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Advances in analytical solutions for time-dependent solute transport model

Rohit Kumar[1], Ayan Chatterjee[2], Mritunjay Kumar Singh[3,*]
and Frank T-C Tsai[4]

[1]*Department of Mathematics and Computing, Indian Institute of Technology (Indian School of Mines), Dhanbad, Jharkhand 826 004, India.*
[2]*Department of Mathematics, The Neotia University, Diamond Harbour, Kolkata, West Bengal 743 368, India.*
[3]*Department of Mathematics and Computing, Indian Institute of Technology (Indian School of Mines), Dhanbad, Jharkhand 826 004, India.*
[4]*Department of Civil and Environmental Engineering, Louisiana State University, Baton Rouge, LA 70803, USA.*
**Corresponding author. e-mail: drmks29@rediffmail.com*

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This study adopts generalized dispersion theory in one-dimensional advection–dispersion equation (ADE), where time-dependent dispersion and velocity are considered. The generalized dispersion theory allows mechanical dispersion to be directly proportional to seepage velocity with power n, where n is any real number. Homotopy analysis method (HAM) that uses a simple algorithm is adopted to handle the non-linearity that occurred in the ADE under the generalized dispersion. A point source is introduced to the entry boundary and a line source is introduced to the entire model domain. Three time-dependent point sources in the form of (i) exponentially decreasing function, (ii) linear function and (iii) sinusoidal function, at the entry boundary are considered. Two-line sources are considered in the form of (i) linear space-dependent function and (ii) nonlinear space-time-dependent function. Using the HAM, semi-analytical solutions for any power n are derived and semi-analytical solutions for $n = 1$ and $n = 1.5$ are discussed in particular. Comparison with the analytical solution is discussed and found good agreement for 6th order of solution obtained by HAM.

Keywords. ADE; time-dependent dispersion and velocity; generalized dispersion theory; semi-analytical solution; HAM.

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1. Introduction

Contaminant transport modelling in groundwater systems plays an important role to understand transport mechanisms and to design for groundwater contamination mitigation (Batu 2006; Todd and Mays 2007). For example, groundwater reservoirs can be directly contaminated from landfill sites by industrial zones such as construction sites, chemical sites, nuclear power plants, etc. Deep wells can be contaminated by high-level toxic wastes such as arsenic and fluoride during agricultural and waste disposal management. High-level toxic wastes disposed under the ground can directly enter aquifers through natural hydrologic processes. Solute transport modelling still presents a challenging task to researchers and scientists working in the field of hydrogeology.

In the last few decades, dispersion theories play an important role in the solute transport

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equation/porous medium, which relate the two physical parameters, mechanical dispersion and seepage velocity, in the 1-D ADE. Large number of laboratory and conceptual experiments have been investigated to assess the value of n in the following expression $D \propto u^n$. Two possible relationships were proposed between dispersion coefficient (D) and seepage velocity (u); (i) $D \propto u$, i.e., the dispersion coefficient is directly proportional to the seepage velocity and (ii) $D \propto u^2$, i.e., the dispersion coefficient is proportional to the square of the seepage velocity (Scheidegger 1957). A general dispersion theory in the porous media was explored by Scheidegger (1961). However, Ebach and White (1958) had found the value of $n = 1.54$. Freeze and Cherry (1979) proposed dispersion theory that dispersion coefficient is proportional to the nth power of the velocity, where n varies between 1 and 2. Later on, Ghosh and Sharma (2006) described that dispersion coefficient is proportional to the seepage velocity with power ranging from 1 to 1.2, which is depending upon the solute movement patterns and porous medium. In the study of Bharati *et al.* (2017, 2018), they considered the fixed values of n and obtained the solution of the system. Whereas in the present study, authors are providing solutions with general n. The present solutions which are mentioned in section 4 may be able to address the different situations observed in groundwater contamination modelling problem in real life. Most of the publications available in the literature are based on the theory of dispersion, where dispersion is directly proportional to the velocity or square of the velocity. The fractional power of seepage velocity is still debatable among the scientific community. The literature of 1-D and 2-D solute transport models is given in table 1.

From the above-mentioned literature review, it is quite clear that most of the studies have considered dispersion to be directly proportional to either velocity or square of velocity. However, dispersion can be proportional to a power of velocity between 1 and 2 (Freeze and Cherry 1979; Ghosh and Sharma 2006). For the purpose of generosity, we keep the general form $D \propto V^n$ (dispersion is proportional to the nth power of the seepage velocity) in this study. Given the general form, we may not be able to find analytical solutions by using the LTT or Fourier transform technique. Instead, we may derive semi-analytical solutions using the homotopy analysis method (HAM). Specifically, the dispersion and velocity coefficients in this study are taken as linear, sinusoidal and exponentially decreasing functions, and the point source of contamination at the entry boundary is taken as a general function of time, which can be implemented to study real problems. Generalised dispersion theory has not been solved by any other authors. Solution for specific values of n, i.e., for 1, 1.5 and 2 has been solved, but the real-life situation is not always as good as the values of n considered. So it may be 1.2/1.1 in this present solution, we provided the solution for any value of n between (1, 2) hopefully this makes the present paper quite interesting and helpful for the researchers working in the field of hydrological modelling.

Regarding the organization of the study, the Introduction section reviews the solute transport modelling under the generalized dispersion theory. A mathematical formulation for 1-D ADE is considered with the generalized dispersion theory. The background of homotopy analysis method section presents the basic idea of the HAM for the semi-analytical solution. Generalized solutions for any power of n are derived in the semi-analytical solution by HAM section. In the Result and discussion section, semi-analytical solutions for some special cases are presented. Then, semi-analytical solution is compared to an analytical solution derived by the Laplace transform method for a specific case. Finally, few conclusions are drawn from the study.

2. Mathematical formulation

Let C be the solute concentration (ML^{-3}) in the liquid phase, $D(t)$ be the time-dependent dispersion coefficient (L^2T^{-1}), $V(t)$ be the time-dependent velocity (LT^{-1}), $G(x,t) = g(x,t)/g_1$ be a dimensionless coefficient function for the source term, where $g(x,t)$ is a function of space and time and $g_1 = g(1,1)$, and c'_0 be the concentration rate at sources ($ML^{-3}T^{-1}$). The 1-D ADE can be written as:

$$\frac{\partial C}{\partial t} = D(t)\frac{\partial^2 C}{\partial x^2} - V(t)\frac{\partial C}{\partial x} + c'_0 G(x,t). \quad (1)$$

The dispersion theory was proposed by Freeze and Cherry (1979) that the dispersion coefficient is proportional to either V^1 or V^2. In this work, we consider a generalized dispersion theory, i.e., $D \propto V^n$, where $1 \le n \le 2$.

Table 1. *1-D and 2-D solute transport models.*

Authors	Transport model	Velocity	Dispersion	Input source	Solution methodology	Note
Van Genuchten (1981)	1-D, Cartesian, transient	Constant	Constant	Plus type and Constant concentration	Laplace transform	Constant velocity and dispersion (assumed as directly proportional)
Kumar (1983)	1-D, Cartesian	Time-dependent (linear or exponentially)	Initial dispersion	Concentration. (1) zero (2) proportional to the flow	Laplace transform technique	Dispersion is directly proportional to the velocity
Barry and Sposito (1989)	1-D, Cartesian	Time-dependent	Time-dependent	Arbitrary boundary flux conditions	Fundamental solution	Velocity and dispersion are defined in terms of spatial moment functions
Basha and El-Habel (1993)	1-D, Cartesian	Constant	Time-dependent	Point source	Fundamental solution	Dispersion is not proportional to the velocity
Aral and Liao (1996)	2-D, Cartesian	Constant	Time-dependent function	Instantaneous and continuous point source	Laplace transform technique	Time-dependent dispersion differs with direction
Logan (1996)	1-D, Cartesian	Uniform	Space-dependent and constant	Periodic type	Laplace transform technique	Scale-dependent dispersion coefficient to incorporate heterogeneity
Runkel (1996)	1-D, Cartesian	Constant	Constant	Constant concentration	Several analytical solutions are mentioned	Constant velocity and dispersion (assumed as directly proportional)
Huang *et al.* (1996)	1-D, Cartesian	Constant	Scale dependent	Constant concentration or third type	Laplace transform , Bessel function	Dispersion is proportional to the pore water velocity
Zoppou and Knight (1997)	1-D, Cartesian	Spatially variable coefficient	Spatially variable coefficient	Constant concentration	Laplace transform technique	Velocity is proportional to the distance and the diffusion coefficient is proportional to the square of the velocity
Sander and Braddock (2005)	1-D, Cartesian	Constant	Scale and time dependent dispersivity	Dirichlet type and constant	Similarity solutions	Dispersion is considered as sum of molecular diffusion and mechanical dispersion

Table 1. *(Continued.)*

Authors	Transport model	Velocity	Dispersion	Input source	Solution methodology	Note
Liu and Si (2008)	1-D, Cartesian, diffusion equation	No velocity term	Position dependent coefficient	Arbitrary	Orthogonal expansion technique	Several examples provided based on position dependent diffusion coefficient
Jaiswal *et al.* (2009)	1-D, Cartesian	Spatially and time-dependent	Spatially and time-dependent	Pulse type	Laplace transform method	Analytical solution in the semi-infinite medium
Kumar *et al.* (2009)	1-D, Cartesian	Constant and spatially dependent	Time-dependent and spatially dependent	Constant and mixed type	Laplace transform technique	Dispersion is proportional to the second power of the velocity
Guerrero and Skaggs (2010)	1-D, Cartesian	Distance dependent	Distance dependent	First, second or third type	Generalized integral transform technique	Hydrodynamic dispersion is considered as sum of molecular diffusion and mechanical dispersion, where mechanical dispersion is proportional to velocity
Kumar *et al.* (2012)	1-D, Cartesian	Space and time dependent	Space and time dependent	Constant source	Laplace transform technique	Analytical solutions were developed under the assumption of mechanical dispersion is square of the velocity
Yadav *et al.* (2012)	2-D, Cartesian	Spatio-temporally dependent	Spatio-temporally dependent	Pulse type	Laplace transform technique	Dispersion is proportional to the second power of the velocity
You and Zhan (2013)	1-D, Cartesian	Constant	Distance-dependent	Time-dependent	Laplace transform technique	Dispersion is defined as linear combination of molecular diffusion and velocity
Guerrero *et al.* (2013)	1-D, Cartesian	Constant	Constant	Mixed type	Classic integral transform technique	Constant velocity and dispersion (assumed as directly proportional)
Gao *et al.* (2013)	1-D, Cartesian	Constant	Constant	Function of time	Laplace transform technique	Constant velocity and dispersion (assumed as directly proportional)
Jia *et al.* (2013)	1-D, Cartesian	Variable and constant	Variable and constant	Saturated concentration	Laplace transform technique	The concentration profile have quite different shapes
Wadi *et al.* (2014)	1-D, Cartesian	Constant	Constant	Function of time	Laplace transform technique	Constant velocity and dispersion (assumed as directly proportional)
Deng *et al.* (2014)	1-D, Cartesian, transient	Constant velocities for each layer	Constant dispersion for each layer	First-type or third-type	Generalized integral transform technique	Constant velocity and dispersion (assumed as directly proportional)

Table 1. *(Continued.)*

Authors	Transport model	Velocity	Dispersion	Input source	Solution methodology	Note
Zamani and Bombardelli (2014)	1-D, Cartesian	Spatio-temporal	Spatio-temporal	Constant	Several analytical solutions are mentioned from the literature	Dispersion is defined in terms of molecular diffusion, intrinsic dispersivity and Darcy velocity
Singh and Das (2015)	1-D, Cartesian	Function of space and time	Function of space and time	Exponentially time-dependent decreasing	Laplace transform technique	Dispersion is directly proportional to the second power of the velocity
Shi *et al.* (2016)	1-D, Cartesian	Constant	Hydrodynamic dispersion coefficient	Function of time	Electrolyte tracer method	Dispersion is directly proportional to the velocity
Singh and Chatterjee (2017)	1-D, Cartesian	Space and space-time dependent	Space and space-time dependent	Source incorporated in the FADE	Homotopy perturbation method	Space-time Fractional ADE, uses dispersion is directly proportional to the velocity
Singh *et al.* (2017)	1-D, Cartesian	Temporally and spatially	Temporally and spatially	Source incorporated in the FADE	Homotopy analysis method	Time Fractional ADE, uses dispersion is directly proportional to the velocity
Das *et al.* (2017)	1-D, Cartesian	Function of time	space and time dependent dispersion with diffusion	Exponentially time-dependent decreasing	Laplace transform technique	Dispersion is directly proportional to the velocity
Sanskrityayn *et al.* (2017)	1-D, Cartesian	Spatially and temporally dependent	Spatially and temporally dependent	Instantaneous and continuous sources	Green's function method	Dispersion is square of the velocity
Chatterjee *et al.* (2020)	1-D, Cartesian	Time-dependent	Constant	Constant source	Laplace transform technique	Analytical solutions were obtained under the assumption of dispersion is directly proportional to the first power of the seepage velocity

Let $V = V_0 f(k_1 t)$ and $D = D_0(f(k_1 t))^n$, where D_0 is the initial dispersion coefficient (L^2T^{-1}), V_0 is the initial seepage velocity (LT^{-1}), $f(k_1 t)$ is a function of time and k_1 is the constant (T^{-1}). Substituting these terms in equation (1), we have:

$$\frac{\partial C}{\partial t} = D_0 f^n(k_1 t)\frac{\partial^2 C}{\partial x^2} - V_0 f(k_1 t)\frac{\partial C}{\partial x} + c_0' \frac{g(x,t)}{g_1}. \tag{2}$$

Initially, the model domain is considered solute-free, and therefore the initial condition is $C(x,0) = 0$. A time-dependent generalized source condition is considered at the entry boundary as $C(0,t) = b_0 t\gamma(t)/\beta(1)$, where b_0 is the constant concentration (ML^{-3}), $\gamma(t)$ is a function of time, and $\beta(1) = t\gamma(t)$ is the function value at $t = 1$. A weak boundary condition $\partial C/\partial x|_{x=L} = 0$ is considered at the exit boundary at all times, where L is the length of the model domain.

3. Background of homotopy analysis method (HAM)

The basic idea of the HAM is adopted from the concept of topology and has been widely used to solve non-linear differential equations. HAM was firstly introduced by Liao (1992) in the PhD thesis to solve the highly non-linear problems. This method was adopted to handle a wide variety of non-linear equations related to hydromechanics or Blasius flow and soon (Liao 1995, 2005; Liao *et al.* 2006; Yu *et al.* 2018a, b, 2019). The conventional mathematical methods have proven to be of numerous benefit: (i) simplicity of the mathematical derivation without complicated concept, (ii) permitting extremely broad independence to choose linear sub-problems equation form, simple solution mechanism and preliminary guess, and thus (iii) an efficient way of obtaining approximate solutions with high correctness and ensuring that solutions converge. The HAM begins with the equation

$$N[C(x,t)] = 0, \tag{3}$$

where N is the non-linear operator, x and t are the independent variables, $C(x,t)$ is the unknown function. Liao (1992) constructed a zeroth-order deformation equation which is as follows:

$$(1-q)Z[\varphi(x,t;q) - C_0(x,t)] = h_0 q N[\varphi(x,t;q)], \tag{4}$$

where Z is the linear operator, N is the nonlinear operator, $q \in [0,1]$ is the homotopy embedded parameter, $h_0 \neq 0$ is the non-zero control-convergence parameter, $C_0(x,t)$ is the initial guess of $C(x,t)$, and $\varphi(x,t;q)$ is the Maclaurin series with respect to q. When $q = 0$, $\varphi(x,t;0)$ is the initially guessed solution, i.e., $\varphi(x,t;0) = C_0(x,t)$. When $q = 1$, $\varphi(x,t,1)$ is the solution for the ADE, i.e., $\varphi(x,t,1) = C(x,t)$. So, it is clearly indicated that the solution of $\varphi(x,t;q)$ varies from initial guess to the original equation when q increases from 0 to 1.

The series expansion of $\varphi(x,t,q)$ with respect to q is given as follows:

$$\varphi(x,t;q) = C_0(x,t) + \sum_{m=1}^{\infty} C_m(x,t)\,q^m, \tag{5}$$

where

$$C_m(x,t) = \frac{1}{m!}\frac{\partial^m N[\varphi(x,t;q)]}{\partial q^m}\bigg|_{q=0}. \tag{6}$$

In equation (5), we select the initial guess, auxiliary linear operator, and non-zero parameter h_0 is properly chosen in the right form, then the approximate analytical solution series (5) converges at $q = 1$ and it gives rise to the new form as follows:

$$\varphi(x,t;1) = C_0(x,t) + \sum_{m=1}^{\infty} C_m(x,t). \tag{7}$$

Equation (7) is one of the solutions of the original equation, which was proved by Liao *et al.* (2006). The homotopy approximation can be determined by higher-order differential equation. Firstly, we define the vector

$$\vec{C}_m = \{C_0(x,t), C_1(x,t), C_2(x,t), \cdots, C_m(x,t)\}.$$

According to the HAM (Liao 1992, 2012) differentiating (4) m times with respect to q with setting $q = 0$ and dividing by $m!$. Then the mth-order deformation equation is obtained as follows:

$$Z[C_m(x,t) - \chi_m C_{m-1}(x,t)] = h_0 \delta_m(\vec{C}_{m-1}(x,t)), \tag{8}$$

where $C_m(x,t)$ is the mth-order homotopy approximation, $\chi_m = 0$ for $m = 1$ and $\chi_m = 1$ for $m > 1$ with

$$\delta_m(\vec{C}_{m-1}(x,t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}(N[\varphi(x,t;q)])\bigg|_{q=0}. \tag{9}$$

By applying inversion of the linear operator on equation (8), $C_m(x,t)$ can be derived as:

$$C_m(x,t) = \chi_m C_{m-1}(x,t) + h_0 Z^{-1}\left[\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}(N[\varphi(x,t;q)])\bigg|_{q=0}\right]. \quad (10)$$

The auxiliary linear operator Z, the initial guess $C_0(x,t)$, non-zero parameter h_0 and auxiliary function $N[\varphi(x, t : q)]$ are properly chosen. Then equation (3) can be easily solved and the Mth order approximation solution of $C(x,t)$ is

$$C(x,t) \approx \sum_{m=0}^{M} C_m(x,t), \quad (11)$$

h_0 play an important role in the series solution of HAM, because h_0 provided a simple way to adjust and control the convergence region of the series solution. By plotting the h_0-curve, the best value of h_0 can be chosen and selected from the proper range of convergence region.

4. Semi-analytical solution by HAM

In this section, HAM applied for solving the one-dimensional ADE with time-dependent solute dispersion and groundwater flow velocity on the assumption that dispersion is nth power of the seepage velocity because of the non-linearity cause in the mathematical formulation (equation 2). In order to derive the solutions semi-analytically, a general time-dependent source condition is considered at the inlet boundary condition and flux type boundary condition is assumed to be zero at the end of the boundary.

An initial guess is considered as follows:

$$C_0(x,t) = b_0 t\left((x-L)^2 + \frac{\gamma(t)}{\beta(1)} - L^2\right). \quad (12)$$

This initial guess satisfies the designated initial and boundary conditions for equation (2).

We consider the linear operator as follows:

$$Z \equiv \frac{\partial}{\partial t}, \quad (13)$$

with the property

$$Z[j] = 0, \quad (14)$$

where j is the integrating constant and the non-linear operator as:

$$N[\varphi(x,t;q)] = \frac{\partial C}{\partial t} - D_0 f^n(k_1 t)\frac{\partial^2 C}{\partial x^2} + V_0 f(k_1 t)\frac{\partial C}{\partial x} - c_0'\frac{g(x,t)}{g_1}. \quad (15)$$

Using the above procedure described in section 3, the mth order deformation equation can be written as follows:

$$Z[C_m(x,t) - \chi_m C_{m-1}(x,t)] = h_0 \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\varphi(x,t;q)]}{\partial q^{m-1}}\bigg|_{q=0}, \quad (16)$$

with initial condition

$$C_m(x, t=0) = 0, \quad (17)$$

where $\chi_m = 0$ for $m = 1$ and $\chi_m = 1$ for $m > 1$.

Applying inversion of the linear operator given by equation (16), $C_m(x,t)$ can be derived as:

$$C_m(x,t) = \chi_m C_{m-1}(x,t) + h_0\int_0^t \left[\frac{1}{(m-1)!}\frac{\partial^{m-1} N[\varphi(x,t;q)]}{\partial q^{m-1}}\bigg|_{q=0}\right] dt + j, \quad (18)$$

where j can be determined by using the initial guess (equation 17).

In this section, we discuss three types of seepage velocity in the form of $V = V_0 f(k_1 t)$, where $f(k_1 t)$ is in the form of (i) linear, (ii) exponentially decreasing, and (iii) a sinusoidal function. These three velocity patterns are considered because they are applicable to practical problems. Linear velocity may exhibit in the coastal regions. Groundwater velocity may decrease exponentially in high mountainous regions, e.g., in the Himalayan aquifer (Singh and Singh 2001; Singh and Das 2018). Sinusoidal velocity patterns may exhibit seasonal forcing to aquifers in tropical regions (Kumar and Kumar 1998; Thangarajan 2006; Jain *et al.* 2007; Singh *et al.* 2009). In order to derive semi-analytical solutions, we conduct the following specific cases to demonstrate the HAM.

4.1 *Case I: Time-invariant linear line source*

Let $g(x,t) = x$ be the time-invariant linear source. Three time-dependent velocity patterns are discussed.

- Consider $f(k_1 t) = e^{-k_1 t}$ for exponentially decreasing time-dependent velocity.

From equation (18), the first six semi-analytical solutions are

$$C_1(x,t) = h_0 C_0 - \frac{2h_0 D_0 b_0}{n^2 k_1^2}\left(1 - k_1 nte^{-k_1 nt} - e^{-k_1 nt}\right) + \frac{2h_0 V_0 b_0 (x-L)}{k_1^2}\left(1 - k_1 te^{-k_1 t} - e^{-k_1 t}\right) - \frac{c_0' h_0 xt}{g_1}, \tag{19}$$

$$\begin{aligned} C_2(x,t) = {} & (1+h_0) C_1(x,t) - \frac{2h_0^2 D_0 b_0}{n^2 k_1^2}\left(1 - k_1 nte^{-k_1 nt} - e^{-k_1 nt}\right) \\ & + \frac{2h_0^2 V_0 b_0 (x-L)}{k_1^2}\left(1 - k_1 te^{-k_1 t} - e^{-k_1 t}\right) + \frac{2h_0^2 V_0^2 b_0}{k_1^3}\left(1 - e^{-k_1 t}\right) - \frac{h_0^2 V_0^2 b_0}{2k_1^3}\left(1 - 2k_1 te^{-2k_1 t} - e^{-2k_1 t}\right) \\ & - \frac{h_0^2 V_0^2 b_0}{k_1^3}\left(1 - e^{-2k_1 t}\right) - \frac{h_0^2 V_0 c_0'}{g_1 k_1^2}\left(1 - k_1 te^{-k_1 t} - e^{-k_1 t}\right), \end{aligned} \tag{20}$$

$$\begin{aligned} C_3(x,t) = {} & (1+h_0) C_2(x,t) - \frac{2h_0^2 D_0 b_0 (1+h_0)}{n^2 k_1^2}\left(1 - k_1 nte^{-k_1 nt} - e^{-k_1 nt}\right) \\ & + \frac{2h_0^2 (1+h_0) V_0 b_0 (x-L)}{k_1^2}\left(1 - k_1 te^{-k_1 t} - e^{-k_1 t}\right) \\ & + \frac{2h_0^2 (1+2h_0) V_0^2 b_0}{k_1^3}\left(1 - e^{-k_1 t}\right) - \frac{h_0^2 (1+2h_0) V_0^2 b_0}{2k_1^3}\left(1 - 2k_1 te^{-2k_1 t} - e^{-2k_1 t}\right) \\ & - \frac{h_0^2 (1+2h_0) V_0^2 b_0}{k_1^3}\left(1 - e^{-2k_1 t}\right) - \frac{h_0^2 (1+h_0) V_0 c_0'}{g_1}\left(1 - tk_1 e^{-k_1 t} - e^{-k_1 t}\right), \end{aligned} \tag{21}$$

$$\begin{aligned} C_4(x,t) = {} & (1+h_0) C_3(x,t) - \frac{2h_0^2 D_0 b_0 (1+h_0)^2}{n^2 k_1^2}\left(1 - k_1 nte^{-k_1 nt} - e^{-k_1 nt}\right) \\ & + \frac{2h_0^2 (1+h_0)^2 V_0 b_0 (x-L)}{k_1^2}\left(1 - k_1 te^{-k_1 t} - e^{-k_1 t}\right) + \frac{2h_0^2 (1+h_0)(1+3h_0) V_0^2 b_0}{k_1^3}\left(1 - e^{-k_1 t}\right) \\ & - \frac{h_0^2 (1+h_0)(1+3h_0) V_0^2 b_0}{2k_1^3}\left(1 - 2k_1 te^{-2k_1 t} - e^{-2k_1 t}\right) - \frac{h_0^2 (1+h_0)(1+3h_0) V_0^2 b_0}{k_1^3}\left(1 - e^{-2k_1 t}\right) \\ & - \frac{h_0^2 (1+h_0)^2 V_0 c_0'}{g_1 k_1^2}\left(1 - tk_1 e^{-k_1 t} - e^{-k_1 t}\right), \end{aligned} \tag{22}$$

$$\begin{aligned} C_5(x,t) = {} & (1+h_0) C_4(x,t) - \frac{2h_0^2 D_0 b_0 (1+h_0)^3}{n^2 k_1^2}\left(1 - k_1 nte^{-k_1 nt} - e^{-k_1 nt}\right) \\ & + \frac{2h_0^2 (1+h_0)^3 V_0 b_0 (x-L)}{k_1^2}\left(1 - k_1 te^{-k_1 t} - e^{-k_1 t}\right) + \frac{2h_0^2 (1+h_0)^2 (1+4h_0) V_0^2 b_0}{k_1^3}\left(1 - e^{-k_1 t}\right) \\ & - \frac{h_0^2 (1+h_0)^2 (1+4h_0) V_0^2 b_0}{2k_1^3}\left(1 - 2k_1 te^{-2k_1 t} - e^{-2k_1 t}\right) - \frac{h_0^2 (1+h_0)^2 (1+4h_0) V_0^2 b_0}{k_1^3}\left(1 - e^{-2k_1 t}\right) \\ & - \frac{h_0^2 (1+h_0)^3 V_0 c_0'}{g_1 k_1^2}\left(1 - tk_1 e^{-k_1 t} - e^{-k_1 t}\right), \end{aligned} \tag{23}$$

$$C_6(x,t) = (1+h_0)\,C_5(x,t) - \frac{2h_0^2 D_0\, b_0 (1+h_0)^4}{n^2 k_1^2}\left(1 - k_1 n t e^{-k_1 n t} - e^{-k_1 n t}\right)$$
$$+ \frac{2h_0^2 (1+h_0)^4\, V_0\, b_0 (x-L)}{k_1^2}\left(1 - k_1 t e^{-k_1 t} - e^{-k_1 t}\right) + \frac{2h_0^2 (1+h_0)^3 (1+5h_0)\, V_0^2\, b_0}{k_1^3}\left(1 - e^{-k_1 t}\right)$$
$$- \frac{h_0^2 (1+h_0)^3 (1+5h_0)\, V_0^2\, b_0}{2k_1^3}\left(1 - 2k_1 t e^{-2k_1 t} - e^{-2k_1 t}\right) - \frac{h_0^2 (1+h_0)^3 (1+5h_0)\, V_0^2\, b_0}{k_1^3}\left(1 - e^{-2k_1 t}\right)$$
$$- \frac{h_0^2 (1+h_0)^4\, V_0\, c_0'}{g_1 k_1^2}\left(1 - t k_1 e^{-k_1 t} - e^{-k_1 t}\right). \tag{24}$$

Authors approximate the series of concentration with respect to the homotopy embedding parameter q as follows:

$$\varphi(x,t;q) = C_0(x,t) + \sum_{m=1}^{\infty} C_m(x,t)\, q^m. \tag{25}$$

The above series converge at, $q \to 1$ where $\varphi(x,t;1)$ is the sum of all $C_m(x,t)$ terms, which represents a semi-analytical solution for $C(x,t)$. For the practical purpose, we may truncate the series up to $m = 6$ and the approximate solution can be written as follows:

$$C(x,t) = C_0(x,t) + C_1(x,t) + C_2(x,t) + C_3(x,t) + C_4(x,t) + C_5(x,t) + C_6(x,t). \tag{26}$$

- Consider $f(k_1 t) = 1 + k_1 t$ for linear time-dependent velocity. From equation (18), the first six semi-analytical solutions are

$$C_1(x,t) = h_0 C_0 - 2h_0 D_0 b_0\left(\frac{t^2}{2} + \frac{n k_1 t^3}{3} + \frac{n(n-1)k_1 t^4}{8}\right) + 2h_0\, V_0\, b_0 (x-L)\left(\frac{t^2}{2} + k_1\frac{t^3}{3}\right) - \frac{h_0\, c_0' x t}{g_1}, \tag{27}$$

$$C_2(x,t) = (1+h_0)\,C_1(x,t) - 2h_0^2 D_0 b_0\left(\frac{t^2}{2} + \frac{n k_1 t^3}{3} + \frac{n(n-1)k_1 t^4}{8}\right) + 2h_0^2\, V_0\, b_0 (x-L)\left(\frac{t^2}{2} + k_1\frac{t^3}{3}\right)$$
$$+ 2h_0^2\, V_0^2\, b_0\left(\frac{t^3}{6} + k_1\frac{5t^4}{24} + k_1^2\frac{t^5}{15}\right) - \frac{h_0^2\, V_0\, c_0'}{g_1}\left(\frac{t^2}{2} + k_1\frac{t^3}{3}\right), \tag{28}$$

$$C_3(x,t) = (1+h_0)\,C_2(x,t) - 2h_0^2 (1+h_0) D_0 b_0\left(\frac{t^2}{2} + \frac{n k_1 t^3}{3} + \frac{n(n-1)k_1 t^4}{8}\right)$$
$$+ 2h_0^2 (1+h_0)\, V_0\, b_0 (x-L)\left(\frac{t^2}{2} + k_1\frac{t^3}{3}\right) + 2h_0^2 (1+2h_0)\, V_0^2\, b_0\left(\frac{t^3}{6} + k_1\frac{5t^4}{24} + k_1^2\frac{t^5}{15}\right)$$
$$- \frac{h_0^2 (1+h_0)\, V_0\, c_0'}{g_1}\left(\frac{t^2}{2} + k_1\frac{t^3}{3}\right), \tag{29}$$

$$C_4(x,t) = (1+h_0)\,C_3(x,t) - 2h_0^2 (1+h_0)^2 D_0 b_0\left(\frac{t^2}{2} + \frac{n k_1 t^3}{3} + \frac{n(n-1)k_1 t^4}{8}\right)$$
$$+ 2h_0^2 (1+h_0)^2\, V_0\, b_0 (x-L)\left(\frac{t^2}{2} + k_1\frac{t^3}{3}\right) + 2h_0^2 (1+h_0)(1+3h_0)\, V_0^2\, b_0\left(\frac{t^3}{6} + k_1\frac{5t^4}{24} + k_1^2\frac{t^5}{15}\right)$$
$$- \frac{h_0^2 (1+h_0)^2\, V_0\, c_0'}{g_1}\left(\frac{t^2}{2} + k_1\frac{t^3}{3}\right), \tag{30}$$

$$C_5(x,t) = (1+h_0)\,C_4(x,t) - 2h_0^2(1+h_0)^3 D_0 b_0\left(\frac{t^2}{2} + \frac{nk_1t^3}{3} + \frac{n(n-1)k_1t^4}{8}\right)$$
$$+ 2h_0^2(1+h_0)^3 V_0 b_0(x-L)\left(\frac{t^2}{2} + k_1\frac{t^3}{3}\right) + 2h_0^2(1+h_0)^2(1+4h_0)\,V_0^2 b_0\left(\frac{t^3}{6} + k_1\frac{5t^4}{24} + k_1^2\frac{t^5}{15}\right)$$
$$- \frac{h_0^2(1+h_0)^3 V_0 c_0'}{g_1}\left(\frac{t^2}{2} + k_1\frac{t^3}{3}\right), \tag{31}$$

$$C_6(x,t) = (1+h_0)\,C_5(x,t) - 2h_0^2(1+h_0)^4 D_0 b_0\left(\frac{t^2}{2} + \frac{nk_1t^3}{3} + \frac{n(n-1)k_1t^4}{8}\right)$$
$$+ 2h_0^2(1+h_0)^4 V_0 b_0(x-L)\left(\frac{t^2}{2} + k_1\frac{t^3}{3}\right) + 2h_0^2(1+h_0)^3(1+5h_0)\,V_0^2 b_0\left(\frac{t^3}{6} + k_1\frac{5t^4}{24} + k_1^2\frac{t^5}{15}\right)$$
$$- \frac{h_0^2(1+h_0)^4 V_0 c_0'}{g_1}\left(\frac{t^2}{2} + k_1\frac{t^3}{3}\right). \tag{32}$$

Similarly, the semi-analytical approximate solution is the same as equation (26).

- Consider $f(k_1t) = 1 - \sin k_1t$ for sinusoidal time-dependent velocity. From equation (18), the first six semi-analytical solutions are

$$C_1(x,t) = h_0\,C_0(x,t) - 2h_0 D_0 b_0 t^2 + \frac{2h_0 D_0 b_0 n}{k_1^2}(\sin k_1t - k_1t\cos k_1t)$$
$$+ h_0 V_0 b_0(x-L)t^2 - \frac{2h_0 V_0 b_0(x-L)}{k_1^2}(\sin k_1t - k_1t\cos k_1t) - \frac{h_0 c_0' xt}{g_1} \tag{33}$$

$$C_2(x,t) = (1+h_0)\,C_1(x,t) - 2h_0^2 D_0 b_0 t^2 + \frac{2h_0^2 D_0 b_0 n}{k_1^2}(\sin k_1t - k_1t\cos k_1t)$$
$$+ 2h_0^2 V_0 b_0(x-L)\left(\frac{t^2}{2} - \frac{1}{k_1^2}(\sin k_1t - k_1t\cos k_1t)\right)$$
$$+ h_0^2 V_0^2 b_0\left(\frac{t^3}{3} + \frac{t^2}{k_1}\cos k_1t - \frac{2t}{k_1^2}\sin k_1t - \frac{2}{k_1^3}\cos k_1t + \frac{2}{k_1^3}\right)$$
$$\times \frac{2h_0^2 V_0^2 b_0}{k_1^2}\left(\begin{array}{l}\left(\frac{1}{k_1} - \frac{\cos k_1t}{k_1}\right) - k_1\left(\frac{t}{k_1}\sin k_1t - \frac{1}{k_1^2}\cos k_1t - \frac{1}{k_1^2}\right)\\ -\left(\frac{t}{2} - \frac{3}{8k_1}\sin 2k_1t + \frac{t}{4}\cos 2k_1t\right)\end{array}\right)$$
$$- \frac{h_0^2 V_0 c_0'}{g_1}\left(\frac{t^2}{2} - \frac{1}{k_1^2}(\sin k_1t - k_1t\cos k_1t)\right), \tag{34}$$

$$C_3(x,t) = (1+h_0)\,C_2(x,t) - 2h_0^2(1+h_0) D_0 b_0 t^2 + \frac{2h_0^2(1+h_0) D_0 b_0 n}{k_1^2}(\sin k_1t - k_1t\cos k_1t)$$
$$+ 2h_0^2(1+h_0) V_0 b_0(x-L)\left(\frac{t^2}{2} - \frac{1}{k_1^2}(\sin k_1t - k_1t\cos k_1t)\right)$$
$$+ h_0^2(1+2h_0) V_0^2 b_0\left(\frac{t^3}{3} + \frac{t^2}{k_1}\cos k_1t - \frac{2t}{k_1^2}\sin k_1t - \frac{2}{k_1^3}\cos k_1t + \frac{2}{k_1^3}\right)$$
$$\times \frac{2h_0^2(1+2h_0) V_0^2 b_0}{k_1^2}\left(\begin{array}{l}\left(\frac{1}{k_1} - \frac{\cos k_1t}{k_1}\right) - k_1\left(\frac{t}{k_1}\sin k_1t - \frac{1}{k_1^2}\cos k_1t - \frac{1}{k_1^2}\right)\\ -\left(\frac{t}{2} - \frac{3}{8k_1}\sin 2k_1t + \frac{t}{4}\cos 2k_1t\right)\end{array}\right)$$
$$- \frac{h_0^2(1+h_0) V_0 c_0'}{g_1}\left(\frac{t^2}{2} - \frac{1}{k_1^2}(\sin k_1t - k_1t\cos k_1t)\right), \tag{35}$$

$$C_4(x,t) = (1+h_0)C_3(x,t) - 2h_0^2(1+h_0)^2 D_0 b_0 t^2 + \frac{2h_0^2(1+h_0)^2 D_0 b_0 n}{k_1^2}(\sin k_1 t - k_1 t\cos k_1 t)$$
$$+ 2h_0^2(1+h_0)^2 V_0 b_0 (x-L)\left(\frac{t^2}{2} - \frac{1}{k_1^2}(\sin k_1 t - k_1 t \cos k_1 t)\right)$$
$$+ h_0^2(1+h_0)(1+3h_0)V_0^2 b_0\left(\frac{t^3}{3} + \frac{t^2}{k_1}\cos k_1 t - \frac{2t}{k_1^2}\sin k_1 t - \frac{2}{k_1^3}\cos k_1 t + \frac{2}{k_1^3}\right)$$
$$\times \frac{2h_0^2(1+h_0)(1+3h_0)V_0^2 b_0}{k_1^2}\left(\begin{array}{l}\left(\frac{1}{k_1} - \frac{\cos k_1 t}{k_1}\right) - k_1\left(\frac{t}{k_1}\sin k_1 t - \frac{1}{k_1^2}\cos k_1 t - \frac{1}{k_1^2}\right)\\ -\left(\frac{t}{2} - \frac{3}{8k_1}\sin 2k_1 t + \frac{t}{4}\cos 2k_1 t\right)\end{array}\right)$$
$$- \frac{h_0^2(1+h_0)^2 V_0 c_0'}{g_1}\left(\frac{t^2}{2} - \frac{1}{k_1^2}(\sin k_1 t - k_1 t\cos k_1 t)\right), \quad (36)$$

$$C_5(x,t) = (1+h_0)C_4(x,t) - 2h_0^2(1+h_0)^3 D_0 b_0 t^2 + \frac{2h_0^2(1+h_0)^3 D_0 b_0 n}{k_1^2}(\sin k_1 t - k_1 t\cos k_1 t)$$
$$+ 2h_0^2(1+h_0)^3 V_0 b_0 (x-L)\left(\frac{t^2}{2} - \frac{1}{k_1^2}(\sin k_1 t - k_1 t \cos k_1 t)\right)$$
$$+ h_0^2(1+h_0)^2(1+4h_0)V_0^2 b_0\left(\frac{t^3}{3} + \frac{t^2}{k_1}\cos k_1 t - \frac{2t}{k_1^2}\sin k_1 t - \frac{2}{k_1^3}\cos k_1 t + \frac{2}{k_1^3}\right)$$
$$\times \frac{2h_0^2(1+h_0)^2(1+4h_0)V_0^2 b_0}{k_1^2}\left(\begin{array}{l}\left(\frac{1}{k_1} - \frac{\cos k_1 t}{k_1}\right) - k_1\left(\frac{t}{k_1}\sin k_1 t - \frac{1}{k_1^2}\cos k_1 t - \frac{1}{k_1^2}\right)\\ -\left(\frac{t}{2} - \frac{3}{8k_1}\sin 2k_1 t + \frac{t}{4}\cos 2k_1 t\right)\end{array}\right)$$
$$- \frac{h_0^2(1+h_0)^3 V_0 c_0'}{g_1}\left(\frac{t^2}{2} - \frac{1}{k_1^2}(\sin k_1 t - k_1 t\cos k_1 t)\right), \quad (37)$$

$$C_6(x,t) = (1+h_0)C_5(x,t) - 2h_0^2(1+h_0)^4 D_0 b_0 t^2 + \frac{2h_0^2(1+h_0)^4 D_0 b_0 n}{k_1^2}(\sin k_1 t - k_1 t\cos k_1 t)$$
$$+ 2h_0^2(1+h_0)^4 V_0 b_0 (x-L)\left(\frac{t^2}{2} - \frac{1}{k_1^2}(\sin k_1 t - k_1 t \cos k_1 t)\right)$$
$$+ h_0^2(1+h_0)^3(1+5h_0)V_0^2 b_0\left(\frac{t^3}{3} + \frac{t^2}{k_1}\cos k_1 t - \frac{2t}{k_1^2}\sin k_1 t - \frac{2}{k_1^3}\cos k_1 t + \frac{2}{k_1^3}\right)$$
$$\times \frac{2h_0^2(1+h_0)^3(1+5h_0)V_0^2 b_0}{k_1^2}\left(\begin{array}{l}\left(\frac{1}{k_1} - \frac{\cos k_1 t}{k_1}\right) - k_1\left(\frac{t}{k_1}\sin k_1 t - \frac{1}{k_1^2}\cos k_1 t - \frac{1}{k_1^2}\right)\\ -\left(\frac{t}{2} - \frac{3}{8k_1}\sin 2k_1 t + \frac{t}{4}\cos 2k_1 t\right)\end{array}\right)$$
$$- \frac{h_0^2(1+h_0)^4 V_0 c_0'}{g_1}\left(\frac{t^2}{2} - \frac{1}{k_1^2}(\sin k_1 t - k_1 t\cos k_1 t)\right), \quad (38)$$

The semi-analytical approximate solution is the same as equation (26).

4.2 *Case II: Time-dependent line source*

Let $g(x,t) = xt$ be the space–time-dependent line source. We also consider the same three temporal patterns for velocity.

- Given $f(k_1 t) = e^{-k_1 t}$, from equation (18), the first six semi-analytical solutions are:

$$C_1(x,t) = h_0 C_0 - \frac{2h_0 D_0 b_0}{n^2 k_1^2}\left(1 - k_1 nte^{-k_1 nt} - e^{-k_1 nt}\right) + \frac{2h_0 V_0 b_0 (x-L)}{k_1^2}\left(1 - k_1 te^{-k_1 t} - e^{-k_1 t}\right) - \frac{c_0' h_0 xt^2}{2g_1}, \tag{39}$$

$$\begin{aligned} C_2(x,t) = {} & (1+h_0) C_1(x,t) - \frac{2h_0^2 D_0 b_0}{n^2 k_1^2}\left(1 - k_1 nte^{-k_1 nt} - e^{-k_1 nt}\right) \\ & + \frac{2h_0^2 V_0 b_0 (x-L)}{k_1^2}\left(1 - k_1 te^{-k_1 t} - e^{-k_1 t}\right) + \frac{2h_0^2 V_0^2 b_0}{k_1^3}\left(1 - e^{-k_1 t}\right) \\ & - \frac{h_0^2 V_0^2 b_0}{2k_1^3}\left(1 - 2k_1 te^{-2k_1 t} - e^{-2k_1 t}\right) - \frac{h_0^2 V_0^2 b_0}{k_1^3}\left(1 - e^{-2k_1 t}\right) \\ & - \frac{h_0^2 V_0 c_0'}{2g_1 k_1^3}\left(2 - k_1^2 t^2 e^{-k_1 t} - 2k_1 te^{-k_1 t} - 2e^{-k_1 t}\right) \end{aligned} \tag{40}$$

$$\begin{aligned} C_3(x,t) = {} & (1+h_0) C_2(x,t) - \frac{2h_0^2 D_0 b_0 (1+h_0)}{n^2 k_1^2}\left(1 - k_1 nte^{-k_1 nt} - e^{-k_1 nt}\right) \\ & + \frac{2h_0^2 (1+h_0) V_0 b_0 (x-L)}{k_1^2}\left(1 - k_1 te^{-k_1 t} - e^{-k_1 t}\right) + \frac{2h_0^2 (1+2h_0) V_0^2 b_0}{k_1^3}\left(1 - e^{-k_1 t}\right) \\ & - \frac{h_0^2 (1+2h_0) V_0^2 b_0}{2k_1^3}\left(1 - 2k_1 te^{-2k_1 t} - e^{-2k_1 t}\right) - \frac{h_0^2 (1+2h_0) V_0^2 b_0}{k_1^3}\left(1 - e^{-2k_1 t}\right) \\ & - \frac{h_0^2 (1+h_0) V_0 c_0'}{2g_1 k_1^3}\left(2 - t^2 k_1^2 e^{-k_1 t} - 2k_1 te^{-k_1 t} - 2e^{-k_1 t}\right) \end{aligned} \tag{41}$$

$$\begin{aligned} C_4(x,t) = {} & (1+h_0) C_3(x,t) - \frac{2h_0^2 D_0 b_0 (1+h_0)^2}{n^2 k_1^2}\left(1 - k_1 nte^{-k_1 nt} - e^{-k_1 nt}\right) \\ & + \frac{2h_0^2 (1+h_0)^2 V_0 b_0 (x-L)}{k_1^2}\left(1 - k_1 te^{-k_1 t} - e^{-k_1 t}\right) + \frac{2h_0^2 (1+h_0)(1+3h_0) V_0^2 b_0}{k_1^3}\left(1 - e^{-k_1 t}\right) \\ & - \frac{h_0^2 (1+h_0)(1+3h_0) V_0^2 b_0}{2k_1^3}\left(1 - 2k_1 te^{-2k_1 t} - e^{-2k_1 t}\right) - \frac{h_0^2 (1+h_0)(1+3h_0) V_0^2 b_0}{k_1^3}\left(1 - e^{-2k_1 t}\right) \\ & - \frac{h_0^2 (1+h_0)^2 V_0 c_0'}{2g_1 k_1^3}\left(2 - t^2 k_1^2 e^{-k_1 t} - 2k_1 te^{-k_1 t} - 2e^{-k_1 t}\right) \end{aligned} \tag{42}$$

$$\begin{aligned} C_5(x,t) = {} & (1+h_0) C_4(x,t) - \frac{2h_0^2 D_0 b_0 (1+h_0)^3}{n^2 k_1^2}\left(1 - k_1 nte^{-k_1 nt} - e^{-k_1 nt}\right) \\ & + \frac{2h_0^2 (1+h_0)^3 V_0 b_0 (x-L)}{k_1^2}\left(1 - k_1 te^{-k_1 t} - e^{-k_1 t}\right) + \frac{2h_0^2 (1+h_0)^2 (1+4h_0) V_0^2 b_0}{k_1^3}\left(1 - e^{-k_1 t}\right) \\ & - \frac{h_0^2 (1+h_0)^2 (1+4h_0) V_0^2 b_0}{2k_1^3}\left(1 - 2k_1 te^{-2k_1 t} - e^{-2k_1 t}\right) - \frac{h_0^2 (1+h_0)^2 (1+4h_0) V_0^2 b_0}{k_1^3}\left(1 - e^{-2k_1 t}\right) \\ & - \frac{h_0^2 (1+h_0)^3 V_0 c_0'}{2g_1 k_1^3}\left(2 - t^2 k_1^2 e^{-k_1 t} - 2k_1 te^{-k_1 t} - 2e^{-k_1 t}\right) \end{aligned} \tag{43}$$

$$C_6(x,t) = (1+h_0)\,C_5(x,t) - \frac{2h_0^2 D_0 b_0 (1+h_0)^4}{n^2 k_1^2}\left(1 - k_1 n t e^{-k_1 n t} - e^{-k_1 n t}\right)$$
$$+ \frac{2h_0^2 (1+h_0)^4 V_0 b_0 (x-L)}{k_1^2}\left(1 - k_1 t e^{-k_1 t} - e^{-k_1 t}\right) + \frac{2h_0^2 (1+h_0)^3 (1+5h_0) V_0^2 b_0}{k_1^3}\left(1 - e^{-k_1 t}\right)$$
$$- \frac{h_0^2 (1+h_0)^3 (1+5h_0) V_0^2 b_0}{2k_1^3}\left(1 - 2k_1 t e^{-2k_1 t} - e^{-2k_1 t}\right) - \frac{h_0^2 (1+h_0)^3 (1+5h_0) V_0^2 b_0}{k_1^3}\left(1 - e^{-2k_1 t}\right)$$
$$- \frac{h_0^2 (1+h_0)^4 V_0 c_0'}{2 g_1 k_1^3}\left(2 - t^2 k_1^2 e^{-k_1 t} - 2k_1 t e^{-k_1 t} - 2e^{-k_1 t}\right) \tag{44}$$

- Given $f(k_1 t) = 1 + k_1 t$, from equation (18), the first six semi-analytical solutions are:

$$C_1(x,t) = h_0 C_0 - 2h_0 D_0 b_0\left(\frac{t^2}{2} + \frac{n k_1 t^3}{3} + \frac{n(n-1) k_1 t^4}{8}\right) + 2h_0 V_0 b_0 (x-L)\left(\frac{t^2}{2} + k_1 \frac{t^3}{3}\right) - \frac{h_0 c_0' x t^2}{2 g_1} \tag{45}$$

$$C_2(x,t) = (1+h_0)\,C_1(x,t) - 2h_0^2 D_0 b_0\left(\frac{t^2}{2} + \frac{n k_1 t^3}{3} + \frac{n(n-1) k_1 t^4}{8}\right) + 2h_0^2 V_0 b_0 (x-L)\left(\frac{t^2}{2} + k_1 \frac{t^3}{3}\right)$$
$$+ 2h_0^2 V_0^2 b_0\left(\frac{t^3}{6} + k_1 \frac{5t^4}{24} + k_1^2 \frac{t^5}{15}\right) - \frac{h_0^2 V_0 c_0'}{2 g_1}\left(\frac{t^2}{2} + k_1 \frac{t^3}{3}\right) \tag{46}$$

$$C_3(x,t) = (1+h_0)\,C_2(x,t) - 2h_0^2 (1+h_0) D_0 b_0\left(\frac{t^2}{2} + \frac{n k_1 t^3}{3} + \frac{n(n-1) k_1 t^4}{8}\right)$$
$$+ 2h_0^2 (1+h_0) V_0 b_0 (x-L)\left(\frac{t^2}{2} + k_1 \frac{t^3}{3}\right) + 2h_0^2 (1+2h_0) V_0^2 b_0\left(\frac{t^3}{6} + k_1 \frac{5t^4}{24} + k_1^2 \frac{t^5}{15}\right)$$
$$- \frac{h_0^2 (1+h_0) V_0 c_0'}{2 g_1}\left(\frac{t^2}{2} + k_1 \frac{t^3}{3}\right) \tag{47}$$

$$C_4(x,t) = (1+h_0)\,C_3(x,t) - 2h_0^2 (1+h_0)^2 D_0 b_0\left(\frac{t^2}{2} + \frac{n k_1 t^3}{3} + \frac{n(n-1) k_1 t^4}{8}\right)$$
$$+ 2h_0^2 (1+h_0)^2 V_0 b_0 (x-L)\left(\frac{t^2}{2} + k_1 \frac{t^3}{3}\right) + 2h_0^2 (1+h_0)(1+3h_0) V_0^2 b_0\left(\frac{t^3}{6} + k_1 \frac{5t^4}{24} + k_1^2 \frac{t^5}{15}\right)$$
$$- \frac{h_0^2 (1+h_0)^2 V_0 c_0'}{2 g_1}\left(\frac{t^3}{3} + k_1 \frac{t^4}{4}\right) \tag{48}$$

$$C_5(x,t) = (1+h_0)\,C_4(x,t) - 2h_0^2 (1+h_0)^3 D_0 b_0\left(\frac{t^2}{2} + \frac{n k_1 t^3}{3} + \frac{n(n-1) k_1 t^4}{8}\right)$$
$$+ 2h_0^2 (1+h_0)^3 V_0 b_0 (x-L)\left(\frac{t^2}{2} + k_1 \frac{t^3}{3}\right) + 2h_0^2 (1+h_0)^2 (1+4h_0) V_0^2 b_0\left(\frac{t^3}{6} + k_1 \frac{5t^4}{24} + k_1^2 \frac{t^5}{15}\right)$$
$$- \frac{h_0^2 (1+h_0)^3 V_0 c_0'}{2 g_1}\left(\frac{t^3}{3} + k_1 \frac{t^4}{4}\right) \tag{49}$$

$$C_6(x,t) = (1+h_0)\,C_5(x,t) - 2h_0^2 (1+h_0)^4 D_0 b_0\left(\frac{t^2}{2} + \frac{n k_1 t^3}{3} + \frac{n(n-1) k_1 t^4}{8}\right)$$
$$+ 2h_0^2 (1+h_0)^4 V_0 b_0 (x-L)\left(\frac{t^2}{2} + k_1 \frac{t^3}{3}\right) + 2h_0^2 (1+h_0)^3 (1+5h_0) V_0^2 b_0\left(\frac{t^3}{6} + k_1 \frac{5t^4}{24} + k_1^2 \frac{t^5}{15}\right)$$
$$- \frac{h_0^2 (1+h_0)^4 V_0 c_0'}{2 g_1}\left(\frac{t^3}{3} + k_1 \frac{t^4}{4}\right) \tag{50}$$

- Given $f(k_1 t) = 1 - \sin k_1 t$, from equation (18), the first six semi-analytical solutions are:

$$C_1(x,t) = h_0 C_0(x,t) - 2h_0 D_0 b_0 t^2 + \frac{2h_0 D_0 b_0 n}{k_1^2}(\sin k_1 t - k_1 t \cos k_1 t) + 2h_0^2 V_0 b_0 (x-L)\left(\frac{t^2}{2} - \frac{1}{k_1^2}(\sin k_1 t - k_1 t\cos k_1 t)\right) - \frac{h_0 c_0' x t^2}{2g_1} \tag{51}$$

$$\begin{aligned} C_2(x,t) = {} & (1+h_0) C_1(x,t) - 2h_0^2 D_0 b_0 t^2 + \frac{2h_0^2 D_0 b_0 n}{k_1^2}(\sin k_1 t - k_1 t\cos k_1 t) \\ & + 2h_0^2 V_0 b_0 (x-L)\left(\frac{t^2}{2} - \frac{1}{k_1^2}(\sin k_1 t - k_1 t \cos k_1 t)\right) \\ & + h_0^2 V_0^2 b_0 \left(\frac{t^3}{3} + \frac{t^2}{k_1}\cos k_1 t - \frac{2t}{k_1^2}\sin k_1 t - \frac{2}{k_1^3}\cos k_1 t + \frac{2}{k_1^3}\right) \\ & \times \frac{2h_0^2 V_0^2 b_0}{k_1^2}\begin{pmatrix} \left(\frac{1}{k_1} - \frac{\cos k_1 t}{k_1}\right) - k_1\left(\frac{t}{k_1}\sin k_1 t - \frac{1}{k_1^2}\cos k_1 t - \frac{1}{k_1^2}\right) \\ -\left(\frac{t}{2} - \frac{3}{8k_1}\sin 2k_1 t + \frac{t}{4}\cos 2k_1 t\right) \end{pmatrix} \\ & - \frac{h_0^2 V_0 c_0'}{2g_1}\left(\frac{t^3}{3} + \frac{t^2}{k_1}\cos k_1 t - \frac{2t}{k_1^2}\sin k_1 t - \frac{2}{k_1^3}\cos k_1 t + \frac{2}{k_1^3}\right) \end{aligned} \tag{52}$$

$$\begin{aligned} C_3(x,t) = {} & (1+h_0) C_2(x,t) - 2h_0^2(1+h_0) D_0 b_0 t^2 + \frac{2h_0^2(1+h_0) D_0 b_0 n}{k_1^2}(\sin k_1 t - k_1 t\cos k_1 t) \\ & + 2h_0^2(1+h_0) V_0 b_0 (x-L)\left(\frac{t^2}{2} - \frac{1}{k_1^2}(\sin k_1 t - k_1 t \cos k_1 t)\right) \\ & + h_0^2(1+2h_0) V_0^2 b_0 \left(\frac{t^3}{3} + \frac{t^2}{k_1}\cos k_1 t - \frac{2t}{k_1^2}\sin k_1 t - \frac{2}{k_1^3}\cos k_1 t + \frac{2}{k_1^3}\right) \\ & \times \frac{2h_0^2(1+2h_0) V_0^2 b_0}{k_1^2}\begin{pmatrix} \left(\frac{1}{k_1} - \frac{\cos k_1 t}{k_1}\right) - k_1\left(\frac{t}{k_1}\sin k_1 t - \frac{1}{k_1^2}\cos k_1 t - \frac{1}{k_1^2}\right) \\ -\left(\frac{t}{2} - \frac{3}{8k_1}\sin 2k_1 t + \frac{t}{4}\cos 2k_1 t\right) \end{pmatrix} \\ & - \frac{h_0^2(1+h_0) V_0 c_0'}{2g_1}\left(\frac{t^3}{3} + \frac{t^2}{k_1}\cos k_1 t - \frac{2t}{k_1^2}\sin k_1 t - \frac{2}{k_1^3}\cos k_1 t + \frac{2}{k_1^3}\right) \end{aligned} \tag{53}$$

$$\begin{aligned} C_4(x,t) = {} & (1+h_0) C_3(x,t) - 2h_0^2(1+h_0)^2 D_0 b_0 t^2 + \frac{2h_0^2(1+h_0)^2 D_0 b_0 n}{k_1^2}(\sin k_1 t - k_1 t\cos k_1 t) \\ & + 2h_0^2(1+h_0)^2 V_0 b_0 (x-L)\left(\frac{t^2}{2} - \frac{1}{k_1^2}(\sin k_1 t - k_1 t \cos k_1 t)\right) \\ & + h_0^2(1+h_0)(1+3h_0) V_0^2 b_0 \left(\frac{t^3}{3} + \frac{t^2}{k_1}\cos k_1 t - \frac{2t}{k_1^2}\sin k_1 t - \frac{2}{k_1^3}\cos k_1 t + \frac{2}{k_1^3}\right) \\ & \times \frac{2h_0^2(1+h_0)(1+3h_0) V_0^2 b_0}{k_1^2}\begin{pmatrix} \left(\frac{1}{k_1} - \frac{\cos k_1 t}{k_1}\right) - k_1\left(\frac{t}{k_1}\sin k_1 t - \frac{1}{k_1^2}\cos k_1 t - \frac{1}{k_1^2}\right) \\ -\left(\frac{t}{2} - \frac{3}{8k_1}\sin 2k_1 t + \frac{t}{4}\cos 2k_1 t\right) \end{pmatrix} \\ & - \frac{h_0^2(1+h_0)^2 V_0 c_0'}{2g_1}\left(\frac{t^3}{3} + \frac{t^2}{k_1}\cos k_1 t - \frac{2t}{k_1^2}\sin k_1 t - \frac{2}{k_1^3}\cos k_1 t + \frac{2}{k_1^3}\right) \end{aligned} \tag{54}$$

$$\begin{aligned} C_5(x,t) &= (1+h_0)\,C_4(x,t) - 2h_0^2(1+h_0)^3 D_0 b_0 t^2 + \frac{2h_0^2(1+h_0)^3 D_0 b_0 n}{k_1^2}(\sin k_1 t - k_1 t\cos k_1 t) \\ &+ 2h_0^2(1+h_0)^3\, V_0 b_0 (x-L)\left(\frac{t^2}{2} - \frac{1}{k_1^2}(\sin k_1 t - k_1 t\cos k_1 t)\right) \\ &+ h_0^2(1+h_0)^2(1+4h_0)\, V_0^2 b_0\left(\frac{t^3}{3} + \frac{t^2}{k_1}\cos k_1 t - \frac{2t}{k_1^2}\sin k_1 t - \frac{2}{k_1^3}\cos k_1 t + \frac{2}{k_1^3}\right) \\ &\times \frac{2h_0^2(1+h_0)^2(1+4h_0)\, V_0^2 b_0}{k_1^2}\left(\begin{array}{l} \left(\frac{1}{k_1} - \frac{\cos k_1 t}{k_1}\right) - k_1\left(\frac{t}{k_1}\sin k_1 t - \frac{1}{k_1^2}\cos k_1 t - \frac{1}{k_1^2}\right) \\ -\left(\frac{t}{2} - \frac{3}{8k_1}\sin 2k_1 t + \frac{t}{4}\cos 2k_1 t\right) \end{array}\right) \\ &- \frac{h_0^2(1+h_0)^3\, V_0 c_0'}{2g_1}\left(\frac{t^3}{3} + \frac{t^2}{k_1}\cos k_1 t - \frac{2t}{k_1^2}\sin k_1 t - \frac{2}{k_1^3}\cos k_1 t + \frac{2}{k_1^3}\right) \end{aligned} \tag{55}$$

$$\begin{aligned} C_6(x,t) &= (1+h_0)\,C_5(x,t) - 2h_0^2(1+h_0)^4 D_0 b_0 t^2 + \frac{2h_0^2(1+h_0)^4 D_0 b_0 n}{k_1^2}(\sin k_1 t - k_1 t\cos k_1 t) \\ &+ 2h_0^2(1+h_0)^4\, V_0 b_0 (x-L)\left(\frac{t^2}{2} - \frac{1}{k_1^2}(\sin k_1 t - k_1 t\cos k_1 t)\right) \\ &+ h_0^2(1+h_0)^3(1+5h_0)\, V_0^2 b_0\left(\frac{t^3}{3} + \frac{t^2}{k_1}\cos k_1 t - \frac{2t}{k_1^2}\sin k_1 t - \frac{2}{k_1^3}\cos k_1 t + \frac{2}{k_1^3}\right) \\ &\times \frac{2h_0^2(1+h_0)^3(1+5h_0)\, V_0^2 b_0}{k_1^2}\left(\begin{array}{l} \left(\frac{1}{k_1} - \frac{\cos k_1 t}{k_1}\right) - k_1\left(\frac{t}{k_1}\sin k_1 t - \frac{1}{k_1^2}\cos k_1 t - \frac{1}{k_1^2}\right) \\ -\left(\frac{t}{2} - \frac{3}{8k_1}\sin 2k_1 t + \frac{t}{4}\cos 2k_1 t\right) \end{array}\right) \\ &- \frac{h_0^2(1+h_0)^4\, V_0 c_0'}{2g_1}\left(\frac{t^3}{3} + \frac{t^2}{k_1}\cos k_1 t - \frac{2t}{k_1^2}\sin k_1 t - \frac{2}{k_1^3}\cos k_1 t + \frac{2}{k_1^3}\right) \end{aligned} \tag{56}$$

5. Results and discussion

In this present work, authors consider distance unit in km, contaminant concentration unit in mg/l, and time unit in years. Following the available literature (Singh and Kumari 2014) input data are selected ($D_0 = 0.8$ km^2/yr, $V_0 = 0.83$ km/yr, $c_0' =$ 1.0 mg/l-yr, $b_0 = 1.0$ mg/l, $k_1 = k_2 = 1$ yr^{-1}).

The initial guess converges to the final solution when $q = 1$, so authors use this condition in the final solution. However, the series solution contains the non-zero control convergence parameter h_0. Therefore, we have to find the non-zero parameter h_0 such that it ensures convergence. The convergence region and rate of convergence can be controlled by adjusting the parameter h_0 in the HAM. In other words, the non-zero control convergence parameter h_0 can be chosen by the h_0-curve. Figure 1 shows the valid range of h_0 between 0 and -0.5. Thus, the analysis of h_0-curve provides us series solution convergence criterion for h_0. Now we select the parameter $h_0 = -0.03$ (control convergence parameter) for this study.

Three specific functions for $\gamma(t)$ at the inlet boundary are considered in this study, which are: (i) exponentially time-dependent function $\gamma(t) = \exp(-k_2 t)$, (ii) sinusoidally time-dependent function $\gamma(t) = 1 + \sin(k_2 t)$, and (iii) linear time-dependent function $\gamma(t) = (1 + k_2 t)$.

5.1 *Case I: Time-invariant linear line source,* $g(x,t) = x$

We depict the contaminant concentration profiles for two exponent values, $n = 1$ and $n = 1.5$ at $t = 0.8$ yr. Figure 2(a) shows the concentration profiles at time $t = 0.8$ yr for velocity pattern $f(k_1 t) = e^{-k_1 t}$. We observe that concentration at the entry boundary is

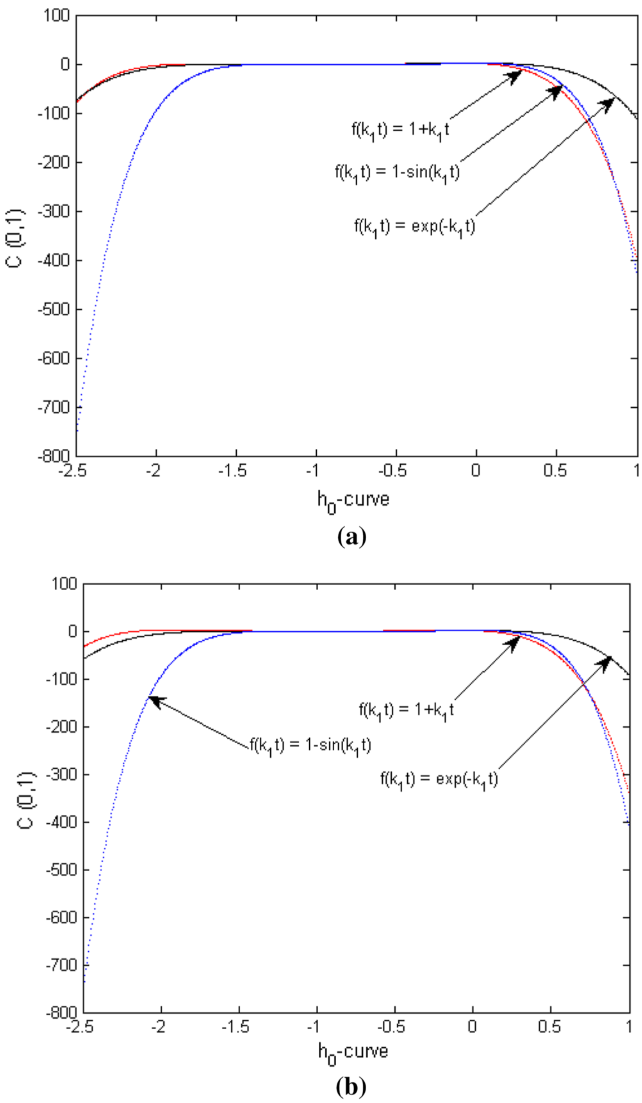

Figure 1. The h_0-curve for the valid region of convergence in the series solution: (**a**) for section 4.1 and (**b**) for section 4.2.

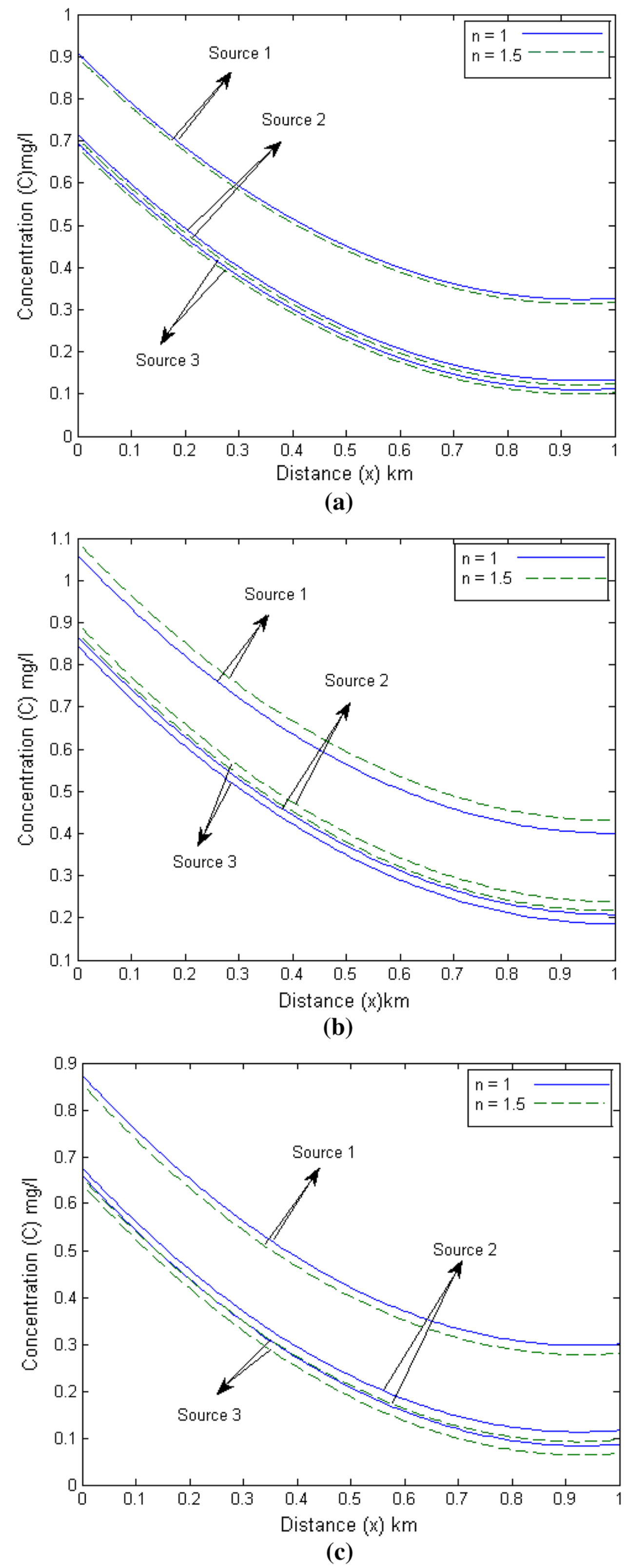

Figure 2. Concentration profiles at time $t = 0.8$ yr given $g(x, t) = x$, entry boundary source 1: $\gamma(t) = \exp(-k_2 t)$, entry boundary source 2: $\gamma(t) = 1 + \sin(k_2 t)$, and entry boundary source 3: $\gamma(t) = 1 + k_2 t$ for (**a**) $f(k_1 t) = e^{-k_1 t}$, (**b**) $f(k_1 t) = 1 + k_1 t$, and (**c**) $f(k_1 t) = 1 - \sin k_1 t$.

approximately 0.9, 0.71 and 0.69 mg/l for the three entry boundary functions $\gamma(t)$, respectively. Similarly, for $f(t) = 1 + k_1 t$, we observe that concentration at the entry boundary is approximately 1.01, 0.87, and 0.85 mg/l for the three entry boundary functions $\gamma(t)$, respectively, shown in figure 2(b). For $f(k_1 t) = 1 - \sin k_1 t$, we observed that concentration is approximately 0.88, 0.67, and 0.65, respectively, shown in figure 2(c). For all these three cases, concentration decreases towards the exit boundary.

For a fixed location at $x = 0.05$ km, figure 3 shows the concentration profiles with respect to time. The two figures, i.e., figure 3(a and c) shows similar results that concentration rapidly increases at the beginning of time (i.e., $0 \le t \le 1$ years) for

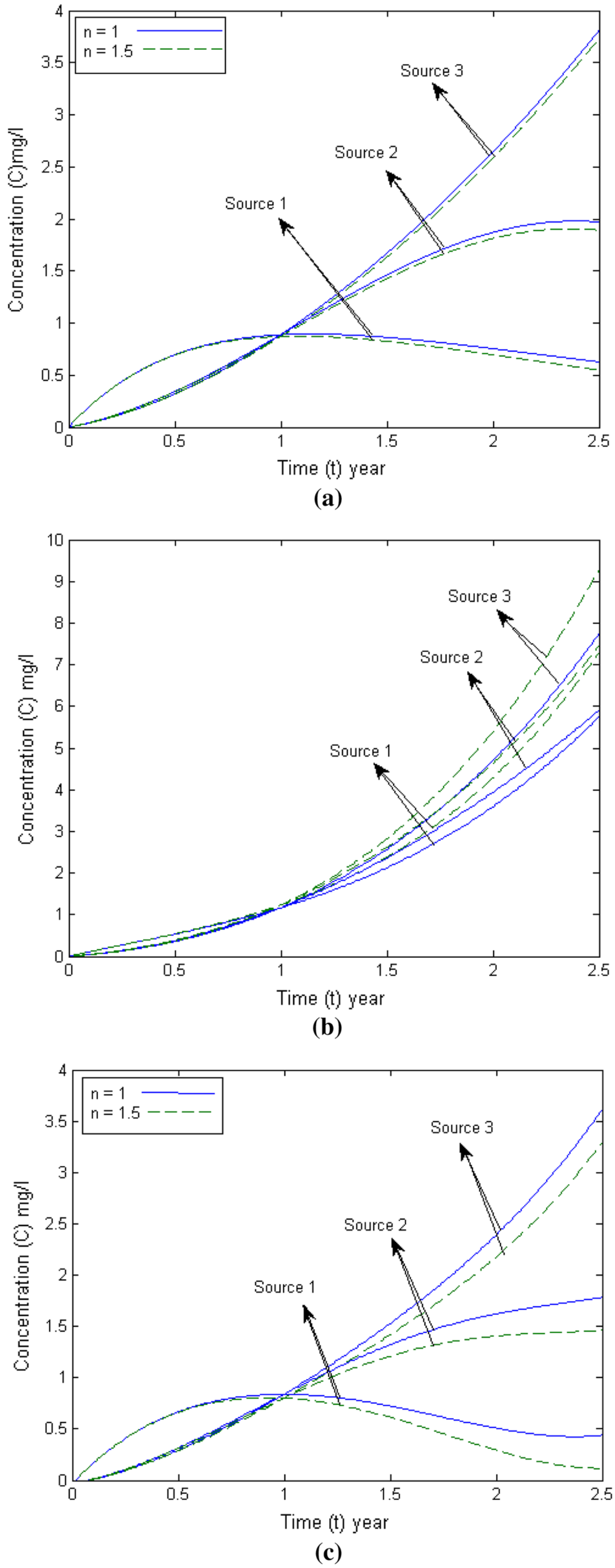

Figure 3. Concentration profiles at location $x = 0.05$ km given $g(x,t) = x$, entry boundary source 1: $\gamma(t) = \exp(-k_2 t)$, entry boundary source 2: $\gamma(t) = 1 + \sin(k_2 t)$, and entry boundary source 3: $\gamma(t) = 1 + k_2 t$ for (**a**) $f(k_1 t) = e^{-k_1 t}$, (**b**) $f(k_1 t) = 1 + k_1 t$, and (**c**) $f(k_1 t) = 1 - \sin k_1 t$.

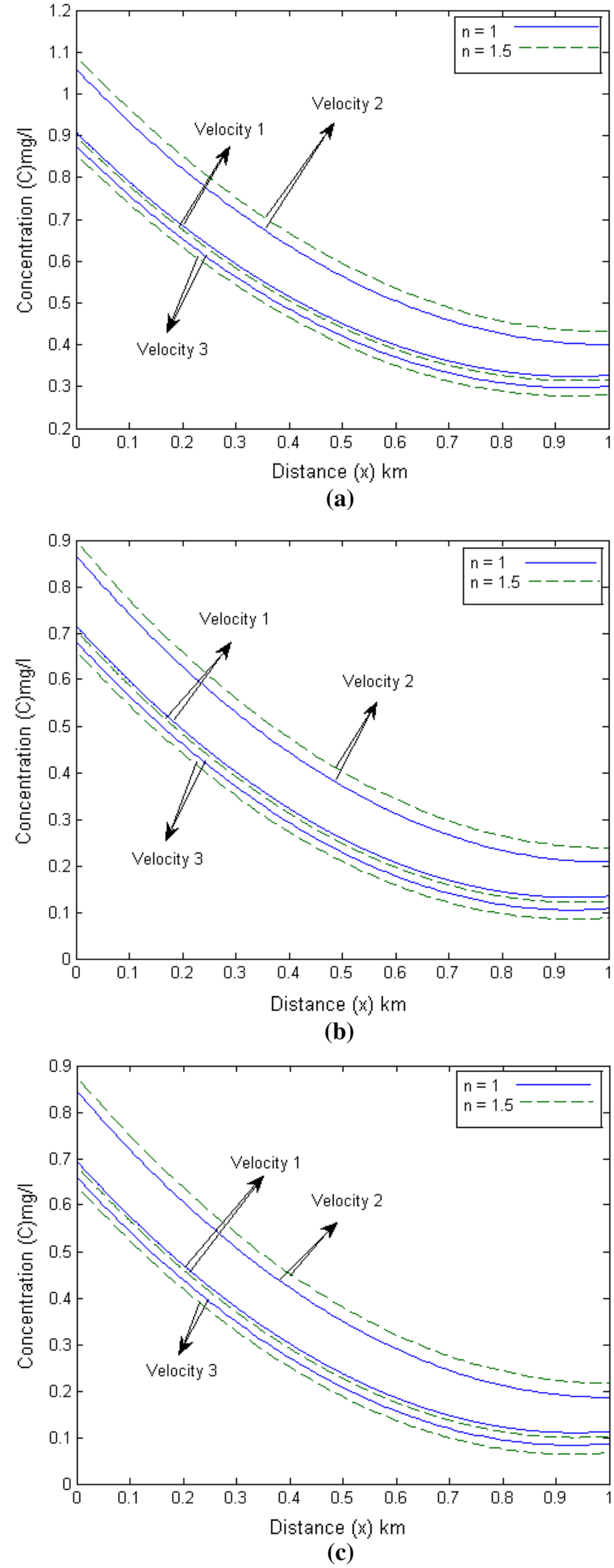

Figure 4. The concentration profiles for the three different velocity patterns in section 4.1 (velocity 1: $f(k_1 t) = \exp(-k_1 t)$, velocity 2: $f(k_1 t) = 1 + k_1 t$, and velocity 3: $f(k_1 t) = 1 - \sin k_1 t$) at the entry source 1: (**a**) $\gamma(t) = \exp(-k_2 t)$), at the entry source 2: (**b**) $\gamma(t) = 1 + \sin(k_2 t)$, and at the entry source 3: (**c**) $\gamma(t) = 1 + k_2 t$.

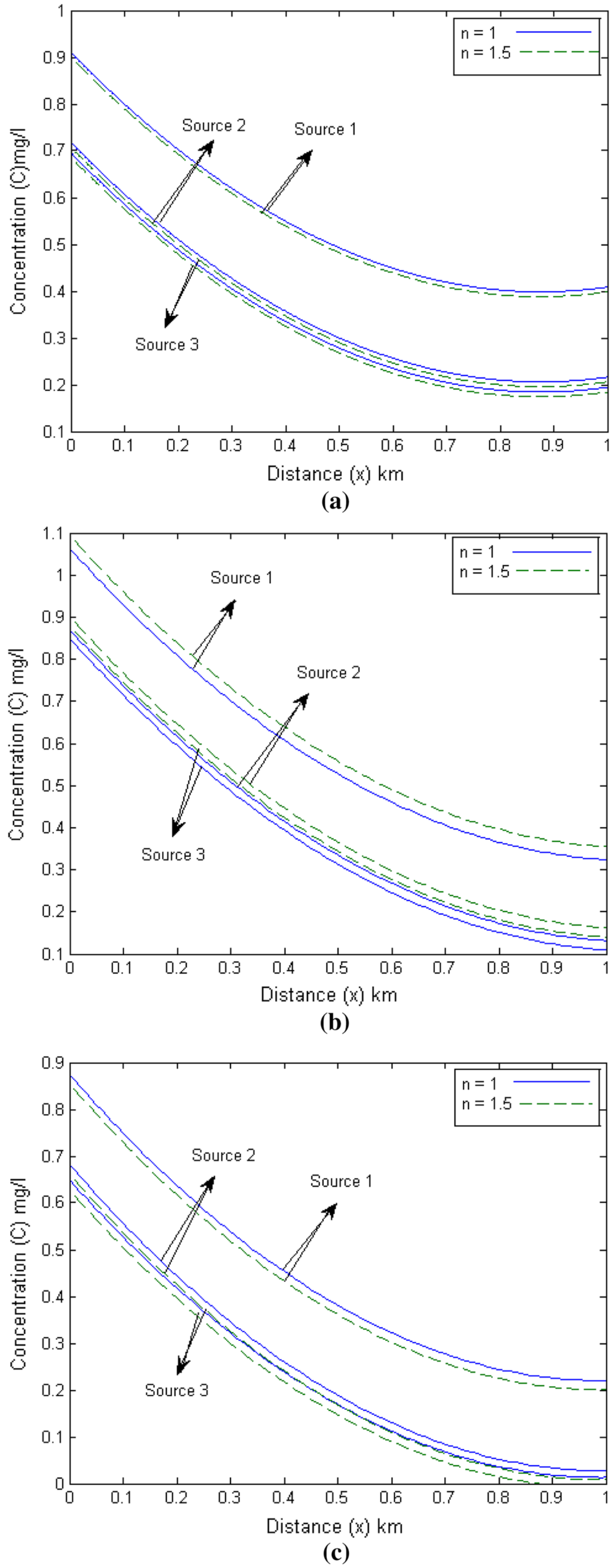

Figure 5. Concentration profiles at time $t = 0.8$ yr given $g(x,t) = xt$, entry boundary source 1: $\gamma(t) = \exp(-k_2 t)$, entry boundary source 2: $\gamma(t) = 1 + \sin(k_2 t)$, and entry boundary source 3: $\gamma(t) = 1 + k_2 t$ for (**a**) $f(k_1 t) = e^{-k_1 t}$, (**b**) $f(k_1 t) = 1 + k_1 t$, and (**c**) $f(k_1 t) = 1 - \sin k_1 t$.

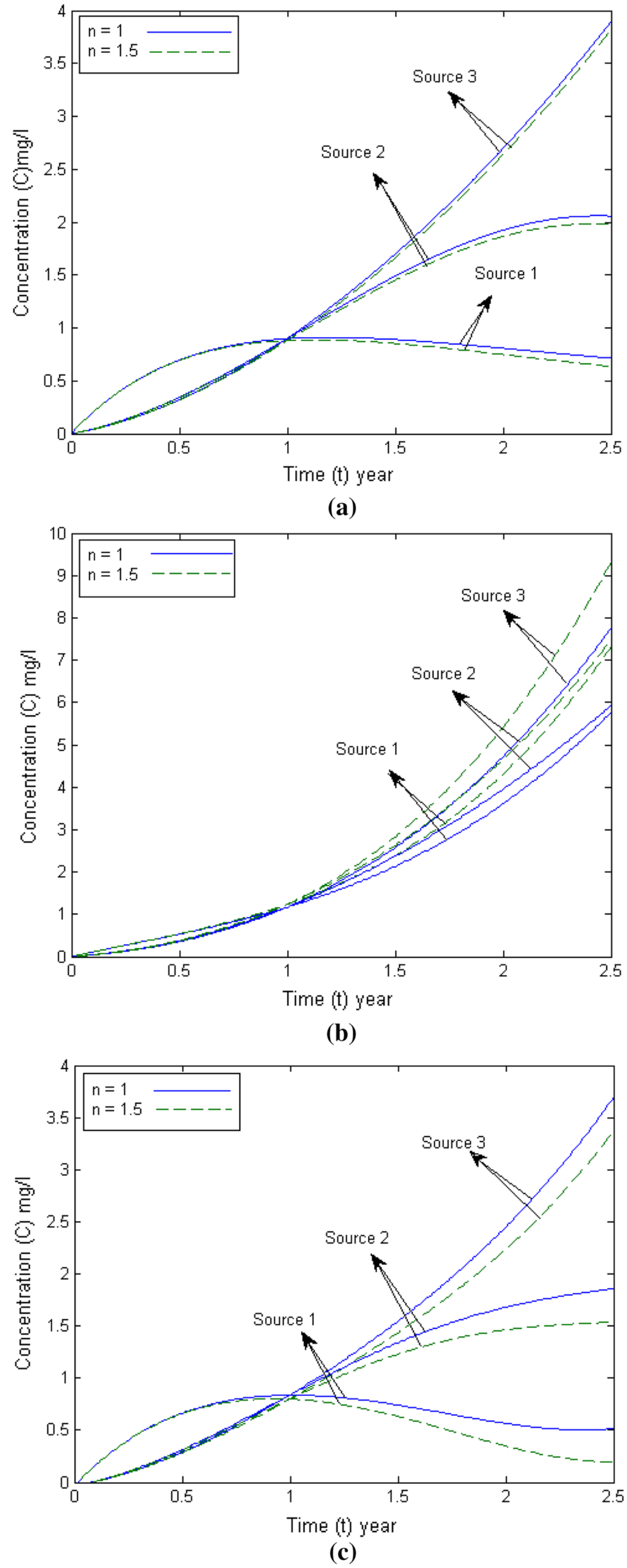

Figure 6. Concentration profiles at location $x = 0.05$ km given $g(x,t) = x$, entry boundary source 1: $\gamma(t) = \exp(-k_2 t)$, entry boundary source 2: $\gamma(t) = 1 + \sin(k_2 t)$, and entry boundary source 3: $\gamma(t) = 1 + k_2 t$ for (**a**) $f(k_1 t) = e^{-k_1 t}$, (**b**) $f(k_1 t) = 1 + k_1 t$, and (**c**) $f(k_1 t) = 1 - \sin k_1 t$.

the case $\gamma(t) = \exp(-k_2 t)$, but after that time i.e., $t = 1$ year, the concentration rapidly decreases with respect to time. At entry source 2 and 3, the concentrations slowly increase at the beginning of time and rapidly increase at a later time as shown in figure 3(a and c). Figure 3(b) shows the concentration profiles for the linear velocity pattern at the entry boundary, concentrations slowly increase at the beginning of time and rapidly increase at a later time. Overall, concentration in these three cases increases towards the exit boundary.

Figure 4(a) shows the concentration profiles for all three velocity patterns at the particular time $t = 0.8$ yr. We observe that concentration at the entry boundary is approximately 1.07, 0.9 and 0.88 mg/l for all velocity profiles, respectively. The concentration given the exponentially decreasing velocity $(f(k_1 t) = \exp(-k_1 t))$ is lower than the linear velocity $(f(k_1 t) = 1 + k_1 t)$, but higher than the sinusoidal velocity $(f(k_1 t) = 1 - \sin(k_1 t))$. Similarly, from figure 4(b), we observe that the concentration initially starts approximately at 0.9, 0.7, and 0.68 mg/l, respectively, for all three velocity profiles at the entry source 2. Furthermore, for entry source 3, we observe that the concentration starts approximately 0.85, 0.7 and 0.65 mg/l, respectively, for all three velocity profiles as shown in figure 4(c). The concentration profiles for all three cases decrease as distance increases. The effect of generalised theory clearly shows that for the two velocity profiles $(f(k_1 t) = \exp(-k_1 t)$ and $f(k_1 t) = 1 - \sin(k_1 t))$, the concentration value for $n = 1$ is higher than that for $n = 1.5$. However, for the linear velocity profile $(f(k_1 t) = 1 + k_1 t)$ the concentration value for $n = 1$ is lower than that for $n = 1.5$.

5.2 *Case II: Time-dependent source, $g(x, t) = xt$*

For the case of $f(k_1 t) = e^{-k_1 t}$, figure 5(a) shows concentration profiles at $t = 0.8$ yr for the three $\gamma(t)$ functions, respectively, given $n = 1$ and $n = 1.5$. The concentration profile at the entry boundary is 0.9, 0.71, and 0.68 mg/l, respectively, for the three $\gamma(t)$ functions as shown in figure 5(a). The concentration decreases when distance increases for all three entry boundaries. Concentration profile for $f(k_1 t) = 1 + k_1 t$ and $f(k_1 t) = 1 - \sin k_1 t$ are shown in figure 5(b and c), respectively. Overall, in all three cases, concentrations rapidly decrease towards the exit boundary. The concentration breakthrough

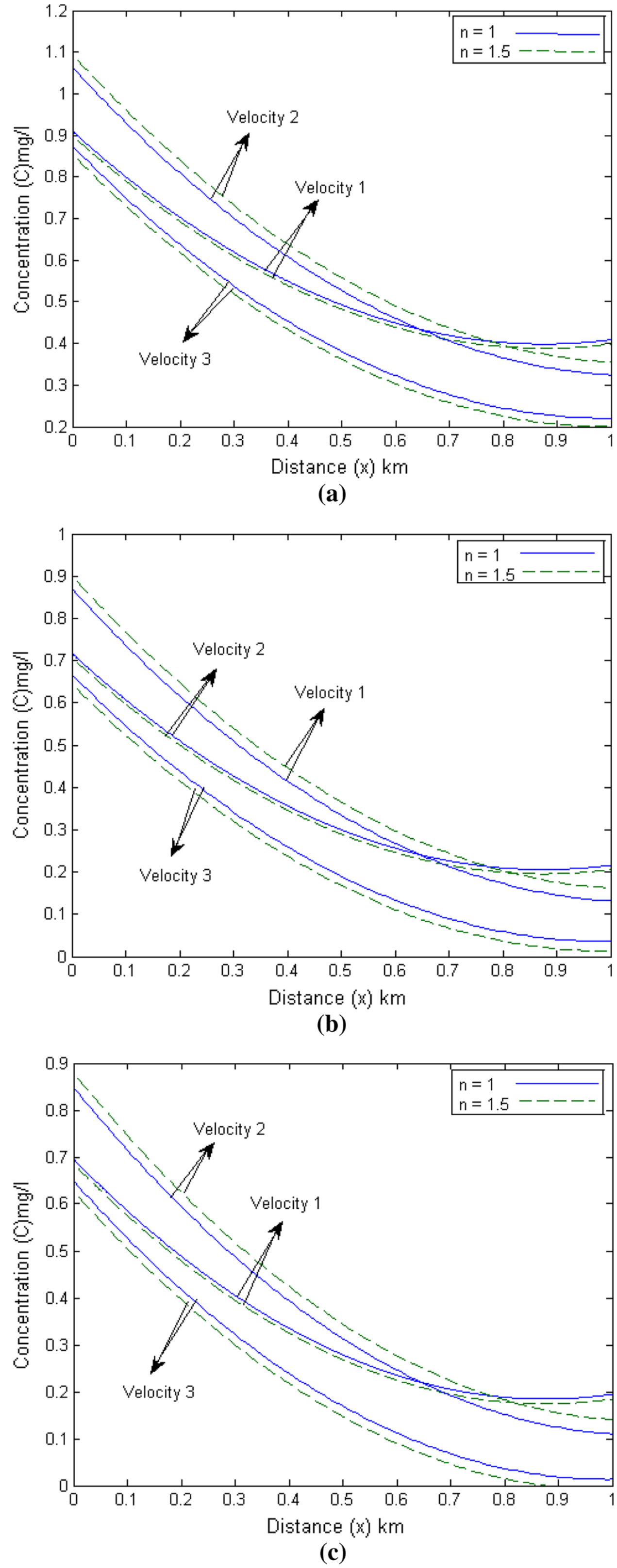

Figure 7. The concentration profiles for the three different velocity patterns in section 4.2 (velocity 1: $f(k_1 t) = \exp(-k_1 t)$, velocity 2: $f(k_1 t) = 1 + k_1 t$, and velocity 3: $f(k_1 t) = 1 - \sin k_1 t$) at the entry source 1: (**a**) $\gamma(t) = \exp(-k_2 t)$ at the entry source 2: (**b**) $\gamma(t) = 1 + \sin(k_2 t)$, and at the entry source 3: (**c**) $\gamma(t) = 1 + k_2 t$.

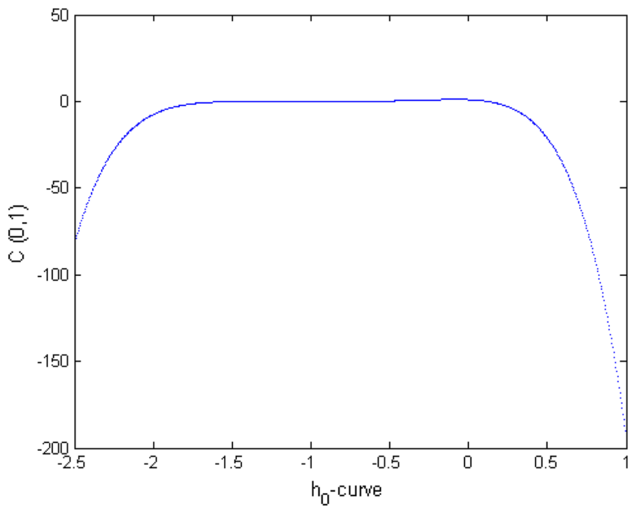

Figure 8. The h_0-curve for section 5.3.

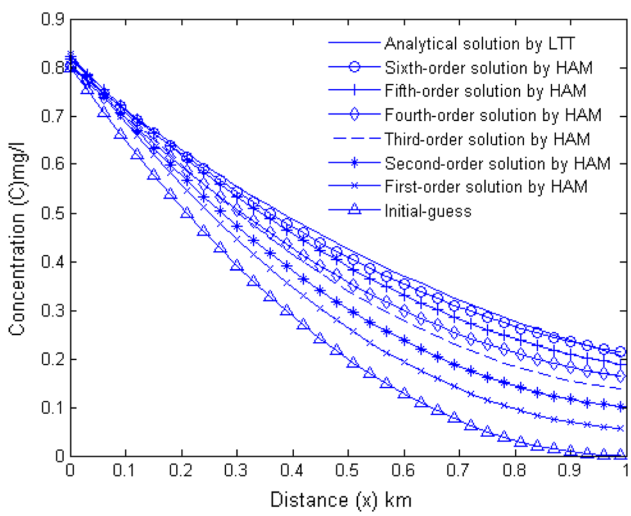

Figure 9. The concentration profiles at $t = 0.8$ yr for $n = 1$, $V_0 = 0.83$ km/year, and $D_0 = 0.8$ km^2/year derived by the Laplace transform method and the HAM.

curves at a fixed location $x = 0.05$ km are shown in figure 6, which are very similar to figure 3.

It has been known that dispersion theory plays an important role in solute transport modelling. Although this study only provides figures to show solutions for the power $n = 1$ and 1.5, the generalised solutions for any exponent value of n between 1 and 2 are given in section 4. The concentration values are higher at each of the positions for $n = 1$ than $n = 1.5$ as shown in figures 2(a, c) and 5(a, c) at particular time $t = 0.8$ yr. But in figures 2(b) and 5(b), the concentration values are higher at each of the positions for $n = 1.5$ than $n = 1$ at the same particular time. Similarly, for the particular location $x = 0.05$ km, concentration values are either higher or lower for $n = 1$ and 1.5 with respect to time as shown in figures 3 and 6, respectively.

In figure 7, the concentration profiles are depicted for all three velocities at the entry source 1, 2, and 3. For entry source 1, concentration values are 1.05, 0.9, and 0.86 mg/l, respectively, for all velocity profiles as shown in figure 7(a). Similarly, for the entry sources 2 and 3, the concentration profiles are shown in figure 7(b and c). From figure 7, it was found that the concentration value for the exponentially decreasing velocity $(f(k_1 t) = \exp(-k_1 t))$ is lower than the linear velocity $(f(k_1 t) = 1 + k_1 t)$ and higher than the sinusoidal velocity $(f(k_1 t) = 1- \sin(k_1 t))$. However, after the distance $x = 0.7$ km, the linear velocity $(f(k_1 t) = 1 + k_1 t)$ is lower than the exponentially decreasing velocity $(f(k_1 t) = \exp(-k_1 t))$ and higher than the sinusoidal velocity $(f(k_1 t) = 1 - \sin(k_1 t))$. The concentration profiles are depicted for two different values $n = 1$ and $n = 1.5$ at $t = 0.8$ year. Overall, the concentration rapidly decreases as distance increases as compared to figure 4.

5.3 *Solution comparison by Laplace transform method*

In this section, authors consider a particular case to compare the semi-analytical solution (HAM) with the analytical solution derived by the Laplace transform method. Now we consider $n = 1$, $c_0' = 0$, and $f(k_1 t) = 1$ in equation (2) with the entry boundary condition $C(0, t) = t$.

The governing equation, the initial condition and the boundary conditions are:

$$\frac{\partial C}{\partial t} = D_0 \frac{\partial^2 C}{\partial x^2} - V_0 \frac{\partial C}{\partial x}, \tag{57}$$

$$C(x, 0) = 0 \quad \text{at} \quad t = 0, \tag{58}$$

$$C(x, t) = t \quad \text{at} \quad x = 0, \tag{59}$$

$$\frac{\partial C}{\partial x} = 0 \quad \text{at} \quad x \to L. \tag{60}$$

Also using the values $k_1 = 0$, $b_0 = 1$ and $c_0' = 0$ in our solution (sections 4.1 or 4.2, 2nd point), we obtain the six-approximate homotopy terms as follows:

$$C_1(x, t) = h_0 C_0 - h_0 D_0 t^2 + h_0 V_0 b_0 (x - L) t^2, \tag{61}$$

$$C_2(x,t) = (1+h_0)C_1(x,t) - h_0^2 D_0 t^2 + h_0^2 V_0 t^2 (x-L) + h_0^2 V_0^2 \frac{t^3}{3}, \quad (62)$$

$$C_3(x,t) = (1+h_0)C_2(x,t) - h_0^2(1+h_0)D_0 t^2 + h_0^2(1+h_0)V_0(x-L)t^2 + h_0^2(1+2h_0)V_0^2\frac{t^3}{3}, \quad (63)$$

$$C_4(x,t) = (1+h_0)C_3(x,t) - h_0^2(1+h_0)^2 D_0 t^2 + h_0^2(1+h_0)^2 V_0(x-L)t^2 + h_0^2(1+h_0)(1+3h_0)V_0^2\frac{t^3}{3}, \quad (64)$$

$$C_5(x,t) = (1+h_0)C_4(x,t) - h_0^2(1+h_0)^3 D_0 t^2 + h_0^2(1+h_0)^3 V_0(x-L)t^2 + h_0^2(1+h_0)^2(1+4h_0)V_0^2\frac{t^3}{3}, \quad (65)$$

$$C_6(x,t) = (1+h_0)C_5(x,t) - h_0^2(1+h_0)^4 D_0 t^2 + h_0^2(1+h_0)^4 V_0(x-L)t^2 + h_0^2(1+h_0)^3(1+5h_0)V_0^2\frac{t^3}{3}. \quad (66)$$

Clearly, this series solution (HAM) contains the non-zero control convergence parameter h_0. The non-zero control convergence parameter h_0 can be chosen and selected by using figure 8. The valid range of h_0 is between 0 and -0.2 (the plateau). Now we select the control convergence parameter $h_0 = -0.11$ for this case.

Using the transformation $C(x,t) = P(x,t) \times \exp\left(\frac{V_0 x}{2D_0} - \frac{V_0^2 t}{4D_0}\right)$ in equations (57–60), location beyond 1 km is treated as infinite. The analytical solution obtained by the Laplace transform method is:

$$C(x,t) = \begin{pmatrix} \frac{1}{4\sqrt{B_1}}\left(2\sqrt{B_1}t - \frac{x}{\sqrt{D_0}}\right)\exp\left(B_1 t - \sqrt{B_1}\frac{x}{\sqrt{D_0}}\right) erfc\left(\frac{x}{2\sqrt{D_0 t}} - \sqrt{B_1 t}\right) \\ + \frac{1}{4\sqrt{B_1}}\left(2\sqrt{B_1}t + \frac{x}{\sqrt{D_0}}\right)\exp\left(B_1 t + \sqrt{B_1}\frac{x}{\sqrt{D_0}}\right) erfc\left(\frac{x}{2\sqrt{D_0 t}} + \sqrt{B_1 t}\right) \end{pmatrix} \times \exp\left(\frac{V_0 x}{2D_0} - \frac{V_0^2 t}{4D_0}\right) \quad (67)$$

where $B_1 = \frac{V_0^2}{4D_0}$.

In figure 9, the input values from the existing literature (Gelhar *et al.* 1992; Singh and Kumari 2014) were used. Gelhar *et al.* (1992) showed the effect of scale-dispersivity relationship in the aquifer and its range may vary from 10^{-3} m to 10^4 m. The semi-analytical solutions for different orders of m with respect to the LTT analytical solution are shown in figure 9. A good agreement (<5%) is observed between LTT and HAM at the 6th order solution. The accuracy of the obtained results is calculated by using the root mean square error (RMSE) method, which is defined as:

$$\text{RMSE} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}|\Delta C_i|^2}, \quad (68)$$

where $\Delta C = C_{\text{analytical}} - C_{\text{semi-analytical}}$ and N is the number of data. The root mean square error is 0.0118.

6. Conclusions

The generalized semi-analytical solutions are derived by the HAM for one-dimensional advection–dispersion equation with generalised time-dependent dispersion theory. The derived semi-analytical solutions are applicable to any (real number) value of power n, which is the most generalization. In this present work, the impact is shown for $1 < n < 2$. The solutions are mostly applicable for any fractional value of n (e.g., $n = 1.3$ or 1.78). So, these solutions add advancement to the existing literature in the field of contaminant transport in aquifers. Although this is the main conclusion of the work, we also have some general concluding remarks as follows:

(1) The effective and valid region for semi-analytical solutions was obtained by plotting the h_0-curve as shown in figures 1 and 8. It was found that h_0 plays an important role in the convergence of the series solution because h_0 provides a convenient way to control and adjust the convergence region of the series solution obtained by the HAM.
(2) Due to the effect of generalised dispersion theory, the concentration is lower for the exponentially decreasing and sinusoidal velocity profiles for $n = 1.5$ than $n = 1$. However, for the linear velocity profile, the concentration is higher for $n = 1.5$ than $n = 1$.
(3) The comparison of the velocity profiles clearly indicates that the concentration for the exponentially decreasing velocity profile is lower than the linear velocity profile and higher than the sinusoidal velocity profile. Similarly, we observed that the comparison of the velocity profiles for some distance are the same, but after a certain distance (e.g., $x = 0.7$ km), the linear velocity profile is lower than the exponentially decreasing velocity profile and higher than the sinusoidal velocity profile.
(4) The obtained solution shows that the contaminant concentration profiles change with fractal power. The power of velocity has a significant influence on the obtained solution.
(5) For a simplified case, the semi-analytical solution is validated with the solution obtained by the LTT and good agreement is found for 6th order of solution obtained by HAM.
(6) The study concludes that the HAM is an efficient and flexible method to solve highly non-linear solute transport equations with general initial conditions.

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Author statement

Rohit Kumar: Conceptualization, methodology, writing – original draft preparation, investigation, validation. Ayan Chatterjee: Formal analysis. Mritunjay Kumar Singh: Supervision, writing – reviewing and editing. Frank T-C Tsai: Reviewing and editing.

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