AStA Wirtsch Sozialstat Arch (2018) 12:235–244
https://doi.org/10.1007/s11943-018-0233-0

ORIGINALVERÖFFENTLICHUNG

A note on how to normalise the Geary-Khamis index of purchasing power parity

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Received: 19 March 2018 / Accepted: 8 November 2018 / Published online: 16 November 2018

Abstract The World Bank regularly publishes GDP-figures of its member countries, compiled at purchasing power parities. It employs the Geary-Khamis index for transforming the aggregates. That index consists of a system of homogeneous equations, which requires an additional non-homogeneous equation, to determine an appropriate unit of value measurement. This is the US-$, at present. The paper discusses that convention, points out its shortcomings, suggests an alternative existing, and adds an interpretation in terms of international economics.

Keywords Index numbers · Geary-Khamis index · Purchasing power parity · International comparison

JEL Classification C43 · C82 · F19

The World Bank regularly publishes GDP and components for each of its member countries. International comparisons of national product figures encounter the difficulty that there is no common currency ruling between nations, and no common unit of measuring economic value, as a result. Transformation of national values to purchasing power parity has been established as a remedy, and the World Bank applies the Geary-Khamis index for the purpose (https://data.worldbanc.org). The Geary-Khamis index is a homogeneous system of linear equations used for making different national accounts internationally comparable in real terms. The system yields a well-defined solution if it satisfies three conditions:

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a) The number of equations equals the number of unknowns.
b) The rank is equal to that number minus one.
c) The system is completed by an additional independent non-homogeneous equation, which defines what is called a “normalisation” in mathematics, and “factor of scale” or “numeraire” in economics.

This note is about the last condition. Geary, in his first, three-page proposal (1958), makes no mention of it, and later index number theorists are hardly concerned with it, either (see, for example, Balk 2008; Diewert 2008). Yet, that condition is just as essential to the system as the other two ones. For it defines the unit of measurement, and with that it determines material content and empirical meaning of the otherwise purely mathematical solution. Tradition has it to normalise the index on one particular country, chosen arbitrarily, namely the US-\$. It does so on the implicit understanding that such a scaling factor serves just as a “numeraire”, and has no theoretical implication or significance. The theory behind that practice is rooted in the Walrasian theory of value. Here the “scaling” of a system of values is no more than choosing a convenient “numeraire”; it makes values expressed in different denominations comparable with each other, but has no economic meaning in itself.

That assumption, however, is only partially true. It is harmless as long as you cope with national accounting figures of one, and the same year. It becomes questionable when you extend the comparison over a span of more than one years, compiling figures of world economic growth, and—possibly in the future—of world monetary inflation. For the choice of an international measurement unit of value has more implications than just the statistical function of a numeraire. It determines what is known as “the real exchange rate” of a national currency in the theory of international economics.

Let x_i^j be the value of a product group i contained in GDP of a country j. It is expressed in the currency of some base country (USA), converted by means of official exchange rates. Let ppp_i^j stand for the corresponding purchasing power parity. You can then define a “quantity index” or “volume” q_i^j, which measures the amount of goods and services you can buy in base country 0 for the price p_i^j, which you would pay for it in country j, at current exchange rates e^j; m stands for the number of product groups and n for the number of countries covered by the dataset:

$$q_i^j\,[\$] = \frac{x_i^j\,[\$]}{ppp_i^j}, i = 1,...,m; j = 1,...,n, \tag{1}$$

with:

$$ppp_i^j = e^j\,[\$/€]\,\frac{p_i^j\,[€/piece]}{p_i^0\,[\$/piece]} \tag{2}$$

The GK-index defines two sets of theoretical variables for these data, namely, an average world price π_i for product group i and a "real" exchange rate ε^j, compiled in contrast to the actual market ("nominal") exchange rate. The real exchange rate is a broad measure of prices of goods and services in one country in relation to another (Krugman and Obstfeld 2004, p. 532):

$$\begin{aligned} &\pi_i \sum_j q_i^j = \sum_j \varepsilon^j x_i^j, i = 1,\ldots,m, \\ &\sum_i \pi_i q_i^j = \varepsilon^j \sum_i x_i^j, j = 1,\ldots,n. \end{aligned} \tag{3}$$

Equations 3 form a homogeneous system of (m + n) linear equations for an equal number of unknowns. In the following I will show, first, that normalisation of the GK-index at a particular national currency may lead to ambivalent results when measuring economic growth. From it follows, second, that the neutral normalisation at the total set off all member countries is a necessary condition for achieving unambiguous growth figures, to which, third, I add the economic perspective that the resulting GK-variables π_i and ε^j may then be interpreted as world price of product i, and real exchange rate of nation j respectively.

The consequences of normalising the GK-index on one particular country may be demonstrated by means of the smallest possible example, namely, the case of two countries A and B producing and exchanging two groups of commodities. Table 1 gives some simple numbers, and it assumes a case where PPPs are all equal to one. The currency exchange rate is 1:1. So both sides of the table show the same numbers.

When entered into the Geary-Khamis system of equations these numbers yield the following homogeneous system of linear equations:

$$\begin{aligned} &3\pi_1 - 1\varepsilon^A - 2\varepsilon^B = 0 \\ &7\pi_2 - 3\varepsilon^A - 4\varepsilon^B = 0 \\ &-1\pi_1 - 3\pi_2 + 4\varepsilon^A = 0 \\ &-2\pi_1 - 4\pi_2 + 6\varepsilon^B = 0 \end{aligned} \tag{4}$$

The system is solvable, by construction of the data. However, the solution is undetermined by a factor of scale. This is the essential issue of normalisation. It is essential because a "scale factor" determines the unit of measurement and with it the economic meaning of the mathematical solution. Following an unhappy tradition established by Walras, such a factor is considered as being of no more significance than an arbitrary "numeraire". Geary, in his original paper, does not refer to the problem, at all. However, the matter is not trivial.

The mathematical solution of the GK-system corresponding to the data of Table 1 is obvious, but it depends on what is chosen as the rule of normalisation. Table 2 shows several options. The first column (a) stipulates that the solution is normalised to country A, which means that its real exchange rate ε^A is set equal to one. The second column (b) shows the alternative with country B chosen as the country of reference. The last column (c) offers an alternative, which has already been used

Table 1 2×2 Example: The Base Case (Bill. \$)

	Current Values (x_i^j)			Quantities (q_i^j)		
	Country A	Country B	World	Country A	Country B	World
Product group 1	1	2	3	1	2	3
Product group 2	3	4	7	3	4	7
GDP	4	6	10	4	6	10

Table 2 GK-Index for the Data of Table 1

	System Normalisation		
Unknowns	(a) $\varepsilon^A = 1$	(b) $\varepsilon^B = 1$	(c) $\sum_{i,j} \varepsilon^j x_i^j = \sum_{i,j} x_i^j$
π_1	1	1	1
π_2	1	1	1
ε^A	1	1	1
ε^B	1	1	1
World PPP-GDP	10	10	10

in other contexts (Diewert and Fox 2017) and is proposed here for adaptation, by the World Bank, as well. It assumes that total world GDP serves as the standard of normalisation so that world GDP at purchasing power parities equals world GDP at exchange rates:

$$\sum_{i,j} \pi_i q_i^j = \sum_{i,j} \varepsilon^j x_i^j = Y = \sum_{i,j} x_i^j = 10 \tag{5}$$

Table 2 is no more than a check of consistency of index number Eqs. (3) and (4), in that two identical countries working with equal currencies solve the system, by definition. Here the choice of a rule of normalisation does not matter, indeed. They all produce the same trivial result for the unknowns.

Let a first modification of the identity case be given by a pure increase in production. In Table 3 the actual (transaction) value of product group 1 in country B grows from 2 to 3 Bill. \$, so does volume q, because exchange rate and prices of Eq. 2 remain the same as before. World GDP rises from 10 to 11 Bill. \$ as a result, both in nominal and in real terms (ppp-values).

Table 3 2×2 Example: The Pure Growth Case (Bill. \$)

	Current Values (x_i^j)			Quantities (q_i^j)		
	Country A	Country B	World	Country A	Country B	World
Product group 1	1	3	4	1	3	4
Product group 2	3	4	7	3	4	7
GDP	4	7	11	4	7	11

Table 4 GK-index for the data of Table 3

	System normalisation		
Unknowns	(a) $\varepsilon^A = 1$	(b) $\varepsilon^B = 1$	(c) $\sum_{i,j} \varepsilon^j x_i^j = \sum_{i,j} x_i^j$
π_1	1	1	1
π_2	1	1	1
ε^A	1	1	1
ε^B	1	1	1
World PPP-GDP	11	11	11

Table 4 shows the solution resulting for the GK-index. Again, normalisation does not matter. Whether you normalise at country A or at country B, or finally at world GDP, all variables remain the same as in the identity case. Calculating growth at constant prices, and constant exchange rates is invariant with respect to the chosen normalisation.

Table 5, in contrast, illustrates a pure change in prices. Prices of product group 1 increase in country B by 50% from 2 to 3 Bill. \$. This raises world GDP in current prices, but not in quantities.

Here the results of the compilation differ with the normalisation rule you apply. Normalising the GK-index at country A yields a quantity index of 10.140, normalising it at country B yields 11.560. One is lower, the other is higher than actual world GDP of 11.000. Only when you normalise the GK-system to that actual value of world GDP you get 10% as the actual rate of world monetary inflation (Table 6).

Table 5 2×2 Example: The Pure Price Change (Bill. \$)

	Current Values (x_i^j)			Quantities (q_i^j)		
	Country A	Country B	World	Country A	Country B	World
Product group 1	1	3	4	1	2	3
Product group 2	3	4	7	3	4	7
GDP	4	7	11	4	6	10

Table 6 GK-index for the data of Table 5

	System normalisation		
Unknowns	(a) $\varepsilon^A = 1$	(b) $\varepsilon^B = 1$	(c) $\sum_{i,j} \varepsilon^j x_i^j = \sum_{i,j} x_i^j$
π_1	1.211	1.3801	1.313
π_2	0.930	1.060	1.009
ε^A	1.000	1.140	1.085
ε^B	0.877	1.000	0.952
World PPP-GDP	10.140	11.560	11.000

For interpreting the difference of these normalisations remember that the GK-Eq. 5 imply that world GDP is world prices multiplied by world volumes of products, where q is the quantity index as defined by Eq. 1:

$$Y = \sum_{i,j} \pi_i q_i^j \tag{6}$$

from which you derive:

$$dY = \sum_{i,j} d\pi_i q_i^j + \sum_{i,j} \pi_i dq_i^j \tag{7}$$

for a differential change in time. The first term measures world inflation, and the second term measures world growth. It does not make sense to let these results depend on the choice of a numeraire country. On the contrary, if the measurement of purchasing power parities is carried out with the purpose of comparing relative positions of countries, the absolute GDP value of all of them together ought not to be the touched by the revaluing operation. Normalisation (c) serves that purpose. It is compatible with the observed fact that world product has increased by 1 Bill. \$ in a pure price change, in Table 5. The nominal difference of world GDP is 1 Bill. \$, between the two periods. That cannot be accounted for by either normalisation (a) or (b). The corresponding increase in world price level comes out either two low (1.4% in case (a)) or too high (15.6% in case (b)). The only normalisation producing a consistent partition between the two movements of price and quantity is normalisation at the actual transactions value of 11 Bill. \$. It exhibits an increase of 10% in current prices, and no change in product.

Finally, let both, prices and volumes, of product group 1 increase in country B raising the transaction value of product group 1 in country B from 3 to 4 Bill. \$ (Table 7).

Again, normalisation matters (Table 8). World GDP, and national GDPs, expressed in purchasing power parity are different depending on which normalisation rule you apply. If normalised to country A world PPP-GDP is smaller than its actual transaction value, and if normalised to country B it is larger. However, in stating that sentence you do not really want to say that one GDP is larger or smaller than the other one, but that the measures in which they are expressed differ. A unit of the first is smaller than is a unit of the second measure; the GDP to which they are applied is always the same set of goods and services; just as a foot is smaller than a meter, but both units can be used to measure the same length.

The example raises the general question of how to compare a national inflation with its worldwide counterpart, especially when an inflationary currency is used as the base of comparison. In Table 8 that applies to case (b). The price increase of product 1 occurs there, and it implies a corresponding increase in national inflation. The traditional solution is to subtract the national inflation from the world figure, dividing the current US-\$ by the US rate of inflation. Here you would calculate a national inflation of 8/7 for country B, and apply it to the world ppp-value of 12.507. The result is a final figure of $7/8 \cdot 12.507 = 10.94$ Bill. "constant" B-\$.

Table 7 2×2 example: Variation in Quantity and price

	Current values (x_i^j)			Quantities (q_i^j)		
	Country A	Country B	World	Country A	Country B	World
Product group 1	1	4	5	1	3	4
Product group 2	3	4	7	3	4	7
GDP	4	8	12	4	7	11

Country A has not experienced inflation; measured in its currency, real world GDP equals nominal world GDP of 11.355 Bill. A-\$. World growth is 9.4% in the first case, and 13.55% in the second case, as a result of those differing normalisations. Only normalisation (c) yields a growth rate of 10% as a country-independent world average.

If you want to understand why present data are found to be inconsistent and difficult to reconcile over extended periods of time (Oulton 2014) the point of normalisation may be of significance, in this respect. Moreover, shifting normalisation of the GK-index from the US national level to the world level, as suggested by Eq. 5, extends the range of theoretical analysis. You are then able to compare figures "at International \$" with figures "at US-\$" for every country, a comparison which does not make sense, at present. If normalised at world GDP, the PPP-GDP of a particular country may be compared with its GDP measured at exchange rates, directly, and the comparison has a significant meaning: A PPP-GDP higher than the exchange rate-GDP ($\varepsilon > 1$) means that the existing national exchange rate is below equality. Its currency purchases less products abroad than at home, and loses value when exchanged abroad. If, in contrast, the real exchange rate is below one ($\varepsilon < 1$) the national currency, when exchanged abroad, buys more products than it does at home; terms of international trade are favourable, an information worthwhile to have in an era of global interconnection of national production and trade.

Modifying present practice in this direction is not a big deal; every user can do so for himself, as it is just a matter of rescaling the results presented in the official statistics. Table 9 presents GDPs of the G20-group of countries for year 2014, valued in US-\$, i.e. at actual exchange rates (first column), and at International \$,

Table 8 GK-index for the data of Table 7

	System normalisation		
Unknowns	(a) $\varepsilon^A = 1$	(b) $\varepsilon^B = 1$	(c) $\sum_{i,j} \varepsilon^j x_i^j = \sum_{i,j} x_i^j$
π_1	1.158	1.275	1.224
π_2	0.947	1.043	1.001
ε^A	1.000	1.101	1.057
ε^B	0.908	1.000	0.959
World PPP-GDP	11.355	12.507	12.000

Table 9 Gross domestic product of G20-countries in year 2014 at US-$ (**a**), at International $ (**b**) and the resulting real exchange rates ε. (Source: https://data.worldbanc.org, and own calculations)

Country	US-$ Billion	(a) Share	International $ Billion	(b) Share	ε = (b) / (a) Real exchange rate
Australia	1460	0.01851	1090	0.00984	0.531
Brazil	2456	0.03114	3307	0.02986	0.959
Canada	1793	0.02273	1602	0.01447	0.637
China	10,482	0.1329	18,335	0.16558	1.246
Germany	3879	0.04918	3811	0.03442	0.700
France	2849	0.03612	2667	0.02408	0.667
Great Britain	2999	0.03802	2630	0.02375	0.625
Indonesia	891	0.0113	2689	0.02428	2.150
India	2035	0.0258	7346	0.06634	2.571
Italy	2152	0.02729	2206	0.01992	0.730
Japan	4849	0.06148	5013	0.04527	0.736
Korea	1411	0.01789	1707	0.01542	0.862
Mexico	1298	0.01646	2157	0.01948	1.183
Russia	2064	0.02617	3722	0.03361	1.284
Turkey	934	0.01184	1780	0.01607	1.357
USA	17,393	0.22053	17,393	0.15707	0.712
Other countries	19,925	0.25263	33,280	0.30054	1.190
World	78,870	1.00000	110,735	1.00000	1.000

expressing purchasing power parities (third column), as an example. Let national GDP y^j of country j at actual exchange rates be given by:

$$y^j = \sum_i x_i^j, j = 1,\ldots,n \tag{8}$$

World GDP of all countries together, at actual exchange rates, has been:

$$Y = \sum_j y^j = 78{,}870 bill.\$ \tag{9}$$

in year 2014. The third column of Table 9 shows the same GDP valued at purchasing power parities, normalised to GDP of the United States ($\varepsilon^{US} = 1$), as is customary at present. It results in a world total of 110,735 "International $":

$$Y_{US}^{\mathrm{PPP}} = \sum_j \varepsilon_{US}^j y^j = 110{,}735 bill.\$ \tag{10}$$

The ratio r of the corresponding world totals is:

$$r = \frac{110{,}735}{78{,}870} \tag{11}$$

Following the argument of this paper, it does not make sense to have a valuation in purchasing power parities have increase total world GDP. For the resulting difference depends on the choice of a numeraire currency, alone; it is arbitrary and meaningless, in terms of economic theory. It follows that in order to equalise actual and PPP world product, according to Eq. 5, we must have:

$$Y^{\mathrm{PPP}} = \sum_j \varepsilon^j y^j = \frac{Y}{Y_{US}^{\mathrm{PPP}}} \sum_j \varepsilon_{US}^j y^j \tag{12}$$

and hence:

$$\varepsilon^j = \varepsilon_{US}^j \frac{Y}{Y_{US}^{\mathrm{PPP}}} = \frac{\varepsilon_{US}^j}{r}, j = 1,\ldots,n \tag{13}$$

Equality is required,—so my argument,—in order to preserve consistency and uniqueness in measuring and comparing countries' economic growth over time. The same idea seems to be embedded in present terminology, implicitly. One speaks of GDP "in US-$" as against BSP in "International $". As both valuations apply to the same set of goods and services, namely world GDP, that distinction implies the notion that the International $ is worth 1 / r of the national US-$, a notion which is equivalent to accepting Eq. 5 as the rule of normalisation.

Table 9 allows a possible further step of economic analysis, enabled by the by the suggested rule of normalisation. It measures what is defined as the "real exchange rate", in contrast to the nominal exchange rate, in the theory of international economics as follows: The share in total world product at actual exchange rates a^j of each country j is given by:

$$a^j = \frac{y^j}{\sum_j y^j} \tag{14}$$

Likewise, share b^j of each country in world PPP-GDP be given by:

$$b^j = \frac{y_{\mathrm{PPP}}^j}{Y_{\mathrm{PPP}}} = \frac{\varepsilon_{US}^j y^j}{\sum_j \varepsilon_{US}^j y^j} \tag{15}$$

It follows, respecting Eq. 13:

$$\frac{b^j}{a^j} = \frac{\varepsilon_{US}^j y^j}{\sum_j \varepsilon_{US}^j y^j} \times \frac{\sum_j y^j}{y^j} = \varepsilon_{US}^j \frac{1}{r} = \varepsilon^j \tag{16}$$

The ratio (last column in Table 9) yields what has been identified as real exchange rates, in the theory of international trade, i. e. imputed rates calculated on the assumption of a universally ruling purchasing power parity. That economic interpretation of the statistical figures, however, is valid only if world GDP at purchasing power parities is scaled to world GDP at current prices. It would not make sense, otherwise.

A brief interpretation of the new figures follows: Real exchange rates vary significantly between countries, from 0.531 for Australia to 2.571 for India, the Australian dollar buying abroad almost twice of what it buys at home, while the Indian rupee, in contrast, buys abroad less than half of what it buys at home. Apparently, real exchange rates differ significantly with actual exchange rates, a lesson which is not new. The data of Table 9 quantify that expectation. Whether the fact itself is good or bad, advantageous or disadvantageous for a country is a matter of economic policy, and not to be discussed here. A low real exchange rate ($\varepsilon < 1$) is an advantage in international competitiveness, and paid for by lower domestic earnings, a high real exchange rate means the opposite, high earnings at low competitiveness. International trade, anyway, does not lead to an internationally homogeneous price level. The fact itself is well known; Table 9 quantifies the national positions and their differences.

In conclusion, the issue of how to normalise the index number formula applied has not been relevant at the beginning of the ICP project. It is only now, with interconnectedness of national economies increasing, and ICP becoming a regular statistical exercise of international organisations, that the issue of scaling the compiled exchange rate parities demands attention, and certain problems connected with the present convention become apparent, in the larger context. While they could not fully be elaborated here (see Reich 2017, chapters 7 and 9 for an extensive treatment), this note may be an enticement for more discussion and research in that direction at national and international levels.

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