$ =constant, the average $\langle\theta\rangle_{\Sigma_T} = 0$. As one can check for the explicit expression (12), θ is strictly positive everywhere but not bounded from below by a positive constant because $\lim_{\rho\to\infty}\theta = 0$. Observe, however, that for finite ρ one has $\lim_{z\to\infty}\theta > 0$ and finite.

Similarly, one can check that the scalar curvature of each Σ_T is given by

$$\bar{R} = 30a^2\cosh^{-4}(aT)\cosh^{-4}(3a\rho) > 0$$

which is positive everywhere. However, $\langle\bar{R}\rangle_{\Sigma_T} = 0$, and analogously $\langle\varrho\rangle_{\Sigma_T} = \langle p\rangle_{\Sigma_T} = 0$. Observe also that

$$a_\mu a^\mu = 9a^2 \frac{\sinh^2(3a\rho)}{\cosh^4(at)\cosh^4(3a\rho)}$$

and thus $\langle a_\mu a^\mu\rangle_{\Sigma_T} = 0$. This implies that in this case $\langle u^\mu\nabla_\mu\theta\rangle_{\Sigma_T} = 0$. The sign of $u^\mu\nabla_\mu\theta - \nabla_\mu a^\mu$ is negative everywhere, as can be easily checked:

$$u^\mu\nabla_\mu\theta = 3a^2 \frac{1 - 3\tanh^2(at)}{\cosh^4(at)\cosh^2(3a\rho)},$$

$$\nabla_\mu a^\mu = 3a^2 \frac{\cosh^2(3a\rho) + 5}{\cosh^4(at)\cosh^4(3a\rho)}.$$

All these are in agreement with, and illustrate, Theorem 2, Propositions 1 and 2, and their corresponding remarks.

This simple model shows that there exist well-founded, well-behaved classical models which expand everywhere, satisfying all energy and causality conditions, and are singularity-free. However, as we have just seen, the model is somehow not 'cosmological' to the extent that the above-mentioned spatial averages vanish.

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