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On the incompatibility of standard quantum mechanics and conventional de Broglie–Bohm theory

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Abstract. It is shown that conventional de Broglie–Bohm quantum theory is incompatible with the standard quantum theory of a system unless the former is ergodic.

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Whereas a standard quantum mechanical system is usually ergodic, the corresponding Bohmian system may not be so, leading to a difference between the space and time averages of a suitably chosen observable over the ensemble in the Bohmian case. In this paper I will give a simple example of this incompatibility.

Let us consider the familiar classical system of two identical simple pendulums of length $l_1 = l_2 = 1$ and mass $m_1 = m_2 = 1$ connected by a weightless spring whose length ℓ is equal to the distance between the points of suspension. If q_1 and q_2 denote the angles of inclination of the pendulums, then for small oscillations the kinetic energy is $T = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2)$ and the potential energy is $U = \frac{1}{2}(q_1^2 + q_2^2 + \alpha(q_1 - q_2)^2)$, where $\alpha(q_1 - q_2)^2$ is the potential energy of the elastic spring. Now define the normal coordinates

$$Q_1 = \frac{q_1 + q_2}{\sqrt{2}} \quad \text{and} \quad Q_2 = \frac{q_1 - q_2}{\sqrt{2}}. \tag{1}$$

Then,

$$T = \frac{1}{2}(\dot{Q}_1^2 + \dot{Q}_2^2) \quad \text{and} \quad U = \frac{1}{2}(\omega_1^2 Q_1^2 + \omega_2^2 Q_2^2), \tag{2}$$

where $\omega_1 = 1$ and $\omega_2 = \sqrt{1 + 2\alpha}$. So, the characteristic oscillations are:

1. $Q_2 = 0$, i.e., $q_1 = q_2$ and the two pendulums oscillate in phase with the original frequency $\omega_1 = 1$, or
2. $Q_1 = 0$, i.e., $q_1 = -q_2$ and the two pendulums oscillate with opposite phase with the increased frequency $\omega_2 > 1$.

The smooth phase-space manifold M on which the motion occurs is the torus T^2, i.e., the orbits are closed curves on this torus and the system is non-ergodic (i.e., the orbits are not everywhere dense on the torus) [1]. Therefore, the space and time means, $\bar{F}$ and F^* respectively, of every complex-valued function F on M cannot be the same [2].

If one regards the system as a two-dimensional oscillator rather than two one-dimensional ones that are coupled, the system will still be non-ergodic provided ω_1/ω_2 is a rational number.

The corresponding system is described in standard quantum theory (SQT) by the two-particle Schrödinger equation

$$i\hbar\frac{\partial\psi(Q_1,Q_2)}{\partial t} = \left[-\frac{\hbar^2}{2}\partial^2_{Q_1} - \frac{\hbar^2}{2}\partial^2_{Q_2} + \frac{1}{2}\omega_1^2 Q_1^2 + \frac{1}{2}\omega_2^2 Q_2^2\right]\psi(Q_1,Q_2). \tag{3}$$

One can then construct two non-dispersive wave-packets oscillating about $Q_1 = a$ and $Q_2 = -a$ [3]. Let

$$\begin{aligned}\psi_A(Q_1,t) = (\omega_1/\pi\hbar)^{1/4}\exp\{&-(\omega_1/2\hbar)(Q_1 - a\cos\omega_1 t)^2 \\ &-(i/2)[\omega_1 t + (\omega_1/\hbar)(2Q_1 a\sin\omega_1 t - \frac{1}{2}a^2\sin 2\omega_1 t)]\}\end{aligned} \tag{4}$$

be the packet initially centred about $Q_1 = a$ and

$$\begin{aligned}\psi_B(Q_2,t) = (\omega_2/\pi\hbar)^{1/4}\exp\Big\{&-(\omega_2/2\hbar)(Q_2 + a\cos\omega_2 t)^2 \\ &-(i/2)\left[\omega_2 t + (\omega_2/\hbar)(-2Q_2 a\sin\omega_2 t - \frac{1}{2}a^2\sin 2\omega_2 t)\right]\Big\}\end{aligned} \tag{5}$$

the packet initially centred about $Q_2 = -a$. Since

$$\begin{aligned}|\psi_A(Q_1,t)|^2 &= |\psi_{A0}(Q_1 - a\cos\omega_1 t)|^2, \\ |\psi_B(Q_2,t)|^2 &= |\psi_{B0}(Q_2 + a\cos\omega_2 t)|^2,\end{aligned} \tag{6}$$

the packets oscillate harmonically without change of shape between the angles $\pm a$.

Let the half-widths $\sigma_1 = (\hbar/2\omega_1)^{1/2}$ and $\sigma_2 = (\hbar/2\omega_2)^{1/2}$ of the packets be small compared to ℓ, the distance between the points of suspension, so that the two packets do not overlap initially. They will not overlap at any time if $\ell \ll 2(a+\sigma_1)$. We will assume this to be the case. Then the two- particle wave function is given by

$$\psi(Q_1,Q_2,t) = \psi_A(Q_1,t)\psi_B(Q_2,t) = R(Q_1,Q_2,t)\exp\frac{i}{\hbar}S(Q_1,Q_2,t), \tag{7}$$

and therefore the phase or action function by

$$\begin{aligned}S(Q_1,Q_2,t) = &-\frac{1}{2}\hbar\omega_1 t - \frac{1}{2}\omega_1\left(2Q_1 a\sin\omega_1 t - \frac{1}{2}a^2\sin 2\omega_1 t\right) \\ &-\frac{1}{2}\hbar\omega_2 t - \frac{1}{2}\omega_2\left(-2Q_2 a\sin\omega_2 t - \frac{1}{2}a^2\sin 2\omega_2 t\right).\end{aligned} \tag{8}$$

The Bohmian trajectory equations are therefore

$$P_1 = \frac{\mathrm{d}Q_1}{\mathrm{d}t} = \partial_{Q_1} S(Q_1, Q_2, t) = -\omega_1 a \sin\omega_1 t, \tag{9}$$

$$P_2 = \frac{\mathrm{d}Q_2}{\mathrm{d}t} = \partial_{Q_2} S(Q_1, Q_2, t) = \omega_2 a \sin\omega_2 t, \tag{10}$$

whose solutions are

$$Q_1(t) = Q_1(0) + a\,(\cos\omega_1 t - 1), \tag{11}$$

$$Q_2(t) = Q_2(0) - a\,(\cos\omega_2 t - 1), \tag{12}$$

where $Q_1(0)$ and $Q_2(0)$ are the initial coordinates.

If one considers an ensemble of such oscillators, their centre points are distributed in a Gaussian fashion.

The characteristic oscillations are again:

1. $Q_2(t) = 0$, i.e., $q_1(t) = q_2(t)$, and the two particles oscillate in phase with the original frequency ω_1 (and hence with the length ℓ of the spring unchanged), or
2. $Q_1(t) = 0$, i.e., $q_1(t) = -q_2(t)$, and the two particles oscillate out of phase with the increased frequency ω_2.

However, the particles moving in the oscillating packets are not conservative systems [3], because the sum of their kinetic and potential energies evaluated along the trajectories (11,12) is not conserved, i.e.

$$\begin{aligned}
\frac{1}{2}(P_1^2(t) + \omega_1^2 Q_1^2(t)) &= \frac{1}{2}\omega_1^2 a^2 + \frac{1}{2}\omega_1^2 (Q_1(0) - a)^2 \\
&\quad + \omega_1^2\, a\,(Q_1(0) - a)\cos\omega_1 t, \\
\frac{1}{2}(P_2^2(t) + \omega_2^2 Q_2^2(t)) &= \frac{1}{2}\omega_2^2 a^2 + \frac{1}{2}\omega_2^2 (Q_2(0) + a)^2 \\
&\quad + \omega_2^2\, a\,(Q_2(0) + a)\cos\omega_2 t
\end{aligned} \tag{13}$$

unless $Q_1(0) = a$ and $Q_2(0) = -a$. Nevertheless, the motion is still on a torus T^2 in each case (1 and 2) with the size of the torus oscillating in time about a mean value. Since the motion is periodic, the system is non-ergodic. This means there is at least one observable of the system whose space and time averages are different.

The corresponding SQT system is, however, ergodic by von Neuman's theorem [4]. A simple proof is given in the next section [5]. Hence the space and time averages of every observable of the system must be the same.

I will now show that the joint detection of the oscillating particles is an observable whose space and time averages are different in the de Broglie–Bohm theory (dBB). One can define the joint distribution function $f(q, p, t)$ in dBB by

$$f(q, p, t) = P(q(t))\,\delta(p - \nabla S(q, t)), \tag{14}$$

$$\int f(q, p, t)\mathrm{d}q\mathrm{d}p = 1, \tag{15}$$

where $P(q(t))$ is the real statistical probability density in dBB that is equivalent to the quantum mechanical probability density $R^2(q, t)$. Take any function $F(q, p)$ on phase-space. Its space average is defined by

$$\bar{F} = \int F(q,p) f(q,p,t) \mathrm{d}q \mathrm{d}p$$
$$= \int F(q, \nabla S) P(q(t)) \mathrm{d}q. \quad (16)$$

Let the mid-point of the points of suspension of the oscillators be taken as the origin. Let us further assume that $\sigma_2 \approx 2a$. Then, since $\sigma_1 > \sigma_2$, the wave-packets are non-zero at all times in the intervals $(-\ell/2 - a, -\ell/2 + a)$ and $(\ell/2 - a, \ell/2 + a)$. Let us consider two detectors D_1 and D_2 of size d much smaller than these intervals, one in each of them, and separated by a distance $D \neq \ell$ and placed asymmetrically about the origin. Then the joint detection probability as space and time means are respectively given by

$$\bar{P}_{12(\mathrm{dBB})} = \int_{D_1, D_2, t} \mathrm{d}Q_1 \mathrm{d}Q_2 P(Q_1(t), Q_2(t)) = \bar{P}_{12(\mathrm{SQT})} \neq 0, \quad (17)$$

$$P^*_{12(\mathrm{dBB})} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} P(\phi_t^n Q)|_{D_1, D_2} = 0, \quad (18)$$

where $\phi_t^n : M \to M$ is a one-parameter group of measure preserving diffeomorphisms, and $Q = (Q_1, Q_2)$ such that

$$\phi_t^n Q = (Q_1(t_n), Q_2(t_n)) \frac{1}{\delta(0)} [\delta(Q_1(t_n)) + \delta(Q_2(t_n))]. \quad (19)$$

These two averages are clearly different in dBB because the system is non-ergodic. Notice that without the constraints imposed by the delta functions in (19), the two averages would be the same.

Now, the space average $\bar{P}_{12(\mathrm{SQT})} = \bar{P}_{12(\mathrm{dBB})}$ by construction, and the space and time averages are the same in SQT ($\bar{P}_{12(\mathrm{SQT})} = P^*_{12(\mathrm{SQT})}$) because the SQT system is ergodic. This completes the demonstration of incompatibility between dBB and SQT in the case of two coupled one-dimensional simple harmonic oscillators (and equivalently one two-dimensional harmonic oscillator with commensurate frequencies).

I will now give a simple proof of ergodicity for two-particle systems in SQT which can be easily generalized to n-particle systems. Let $\Psi(x_1, x_2, t) = \exp(-iHt/\hbar)\, \psi(x_1, x_2)$ be a normalized solution of the time-dependent Schrödinger equation, and let $\psi(x_1, x_2) = \sum_n c_n \phi_n(x_1, x_2)$, where $\phi_n(x_1, x_2)$ are a complete set of orthonormal energy eigenfunctions. Consider the time average of any observable $\hat{F}$ in the state $\Psi(x_1, x_2, t)$:

$$\begin{aligned} F^* &= \lim_{T \to \infty} \frac{1}{T} \int_0^T \mathrm{d}t \int \mathrm{d}x_1 \mathrm{d}x_2\, \Psi^*(x_1, x_2, t) \hat{F} \Psi(x_1, x_2, t) \\ &= \lim_{T \to \infty} \frac{1}{T} \int_0^T \mathrm{d}t \int \mathrm{d}x_1 \mathrm{d}x_2 \left(\sum_n |c_n|^2 \phi_n^*(x_1, x_2) \hat{F} \phi_n(x_1, x_2) \right. \\ &\quad \left. + \sum_{n,m} c_n^* c_m\, \mathrm{e}^{i(E_n - E_m)t} \phi_n^*(x_1, x_2)\, \phi_m(x_1, x_2) \right) \\ &= \sum_n |c_n|^2 \int \mathrm{d}x_1 \mathrm{d}x_2 \phi_n^*(x_1, x_2)\, \hat{F}\, \phi_n(x_1, x_2) \\ &= \mathrm{Tr}(\hat{\rho} \hat{F}) \\ &= \bar{F}, \end{aligned} \quad (20)$$

where $\hat{\rho}$ is the reduced density matrix. This is independent of time. This makes clear the conditions under which ergodicity holds in SQT.

Before concluding, I will discuss another system for which dBB and SQT are incompatible. Consider a source of two momentum-correlated identical particles of mass m (described by wave packets) set up in such a fashion that a large number of them simultaneously pass through two point slits A and B situated on the y axis and separated by a distance $2a$. Let only one pair of packets pass through the slits at a time. Let the line bisecting the line joining the two slits be the x axis (i.e., $y = 0, x \geq 0$). It is a natural symmetry axis of the system. After passing through the slits, the two probability amplitudes propagate with uniform speed v in spherical waves. In a region in which these waves do not overlap, the normalized two-particle wave function in the xy plane is given by

$$\psi(r_{1A}, r_{2B}, t) = \frac{1}{2\pi} \frac{\mathrm{e}^{ik(r_{1A}+r_{2B})}}{r_{1A} r_{2B}} \frac{\delta(r_{1A} - vt)\delta(r_{2B} - vt)}{\delta(0)}, \tag{21}$$

where $r_{1A} = \sqrt{x_1^2 + (y_1 - a)^2}$ and $r_{2B} = \sqrt{x_2^2 + (y_2 + a)^2}$ are the radius vectors of points on the wave fronts measured from the two slits. This wave function is symmetric under reflection about the x axis together with the interchange of the particle labels $1 \leftrightarrow 2$. The phase $S(r_{1A}, r_{2B}, t)$ of the wave function is

$$S(r_{1A}, r_{2B}, t) = \hbar k(r_{1A} + r_{2B})|_{r_{1A}=r_{2B}=vt}. \tag{22}$$

It is clear from this that the Bohmian trajectories fan out radially with the slits as the initial positions. (Note that a spherical wave function is singular at its origin. Hence, the point nature of the slits must be understood in the sense of a limit. This is also necessary because otherwise one would get a single trajectory corresponding to a single initial position rather than trajectories normal to every point of the spherical wave front, corresponding to a Gibbs ensemble of initial positions at the slit. This is necessary for the compatibility of dBB and SQT for Gibbs ensembles.) The x and y components of the Bohmian velocities are given by

$$v_{x_1} = \frac{1}{m} \frac{\partial S}{\partial r_{1A}} \frac{\partial r_{1A}}{\partial x_1} = \left. \frac{\hbar k x_1}{m r_{1A}} \right|_{r_{1A}=vt}, \tag{23}$$

$$v_{x_2} = \frac{1}{m} \frac{\partial S}{\partial r_{2B}} \frac{\partial r_{2B}}{\partial x_2} = \left. \frac{\hbar k x_2}{m r_{2B}} \right|_{r_{2B}=vt}, \tag{24}$$

$$v_{y_1} = \frac{1}{m} \frac{\partial S}{\partial r_{1A}} \frac{\partial r_{1A}}{\partial y_1} = \left. \frac{\hbar k (y_1 - a)}{m r_{1A}} \right|_{r_{1A}=vt}, \tag{25}$$

$$v_{x_2} = \frac{1}{m} \frac{\partial S}{\partial r_{2B}} \frac{\partial r_{2B}}{\partial x_2} = \left. \frac{\hbar k (y_2 + a)}{m r_{2B}} \right|_{r_{2B}=vt}. \tag{26}$$

One therefore obtains

$$v_{x_1} - v_{x_2} = \frac{\mathrm{d}(x_1 - x_2)}{\mathrm{d}t} = \frac{1}{t}(x_1 - x_2) \tag{27}$$

and

$$v_{y_1} + v_{y_2} = \frac{\mathrm{d}(y_1 + y_2)}{\mathrm{d}t} = \frac{1}{t}(y_1 + y_2). \tag{28}$$

Solving these equations and using the initial condition $x_1(t_0) = x_2(t_0) = 0$ and $y_1(t_0) + y_2(t_0) = 0$, one obtains

$$x_1(t) = x_2(t), \tag{29}$$

$$y_1(t) = -y_2(t), \tag{30}$$

at all times t. (The choice of the plus sign in eq. (27) would have led to the solution $x_1(t) = -x_2(t)$ which is unacceptable because the motion occurs in the region $x \geq 0$.) This shows that the trajectories of the two particles are at all times symmetrical about the x axis.

If one considers the region where the two spherical waves overlap, and the particles are bosons, the wave function (21) must be replaced by

$$\psi(r_1, r_2, t) = \frac{1}{N}\left[\frac{\mathrm{e}^{ik(r_{1A}+r_{2B})}}{r_{1A}r_{2B}}\frac{\delta(r_{1A} - vt)\delta(r_{2B} - vt)}{\delta(0)} + \frac{\mathrm{e}^{ik(r_{1B}+r_{2A})}}{r_{1B}r_{2A}}\frac{\delta(r_{1B} - vt)\delta(r_{2A} - vt)}{\delta(0)}\right], \tag{31}$$

where N is a normalization factor, $r_{1B} = \sqrt{x_1^2 + (y_1 + a)^2}$ and $r_{2A} = \sqrt{x_2^2 + (y_2 - a)^2}$. This is separately symmetric under reflection about the x axis and the interchange of the two particles. It follows from the conditions $r_{1A} = r_{2B} = vt$ and $r_{1B} = r_{2A} = vt$ which must be satisfied simultaneously that the conditions (29) and (30)) must still hold. Hence, the Bohmian trajectories of the two particles are symmetric about the x axis in this case too.

Furthermore, the y components of the velocities of the particles are given by

$$v_{y_1} = \frac{\hbar}{m}\mathrm{Im}\frac{\partial_{y_1}\psi(r_1, r_2, t)}{\psi(r_1, r_2, t)}, \tag{32}$$

$$v_{y_2} = \frac{\hbar}{m}\mathrm{Im}\frac{\partial_{y_2}\psi(r_1, r_2, t)}{\psi(r_1, r_2, t)}, \tag{33}$$

and therefore

$$v_{y_1}(x_1(t), y_1(t), x_2(t), y_2(t)) = -v_{y_1}(x_1(t), -y_1(t), x_2(t), -y_2(t)), \tag{34}$$

$$v_{y_2}(x_1(t), y_1(t), x_2(t), y_2(t)) = -v_{y_2}(x_1(t), -y_1(t), x_2(t), -y_2(t)). \tag{35}$$

This shows that by virtue of condition (30) the y components of the velocities of the particles must vanish on the x axis. This implies that the trajectories of the particles are not only symmetrical about the x axis, they also do not cross this axis in this case.

This has nontrivial empirical consequences. If two detectors D_1 and D_2 are placed anywhere perpendicular to the x axis such that they are asymmetrical about this axis, the joint detection probability as a time average will vanish, and

$$P^{*}_{12(\mathrm{dBB})} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} P(\phi_t^n Y)\bigg|_{\mathrm{D}_1,\mathrm{D}_2} = 0, \tag{36}$$

where $Y = (y_1, y_2)$. On the other hand, the space average is non-vanishing:

$$\bar{P}_{12(\mathrm{dBB})} = \int_{\mathrm{D}_1,\mathrm{D}_2,t} \mathrm{d}y_1 \mathrm{d}y_2 P(y_1(t), y_2(t)) = \bar{P}_{12(\mathrm{SQT})} \neq 0. \tag{37}$$

This shows that the Bohmian motion in this case is also non-ergodic, and therefore incompatible with SQT.

What we have shown above is generic. One can, in fact, state a general theorem:

Theorem. *Conventional dBB is incompatible with SQT unless the Bohmian system corresponding to an SQT system is ergodic*.

I must emphasize that this theorem holds only for conventional dBB as originally proposed by Bohm [6] and elaborated, for example, by Holland [3]. The key feature of this theory is the ontology of unique deterministic trajectories of particles corresponding to given initial positions. An extension of this theory has been proposed [7] that randomizes the position coordinates and claims to make the theory consistent with SQT for every experiment. Since 'absolute uncertainty' is built into this extended theory, its interpretation must be very similar to the standard one, except that position is given an ontology. In any case, its spirit is very different from that of Bohm who did not wish to make his theory completely equivalent to SQT in every conceivable situation. This is clearly borne out by the following statement of his about the standard interpretation of quantum theory and his own interpretation [6]:

> "An experimental choice between these two interpretations cannot be made in a domain in which the present mathematical formulation of the quantum theory is a good approximation; but such a choice is conceivable in domains, such as those associated with dimensions of the order of 10^{-13} cm, where the extrapolation of the present theory seems to break down and where our suggested new interpretation can lead to completely different kinds of predictions."

The fact that the particular domain referred to by Bohm still continues to be described very accurately by SQT is irrelevant in this context. What is significant is that even in domains where SQT is supposed to be an excellent theory, dBB can be in conflict with it, and that this difference can only be discovered through time averages of observables whenever the Bohmian system is non-ergodic, a feature of his own theory that Bohm seems to have ignored. Such experiments in the time domain have not been done so far, but one is under preparation at Pavia.

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