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On Out of Plane Equilibrium Points in Photo-Gravitational Restricted Three-Body Problem

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Abstract. We have investigated the out of plane equilibrium points of a passive micron size particle and their stability in the field of radiating binary stellar systems *Krüger*-60, *RW-Monocerotis* within the framework of photo-gravitational circular restricted three-body problem. We find that the out of plane equilibrium points (L_i, $i = 6, 7, 8, 9$) may exist for range of β_1 (ratio of radiation to gravitational force of the massive component) values for these binary systems in the presence of Poynting–Robertson drag (hereafter PR-drag). In the absence of PR-drag, we find that the motion of a particle near the equilibrium points $L_{6,7}$ is stable in both the binary systems for a specific range of β_1 values. The PR-drag is shown to cause instability of the various out of plane equilibrium points in these binary systems.

Key words. Radiation—Poynting–Robertson drag—binary stellar system—equilibrium points—stability.

1. Introduction

The photo-gravitational circular restricted three-body problem was first studied by Radzievskii (1950, 1953). In this work, besides the coplanar libration points L_i, $i = 1$ to 5, the effect of radiation was shown to result in the libration points $L_{6,7}$ that exist in a plane perpendicular to the orbital plane of the radiating primaries. Since then several authors (cf. Chernikov 1970; Perezhogin 1976; Scheurman 1980; Simmons *et al.* 1985; Ragos & Zagouras 1988; Murray 1994; Ragos & Zafiropoulos 1995; Ragos *et al.* 1995; Roman 2001; Kunitsyn & Chudayeva 2003; Kushvah & Ishwar 2004; Das *et al.* 2008a) extended the work to understand various issues related to the dynamics of a particle around radiating primaries. However, majority of these works involve the use of independent quantities $q_1 = 1 - \beta_1$ and $q_2 = 1 - \beta_2$, where β_i corresponds to the ratio of radiation pressure force to the gravitational force of i-th binary component. Since β_i does depend on the size, density of the particle and mass and luminosity of the respective binary component, we apply a realistic relation connecting the parameters β_1 and β_2 and study the location and stability of out of plane equilibrium points of a micron size particle moving around a radiating binary stellar system. Incorporating the PR-drag effect, we observe that the libration points $L_{6,7}$ exist for certain range of values of β_1 for the binary systems *Krüger*-60 and *RW-Monocerotis*. Further, certain β_1 values exist for which it is also possible to have four libration points, i.e., L_i, $i = 6, 7, 8, 9$ in these binary stars. Using linear stability analysis, it is observed that the stability of motion around any of these points depends on the parameter β_1 and β_2 involving physical parameters, i.e., mass and luminosity of the given binary system. For the binary stellar systems considered here, we find that all such equilibrium points are unstable. However, in the absence of PR-drag we observe that it is possible to have linearly stable motion around $L_{6,7}$ for certain β_1 values in both the binary systems.

2. The location and stability of out of plane equilibrium points

Following Ragos & Zafiropoulos (1995) and Ragos *et al.* (1995) the equation of motion of an infinitesimal mass moving in the radiation and gravitational field of the binary system, in a rotating barycentric co-ordinate system (cf. Szebeheley 1967; Hénon 1983), could be written as:

$$\ddot{X} - 2\dot{Y} = \Omega_x, \quad \ddot{Y} + 2\dot{X} = \Omega_y, \quad \ddot{Z} = \Omega_z, \tag{1}$$

where

$$\begin{aligned}\Omega_x = X &- \frac{Q_1(X+\mu)}{r_1^3} - \frac{Q_2(X+\mu-1)}{r_2^3} \\ &- \frac{W_1}{r_1^2}\left[\frac{X+\mu}{r_1^2}((X+\mu)\dot{X} + Y\dot{Y} + Z\dot{Z}) + \dot{X} - Y\right] \\ &- \frac{W_2}{r_2^2}\left[\frac{X+\mu-1}{r_2^2}((X+\mu-1)\dot{X} + Y\dot{Y} + Z\dot{Z}) + \dot{X} - Y\right]\end{aligned} \tag{2}$$

$$\begin{aligned}\Omega_y = Y &\left(1 - \frac{Q_1}{r_1^3} - \frac{Q_2}{r_2^3}\right) \\ &- \frac{W_1}{r_1^2}\left[\frac{Y}{r_1^2}((X+\mu)\dot{X} + Y\dot{Y} + Z\dot{Z}) + \dot{Y} + X + \mu\right] \\ &- \frac{W_2}{r_2^2}\left[\frac{Y}{r_2^2}((X+\mu-1)\dot{X} + Y\dot{Y} + Z\dot{Z}) + \dot{Y} + X + \mu - 1\right]\end{aligned} \tag{3}$$

$$\begin{aligned}\Omega_z = &\left[-\frac{Q_1}{r_1^3} - \frac{Q_2}{r_2^3}\right] Z \\ &- \frac{W_1}{r_1^2}\left[\frac{Z}{r_1^2}((X+\mu)\dot{X} + Y\dot{Y} + Z\dot{Z}) + \dot{Z}\right] \\ &- \frac{W_2}{r_2^2}\left[\frac{Z}{r_2^2}((X+\mu-1)\dot{X} + Y\dot{Y} + Z\dot{Z}) + \dot{Z}\right]\end{aligned} \tag{4}$$

$$\mu = \frac{M_2}{M_1 + M_2}, \quad Q_1 = q_1(1-\mu), \quad Q_2 = q_2\mu, \tag{5}$$

$$W_1 = \frac{(1-q_1)(1-\mu)}{C_d}, \quad W_2 = \frac{(1-q_2)\mu}{C_d}, \tag{6}$$

$$r_1^2 = (X+\mu)^2 + Y^2 + Z^2, \quad r_2^2 = (X+\mu-1)^2 + Y^2 + Z^2. \tag{7}$$

Here, M_1 and M_2 refer to masses of the respective binary component; $C_d = c/v_{\rm in}$ corresponds to the dimensionless velocity of light and depends on the physical masses of primaries and distance between them; $q_{1,2} = 1 - \beta_{1,2}$ corresponds to radiation parameters from the respective primaries; r_1 and r_2 correspond to the distances between the third body and primaries. Further, β_i corresponds to the ratio of force due to radiation and the gravitational force of the i-th binary component (cf. Das *et al.* 2008b) from the i-th binary component. It is important to note that for solar dust particles less than a μm comprising spherical silicate BPCA, carbon BPCA, silicate compact, asteroidal dust, young and cometary dust grains, β may vary in the range $\sim 10^{-2}$–5.0 (cf. Wilck & Mann 1996; Krivov *et al.* 1998; Kimura *et al.* 2002). Therefore in a real situation, it is possible to have $\beta_i \geq 1$. However, there exists a relation:

$$\beta_2 = \beta_1 \frac{L_2}{L_1} \frac{M_1}{M_2}, \tag{8}$$

which connects the radiation parameters of respective binary components in terms of their luminosities and masses. We use the above relation to fix the value of the parameter β_2 in terms of the mass and luminosity of the binary components for a given $\beta_1 > 1$. Therefore, the quantities q_1 and q_2 are not independent. It may be noted that several authors (cf. Ragos & Zafiropoulos 1995; Ragos *et al.* 1995 and references quoted therein) have used the radiation parameters q_1 and q_2 as independent. It is in this sense their results are of limited applicability to the motion of a particle in stellar binary systems in general. Besides the classical coplanar equilibrium points (cf. Ragos & Zafiropoulos 1995; Das *et al.* 2008b), it is possible to have out of plane equilibrium points exclusively due to radiation from binary components. Such points do not have any classical analogue. In the following, we discuss the effect of radiation on the location and stability of possible equilibrium points in two steps. First only the major radial component of the pressure force is considered so that the problem is reduced to that of a central force only (cf. Radzievskii 1950, 1953). In fact this approach to the problem is already an approximate one: for particles with velocity $\mathbf{v}$, terms of order $\mathbf{v}/c$ and higher in the general radiation force term are neglected. In fact, due to radiation, the radiation force F on a particle may be written as $F = F_p + F_{PR}$ (cf. Robertson 1937) where F_{PR} is the Poynting–Robertson drag and corresponds to a first order term in $\mathbf{v}/c$. For a 1 μm dust particle at a distance of 1 AU from sun $F_{PR}/F_P \sim 10^{-4}$ and therefore significant changes in the location of various equilibrium points are unlikely by the inclusion of F_{PR} terms. However, the inclusion of F_{PR} drag term change the nature of problem from purely a central force to a dissipative one. Therefore, in the second step, we incorporate F_{PR} term as well and show that the various out-of plane equilibrium points of the binary system become unstable.

2.1 *Location and stability of equilibrium points in the absence of PR-drag*

In the absence of PR-drag, the stationary solutions of equation (1), for the case $Z \neq 0$, results in the following conditions:

$$X_0 - \frac{Q_1(X_0+\mu)}{r_{10}^3} - \frac{Q_2(X_0+\mu-1)}{r_{20}^3} = 0, \tag{9}$$

$$Y_0 = 0, \tag{10}$$

$$\left[\frac{Q_1}{r_{10}^3} + \frac{Q_2}{r_{20}^3}\right] = 0, \tag{11}$$

where

$$r_{10}^2 = (X_0+\mu)^2 + Z_0^2, \quad r_{20}^2 = (X_0+\mu-1)^2 + Z_0^2 \tag{12}$$

and the subscript '0' is used to denote the equilibrium values. Since $Z_0 \neq 0$, we observe that equations (9)–(11) are satisfied if:

$$X_0 = \frac{Q_1}{r_{10}^3} = -\frac{Q_2}{r_{20}^3}. \tag{13}$$

Obviously the last equation results in either $Q_1 < 0,\ Q_2 > 0$ or $Q_1 > 0,\ Q_2 < 0$, i.e., $Q_1 Q_2 < 0$. In view of the relation $q_{1,2} = 1 - \beta_{1,2}$, we observe that $Q_1 Q_2 < 0$ implies that either $\beta_1 > 1,\ \beta_2 < 1$ or $\beta_1 < 1,\ \beta_2 > 1$. Since for particles around 0.1 μm it is possible to have $\beta_1 > 1$, we find the possibilities of having $(1-\beta_1) < 0$ and $(1-\beta_2) > 0$ in some binary stellar system.

On elimination of Z_0, we may rewrite equation (13) as:

$$\Phi(X_0) = 8X_0^5 + 12(2\mu-1)X_0^4 + 6(2\mu-1)^2 X_0^3 + (2\mu-1)^3 X_0^2 - (Q_1^{2/3} - Q_2^{2/3})^3 = 0. \tag{14}$$

The solution of equation (14) for real X_0 along with real Z_0 obtained from:

$$Z_0 = \pm\left[\left(\frac{Q_1}{X_0}\right)^{2/3} - (X_0+\mu)^2\right]^{1/2}, \tag{15}$$

using equation (12), provides the equilibrium point in the X–Z plane.

It is readily observed that $X_0 = (1-2\mu)/2$ is a solution of equation (14) in case $|Q_1| = |Q_2|$. Further, from the plot of $\Phi(X_0)$ *vs.* X_0 (cf. Fig. 1), we observe that for certain β_1 value, if $|Q_1| < |Q_2|$ or $|Q_1| > |Q_2|$, only one real solution of equation (14) occurs in the region $X_0 < 0$ and $X_0 > 0$, respectively for *Krüger*-60 (curves 1 and 3 of Fig. 1). The equilibrium point $(X_0, 0, \pm Z_0)$ lying in the region $X_0 < 0$ and referred to as $L_{6,7}$ (cf. Radzievskii 1953) was shown to exist for a range of values of β_1 for binary system *RW-Monocerotis* and *Krüger*-60 (Das *et al.* 2008b). However, it is interesting to observe that for certain β_1 values three possible real root occurs (curve 2, Fig. 1). Of these three roots, one lies in the region $X_0 < 0$, and two roots in the region $X_0 > 0$. For such β_1 values, the root of equation (14) lying in the region $X_0 < 0$ results in

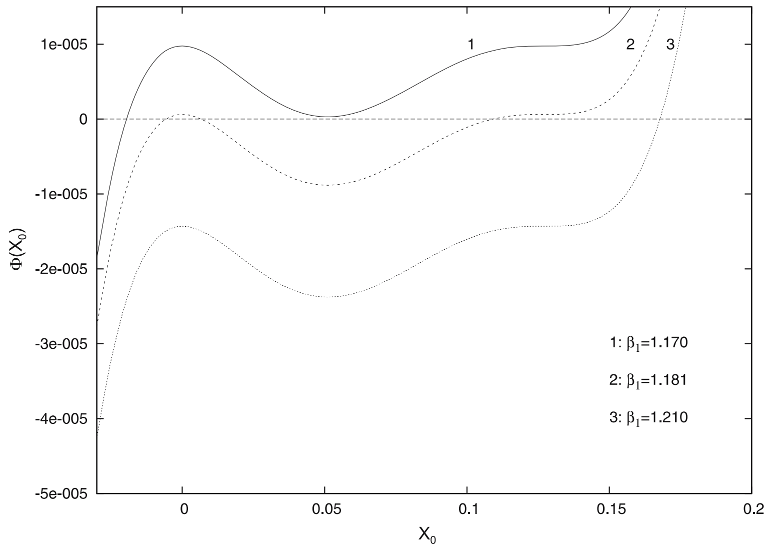

Figure 1. Variation of $\Phi(X_0)$ with X_0 in *Krüger*-60.

two equilibrium points $L_{6,7}$ while the two roots in the region $X_0 > 0$ provide four equilibrium points L_6, L_7, L_8 and L_9 for the two binary systems considered here. Earlier Lukyanov (1984) and Simmons *et al.* (1985) reported the possible existence of such equilibrium points in the general photo-gravitational restricted circular three-body problem in the absence of PR-drag. It may be noted that the application of equation (8) as a relation between β_i's $(i = 1, 2)$ results in the location of various equilibrium points dependent on a single parameter β_1 rather than two independent parameters β_1 and β_2 considered earlier by Lukyanov (1984), Simmons *et al.* (1985) and Ragos & Zagouras (1988). Further since β_2 depends not only on β_1 but also on the physical parameters like mass and luminosity of the binary components, the present computational results of Fig. 2 showing the relationship between various components of L_6 and L_8 along with their variation with β_1 are expected to be more realistic for *Krüger*-60 and *RW-Monocerotis*.

In this work we have confined ourselves to linear stability analysis. The characteristic equation used for computing the eigenvalues for a given binary system is same as in Das *et al.* (2008b). Table 1 listing the real and imaginary components of the characteristic equation for various β_1 values for both the binary system clearly shows the possibility of having stable motion around $L_{6,7}$ in the domain $X_0 > 0$ (as all the eigen values are purely imaginary). However, the motion around $L_{6,7}$ in the domain $X_0 < 0$ is unstable. Further for all β_1 values considered, the motion around $L_{8,9}$ is also found to be unstable in these binary systems.

2.2 *Location and stability of the out of plane equilibrium points in the presence of PR-drag*

Following Ragos *et al.* (1995), we find the out of plane equilibrium points in the presence of PR-drag as solution of the following equations.

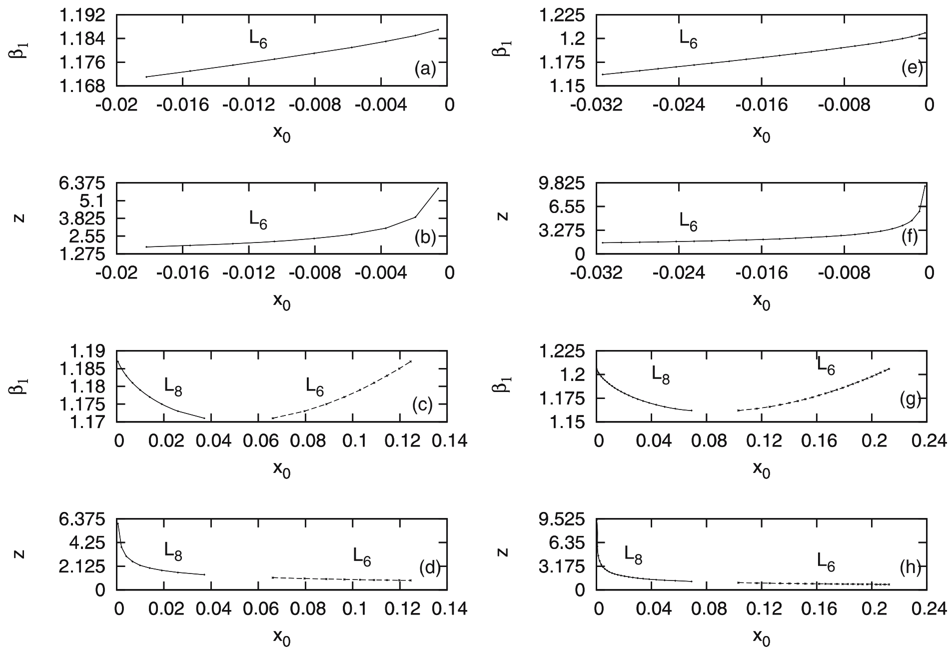

Figure 2. **(a), (e)** Variation of L_{6_x} with β_1 in the region $X_0 < 0$, **(b), (f)** variation of L_{6_z} with L_{6_x} in the region $X_0 < 0$, **(c), (g)** variation of L_{8_x} and L_{6_x} with β_1 in the region $X_0 > 0$, **(d)–(h)** variation of L_{8_z} and L_{6_z} with L_{6_x} in the region $X_0 > 0$ for *Krüger*-60 and *RW-Monocerotis*, respectively.

Table 1. Variation of eigenvalues of $L_{6,7}$ with β_1 for $X_0 > 0$.

Binary	β_1	λ_1	λ_2	λ_3
RW-Monocerotis	1.162	(0, 0.36217024)	(0, 0.84961841)	(0, 1.0709721)
	1.164	(0, 0.42888167)	(0, 0.76875324)	(0, 1.1068328)
	1.166	(0, 0.50137373)	(0, 0.68684667)	(0, 1.1299850)
Krüger-60	1.171	(0, 0.19666655)	(0, 0.92009171)	(0, 1.0558189)
	1.177	(0, 0.29345436)	(0, 0.79985195)	(0, 1.1287698)
	1.183	(0, 0.37515521)	(0, 0.70016981)	(0, 1.1700516)
	1.187	(0, 0.45360428)	(0, 0.61042658)	(0, 1.1923181)

$$X_0 - \frac{Q_1(X+\mu)}{r_1^3} - \frac{Q_2(X_0+\mu-1)}{r_2^3} + \left[\frac{W_1}{r_1^2} + \frac{W_2}{r_2^2}\right] Y_0 = 0, \tag{16}$$

$$\left[1 - \frac{Q_1}{r_1^3} - \frac{Q_2}{r_2^3}\right] Y_0 - \frac{W_1(X_0+\mu)}{r_1^2} - \frac{W_2(X_0+\mu-1)}{r_2^2} = 0, \tag{17}$$

$$\frac{Q_1}{r_1^3} + \frac{Q_2}{r_2^3} = 0. \tag{18}$$

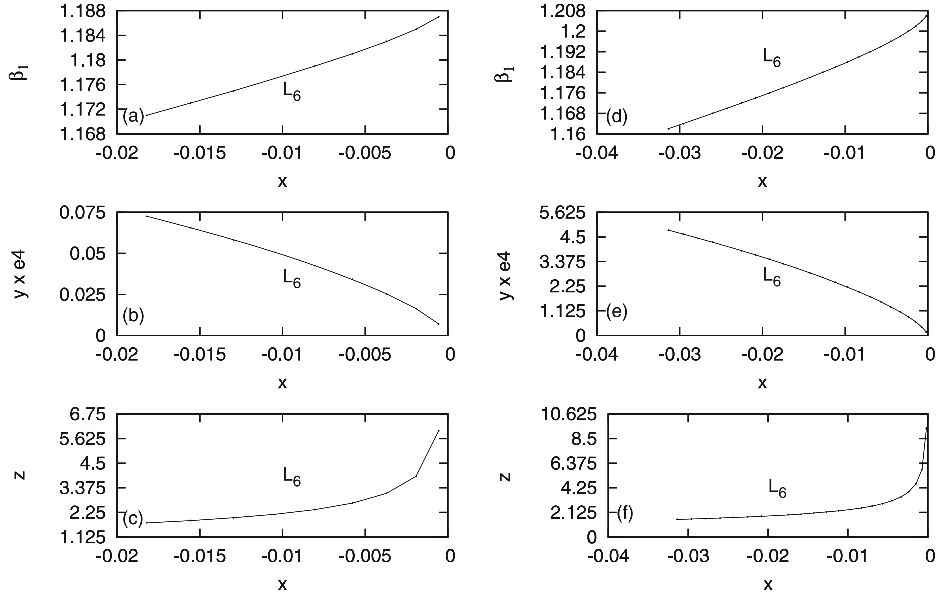

Figure 3. **(a), (d)** Variation of L_{6_x} with β_1; **(b), (e)** variation of L_{6_y} with L_{6_x}; **(c), (f)** variation of L_{6_z} with L_{6_x} in the region $X < 0$ for *Krüger*-60 and *RW-Monocerotis*, respectively.

The foregoing equations could be easily obtained by considering $\ddot{X} = \ddot{Y} = \ddot{Z} = 0$, $\dot{X} = \dot{Y} = \dot{Z} = 0$ in equation (1). Using the last equation, we note that $Q_1 Q_2 < 0$ or $q_1 q_2 < 0$ must be satisfied for the existence of the out of plane equilibrium points. Further, it may be observed that the presence of PR-drag results in the possibility of having $Y \neq 0$, unlike the situation discussed in section 2.2. From equations (17)–(19), we find that:

$$P(r_1) = a_6 r_1^6 + a_4 r_1^4 + a_2 r_1^2 + a_1 r_1 + a_0 = 0, \tag{19}$$

$$X_0 = \frac{1}{2}\left[1 - \left(\frac{Q_2}{Q_1}\right)^{2/3}\right] r_1^2 + \frac{1}{2} - \mu, \tag{20}$$

$$Y_0 = \frac{1}{2}\left[W_1 - W_2\left(\frac{Q_1}{Q_2}\right)^{2/3}\right] r_1^{-2},$$

$$+ \frac{1}{2}\left[W_1 - W_2 - \left(W_1\left(\frac{Q_2}{Q_1}\right)^{2/3} - W_2\left(\frac{Q_1}{Q_2}\right)^{2/3}\right)\right], \tag{21}$$

$$Z_0 = \pm\sqrt{r_1^2 - (X_0 + \mu)^2 - Y^2}. \tag{22}$$

Using equation (8) and the foregoing equations with coefficients $a_i, i = 0, 1, 2, 4, 6$ as defined in Ragos *et al.* (1995), we have computed the co-ordinates X_0, Y_0, and Z_0 for the out of plane equilibrium points of a binary system. In the presence of PR-drag also we observe that a critical β_1 exists for which it is possible to have two equilibrium

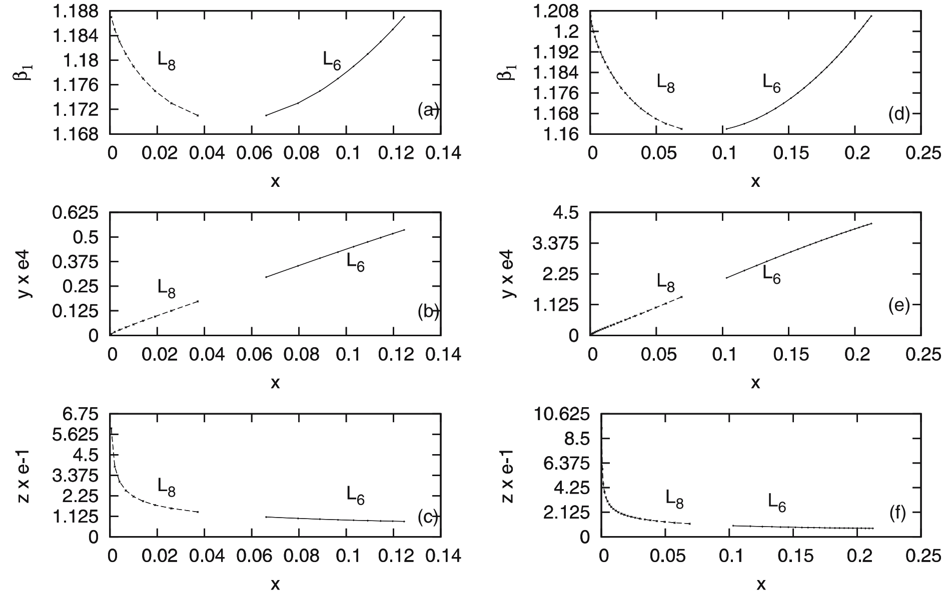

Figure 4. **(a), (d)** Variation of L_{6_x} and L_{8_x} with β_1; **(b), (e)** variation of L_{8_y} and L_{6_y} with L_{6_x}; **(c), (f)** variation of L_{8_z} and L_{6_z} with L_{6_x} in the region $X > 0$ for *Krüger*-60 and *RW-Monocerotis*, respectively.

points $L_{6,7}$ in the region $X_0 < 0$ and four equilibrium points, i.e., $L_{6,7,8,9}$ in the region $X > 0$. Figure 3(a) clearly shows that L_{6_x} increases with increase in β_1 in the region $X < 0$. In such a region, for *Krüger*-60, an increase in X_0 is accompanied by a decrease in Y while there is an increase in the Z values (Fig. 3b-c). Similar results are obtained for *RW-Monocerotis* (Fig. 3d–f). In the region $X > 0$, for *Krüger*-60 an increase in β_1 results in a decrease in the values of L_{8_x} while L_{6_x} increases (Fig. 4a). Further, it is observed that with increase in L_{6_x}, L_{8_y} and L_{6_y} increases while L_{8_z} and L_{6_z} decrease (Fig. 4b–c) in *Krüger*-60. For the binary system *RW-Monocerotis* we observe similar trends in variation of L_6 and L_8 co-ordinates (Fig. 4d–f).

The inclusion of PR-drag induces instability in motion. This is evident from the fact that all the roots of the characteristic equation for equilibrium points $L_{6,7,8,9}$ in entire domain considered here have finite (non-zero) real and imaginary components.

3. Results

The problem concerning the location of out of plane equilibrium points in the photo-gravitational restricted three-body problem has been investigated. We have incorporated the effect of PR-drag in our analysis. Unlike the work of Ragos *et al.* (1995) (and references quoted therein), we considered a realistic relation connecting the parameters β_1 and β_2 with the physical parameters such as mass and luminosity of the binary components and investigated the locations of equilibrium points and their stability using linear analysis for the binary systems *Krüger*-60 and *RW-Monocerotis*. In the absence of PR-drag unlike Roman (2001), we observe that the equilibrium points $L_{6,7}$ exist for several values of $\beta_1 > 1$ for $X_0 < 0$ in both the binary systems in the X–Z

plane. We also find that for $|Q_1| < |Q_2|$, there exists a range of β_1 for which it is also possible to have four equilibrium points L_6, L_7, L_8 and L_9 for $X_0 > 0$ in both the binaries. Increase in β_1 results in an increase in L_{6_x} while L_{8_x} tends to decrease in the region $X_0 > 0$ for both the binary systems in the absence of PR-drag. Further in the absence of PR-drag in both binary systems the components L_{8_z} and L_{6_z} show a decreasing trend with increase in L_{8_x} and L_{6_x} respectively. Similar trends are observed when PR-drag is incorporated in the analysis. However, inclusion of PR-drag in the analysis results in a small but finite y-component to the possible equilibrium points of the system. It is observed that both L_{8_y} and L_{6_y} increase with increase in L_{8_x} and L_{6_x}, respectively in both the binaries.

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