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Basic results for fractional anisotropic spaces and applications

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Abstract

In this paper, we introduce a new space that generalizes the ϕ-Hilfer space with the $\xi(\cdot)$-Laplacian operator, denoted $(\phi, \xi(\cdot))$-HFDS. We refer to this new space as the ϕ-fractional space with anisotropic $\overrightarrow{\xi}(\cdot)$-Laplacian operator, abbreviated as $(\phi, \overrightarrow{\xi}(\cdot))$-HFDAS. We prove that $(\phi, \overrightarrow{\xi}(\cdot))$-HFDAS is a separable, and reflexive Banach space. Furthermore, we extend some well-known properties and embedding results of the $(\phi, \xi(\cdot))$-HFDS space to $(\phi, \overrightarrow{\xi}(\cdot))$-HFDAS. Moreover, we illustrate an application of $(\phi, \overrightarrow{\xi}(\cdot))$-HFDAS by solving a differential equation via variational methods.

Keywords ϕ-Hilfer derivative · $\xi(\cdot)$-Laplacian operator · Anisotropic $\overrightarrow{\xi}(\cdot)$-Laplacian operator

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1 Introduction

Lately, there has been an increasing interest in the study of partial differential equations [7, 10–12, 15, 16, 26, 28], particularly the ones that involves the ϕ-HFDS spaces, as highlighted in [20] and its related literature. Naturally, there's a notable focus on issues pertaining to ϕ-HFD with the $\xi(\cdot)$-Laplacian operator given by

$$\mathbb{L}(\cdot) := \mathfrak{D}_{T}^{\alpha,\beta;\phi}\left(\left|\,\mathfrak{D}_{0^+}^{\alpha,\beta;\phi}(\cdot)\right|^{\xi(x)-2}\,\mathfrak{D}_{0^+}^{\alpha,\beta;\phi}(\cdot)\right). \tag{1.1}$$

Problems involving the above opertator have been a subject of recent few references in the literarure. For instance, in [20] Srivastava and Sousa, focused on exploring quasi-linear fractional-order problems with variable exponents of the form

$$\begin{cases} \mathfrak{D}_{T}^{\alpha,\beta;\phi}\left(\left|\,\mathfrak{D}_{0^+}^{\alpha,\beta;\phi}u\right|^{\xi(x)-2}\,\mathfrak{D}_{0^+}^{\alpha,\beta;\phi}u\right) = p|u|^{\beta(x)-2}u + \mathcal{A}(x,u) & \text{in } \Omega \\ \quad = [0,T], \times[0,T]\times[0,T] & \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

The authors of [20] employed the Genus theory in conjunction with the Concentration-Compactness Principle and the Mountain Pass Theorem to prove the existence and multiplicity of solutions for problem (1.2). Additionally, Sousa et al. [22] utilized the fibering method in conjunction with the Nehari manifold to demonstrate the existence of at least two weak solutions for the following fractional singular double phase problem

$$\begin{cases} \mathfrak{D}_{T}^{\alpha,\beta;\phi}\left(\left|\,\mathfrak{D}_{0^+}^{\alpha,\beta;\phi}u\right|^{\xi-2}\mathfrak{D}_{0^+}^{\alpha,\beta;\phi}u + \mu(x)\left|\mathfrak{D}_{0^+}^{\alpha,\beta;\phi}u\right|^{\xi-2}\mathfrak{D}_{0^+}^{\alpha,\beta;\phi}u\right) & \\ \quad = \xi u^{-\sigma} + pu^{r-1} & \text{in } \Omega = [0,T], \times[0,T] \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

in the case that ζ is sufficiently small. Again, Sousa et al. in [21] established the existence and multiplicity results by utilizing the Nehari manifold technique to the following curvature problem

$$\begin{cases} \mathfrak{D}_{T}^{\alpha,\beta;\phi}\left(\left(1+\dfrac{\left|\mathfrak{D}_{0^+}^{\alpha,\beta;\phi}u\right|^{\xi(x)}}{\sqrt{1+\left|\mathfrak{D}_{0^+}^{\alpha,\beta;\phi}u\right|^{2\xi(x)}}}\right)\left|\,\mathfrak{D}_{0^+}^{\alpha,\beta;\phi}u\right|^{\xi(x)-2}\,\mathfrak{D}_{0^+}^{\alpha,\beta;\phi}u\right) & \\ = |u|^{\beta(x)-2}u + p(x)\mathcal{A}_u(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Other interesting works, related to (1.2), can be consulted [6, 19, 21–24].

Before presenting the problem to be addressed and the main results of this article, it will be presented some preliminary aspects regarding fractional operators.

Let $A := [c, d]$ $(-\infty \leq c < d \leq \infty)$, $n-1 < \alpha < n, n \in \mathbb{N}$ and $\mathbf{g}, \phi \in C^n(A, \mathbb{R})$ such that ϕ is increasing and $\phi'(x) \neq 0$, for all $x \in A$. Consider $Q = A_1 \times \cdots \times A_N := [c_1, d_1] \times \cdot\cdot \times [c_N, d_N]$ where $0 < c_i < d_i$, for all $i \in \{1, ..., N\}$, and $\alpha = (\alpha_1, ..., \alpha_N)$ where $0 < \alpha_1, ..., \alpha_N < 1$.

The ϕ-Riemann-Liouville fractional partial integral (resp. derivative) of order α of N-variables $\mathbf{g} = (\mathbf{g}_1, ..., \mathbf{g}_N)$ are defined by

$$\mathbf{I}_{c,x_i}^{\alpha;\phi}\mathbf{g}(x) = \frac{1}{\Gamma(\alpha_i)} \int_A \phi'(y_i)(\phi(x_i) - \phi(y_i))^{\alpha_i - 1}\mathbf{g}(y_i)\mathrm{d}y_i$$

and

$$\mathfrak{D}_{c,x_i}^{\alpha,\beta;\phi}\mathbf{g}(x) = \mathbf{I}_{c,x_i}^{\beta(n-\alpha);\phi}\left(\frac{1}{\phi'(x_i)}\frac{\partial^N}{\partial x_i}\right)\mathbf{I}_{c,x_i}^{(1-\beta)(n-\alpha);\phi}\mathbf{g}(x_i),$$

where $\phi'(y_i)(\phi(x_i) - \phi(y_i))^{\alpha_i-1} = \phi'(y_1)(\phi(x_1) - \phi(y_1))^{\gamma_1-1} \ldots \phi'(y_N)(\phi(x_N) - \phi(y_N))^{\gamma_N-1}$ and $\Gamma(\alpha_i) = \Gamma(\alpha_1)\Gamma(\alpha_2)\ldots\Gamma(\alpha_N)$, $\mathbf{g}(x_i) = \mathbf{g}(x_1)\mathbf{g}(x_2)\ldots\mathbf{g}(x_N)$ and $\mathrm{d}y_i = \mathrm{d}y_1\mathrm{d}y_2\ldots\mathrm{d}y_N, \partial x_i = \partial x_1, \partial x_2, ..., \partial x_N$ and $\phi'(x_i) = \phi'(x_1)\phi'(x_2)\ldots\phi'(x_N)$ for all $i \in \{1, 2, .., N\}$. Analogously, it is defined $\mathbf{I}_{c,x_i}^{\alpha;\phi}\mathbf{g}(x)$ resp. $\mathfrak{D}_{d,x_i}^{\alpha,\beta;\phi}(\cdot)$.

On the other hand, we point out that partial differential equations involving anisotropic operators are highly relevant across various fields of technology and science. For example, in reference [13], a mathematical model for image enhancement and denoising was proposed, which effectively preserved key characteristics of the images by considering anisotropic operators. Additionally, anisotropic equations are applied in models that describe the spread of epidemic diseases in heterogeneous environments. In Physics, these operators contribute to describing the dynamics of fluids with varying conductivities in different directions. For more details regarding the mentioned applications, refer to [1–3, 13]. Regarding the interest from mathematical point of view, we quote for example [25], where it was obtained existence and multiplicity of solutions, via sub-supersolutions and variational methods, for the anisotropic problem with variable exponents given by

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right) = a(x)u^{\alpha(x)-1} + \lambda f(x,u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 3)$ with smooth boundary, $a, \alpha, f, p_i, i = 1, \ldots, N$ are functions satisfying adequate conditions, and λ is parameter. For other references that consider anisotropics problem we mention [4, 5, 8] and its references.

Thus, based in the previous comments the main goal of this manuscript is to introduce the opertaor general operator considered in (1.1), which will be referred as $(\phi, \overrightarrow{\xi}(\cdot))$-HFDAS, given as

$$\mathbb{A}(\cdot) := \sum_{i=1}^{N} \mathfrak{D}_{T,x_i}^{\alpha,\beta;\phi}\left(\left|\mathfrak{D}_{0^+,x_i}^{\alpha,\beta;\phi}(\cdot)\right|^{\xi_i(x)-2} \mathfrak{D}_{0^+,x_i}^{\alpha,\beta;\phi}(\cdot)\right), \tag{1.4}$$

Furthermore, we introduce an adequate space to deal with the operator (1.4), that will be called as the ϕ-HFD and denoted by $\mathcal{H}_{\overrightarrow{\xi}(x)}(Q)$, which is defined as

$$\mathcal{H}_{\overrightarrow{\xi}(x)}(Q) = \Big\{\omega \in L^1_{loc}(Q) : \omega \in \mathscr{L}^{\xi_i(x)}(Q) \text{ and } \mathfrak{D}_{0^+,x_i}^{\alpha,\beta;\phi}\omega \in \mathscr{L}^{\xi_i(x)}(Q)$$
$$\text{for all } i \in \{1, \dots, N\}\Big\},$$

that can be equipped with the norm

$$\|\omega\|_{\mathcal{H}_{\overrightarrow{\xi}(x)}(Q)} := \|\omega\|_{\mathscr{L}^{\xi_M(x)}(Q)} + \sum_{i=1}^{N}\left\|\mathfrak{D}_{0^+,x_i}^{\alpha,\beta;\phi}\omega\right\|_{\mathscr{L}^{\xi_i(x)}(Q)} \text{ for all } \omega \in \mathcal{H}_{\overrightarrow{\xi}(x)}(Q),$$

where $0 < \alpha < 1$, $0 \le \beta \le 1$, $1 < \alpha\xi_M(x) < N$, $\overrightarrow{\xi} : \overline{Q} \longrightarrow \mathbb{R}^N$ is a vector function defined as $\overrightarrow{\xi}(x) = (\xi_1(x), \dots, \xi_N(x))$, such that $\xi_i \in C^+(\overline{Q})$ satisfying $1 < \xi_i^- \le \xi_i^+ < N < \infty$ for all $i \in \{1, \dots, N\}$ with $\xi_m(x) = \min\{\xi_1(x), \dots, \xi_N(x)\}$, $\xi_M(x) = \max\{\xi_1(x), \dots, \xi_N(x)\}$, $\xi_m^+ = \sup_{x\in Q}\xi_m(x)$ and $\xi_M^+ = \sup_{x\in Q}\xi_M(x)$. We denote that

$$\overline{\xi}(x) = \frac{N}{\sum_{i=1}^{N} 1/\xi_i(x)} \quad \text{and} \quad \overline{\xi}^*(x) = \begin{cases} \frac{N\overline{\xi}}{N-\alpha\overline{\xi}} & \text{if } \alpha\overline{\xi} < N, \\ +\infty & \text{if } \alpha\overline{\xi} \ge N. \end{cases}$$

At this point, we can describe the obtained properties of the space $\mathcal{H}_{\overrightarrow{\xi}(x)}(Q)$.

Proposition 1.1 *The space* $\left(\mathcal{H}_{\overrightarrow{\xi}(x)}(Q), \|\cdot\|_{\mathcal{H}_{\overrightarrow{\xi}(x)}(Q)}\right)$ *is a separable and reflexive Banach space.*

Remark 1.2 The $(\phi, \overrightarrow{\xi}(\cdot))$-HFDAS with zero boundary values $\mathcal{H}_{\overrightarrow{\xi}(x),0}(Q)$ is defined as the closure of $C_0^\infty(Q)$, with the norm

$$\|\omega\| := \|\omega\|_{\mathcal{H}_{\overrightarrow{\xi}(x),0}(Q)} = \sum_{i=1}^{N}\left\|\mathfrak{D}_{0^+,x_i}^{\alpha,\beta;\phi}\omega\right\|_{\mathscr{L}^{\xi_i(x)}(Q)}.$$

Also, $\mathcal{H}_{\overrightarrow{\xi}(x)}(Q)$ satisfies the following embedding described below.

Theorem 1.3 *Consider* $\xi_i \in C^+(\overline{Q})$ *for all* $i \in \{1, \dots, N\}$, *with* $\xi_m \in C^+_{\log}(\overline{Q})$ *such that* $\xi_m^+\alpha < N$. *Assume* $h \in C(\overline{Q})$ *satisfies* $1 \le h(x) \le \max\{\overline{\xi}^*(x), \xi_m^+\}$ *for all* $x \in \overline{Q}$. *Under these conditions, there exists a continuous embedding* $\mathcal{H}_{\overrightarrow{\xi}(x)}(Q) \hookrightarrow$

$\mathscr{L}^{h(x)}(Q)$. Moreover, if we additionally assume $1 \leq h(x) < \max\{\overline{\xi}^*(x), \xi_m^+\}$ for all $x \in \overline{Q}$, then this embedding is also compact.

Additionally, we have the following Poincaré inequality:

Proposition 1.4 *Consider $\xi_i \in C_+(\overline{Q})$ for all $i \in \{1, \ldots, N\}$ such that $\xi_M(x) < \xi_m^*(x)$ for all $x \in \overline{Q}$. For every $\omega \in \mathcal{H}_{\overrightarrow{\xi}(x),0}(Q)$, the Poincaré-type inequality*

$$\|\omega\|_{\mathscr{L}^{\xi_M(x)}(Q)} \leq C \sum_{l=1}^{N} \left\| \mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i} \omega \right\|_{\mathscr{L}^{\xi_i(x)}(Q)} \quad \text{for all } \omega \in \mathcal{H}_{\overrightarrow{\xi}(x),0}(Q), \tag{1.5}$$

holds with $C > 0$ independent of ω.

As an application of the previous results, it will be established the existence of a weak solution for the following fractional differential equation involving the new $(\phi, \overrightarrow{\xi}(\cdot))$-HFDAS

$$\begin{cases} \sum_{i=1}^{N} \mathfrak{D}^{\alpha,\beta;\phi}_{T,x_i} \left(\left| \mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i} \omega \right|^{\xi_i(x)-2} \mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i} \omega \right) + |\omega|^{\xi_M(x)-2}\omega = g(x,\omega), & \text{for } x \in Q, \\ \omega = 0 \quad \text{on } \partial Q, \end{cases} \tag{1.6}$$

where $Q \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with a Lipschitz boundary ∂Q and g: $Q \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function with the potential $G(x, \xi) = \int_0^\xi g(x,t)dt$, that satisfies the hypotheses below.

- **(H1)** There exist $C > 0$ and $q \in C_+(\overline{Q})$ with $\xi_M^+ < q^- \leq q^+ < \overline{\xi}^*(x)$ for all $x \in \overline{Q}$, such that g verifies
$$|g(x,s)| \leq C\left(1 + |s|^{q(x)-1}\right),$$
for all $x \in Q$ and all $s \in \mathbb{R}$ and $g(x,t) = g(x,0) = 0$ for all $x \in Q$, $t \leq 0$.
- **(H2)** $\lim_{t\to 0} \dfrac{g(x,t)}{|t|^{\xi_M^+ - 1}} = l_1 < \infty$, and $\lim_{t\to\infty} \dfrac{g(x,t)t}{|t|^{\xi_M^+}} = \infty$, uniformly for $x \in Q$.
- **(H3)** For a.e. $x \in Q$, $\dfrac{g(x,t)}{t^{\xi_M^+ - 1}}$ is nondecreasing with respect to $t \geq 0$.
- **(H4)** $\limsup_{|t|\to+\infty} \dfrac{\xi_M(x)}{|t|\xi_M(x)} G(x,t) < \zeta_1$, uniformly for a.e. $x \in Q$, with

$$\zeta_1 := \inf_{\omega\in\mathcal{H}_{\overrightarrow{\xi}(x),0}(Q)} \frac{\sum_{i=1}^{N} \int_Q \frac{1}{\xi_i(x)} \left| \mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i} \omega \right|^{\xi_i(x)} dx + \int_Q \frac{1}{\xi_M(x)} |\omega|^{\xi_M(x)} dx}{\int_Q \frac{1}{\xi_M(x)} |\omega|^{\xi_M(x)} dx} > 0.$$

(**H5**) There exist $a_0 > 0$ and $\delta > 0$ such that

$$G(x,t) \geq a_0|t|^{q_0}, \text{ for all } x \in Q, \ |t| < \delta,$$

where $q_0 \in C(\overline{Q})$ with $q_0 < \xi_m^-$.

In what follows we describe the application concerned problem (1.6).

Theorem 1.5 (**i**) *Under hypotheses* (**H1**)–(**H3**), *problem* (1.6) *has at least one nontrivial solution in* $\mathcal{H}_{\overrightarrow{\xi}(x),0}(Q)$.

(**ii**) *With hypotheses* (**H4**)–(**H5**), *problem* (1.6) *possesses at least one nontrivial solution in* $\mathcal{H}_{\overrightarrow{\xi}(x),0}(Q)$.

The remainder of the manuscript is organized as follows. In Sect. 2, we provide a brief overview of the key features of variable exponent Lebesgue spaces. In Sect. 3, we prove the properties of the $\left(\phi, \overrightarrow{\xi}(\cdot)\right)$-HFDAS. Moving on to Sect. 4 we present the application described in Theorem 1.5.

2 Mathematical background

In this paper, we assume that Q is a bounded Lipschitz domai in $\mathbb{R}^N$ $(N \geq 2)$. For the definitions and notation that we will present below, we use [9, 14] and the references therein.

We define the set $C^+(\overline{Q})$ as

$$C^+(\overline{Q}) = \left\{\xi : \xi \in C(\overline{Q}), \ \xi(x) > 1 \text{ for a.e. } x \in \overline{Q}\right\}.$$

Consider $C^+_{\log}(\overline{Q})$ the set of functions $\xi \in C^+(\overline{Q})$ that satisfy the log-Holder continuity condition

$$\sup\left\{|\xi(x) - \xi(y)| \log \frac{1}{|x-y|} : x, y \in \overline{Q}, 0 < |x-y| < \frac{1}{2}\right\} < \infty.$$

For any $\xi \in C^+(\overline{Q})$, we define $\xi^+ = \sup_{x \in Q} \xi(x)$, $\xi^- = \inf_{x \in Q} \xi(x)$ and the modular $\varrho_\xi : \mathscr{L}^{\xi(x)}(Q) \longrightarrow \mathbb{R}$ as $\varrho_\xi(\omega) := \int_Q |\omega(x)|^{\xi(x)} dx$. Then, the variable exponent Lebesgue space is defined as

$$\mathscr{L}^{\xi(x)}(Q) = \left\{\omega \in \mathbf{U}(Q), \quad \varrho_p(\omega) < \infty\right\},$$

where $\mathbf{U}(Q)$ is the set of all real-valued measurable functions defined in Q. We endow the space $\mathscr{L}^{\xi(x)}(Q)$ with the Luxemburg norm

$$\|\omega\|_{\mathscr{L}^{\xi(x)}(Q)} := \inf\left\{\tau > 0 : \varrho_\xi\left(\frac{|\omega(x)|}{\tau}\right) \leq 1\right\}.$$

Then, the variable exponent Lebesgue space $\left(\mathscr{L}^{\xi(x)}(Q), \|\cdot\|_{\mathscr{L}^{\xi(x)}(Q)}\right)$ becomes a Banach space. Let us now revisit some fundamental properties associated with Lebesgue spaces involving variable exponents.

Proposition 2.1 [14] *Let $q, h \in C^+(\overline{Q})$ such that $q \leq h$ within the domain Q. Under these conditions, the embedding $\mathscr{L}^{h(x)}(Q) \hookrightarrow \mathscr{L}^{q(x)}(Q)$ is continuous.*

Furthermore, the following Hölder-type inequality holds for all $\omega \in \mathscr{L}^{\xi(x)}(Q)$ and $v \in \mathscr{L}^{\xi'(x)}(Q)$

$$\left|\int_Q \omega(x)v(x)dx\right| \leq \left(\frac{1}{\xi^-} + \frac{1}{(\xi')^-}\right) \|\omega\|_{\mathscr{L}^{\xi(x)}(Q)} \|v\|_{\mathscr{L}^{\xi'(x)}(Q)}$$
$$\leq 2\|\omega\|_{\mathscr{L}^{\xi(x)}(Q)} \|v\|_{\mathscr{L}^{\xi'(x)}(Q)}. \tag{2.1}$$

Moreover, if $\omega \in \mathscr{L}^{\xi(x)}(Q)$ and $\xi < \infty$, then from [9] we have

$$\|\omega\|_{\mathscr{L}^{\xi(x)}(Q)} < 1 (= 1; > 1) \text{ if and only if } \varrho_\xi(\omega) < 1 (= 1; > 1), \tag{2.2}$$

$$\text{if } \|\omega\|_{\mathscr{L}^{\xi(x)}(Q)} > 1 \text{ then } \|\omega\|^{\xi^-}_{\mathscr{L}^{\xi(x)}(Q)} \leq \varrho_\xi(\omega) \leq \|\omega\|^{\xi^+}_{\mathscr{L}^{\xi(x)}(Q)}, \tag{2.3}$$

and

$$\text{if } \|\omega\|_{\mathscr{L}^{\xi(x)}(Q)} < 1 \text{ then } \|\omega\|^{\xi^+}_{\mathscr{L}^{\xi(x)}(Q)} \leq \varrho_\xi(\omega) \leq \|\omega\|^{\xi^-}_{\mathscr{L}^{\xi(x)}(Q)}. \tag{2.4}$$

As a result, we get

$$\|\omega\|^{\xi^-}_{\mathscr{L}^{\xi(x)}(Q)} - 1 \leq \varrho_\xi(\omega) \leq \|\omega\|^{\xi^+}_{\mathscr{L}^{\xi(x)}(Q)} + 1, \text{ for all } \omega \in \mathscr{L}^{\xi(x)}(Q). \tag{2.5}$$

This leads to an important result that norm convergence and modular convergence are equivalent.

$$\|\omega\|_{\mathscr{L}^{\xi(x)}(Q)} \longrightarrow 0 \ (\longrightarrow \infty) \text{ if and only if } \varrho_\xi(\omega) \longrightarrow 0 \ (\longrightarrow \infty). \tag{2.6}$$

Remark 2.2 The above properties of the modular and norm hold for all $L^{\xi(x)}_\mu(Q) := \left\{\omega : \omega \text{ is } \mu\text{-measurable real-valued function and } \int_Q |\omega(x)|^{\xi(x)} d\mu < \infty\right\}$, where $Q \subset \mathbb{R}^N (N \geq 2)$ is a bounded open subset, μ is a measure on Q, and $\xi \in C^+(\overline{Q})$.

Now, we recall the definition of ϕ-HFDS, denoted by $\mathcal{H}_{\xi(x)}(Q)$, which is consists of functions ω belonging to $\mathscr{L}^{\xi(x)}(Q)$ whose derivatives $\mathfrak{D}^{\alpha,\beta;\phi}_{0+}\omega$ are also in $\mathscr{L}^{\xi(x)}(Q)$, i.e,

$$\mathcal{H}_{\xi(x)}(Q) = \left\{\omega \in \mathscr{L}^{\xi(x)}(Q) : \left|\mathfrak{D}^{\alpha,\beta;\phi}_{0+}\omega\right| \in \mathscr{L}^{\xi(x)}(Q)\right\},$$

endowed with the norm

$$\|\omega\|_{\mathcal{H}_{\xi(x)}(Q)} = \|\omega\|_{\mathscr{L}^{\xi(x)}(Q)} + \left\|\mathfrak{D}_{0+}^{\alpha,\beta;\phi}\omega\right\|_{\mathscr{L}^{\xi(x)}(Q)} \quad \text{for all } \omega \in \mathcal{H}_{\xi(x)}(Q).$$

Remark 2.3 We can define $\mathcal{H}_{\xi(x),0}(Q)$ as the closure of $C_0^\infty(\mathbb{R}^N)$ in $\mathcal{H}_{\xi(x)}(Q)$ which can be equipped with the equivalent norm

$$\|\omega\|_{\mathcal{H}_{\xi(x),0}(Q)} = \left\|\mathfrak{D}_{0+}^{\alpha,\beta;\phi}\omega\right\|_{\mathscr{L}^{\xi(x)}(Q)} \quad \text{for all } \omega \in \mathcal{H}_{\xi(x),0}(Q).$$

Note that $\mathcal{H}_{\xi(x)}(Q)$ and $\mathcal{H}_{\xi(x),0}(Q)$ are separable and reflexive Banach spaces, as established in [27].

Now, let us highlight the crucial embeddings of the space $\mathcal{H}_{\xi(x)}(Q)$.

Proposition 2.4 [27] *Let Q be a Lipschitz bounded domain in $\mathbb{R}^N$, and $\xi \in C^0(\overline{Q})$. If $r : \overline{Q} \longrightarrow (1, +\infty)$ satisfy*

$$1 \le r(x) < \xi^*(x) = \begin{cases} \frac{N\xi(x)}{N-\alpha\xi(x)}, & \text{if } \alpha\xi(x) < N, \\ \infty, & \text{if } \alpha\xi(x) \ge N, \end{cases} \quad \text{for all} \quad x \in \overline{Q},$$

then, the embedding

$$\mathcal{H}_{\xi(x)}(Q) \hookrightarrow \mathscr{L}^{r(x)}(Q) \tag{2.7}$$

is compact and there is a constant $c_0 > 0$, such that $\|\omega\|_{\mathscr{L}^{r(x)}} \le c_0\|\omega\|$.

Proposition 2.5 [17] *Consider a real Banach space Y and its dual Y^*. Assume that $\psi \in C^1(Y, \mathbb{R})$ satisfies the condition*

$$\max\Big(\psi(0), \psi(e)\Big) \le \nu \le \inf_{\|\omega\|=\rho} \psi(\omega),$$

for some $\vartheta < \nu$, $\rho > 0$, and $e \in Y$ with $\|e\| > \rho$. Let $c \le \nu$ be such that

$$c = \inf_{\gamma\in\Gamma} \max_{\tau\in[0,1]} \psi(\gamma(\tau)),$$

where $\Gamma = \Big\{\gamma \in C([0,1], Y) : \gamma(0) = \gamma(1) = e\Big\}$ is the set of continuous paths joining 0 and e. Then, there exists a sequence $\{\omega_n\}_{n\in\mathbb{N}}$ in Y such that

$$\phi(\omega_n) \longrightarrow c \ge \nu \quad \text{and} \quad (1 + \|\omega_n\|)\left\|\psi'(\omega_n)\right\|_{Y^*} \longrightarrow 0.$$

3 Proof of the propert of the ϕ-HFD space with anisotropic operator

This section will be devoted to provide the proofs of propositions 1.1, 1.4, and Theorem 1.3.

Proof of Proposition 1.1 Since $\mathscr{L}^{\xi_i(x)}(Q)$ is a reflexive and separable space, it follows that the space $\prod_{i=1}^{N}\mathscr{L}^{\xi_i(x)}(Q)$ with respect to the norm

$$\|\omega\|_{\mathscr{L}^{\xi_i}} = \left(\sum_{i=1}^{N}\|\omega_i\|_{\mathscr{L}^{\xi_i}}^{\xi_i}\right)^{\frac{1}{\xi_i}}, \tag{3.1}$$

where $\omega = (\omega_1, \omega_2, ..., \omega_N) \in \prod_{i=1}^{N}\mathscr{L}^{\xi_i(x)}(Q)$ is also reflexive and separable space. On the other hand, let us consider the space $\Upsilon = \left\{\left(\omega, \mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega\right) : \omega \in \mathcal{H}_{\overrightarrow{\xi}(x)}(Q)\right\}$, which is a closed subset of $\prod_{i=1}^{N}\mathscr{L}^{\xi_i(x)}(Q)$ as $\mathcal{H}_{\overrightarrow{\xi}(x)}(Q)$ is closed. Therefore, Υ is also reflexive and separable Banach space with respect to the norm (3.1) for $\omega = (\omega_1, \omega_2, ..., \omega_N) \in \Upsilon$.

Define the operator $\mathbf{A} : \mathcal{H}_{\overrightarrow{\xi}(x)}(Q) \longrightarrow \Upsilon$ given by

$$\mathbf{A}(\omega) =: \left(\omega, \mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega\right), \quad \omega \in \mathcal{H}_{\overrightarrow{\xi}(x)}(Q).$$

Therefore, it follows that $\|\omega\|_{\mathcal{H}_{\overrightarrow{\xi}(x)}(Q)} = \sum_{i=1}^{N}\|\mathbf{A}\omega\|_{\mathscr{L}^{\xi_i}}$, which means that the operator $\mathbf{A} : \omega \mapsto \left(\omega, \mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega\right)$ is a isometric isomorphic mapping, which implies that the space $\mathcal{H}_{\overrightarrow{\xi}(x)}(Q)$ is isometric to the space Υ. Hence $\mathcal{H}_{\overrightarrow{\xi}(x)}(Q)$ is a reflexive and separable Banach space and this completes the proof. □

Proof of Theorem 1.3 Consider $\omega \in \mathcal{H}_{\overrightarrow{\xi}(x)}(Q)$. From Proposition 2.1, we deduce that $\omega \in \mathcal{H}_{\xi_m(x)}(Q)$. As $h(x) \leq \xi^*_m(x)$ for all $x \in \overline{Q}$, Proposition 2.4 ensures the existence of a positive constant $c > 0$ such that

$$\|\omega\|_{\mathscr{L}^{h(x)}(Q)} \leq c\left(\|\omega\|_{\mathscr{L}^{\xi_m(x)}(Q)} + \sum_{i=1}^{N}\left\|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega\right\|_{\mathscr{L}^{\xi_m(x)}(Q)}\right). \tag{3.2}$$

As $\xi_m \leq \xi_i \leq \xi_M$ holds for all $i \in \{1, \ldots, N\}$, we can once more utilize Proposition 2.1 to obtain positive constants c_i such that

$$\|\omega\|_{\mathscr{L}^{\xi_m(x)}(Q)} \leq c_0\|\omega\|_{\mathscr{L}^{\xi_i(x)}(Q)}, \quad \text{and}$$
$$\|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega\|_{\mathscr{L}^{\xi_m(x)}(Q)} \leq c_i\|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega\|_{\mathscr{L}^{\xi_i(x)}(Q)}, \tag{3.3}$$

for all $i \in \{1,\ldots,N\}$. Combining (3.2) with (3.3), yields $\|\omega\|_{\mathscr{L}^{h(x)}(Q)} \leq c\|\omega\|_{\mathcal{H}_{\overrightarrow{\xi}(x)}(Q)}$. Since Proposition 2.4 establishes that the embedding

$$\mathcal{H}_{\xi_m(x)}(Q) \hookrightarrow \mathscr{L}^{h(x)}(Q),$$

is compact if $1 \leq h(x) < \xi_m^*(x)$ for all $x \in Q$, we can conclude that the embedding $\mathcal{H}_{\overrightarrow{\xi}(x)}(Q) \hookrightarrow \mathscr{L}^{h(x)}(Q)$ is both continuous and compact. □

Proof of Proposition 1.4 Let us consider a contradiction case where (1.5) is not satisfied. This implies the existence of a sequence $\{\omega_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_{\overrightarrow{\xi}(x),0}(Q)$, which we can assume, without loss of generality, that $\|\omega_n\|_{\mathscr{L}^{\xi_M(x)}(Q)} = 1$,and

$$\sum_{l=1}^{N} \left\| \mathfrak{D}_{0^+,x_i}^{\alpha,\beta;\phi} \omega_n \right\|_{\mathscr{L}^{\xi_i(x)}(Q)} \leq \frac{1}{n}, \quad \text{for all } n = 1, 2, \ldots \text{ and } \{\omega_n\}_{n\in\mathbb{N}}$$

is bounded in $\mathcal{H}_{\overrightarrow{\xi}(x),0}(Q)$.

Using Theorem 1.3, there exists a subsequence of $\{\omega_n\}_{n\in\mathbb{N}}$, denoted by $\{\omega_n\}_{n\in\mathbb{N}}$, which converges in $\mathscr{L}^{\xi_M(x)}(Q)$. Therefore, $\{\omega_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}_{\overrightarrow{\xi}(x),0}(Q)$, and then there exists $\omega_0 \in \mathcal{H}_{\overrightarrow{\xi}(x),0}(Q)$ such that $\omega_n \longrightarrow \omega$ as $n \longrightarrow \infty$. Since

$$\sum_{i=1}^{N} \left\| \mathfrak{D}_{0^+,x_i}^{\alpha,\beta;\phi} \omega_n \right\|_{\mathscr{L}^{\xi_i(x)}(Q)} \leq \frac{1}{n}, \quad \text{and} \quad \mathfrak{D}_{0^+,x_i}^{\alpha,\beta;\phi} \omega_n \longrightarrow \mathfrak{D}_{0^+,x_i}^{\alpha,\beta;\phi} \omega_0, \text{ in } \mathscr{L}^{\xi_i(x)}(Q),$$

for all $i \in \{1, 2, \ldots, N\}$, yields that

$$\sum_{i=1}^{N} \left\| \mathfrak{D}_{0^+,x_i}^{\alpha,\beta;\phi} \omega_0 \right\|_{\mathscr{L}^{\xi_i(x)}(Q)} = \lim_{n\to\infty} \sum_{i=1}^{N} \left\| \mathfrak{D}_{0^+,x_i}^{\alpha,\beta;\phi} \omega_n \right\|_{\mathscr{L}^{\xi_i(x)}Q} = 0,$$

and consequently $\mathfrak{D}_{0^+}^{\alpha,\beta;\phi} \omega_0 = 0$. Since $\omega_0 \in \mathcal{H}_{\overrightarrow{\xi}(x),0}(Q)$, we have $\omega_0 = 0$, which contradicts the fact that $\|\omega_0\|_{\mathscr{L}^{\xi_i(x)}(Q)} = \lim_{n\to\infty} \|\omega_n\|_{\mathscr{L}^{\xi_i(x)}(Q)} = 1$. □

4 Application

In what follows we provide the definition of weak solution that will be considered for (1.6).

Definition 4.1 We say that $\omega \in \mathcal{H}_{\overrightarrow{\xi}(x),0}(Q)$ is a weak solution of (1.6), if for every $v \in \mathcal{H}_{\overrightarrow{\xi}(x),0}(Q)$ the following holds

$$
\sum_{i=1}^{N}\int_Q |\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega|^{\xi_i(x)-2}\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega\,\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}v\,dx+\int_Q|\omega|^{\xi_M(x)-2}\omega v\,dx
-\int_Q g(x,\omega)v\,dx=0.
$$

Now, let us introduce the energy functional $\mathfrak{E}:\mathcal{H}_{\overrightarrow{\xi}(x),0}(Q)\longrightarrow\mathbb{R}$ associated to problem (1.6), which is defined as

$$
\mathfrak{E}(\omega)=\sum_{i=1}^{N}\int_Q\frac{1}{\xi_i(x)}\left|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega\right|^{\xi_i(x)}dx+\int_Q\frac{1}{\xi_M(x)}|\omega|^{\xi_M(x)}dx-\int_Q G(x,\omega)dx. \tag{4.1}
$$

We have $\mathfrak{E}\in C^1\left(\mathcal{H}_{\overrightarrow{\xi}(x),0}(Q),\mathbb{R}\right)$ and it is noteworthy that the critical points of $\mathfrak{E}$ correspond to weak solutions of (1.6) and its Gateaux derivative is

$$
\langle\mathfrak{E}'(\omega),\omega\rangle=\sum_{i=1}^{N}\int_Q\left|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega\right|^{\xi_i(x)}dx+\int_Q|\omega|^{\xi_M(x)}dx-\int_Q g(x,\omega)\omega\,dx.
$$

Mountain-pass geometry

Next, we prove that the energy functional (4.1) satisfy the mountain pass geometry.

Lemma 4.2 *If conditions* (**H1**)–(**H3**) *are satisfied, then the following assertions hold:*

(i): There exists $v\in\mathcal{H}_{\overrightarrow{\xi}(x),0}(Q)$ *with* $v>0$ *such that* $\mathfrak{E}(tv)\longrightarrow-\infty$ *as* $t\longrightarrow\infty$.
(ii): There exist $\vartheta,\nu>0$ *such that* $\mathfrak{E}(\omega)\geq\nu$ *for all* $\omega\in\mathcal{H}_{\overrightarrow{\xi}(x),0}(Q)$ *with* $\|\omega\|=\vartheta$.

Proof (**i**) Using the condition (**H2**), it can be inferred that for all $K>0$, there exist $C_K>0$, such that

$$
G(x,s)>K|s|^{\xi_M^+}\quad\text{for all } x\in Q,\ \text{and } |s|>C_K. \tag{4.2}
$$

Let $t>1$ large enough and $v\in\mathcal{H}_{\overrightarrow{\xi}(x),0}(Q)$ with $v>0$. From (4.2), one has

$$
\begin{aligned}
\mathfrak{E}(tv)&=\sum_{i=1}^{N}\int_Q\frac{1}{\xi_i(x)}|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}tv|^{\xi_i(x)}dx+\int_Q\frac{1}{\xi_M(x)}|tv|^{\xi_M(x)}dx-\int_Q G(x,tv)dx\\
&\leq t^{\xi_M^+}\sum_{i=1}^{N}\int_Q\frac{1}{\xi_i(x)}|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}v|^{\xi_i(x)}dx+t^{\xi_M^+}\int_Q\frac{1}{\xi_M(x)}|v|^{\xi_M(x)}dx\\
&\quad-\int_{|tv|>C_K}G(x,tv)dx-\int_{|tv|\leq C_K}G(x,tv)dx
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{t^{\xi_M^+}}{\xi_m^-}\sum_{i=1}^{N}\int_Q |\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i} v|^{\xi_i(x)}dx + \frac{t^{\xi_M^+}}{\xi_M^-}\int_Q |v|^{\xi_M(x)}dx \\
&- \quad Kt^{\xi_M^+}\int_Q |v|^{\xi_M^+}dx - \int_{|tv|\leq C_K} G(x,tv)dx \\
&\leq \frac{t^{\xi_M^+}}{\xi_m^-}\sum_{i=1}^{N}\int_Q |\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i} v|^{\xi_i(x)}dx + \frac{t^{\xi_M^+}}{\xi_M^-}\int_Q |v|^{\xi_M(x)}dx - Kt^{\xi_M^+}\int_Q |v|^{\xi_M^+}dx + C_1,
\end{aligned}
$$

where $C_1 > 0$ is a constant. Choosing K to be sufficiently large to guarantee a specific condition

$$
\frac{1}{\xi_m^-}\sum_{i=1}^{N}\int_Q \left|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i} v\right|^{\xi_i(x)} dx + \frac{1}{\xi_M^-}\int_Q |v|^{\xi_M(x)}dx - K\int_Q |v|^{\xi_M^+}dx < 0,
$$

we have that

$$
\mathfrak{E}(tv) \longrightarrow -\infty \quad \text{as} \quad t \longrightarrow +\infty,
$$

which concludes the proof of (**i**).

(**ii**) It follows from (**H1**) and (**H2**) that for a given $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$
|g(x,\omega)| \leq \varepsilon|\omega|^{\xi_M^+ - 1} + C_\varepsilon|\omega|^{q(x)-1}, \quad \text{for all } (x,\omega) \in Q \times \mathbb{R}.
$$

Furthermore, according to the continuous embeddings $\mathcal{H}_{\overrightarrow{\xi}(x),0}(Q) \hookrightarrow \mathscr{L}^{q(x)}(Q)$ and $\mathcal{H}_{\overrightarrow{\xi}(x),0}(Q) \hookrightarrow \mathscr{L}^{\xi_M^+}(Q)$, it follows that there exist constants $C_1, C_2 > 0$ such that

$$
\|\omega\|_{\mathscr{L}^{\xi_M^+}(Q)} \leq C_1\|\omega\| \quad \text{and} \quad \|\omega\|_{\mathscr{L}^{q(x)}(Q)} \leq C_2\|\omega\|, \tag{4.3}
$$

for all $\omega \in \mathcal{H}_{\overrightarrow{\xi}(x),0}(Q)$. Therefore,

$$
\begin{aligned}
\int_Q G(x,\omega)dx &\leq \int_Q \frac{\varepsilon}{\xi_M^+}|\omega|^{\xi_M^+}dx + \int_Q \frac{C_\varepsilon}{q(x)}|\omega|^{q(x)}dx \\
&\leq \varepsilon\frac{C_1^{\xi_M^+}}{\xi_M^+}\|\omega\|^{\xi_M^+} + C_\varepsilon\frac{C_2^{q^-}}{q^-}\|\omega\|^{q^-},
\end{aligned} \tag{4.4}
$$

for all $x \in Q$ and all $\omega \in \mathcal{H}_{\overrightarrow{\xi}(x),0}(Q)$. Thus, for $\|\omega\| < 1$, we have from (4.3) and (4.4) that

$$
\begin{aligned}
\mathfrak{E}(\omega) &= \sum_{i=1}^{N} \int_Q \frac{1}{\xi_i(x)} \left|\mathfrak{D}_{0^+,x_i}^{\alpha,\beta;\phi}\omega\right|^{\xi_i(x)} dx + \int_Q \frac{1}{\xi_M(x)} |\omega|^{\xi_M(x)} dx - \int_Q G(x,\omega)dx \\
&\geq \frac{1}{\xi_M^+} \sum_{i=1}^{N} \int_Q \left|\mathfrak{D}_{0^+,x_i}^{\alpha,\beta;\phi}\omega\right|^{\xi_i(x)} dx + \frac{1}{\xi_M^+} \int_Q |\omega|^{\xi_M(x)} dx - \int_Q G(x,\omega)dx \\
&\geq \frac{1}{\xi_M^+} \sum_{i=1}^{N} \left\|\mathfrak{D}_{0^+,x_i}^{\alpha,\beta;\phi}\omega\right\|_{\mathscr{L}^{\xi_i(\cdot)}(Q)}^{\xi_M^+} + \frac{1}{\xi_M^+} \|\omega\|_{\mathscr{L}^{\xi_M(\cdot)}(Q)}^{\xi_M^+} - \int_Q G(x,\omega)dx \\
&\geq \frac{1}{\xi_M^+} \|\omega\|^{\xi_M^+} - \int_Q G(x,\omega)dx \\
&\geq \left(\frac{1}{2(N+1)^{\xi_m^+} \xi_M^+} - \varepsilon \frac{C_1^{\xi_M^+}}{\xi_M^+} + C_\varepsilon \frac{C_2^{q^-}}{q^-} \|\omega\|^{q^- - \xi_M^+} \right) \|\omega\|^{\xi_M^+}.
\end{aligned}
$$

Since $1 < \xi_M^+ < q^-$, for sufficiently small values of ϑ, we choose $\nu > 0$ such that

$$
\mathfrak{E}(\omega) \geq \nu, \quad \text{for all } \omega \in \mathcal{H}_{\overrightarrow{\xi}(x),0}(Q) \quad \text{with } \|\omega\| = \vartheta.
$$

□

Lemma 4.3 *Given the hypotheses* (**H1**) *and* (**H3**), *and a sequence* $\{\omega_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_{\overrightarrow{\xi}(x),0}(Q)$ *such that*

$$
\langle \mathfrak{E}'(\omega_n), \omega_n \rangle \longrightarrow 0 \ \text{as} \ n \longrightarrow \infty, \tag{4.5}
$$

then there is a subsequence, still denoted by $\{\omega_n\}_{n\in\mathbb{N}}$, *such that for all* $t > 0$, *it holds*

$$
\mathfrak{E}(t\omega_n) \leq \frac{t^{\xi_m^-}}{\xi_m^-} \left[\frac{1}{n} + \int_Q \frac{1}{\xi_m^-} g(x,\omega_n)\,\omega_n dx \right] - \int_Q G(x,\omega_n)\,dx.
$$

Proof Let φ be a function such that

$$
\varphi(t) = \frac{t^{\xi_m^-}}{\xi_m^-} g(x,\omega_n)\,\omega_n - G(x,t\omega_n).
$$

Thus,

$$
\begin{aligned}
\varphi'(t) &= t^{\xi_m^- - 1} g(x,\omega_n)\,\omega_n - g(x,t\omega_n)\,\omega_n \\
&= t^{\xi_m^- - 1} \omega_n \left(g(x,\omega_n) - \frac{g(x,t\omega_n)}{t^{\xi_m^- - 1}} \right),
\end{aligned}
$$

which implies that $\varphi'(t) \geq 0$ for $t \in]0, 1]$, and $\varphi'(t) \leq 0$ when $t \geq 1$, which leads to

$$\varphi(t) \leq \varphi(1), \quad \text{for all } t > 0. \tag{4.6}$$

According to (4.5), one has

$$\left|\left\langle \mathfrak{E}'(\omega_n), \omega_n \right\rangle\right| < \frac{1}{n}.$$

Hence,

$$\begin{aligned} -\frac{1}{n}\left\langle \mathfrak{E}'(\omega_n), \omega_n \right\rangle &= \sum_{i=1}^{N} \int_Q \left|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega_n\right|^{\xi_i(x)} dx + \int_Q |\omega_n|^{P_M(x)} dx \\ &\quad - \int_Q g(x, \omega_n)\,\omega_n dx < \frac{1}{n}. \end{aligned} \tag{4.7}$$

By utilizing (4.6) and (4.7), we derive

$$\begin{aligned} \mathfrak{E}(t\omega_n) = \sum_{i=1}^{N} \int_Q \frac{1}{\xi_i(x)} \left|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i} t\omega_n\right|^{\xi_i(x)} dx + \int_Q \frac{1}{\xi_M(x)} |t\omega_n|^{P_M(x)} dx \\ - \int_Q G(x, t\omega_n)\,dx \leq \frac{t^{\xi_m^-}}{\xi_m^-}\left[\frac{1}{n} + \int_Q \frac{1}{\xi_m^-} g(x, \omega_n)\,\omega_n dx\right] \\ - \int_Q G(x, \omega_n)\,dx, \end{aligned} \tag{4.8}$$

which completes the proof. □

Proof of 1.5 (**i**) If $\{\omega_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_{\overrightarrow{\xi}(x),0}(Q)$ satisfy Proposition 2.5, then

$$\begin{aligned} \mathfrak{E}(\omega_n) &= \sum_{i=1}^{N} \int_Q \frac{1}{\xi_i(x)} \left|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega_n\right|^{\xi_i(x)} + \int_Q \frac{1}{\xi_M(x)} |\omega|^{\xi_M(x)} dx \\ &\quad - \int_Q G(x, \omega_n)\,dx = c + o(1), \end{aligned}$$

and

$$\left(1 + \|\omega_n\|\right)\left\|\phi'(\omega_n)\right\| \longrightarrow 0.$$

Hence,

$$\|\omega_n\| - \int_Q g(x, \omega_n)\,\omega_n dx = o(1).$$

Moreover,

$$\sum_{i=1}^{N}\int_Q \left|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega_n\right|^{\xi_i(x)-2}\ \mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega_n\ \mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}v+\int_Q |\omega|^{\xi_M(x)-2}v dx$$
$$-\int_Q g\left(x,\omega_n\right)v=o(1),\text{ for all } v\in\mathcal{H}_{\overrightarrow{\xi}(x),0}(Q).$$

□

Claim 1: The sequence $\{\omega_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_{\overrightarrow{\xi}(x),0}(Q)$. Indeed, let us define

$$t_n=\frac{\left(2\xi_M^+c\right)^{1/\xi_M^+}}{\|\omega_n\|}>0 \quad\text{and}\quad \mathbf{v}_n=t_n\omega_n.$$

Since $\|\mathbf{v}_n\| = \left(2\xi_M^+c\right)^{1/\xi_M^+}$, then $\mathbf{v}_n$ is bounded in $\mathcal{H}_{\overrightarrow{\xi}(x),0}(Q)$. Hence, up to a subsequence still denoted by $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$, we have

$$\begin{cases}\mathbf{v}_n\rightharpoonup\mathbf{v}\text{ in }\mathcal{H}_{\overrightarrow{\xi}(x),0}(Q),\\ \mathbf{v}_n\longrightarrow\mathbf{v}\text{ in }\mathscr{L}^{q(x)}(Q),\ \text{ for } q(x)\in\left(1,\max\left\{\overline{\xi}^*(x),\xi_M(x)\right\}\right),\\ \mathbf{v}_n\longrightarrow\mathbf{v}\text{ a.e. in }Q.\end{cases}$$

If $\|\omega_n\|\longrightarrow\infty$, we obtain $\mathbf{v}\equiv 0$. In fact, let

$$Q_1=\left\{x\in Q:\mathbf{v}(x)=0\right\}\quad\text{and}\quad Q_2=\left\{x\in Q:\mathbf{v}(x)\neq 0\right\}.$$

Since $|\omega_n| = |\mathbf{v}_n|\,\|\omega_n\|\left(2\xi_M^+c\right)^{-1/\xi_M^+}$, it follows that $|\omega_n(x)|\longrightarrow\infty$ a.e. in Q_2. Based on hypothesis (**H2**) and for a sufficiently large n, we deduce that

$$\frac{g\left(x,\omega_n\right)\omega_n}{|\omega_n|_M^{\xi_M^+}}>k\quad\text{uniformly } x\in Q_2,$$

for a large enough k. Then,

$$\begin{aligned}2\xi_M^+c&=\lim_{n\to\infty}\|\mathbf{v}_n\|^{\xi_M^+}\\ &=\lim_{n\to\infty}|t_n|^{\xi_M^+}\,\|\omega_n\|^{\xi_M^+}\\ &=\lim_{n\to\infty}|t_n|^{\xi_M^+}\int_Q\frac{|g\left(x,\omega_n\right)\omega_n|^{\xi_M^+}}{|\omega_n|^{\xi_M^+}}|\omega_n|^{\xi_M^+}dx\\ &>k\lim_{n\to\infty}\int_{Q_2}|\mathbf{v}_n|^{\xi_M^+}dx=k\int_{Q_2}|\mathbf{v}|^{\xi_M^+}dx.\end{aligned}\tag{4.9}$$

Given the constant of $2\xi_M^+ c$ and the sufficiently large value of k, we can conclude that $|Q_2| = 0$, implying $\mathbf{v} \equiv 0$ in Q. Moreover, with $\mathbf{v} = 0$ and considering the continuity of the Nemitskii operator, we obtain

$$G(\cdot, \mathbf{v}_n) \longrightarrow 0 \quad \text{in } \mathscr{L}^1(Q),$$

which implies that

$$\lim_{n \longrightarrow \infty} G(x, \mathbf{v}_n) = 0.$$

Therefore,

$$\begin{aligned}\mathfrak{E}(\mathbf{v}_n) &\geq \frac{t_n^{\xi_M^+}}{\xi_M^+} \left[\sum_{i=1}^{N} \int_Q \left| \mathfrak{D}_{0^+, x_i}^{\alpha, \beta; \phi} \omega \right|^{\xi_i(x)} dx + \int_Q |\omega|^{\xi_M(x)} dx \right] - o(1) \\ &\geq \frac{2\xi_M^+ c}{\xi_M^+} - o(1) = 2c - o(1) > c. \end{aligned} \tag{4.10}$$

Similarly to (4.7), for some $n > 1$, we find

$$\frac{-1}{n} < \frac{\xi_m^-}{\xi_M^+} \left\langle \mathfrak{E}'(\omega_n), \omega_n \right\rangle < \frac{1}{n}.$$

Hence,

$$\begin{aligned}\mathfrak{E}(\omega_n) &= \sum_{i=1}^{N} \int_Q \left| \mathfrak{D}_{0^+, x_i}^{\alpha, \beta; \phi} \omega_n \right|^{\xi_i(x)} dx + \int_Q \frac{1}{\xi_M(x)} |t_n \omega_n|^{\xi_M(x)} dx - \int_Q G(x, \omega_n) dx \\ &\geq \frac{1}{\xi_M^+} \frac{\xi_M^+}{\xi_m^-} \left(\frac{-1}{n} + \int_Q g(x, \omega_n) \omega_n dx \right) - \int_Q G(x, \omega_n) dx, \end{aligned} \tag{4.11}$$

that is,

$$\mathfrak{E}(\omega_n) + \frac{1}{n\xi_m^-} \geq \int_Q \left(\frac{1}{\xi_m^-} g(x, \omega_n) \omega_n - G(x, \omega_n) \right) dx. \tag{4.12}$$

Furthermore, according to Lemma 4.3, one has

$$\mathfrak{E}(t\omega_n) \leq \frac{t^{\xi_m^-}}{n\xi_m^-} + \int_Q \left(\frac{1}{\xi_m^-} g(x, \omega_n) \omega_n - G(x, \omega_n) \right) dx. \tag{4.13}$$

Due to (4.12) and (4.13), we obtain

$$\mathfrak{E}(\mathbf{v}_n) \leq \frac{t^{\xi_m^-} + 1}{n\xi_m^-} + \phi(\omega_n) \longrightarrow c,$$

that contradicts (4.10). Hence, $\{\omega_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_{\overrightarrow{\xi}(x),0}(Q)$.

Claim 2: The sequence $\{\omega_n\}_{n\in\mathbb{N}}$ converges strongly to ω in $\mathcal{H}_{\overrightarrow{\xi}(x),0}(Q)$. In fact, given the boundedness of $\{\omega_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}_{\overrightarrow{\xi}(x),0}(Q)$ and the reflexivity of $\mathcal{H}_{\overrightarrow{\xi}(x),0}(Q)$, there exists $\omega \in \mathcal{H}_{\overrightarrow{\xi}(x),0}(Q)$ such that $\omega_n \rightharpoonup \omega$. Since, the space $\mathcal{H}_{\overrightarrow{\xi}(x),0}(Q)$ is compactly embedded in $\mathscr{L}^{\xi_M}(Q)$, we obtain, for a subsequence still denoted by ω_n, that $\omega_n \longrightarrow \omega$ in $\mathscr{L}^{\xi_M}(Q)$. Then, employing Holder's inequality, we conclude

$$\lim_{n\to\infty} \int_Q |\omega_n|^{\xi_M(x)-2}\, \omega_n\, (\omega_n - \omega)\, dx = 0.$$

On the other hand, utilizing (4.5), yields

$$\lim_{n\to\infty} \left\langle \mathfrak{E}'(\omega_n), \omega_n - \omega \right\rangle = 0.$$

Therefore, employing the aforementioned equations, we obtain

$$\lim_{n\to\infty} \sum_{i=1}^{N} \int_Q \left| \mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i} \omega_n \right|^{\xi_i(x)-2} \mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i} \omega_n \left(\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i} \omega_n - \mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i} \omega \right) dx = 0. \tag{4.14}$$

Moreover, (4.14) combined with the weak convergence of $\{\omega_n\}_{n\in\mathbb{N}}$ to ω in $\mathcal{H}_{\overrightarrow{\xi}(x),0}(Q)$ implies

$$\lim_{n\to\infty} \sum_{i=1}^{N} \int_Q \left(\left| \mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i} \omega_n \right|^{\xi_i(x)-2} \mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i} \omega_n - \left| \mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i} \omega \right|^{\xi_i(x)-2} \mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i} \omega \right) \times \left(\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i} \omega_n - \mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i} \omega \right) dx = 0. \tag{4.15}$$

Thus, by using Simon Inequality [18], we obtain

$$\lim_{n\to\infty} \sum_{i=1}^{N} \int_Q \left| \mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i} \omega_n - \mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i} \omega \right|^{\xi_i(x)} dx = 0.$$

Hence, $\{\omega_n\}_{n\in\mathbb{N}}$ converges strongly to ω in $\mathcal{H}_{\overrightarrow{\xi}(x),0}(Q)$. Then, from Lemma 4.2, 4.3, **Claim 1** and **Claim 2**, it follows that $\mathfrak{E}$ satisfies Mountain-pass geometry.

(**ii**) Remember that by utilizing Jensen's inequality with the convex function $t \mapsto t^{\xi_m^-}$, one has

$$\sum_{i=1}^{N} \left\| \mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i} \omega \right\|^{\xi_m^-}_{\mathscr{L}^{\xi_i(\cdot)} Q} \geq \frac{1}{N^{\xi_m^-} - 1} \left[\sum_{i=1}^{N} \left\| \mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i} \omega \right\|_{\mathscr{L}^{\xi_i(\cdot)}(Q)} \right]^{\xi_m^-}.$$

Let introduce the following notation

$$\mathcal{A}_1=\left\{\left\|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega_n\right\|_{\mathscr{L}^{\xi_i(\cdot)}(Q)}\leq 1,\ i\in\{1,\ldots,N\}\right\},$$
$$\mathcal{A}_2=\left\{\left\|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega_n\right\|_{\mathscr{L}^{\xi_i(\cdot)}(Q)}> 1,\ i\in\{1,\ldots,N\}\right\}.$$

Subsequently, we obtain

$$\begin{aligned}\sum_{i=1}^{N}\int_Q\left|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega_n\right|^{\xi_i(x)}dx&=\sum_{i\in\mathcal{A}_1}\int_Q\left|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega_n\right|^{\xi_i(\cdot)}dx+\sum_{i\in\mathcal{A}_2}\int_Q\left|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega_n\right|^{\xi_i(.)}dx\\&\geq\sum_{i\in\mathcal{A}_1}\left\|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega_n\right\|^{\xi_M}_{\mathscr{L}^{\xi_i(\cdot)}(Q)}+\sum_{i\in\mathcal{A}_2}\left\|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega_n\right\|^{\xi_m^-}_{\mathscr{L}^{\xi_i(\cdot)}(Q)}\\&\geq\sum_{i=1}^{N}\left\|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega_n\right\|^{\xi_m^-}_{\mathscr{L}^{\xi_i(\cdot)}(Q)}-\sum_{i\in\mathcal{A}_1}\left\|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega_n\right\|^{\xi_-^-}_{\mathscr{L}^{\xi_i(\cdot)}(Q)}.\end{aligned}$$

Therefore,

$$\sum_{i=1}^{N}\int_Q\left|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega_n\right|^{\xi_i(x)}dx\geq\sum_{i=1}^{N}\left\|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega_n\right\|^{\xi_m^-}_{\mathscr{L}^{\xi_i(\cdot)}(Q)}-N,\tag{4.16}$$

and

$$\begin{aligned}\sum_{i=1}^{N}\frac{1}{\xi_i(x)}\left\|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega\right\|_{\mathscr{L}^{\xi_i(\cdot)}(Q)}&\geq\frac{1}{\xi_M^+}\sum_{i\in\mathcal{A}_1}\left\|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega\right\|^{\xi_i^+}_{\mathscr{L}^{\xi_i(\cdot)}(Q)}\\&\quad+\frac{1}{\xi_M^+}\sum_{i\in\mathcal{A}_2}\left\|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega\right\|^{\xi_i^-}_{\mathscr{L}^{\xi_i(\cdot)}(Q)}\\&\geq\frac{1}{\xi_M^+}\left(\frac{1}{\xi_M^+}\sum_{i=1}^{N}\left\|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega\right\|^{\xi_m^-}_{\mathscr{L}^{\xi_i(\cdot)}(Q)}-N\right)\\&\geq\frac{1}{\xi_M^+}\left[\frac{1}{N^{\xi_m^-}-1}\left(\sum_{i=1^N}\left\|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega\right\|_{\mathscr{L}^{\xi_i(\cdot)}(Q)}\right)^{\xi_m^-}-N\right].\end{aligned}\tag{4.17}$$

- If $\|\omega\|_{\mathscr{L}^{\xi_M(\cdot)}(Q)}\geq 1$, one has

$$\begin{aligned}\sum_{i=1}^{N}\int_{Q}\frac{1}{\xi_i(x)}|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega|^{\xi_i(x)}dx &\geq \frac{1}{\xi_M^+}\left[\frac{1}{N^{\xi_m^-}-1}\left(\sum_{i=1}^{N}\left\|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega\right\|_{\mathscr{L}^{\xi_i(\cdot)}(Q)}\right)^{\xi_m^-}\right.\\ &\quad\left.-N+\frac{1}{\xi_M^+}\|\omega\|^{\xi_m^-}_{\mathscr{L}^{\xi_M(\cdot)}(Q)}\right]\\ &\geq \frac{1}{\xi_M^+}\left[\frac{1}{N^{\xi_m^-}-1}\left(\sum_{i=1}^{N}\left\|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega\right\|_{\mathscr{L}^{\xi_i(\cdot)}(Q)}\right)^{\xi_m^-}\right.\\ &\quad\left.+\|\omega\|^{\xi_m^-}_{\mathscr{L}^{\xi_M(\cdot)}(Q)}\right]-\frac{N}{\xi_M^+}\\ &\geq\frac{1}{2^{\xi_m^--1}\xi_M^+}\inf\left\{1,\frac{1}{N^{\xi_m^-}-1}\right\}\left[\sum_{i=1}^{N}\left\|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega\right\|_{\mathscr{L}^{\xi_i(\cdot)}(Q)}\right.\\ &\quad\left.+\|\omega\|_{\mathscr{L}^{\xi_M(\cdot)}(Q)}\right]^{\xi_m^-}-\frac{N}{\xi_M^+}\\ &\geq \frac{1}{2^{\xi_m^--1}\xi_M^+}\inf\left\{1,\frac{1}{N^{\xi_m^-}-1}\right\}\|\omega\|^{\xi_m^-}-\frac{N}{\xi_M^+}.\end{aligned}\tag{4.18}$$

• If $\|\omega\|_{\mathscr{L}^{\xi_M(\cdot)}(Q)}<1$, we have

$$\begin{aligned}\sum_{i=1}^{N}\int_{Q}\frac{1}{\xi_i(x)}|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega|^{\xi_i(x)}dx &\geq \frac{1}{\xi_M^+}\left[\frac{1}{N^{\xi_m^-}-1}\left(\sum_{i=1}^{N}\left\|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega\right\|_{\mathscr{L}^{\xi_i(\cdot)}(Q)}\right)^{\xi_m^-}\right.\\ &\quad\left.+\|\omega\|^{\xi_m^-}_{\mathscr{L}^{\xi_M(\cdot)}(Q)}-1-N\right]\\ &\geq \frac{1}{\xi_M^+}\left[\frac{1}{N^{\xi_m^-}-1}\left(\sum_{i=1}^{N}\left\|\mathfrak{D}^{\alpha,\beta;\phi}_{0^+,x_i}\omega\right\|_{\mathscr{L}^{\xi_i(\cdot)}(Q)}\right)^{\xi_m^-}\right.\\ &\quad\left.+\|\omega\|^{\xi_m^-}_{\mathscr{L}^{\xi_M(\cdot)}(Q)}\right]-\frac{N-1}{\xi_M^+}\\ &\geq \frac{1}{2^{\xi_m^--1}\xi_M^+}\inf\left\{1,\frac{1}{N^{\xi_m^-}-1}\right\}\|\omega\|^{\xi_m^-}-\frac{N-1}{\xi_M^+}.\end{aligned}\tag{4.19}$$

Next, we will prove that $\mathfrak{E}$ is coercive. In fact, for $\|\omega\|>1$, by considering (**H4**), in both cases (4.18) or (4.19), we have

$$
\begin{aligned}
\mathfrak{E}(\omega) &= \sum_{i=1}^{N} \int_Q \frac{1}{\xi_i(x)} \left|\mathfrak{D}_{0^+,x_i}^{\alpha,\beta;\phi}\omega\right|^{\xi_i(x)} dx + \int_Q \frac{|\omega|^{\xi_M(x)}}{\xi_M(x)} dx - \int_Q G(x,u)dx \\
&\geq \sum_{i=1}^{N} \int_Q \frac{1}{\xi_i(x)} \left|\mathfrak{D}_{0^+,x_i}^{\alpha,\beta;\phi}\omega\right|^{\xi_i(x)} dx + \int_Q \frac{|\omega|^{\xi_M(x)}}{\xi_M(x)} dx - (\zeta_1-\varepsilon)\int_Q \frac{|\omega|^{\xi_M(x)}}{\xi_M(x)} dx \\
&\geq \sum_{i=1}^{N} \int_Q \frac{1}{\xi_i(x)} \left|\mathfrak{D}_{0^+,x_i}^{\alpha,\beta;\phi}\omega\right|^{\xi_i(x)} + \int_Q \frac{|\omega|^{\xi_M(x)}}{\xi_M(x)} dx \\
&\quad - \frac{(\zeta_1-\epsilon)}{\zeta_1} \sum_{i=1}^{N} \int_Q \frac{1}{\xi_i(x)} \left|\mathfrak{D}_{0^+,x_i}^{\alpha,\beta;\phi}\omega\right|^{\xi_i(x)} dx + \int_Q \frac{1}{\xi_M(x)} |\omega|^{\xi_M(x)} dx \\
&\geq \frac{1}{2^{\xi_m^- -1}\xi_M^+} \inf\left\{1, \frac{1}{N^{\xi_m^- -1}}\right\}\left(1-\frac{(\zeta_1-\epsilon)}{\zeta_1}\right)\|\omega\|^{\xi_m^-} - c.
\end{aligned}
\tag{4.20}
$$

Hence, $\mathfrak{E}$ is coercive and possesses a global minimizer $\overline{\omega}$, which implies that $\langle \mathfrak{E}'(\overline{\omega}), \overline{\omega}\rangle = 0$, which is nontrivial. Thus, by considering $v_0 \in \mathcal{H}_{\overrightarrow{\xi}(x),0}(Q)$, $t > 0$ small enough small, and using the inequality $q_0 < \xi_m^-$, we obtain from (**H5**) that

$$
\begin{aligned}
\mathfrak{E}(tv_0) &\leq C_2 \left(\int_Q \frac{t^{\xi_i(x)}}{\xi_i(x)} |v_0|^{\xi_i(x)} dx + \int_Q \frac{t^{\xi_M(x)}}{\xi_M(x)} |v_0|^{\xi_M(x)} dx \right) - \int_Q G(x, tv_0)\, dx \\
&\leq C_3 t^{\xi_m^-} - C_4 t^{q_0} \\
&< 0,
\end{aligned}
$$

which completes completes the proof.

Data availability statement Data sharing not applicable to this paper as no data sets were generated or analysed during the current study.

Declarations

Conflict of interest There is no conflict of interest.

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