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Estimates for the Berezin number inequalities

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Abstract
Our goal in this paper is to provide generalizations and improvements related to some newly Berezin number inequalities of bounded linear operators defined on a reproducing kernel Hilbert space.

Keywords Berezin number · Berezin norm · Reproducing kernel Hilbert space · Inequalities

Mathematics Subject Classification 47A05 · 47A55 · 47B15

1 Introduction and preliminaries

Let $\mathcal{B}(\mathcal{H})$ denote the C^*- algebra of all bounded linear operators acting on a non trivial complex Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\|\cdot\|$. For $T \in \mathcal{B}(\mathcal{H})$, T^* denotes the adjoint of T and $|T| = (T^*T)^{\frac{1}{2}}$.

Recall that the numerical range of $T \in \mathcal{B}(\mathcal{H})$ is defined by

$$W(T) = \{\langle Tx, x \rangle : x \in \mathcal{H},\ \|x\| = 1\},$$

while the numerical radius is defined as

$$\omega(T) = \sup\{|\langle Tx, x \rangle| : x \in \mathcal{H},\ \|x\| = 1\}.$$

For some results about the numerical radius inequalities and their applications, we refer to [9, 13, 20, 21].

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Let Ω be a nonempty set. A functional Hilbert space $\mathcal{H}(\Omega)$ is a Hilbert space of complex valued functions, which has the property that point evaluations are continuous in the sense that for each $\lambda \in \Omega$ the map $f \longmapsto f(\lambda)$ is a continuous linear functional on $\mathcal{H}(\Omega)$. The Riesz representation theorem ensures that for each $\lambda \in \Omega$ there exists a unique element $k_\lambda \in \mathcal{H}(\Omega)$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathcal{H}(\Omega)$. The set $\{k_\lambda : \lambda \in \Omega\}$ is called the reproducing kernel of the space $\mathcal{H}(\Omega)$. If $\{e_n\}_{n\geq 0}$ is an orthonormal basis for a functional Hilbert space $\mathcal{H}(\Omega)$, then the reproducing kernel of $\mathcal{H}(\Omega)$ is given by $k_\lambda(z) = \sum_{n=0}^{+\infty} \overline{e_n(\lambda)} e_n(z)$ (see [16]). For $\lambda \in \Omega$, let $\hat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|}$ be the normalized reproducing kernel of $\mathcal{H}(\Omega)$.

Let T be a bounded linear operator on $\mathcal{H}(\Omega)$, the Berezin symbol of T, which firstly has been introduced by Berezin [4, 5] is the function $\tilde{T}$ on Ω defined by

$$\tilde{T}(\lambda) := \left\langle T\hat{k}_\lambda, \hat{k}_\lambda \right\rangle.$$

The Berezin set and the Berezin number of the operator T are defined respectively by

$$\mathbf{Ber}(T) := \left\{ \left\langle T\hat{k}_\lambda, \hat{k}_\lambda \right\rangle : \lambda \in \Omega \right\}$$

and

$$\mathbf{ber}(T) := \sup \left\{ \left| \left\langle T\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right| : \lambda \in \Omega \right\}.$$

It is clear that the Berezin symbol $\tilde{T}$ is the bounded function on Ω whose value lies in the numerical range of the operator T and hence for any $T \in \mathcal{B}(\mathcal{H}(\Omega))$,

$$\mathbf{Ber}(T) \subset W(T) \text{ and } \mathbf{ber}(T) \leq \omega(T).$$

Moreover, the Berezin number of an operator T satisfies the following properties:

(i) $\mathbf{ber}(T) = \mathbf{ber}(T^*)$.
(ii) $\mathbf{ber}(T) \leq \|T\|$.
(iii) $\mathbf{ber}(\alpha T) = |\alpha| \, \mathbf{ber}(T)$ for all $\alpha \in \mathbb{C}$.
(iv) $\mathbf{ber}(T + S) \leq \mathbf{ber}(T) + \mathbf{ber}(S)$ for all $T, S \in \mathcal{B}(\mathcal{H}(\Omega))$.

Notice that, in general, the Berezin number does not define a norm. However, if $\mathcal{H}(\Omega)$ is a reproducing kernel Hilbert space of analytic functions, (for instance on the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$), then $\mathbf{ber}(\cdot)$ defines a norm on $\mathcal{B}(\mathcal{H}(D))$ (see [17, 18]). The Berezin symbol has been studied in detail for Toeplitz and Hankel operators on Hardy and Bergman spaces. A nice property of the Berezin symbol is mentioned next. If $\tilde{T}(\lambda) = \tilde{S}(\lambda)$ for all $\lambda \in \Omega$, then $T = S$. Therefore, the Berezin symbol uniquely determines the operator. For more facts about the Berezin symbol and Berezin number we refer the reader to [1, 6, 10, 15, 24–27].

Now, for any operator $T \in \mathcal{B}(\mathcal{H}(\Omega))$, the Berezin norm of T denoted as $\|T\|_{ber}$ is defined by

$$\|T\|_{ber} := \sup_{\lambda \in \Omega} \left\| T\hat{k}_\lambda \right\|,$$

where $\hat{k}_\lambda$ is normalized reproducing kernel for $\lambda \in \Omega$.

For $T, S \in \mathcal{B}(\mathcal{H}(\Omega))$ it is clear from the definition of the Berezin norm that the following properties hold:

(i) $\|\lambda T\|_{ber} = |\lambda| \|T\|_{ber}$ for all $\lambda \in \mathbb{C}$.
(ii) $\|T + S\|_{ber} \leq \|T\|_{ber} + \|S\|_{ber}$.
(iii) $\mathbf{ber}(T) \leq \|T\|_{ber} \leq \|T\|$.

In [24], Taghavi et al. improved the inequality $\mathbf{ber}(T) \leq \|T\|$, and obtained the following result

$$\mathbf{ber}^r(T) \leq \frac{1}{2}\| |T|^r + |T^*|^r \|_{ber}, \text{ for any } r \geq 1. \tag{1.1}$$

Also, they generalized (1.1) for product of two operators, if $T, S \in \mathcal{B}(\mathcal{H}(\Omega))$ and $r \geq 1$, then

$$\mathbf{ber}^r(S^*T) \leq \frac{1}{2}\| |T|^{2r} + |S|^{2r} \|_{ber}. \tag{1.2}$$

Recently, Guesba in [13] proved that

$$\mathbf{ber}^2(T) \leq \frac{1}{4}\left\| |T|^2 + |T^*|^2 \right\|_{ber} + \frac{1}{2}\mathbf{ber}\left(T^2\right). \tag{1.3}$$

After that, in [12] Garayev and Guesba generalized (1.3) as follows

$$\mathbf{ber}^{2r}(T) \leq \frac{1}{4}\left\| |T|^{2r} + |T^*|^{2r} \right\|_{ber} + \frac{1}{2}\mathbf{ber}^r\left(T^2\right), \text{ for any } r \geq 1. \tag{1.4}$$

For futher results about the Berezin norm inequalities and their applications, we refer to see [2, 6, 7, 11] and references therein.

In this paper, some refinements and generalizations of Berezin number inequalities of bounded linear operators defined on a reproducing kernel Hilbert space are established. In particular, we establish some refinements and generalizations of the inequalities (1.1), (1.2) and (1.4).

The following lemmas will be needed in our analysis.

Lemma 1.1 [8] *If a, b, e are vectors in $\mathcal{H}$ with $\|e\| = 1$, then*

$$|\langle a, e\rangle \langle e, b\rangle| \leq \frac{1}{2}(\|a\| \|b\| + |\langle a, b\rangle|).$$

Lemma 1.2 [23] *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator and let $x \in \mathcal{H}$ with $\|x\| = 1$. Then*

(i) $\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle$ for $r \geq 1$.
(ii) $\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r$ for $0 < r \leq 1$.

Lemma 1.3 [20] *Let $A \in \mathcal{B}(\mathcal{H})$ and let f and g be non-negative continuous functions on $[0, \infty)$ such that $f(t)\, g(t) = t$ for all $t \in [0, \infty)$. Then*

$$|\langle Ax, y\rangle|^2 \leq \left\langle f^2(|A|)\, x, x\right\rangle \left\langle g^2\left(\left|A^*\right|\right) y, y\right\rangle,$$

for all $x, y \in \mathcal{H}$.

In particular, if $f(t) = g(t) = \sqrt{t}$, then we have

$$|\langle Ax, y\rangle|^2 \leq \langle |A|\, x, x\rangle \left\langle \left|A^*\right| y, y\right\rangle.$$

Lemma 1.4 [22] *Let $a, b > 0$ and $0 \leq \alpha \leq 1$. Then*

$$a^{\alpha} b^{1-\alpha} + \min\{\alpha, 1-\alpha\} \left(\sqrt{a} - \sqrt{b}\right)^2 \leq \alpha a + (1-\alpha)\, b.$$

Lemma 1.5 [19] *Let $a, b, e \in \mathcal{H}$ with $\|e\| = 1$ and $\alpha \in \mathbb{C} \setminus \{0\}$. Then*

$$|\langle a, e\rangle \langle e, b\rangle| \leq \frac{1}{|\alpha|} \left(\max\{1, |1-\alpha|\} \|a\| \|b\| + |\langle a, b\rangle|\right).$$

Lemma 1.6 *If a, b, e are vectors in $\mathcal{H}$ and $\|e\| = 1$, then*

$$|\langle a, e\rangle \langle e, b\rangle|^2 \leq \frac{4\alpha+3}{4\alpha+4} \|a\|^2 \|b\|^2 + \frac{1}{4\alpha+4} |\langle a, b\rangle|^2,$$

for any $\alpha \geq 0$.

Proof We have

$$\begin{aligned}
|\langle a, e\rangle \langle e, b\rangle|^2 &\leq \frac{1}{4} \left(\|a\|^2 \|b\|^2 + |\langle a, b\rangle|^2 + 2 \|a\| \|b\| |\langle a, b\rangle|\right) \\
&= \frac{1}{4} \left(\|a\|^2 \|b\|^2 + \frac{1}{\alpha+1} |\langle a, b\rangle|^2 + \frac{\alpha}{\alpha+1} |\langle a, b\rangle|^2 + 2 \|a\| \|b\| |\langle a, b\rangle|\right) \\
&\leq \frac{1}{4} \left(\|a\|^2 \|b\|^2 + \frac{1}{\alpha+1} |\langle a, b\rangle|^2 + \frac{\alpha}{\alpha+1} \|a\|^2 \|b\|^2 + 2 \|a\|^2 \|b\|^2\right) \\
&= \frac{1}{4} \left(3 \|a\|^2 \|b\|^2 + \frac{1}{\alpha+1} |\langle a, b\rangle|^2 + \frac{\alpha}{\alpha+1} \|a\|^2 \|b\|^2\right) \\
&= \frac{1}{4} \left(\frac{4\alpha+3}{\alpha+1} \|a\|^2 \|b\|^2 + \frac{1}{\alpha+1} |\langle a, b\rangle|^2\right).
\end{aligned}$$

□

Lemma 1.7 *If a, b, e are vectors in $\mathcal{H}$ and $\|e\| = 1$, then*

$$|\langle a, e\rangle \langle e, b\rangle|^r \leq \frac{2\alpha+1}{2\alpha+2} \|a\|^r \|b\|^r + \frac{1}{2\alpha+2} |\langle a, b\rangle|^r,$$

for any $\alpha \geq 0$ and $r \geq 1$.

Proof By Lemma 1.1 and the convexity of t^r, $r \geq 1$, we have

$$\begin{aligned}
|\langle a, e\rangle \langle e, b\rangle|^r &\leq \left(\frac{\|a\| \|b\| + |\langle a, b\rangle|}{2}\right)^r \\
&\leq \frac{1}{2}\left(\|a\|^r \|b\|^r + |\langle a, b\rangle|^r\right) \\
&= \frac{1}{2}\left(\|a\|^r \|b\|^r + \frac{1}{\alpha+1}|\langle a, b\rangle|^r + \frac{\alpha}{\alpha+1}|\langle a, b\rangle|^r\right) \\
&\leq \frac{1}{2}\left(\|a\|^r \|b\|^r + \frac{1}{\alpha+1}|\langle a, b\rangle|^r + \frac{\alpha}{\alpha+1}\|a\|^r \|b\|^r\right) \\
&= \frac{1}{2}\left(\frac{2\alpha+1}{\alpha+1}\|a\|^r \|b\|^r + \frac{1}{\alpha+1}|\langle a, b\rangle|^r\right) \\
&= \frac{2\alpha+1}{2\alpha+2}\|a\|^r \|b\|^r + \frac{1}{2\alpha+2}|\langle a, b\rangle|^r .
\end{aligned}$$

Therefore,

$$|\langle a, e\rangle \langle e, b\rangle|^r \leq \frac{2\alpha+1}{2\alpha+2}\|a\|^r \|b\|^r + \frac{1}{2\alpha+2}|\langle a, b\rangle|^r .$$

□

Remark 1.8 It was shown in [3, Lemma 3.1] that for any $a, b, e \in \mathcal{H}$ with $\|e\| = 1$, it holds

$$|\langle a, e\rangle \langle e, b\rangle|^2 \leq \frac{3}{4}\|a\|^2 \|b\|^2 + \frac{1}{4}\|a\| \|b\| |\langle a, b\rangle| . \tag{1.5}$$

This follows from Lemma 1.7 by letting $r = 2$, $\alpha = 1$.

2 Main results

In this section, we present our results. Firstly, we introduce a new refinement of the inequality (1.1) for the case $r = 4$.

Theorem 2.1 *Let $T \in \mathcal{B}(\mathcal{H}(\Omega))$ and let $\alpha \geq 0$. Then*

$$\mathbf{ber}^4(T) \leq \frac{4\alpha+3}{8\alpha+8}\left\||T|^4 + |T^*|^4\right\|_{ber} + \frac{1}{4\alpha+4}\mathbf{ber}^2\left(T^2\right).$$

Proof Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}(\Omega)$. Then, by using Lemma 1.6 we have

$$\left|\left\langle T\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^4 = \left|\left\langle T\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 \left|\left\langle \hat{k}_\lambda, T^*\hat{k}_\lambda\right\rangle\right|^2$$

$$
\begin{aligned}
&\le \frac{4\alpha+3}{4\alpha+4}\left\|T\hat{k}_\lambda\right\|^2\left\|T^*\hat{k}_\lambda\right\|^2+\frac{1}{4\alpha+4}\left|\left\langle T\hat{k}_\lambda, T^*\hat{k}_\lambda\right\rangle\right|^2\\
&= \frac{4\alpha+3}{4\alpha+4}\left\langle |T|^2\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\left\langle |T^*|^2\hat{k}_\lambda,\hat{k}_\lambda\right\rangle+\frac{1}{4\alpha+4}\left|\left\langle T^2\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^2\\
&\le \frac{4\alpha+3}{8\alpha+8}\left(\left\langle |T|^2\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^2+\left\langle |T^*|^2\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^2\right)+\frac{1}{4\alpha+4}\left|\left\langle T^2\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^2\\
&\quad\text{(by the arithmetic-geometric mean inequality)}\\
&\le \frac{4\alpha+3}{8\alpha+8}\left\langle\left(|T|^4+|T^*|^4\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle+\frac{1}{4\alpha+4}\left|\left\langle T^2\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^2\\
&\quad\text{(by Lemma 1.2)}\\
&\le \frac{4\alpha+3}{8\alpha+8}\left\||T|^4+|T^*|^4\right\|_{ber}+\frac{1}{4\alpha+4}\mathbf{ber}^2\left(T^2\right).
\end{aligned}
$$

Now, by taking supremum over $\lambda \in \Omega$ in the above inequality, we get

$$
\mathbf{ber}^4(T) \le \frac{4\alpha+3}{8\alpha+8}\left\||T|^4+|T^*|^4\right\|_{ber}+\frac{1}{4\alpha+4}\mathbf{ber}^2\left(T^2\right),
$$

as required. □

Corollary 2.2 *If $T \in \mathcal{B}(\mathcal{H}(\Omega))$ and $\alpha \ge 0$, then*

$$
\begin{aligned}
\mathbf{ber}^4(T) &\le \frac{4\alpha+3}{8\alpha+8}\left\||T|^4+|T^*|^4\right\|_{ber}+\frac{1}{4\alpha+4}\mathbf{ber}^2\left(T^2\right)\\
&\le \frac{1}{2}\left\||T|^4+|T^*|^4\right\|_{ber}.
\end{aligned}
$$

Proof By using the inequality (1.2), we have

$$
\begin{aligned}
\mathbf{ber}^4(T) &\le \frac{4\alpha+3}{8\alpha+8}\left\||T|^4+|T^*|^4\right\|_{ber}+\frac{1}{4\alpha+4}\mathbf{ber}^2\left(T^2\right)\\
&\le \frac{4\alpha+3}{8\alpha+8}\left\||T|^4+|T^*|^4\right\|_{ber}+\frac{1}{4\alpha+4}\frac{1}{2}\left\||T|^4+|T^*|^4\right\|_{ber}\\
&= \frac{1}{2}\left\||T|^4+|T^*|^4\right\|_{ber}.
\end{aligned}
$$

□

This corollary follows directly from Theorem 2.1 by setting $\alpha = 0$.

Corollary 2.3 *Let $T \in \mathcal{B}(\mathcal{H}(\Omega))$. Then*

$$
\mathbf{ber}^4(T) \le \frac{3}{8}\left\||T|^4+|T^*|^4\right\|_{ber}+\frac{1}{4}\mathbf{ber}^2\left(T^2\right).
$$

Remark 2.4 In [14] the authors proved the following inequality

$$\mathbf{ber}^4(T) \le \frac{3}{8}\left\| |T|^4 + |T^*|^4 \right\|_{ber} + \frac{1}{8}\mathbf{ber}\left(T^2\right)\left\| |T|^4 + |T^*|^4 \right\|_{ber}. \tag{2.1}$$

Using the inequality (1.2), it follows that

$$\begin{aligned}
\mathbf{ber}^4(T) &\le \frac{3}{8}\left\| |T|^4 + |T^*|^4 \right\|_{ber} + \frac{1}{4}\mathbf{ber}^2\left(T^2\right) \\
&= \frac{3}{8}\left\| |T|^4 + |T^*|^4 \right\|_{ber} + \frac{1}{4}\mathbf{ber}\left(T^2\right)\mathbf{ber}\left(T^2\right) \\
&\le \frac{3}{8}\left\| |T|^4 + |T^*|^4 \right\|_{ber} + \frac{1}{8}\mathbf{ber}\left(T^2\right)\left\| |T|^4 + |T^*|^4 \right\|_{ber}.
\end{aligned}$$

Hence, the inequality in Corollary 2.3 is a refinement of the inequality (2.1). Next, we obtain a refinement of the inequality (1.4).

Theorem 2.5 *Let* $T \in \mathcal{B}(\mathcal{H}(\Omega))$ *and* $\alpha \ge 0$. *Then*

$$\mathbf{ber}^{2r}(T) \le \frac{2\alpha+1}{4\alpha+4}\left\| |T|^{2r} + |T^*|^{2r} \right\|_{ber} + \frac{1}{2\alpha+2}\mathbf{ber}^r\left(T^2\right),$$

for any $r \ge 1$.

Proof Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}(\Omega)$. Then, by using Lemma 1.7 we have

$$\begin{aligned}
\left|\left\langle T\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^{2r} &= \left(\left|\left\langle T\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\left|\left\langle \hat{k}_\lambda, T^*\hat{k}_\lambda\right\rangle\right|\right)^r \\
&= \frac{2\alpha+1}{2\alpha+2}\left\|T\hat{k}_\lambda\right\|^r\left\|T^*\hat{k}_\lambda\right\|^r + \frac{1}{2\alpha+2}\left|\left\langle T\hat{k}_\lambda, T^*\hat{k}_\lambda\right\rangle\right|^r \\
&\le \frac{2\alpha+1}{4\alpha+4}\left(\left\|T\hat{k}_\lambda\right\|^{2r} + \left\|T^*\hat{k}_\lambda\right\|^{2r}\right) + \frac{1}{2\alpha+2}\left|\left\langle T^2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^r \\
&\quad \text{(by the arithmetic-geometric mean inequality)} \\
&= \frac{2\alpha+1}{4\alpha+4}\left(\left\langle |T|^2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^r + \left\langle |T^*|^2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^r\right) + \frac{1}{2\alpha+2}\left|\left\langle T^2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^r \\
&\le \frac{2\alpha+1}{4\alpha+4}\left(\left\langle |T|^{2r}\hat{k}_\lambda, \hat{k}_\lambda\right\rangle + \left\langle |T^*|^{2r}\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right) + \frac{1}{2\alpha+2}\left|\left\langle T^2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^r \\
&\quad \text{(by Lemma 1.2)} \\
&= \frac{2\alpha+1}{4\alpha+4}\left\langle\left(|T|^{2r} + |T^*|^{2r}\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle + \frac{1}{2\alpha+2}\left|\left\langle T^2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^r \\
&\le \frac{2\alpha+1}{4\alpha+4}\left\| |T|^{2r} + |T^*|^{2r} \right\|_{ber} + \frac{1}{2\alpha+2}\mathbf{ber}^r\left(T^2\right).
\end{aligned}$$

Taking the the supremum over $\lambda \in \Omega$, we get the desired inequality. □

Corollary 2.6 *Let $T \in \mathcal{B}(\mathcal{H}(\Omega))$ and $\alpha \geq 0$. Then, for any $r \geq 1$ we have*

$$\begin{aligned}\mathbf{ber}^{2r}(T) &\leq \frac{2\alpha+1}{4\alpha+4}\left\||T|^{2r}+\left|T^*\right|^{2r}\right\|_{ber}+\frac{1}{2\alpha+2}\mathbf{ber}^r\left(T^2\right)\\ &\leq \frac{1}{2}\left\||T|^{2r}+\left|T^*\right|^{2r}\right\|_{ber}.\end{aligned}$$

Proof By using the inequality (1.2), we have

$$\begin{aligned}\mathbf{ber}^{2r}(T) &\leq \frac{2\alpha+1}{4\alpha+4}\left\||T|^{2r}+\left|T^*\right|^{2r}\right\|_{ber}+\frac{1}{2\alpha+2}\mathbf{ber}^r\left(T^2\right)\\ &\leq \frac{2\alpha+1}{4\alpha+4}\left\||T|^{2r}+\left|T^*\right|^{2r}\right\|_{ber}+\frac{1}{4\alpha+4}\left\||T|^{2r}+\left|T^*\right|^{2r}\right\|_{ber}\\ &= \frac{1}{2}\left\||T|^{2r}+\left|T^*\right|^{2r}\right\|_{ber}.\end{aligned}$$

□

Remark 2.7 By taking in Theorem 2.5, for $\alpha = 0$ we get the inequality (1.4). Hence, the inequality in Theorem 1.7 is a generalization and refinement of the inequality (1.4).

The following theorem is a remarkable extension and improvement of [6, Theorem 2.15].

Theorem 2.8 *Let $T \in \mathcal{B}(\mathcal{H}(\Omega))$ and let f and g be non-negative continuous functions on $[0,\infty)$ satisfying the relation $f(t)g(t) = t$ ($t \in [0,\infty)$). Then*

$$\mathbf{ber}^{2r}(T) \leq \frac{2\alpha+1}{4\alpha+4}\left\|f^{4r}(|T|)+g^{4r}\left(\left|T^*\right|\right)\right\|_{ber}+\frac{1}{2\alpha+2}\mathbf{ber}^r\left(g^2\left(\left|T^*\right|\right)f^2(|T|)\right),$$

for any $r \geq 1$.

Proof Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}(\Omega)$. Then

$$\begin{aligned}\left|\left\langle T\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^{2r} &\leq \left(\left\langle f^2(|T|)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\left\langle g^2\left(\left|T^*\right|\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right)^r\\ &\quad\text{(by Lemma 1.3)}\\ &= \left(\left\langle f^2(|T|)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\left\langle \hat{k}_\lambda,g^2\left(\left|T^*\right|\right)\hat{k}_\lambda\right\rangle\right)^r\\ &\leq \frac{2\alpha+1}{2\alpha+2}\left\|f^2(|T|)\hat{k}_\lambda\right\|^r\left\|g^2\left(\left|T^*\right|\right)\hat{k}_\lambda\right\|^r\\ &\qquad+\frac{1}{2\alpha+2}\left|\left\langle f^2(|T|)\hat{k}_\lambda,g^2\left(\left|T^*\right|\right)\hat{k}_\lambda\right\rangle\right|^r\\ &\quad\text{(by Lemma 1.7)}\\ &\leq \frac{2\alpha+1}{4\alpha+4}\left(\left\|f^2(|T|)\hat{k}_\lambda\right\|^{2r}+\left\|g^2\left(\left|T^*\right|\right)\hat{k}_\lambda\right\|^{2r}\right)\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2\alpha+2}\left|\left\langle g^{2}\left(\left|T^{*}\right|\right) f^{2}(|T|) \hat{k}_{\lambda}, \hat{k}_{\lambda}\right\rangle\right|^{r} \\
&\text{(by the arithmetic-geometric mean inequality)} \\
&=\frac{2\alpha+1}{4\alpha+4}\left(\left\langle f^{4}(|T|) \hat{k}_{\lambda}, \hat{k}_{\lambda}\right\rangle^{r}+\left\langle g^{4}\left(\left|T^{*}\right|\right) \hat{k}_{\lambda}, \hat{k}_{\lambda}\right\rangle^{r}\right) \\
&+\frac{1}{2\alpha+2}\left|\left\langle g^{2}\left(\left|T^{*}\right|\right) f^{2}(|T|) \hat{k}_{\lambda}, \hat{k}_{\lambda}\right\rangle\right|^{r} \\
&\leq \frac{2\alpha+1}{4\alpha+4}\left\langle\left(f^{4r}(|T|)+g^{4r}\left(\left|T^{*}\right|\right)\right) \hat{k}_{\lambda}, \hat{k}_{\lambda}\right\rangle \\
&+\frac{1}{2\alpha+2}\left|\left\langle g^{2}\left(\left|T^{*}\right|\right) f^{2}(|T|) \hat{k}_{\lambda}, \hat{k}_{\lambda}\right\rangle\right|^{r} \\
&\text{(by Lemma 1.2)} \\
&\leq \frac{2\alpha+1}{4\alpha+4}\left\|f^{4r}(|T|)+g^{4r}\left(\left|T^{*}\right|\right)\right\|_{ber} \\
&+\frac{1}{2\alpha+2}\mathbf{ber}^{r}\left(g^{2}\left(\left|T^{*}\right|\right) f^{2}(|T|)\right).
\end{aligned}
$$

Taking the the supremum over $\lambda \in \Omega$, we get

$$
\mathbf{ber}^{2r}(T) \leq \frac{2\alpha+1}{4\alpha+4}\left\|f^{4r}(|T|)+g^{4r}\left(\left|T^{*}\right|\right)\right\|_{ber}+\frac{1}{2\alpha+2}\mathbf{ber}^{r}\left(g^{2}\left(\left|T^{*}\right|\right) f^{2}(|T|)\right).
$$

□

Corollary 2.9 *Let* $T \in \mathcal{B}(\mathcal{H}(\Omega))$ *and let* f *and* g *be non-negative continuous functions on* $[0, \infty)$ *satisfying the relation* $f(t)\, g(t) = t$ $(t \in [0, \infty))$. *Then*

$$
\begin{aligned}
\mathbf{ber}^{2}(T) &\leq \frac{2\alpha+1}{4\alpha+4}\left\|f^{4}(|T|)+g^{4}\left(\left|T^{*}\right|\right)\right\|_{ber}+\frac{1}{2\alpha+2}\mathbf{ber}\left(g^{2}\left(\left|T^{*}\right|\right) f^{2}(|T|)\right) \\
&\leq \frac{1}{2}\left\|f^{4}(|T|)+g^{4}\left(\left|T^{*}\right|\right)\right\|_{ber},
\end{aligned}
$$

for any $r \geq 1$.

Proof By using the inequality (1.2), we observe that

$$
\begin{aligned}
\mathbf{ber}^{2r}(T) &\leq \frac{2\alpha+1}{4\alpha+4}\left\|f^{4}(|T|)+g^{4}\left(\left|T^{*}\right|\right)\right\|_{ber}+\frac{1}{2\alpha+2}\mathbf{ber}\left(g^{2}\left(\left|T^{*}\right|\right) f^{2}(|T|)\right) \\
&\leq \frac{2\alpha+1}{4\alpha+4}\left\|f^{4}(|T|)+g^{4}\left(\left|T^{*}\right|\right)\right\|_{ber}+\frac{1}{4\alpha+4}\left\|f^{4}(|T|)+g^{4}\left(\left|T^{*}\right|\right)\right\|_{ber} \\
&=\frac{1}{2}\left\|f^{4r}(|T|)+g^{4r}\left(\left|T^{*}\right|\right)\right\|_{ber}.
\end{aligned}
$$

□

Remark 2.10 From Corollary 2.9 we note that the inequality in Theorem 2.8 improves and generalizes the inequality (1.2).

Taking $f(t) = g(t) = t^{\frac{1}{2}}$, in Theorem 2.8 we get the following corolloay.

Corollary 2.11 *Let $T \in \mathcal{B}(\mathcal{H}(\Omega))$. Then*

$$\mathbf{ber}^{2r}(T) \leq \frac{1}{4}\left\| |T|^{2r} + \left|T^*\right|^{2r} \right\|_{ber} + \frac{1}{2}\mathbf{ber}^r\left(\left|T^*\right||T|\right),$$

for any $r \geq 1$.

Remark 2.12 Considering $r = 1$ in Corollary 2.9 we get the following inequality

$$\mathbf{ber}^{2}(T) \leq \frac{1}{4}\left\| |T|^{2} + \left|T^*\right|^{2} \right\|_{ber} + \frac{1}{2}\mathbf{ber}\left(\left|T^*\right||T|\right),$$

which is obtined in [6].

Theorem 2.13 *Let $T \in \mathcal{B}(\mathcal{H}(\Omega))$ and let $\alpha \in \mathbb{C} \setminus \{0\}$. Then*

$$\mathbf{ber}^2(T) \leq \frac{1}{2|\alpha|}\max\{1, |1-\alpha|\}\left\|T^*T + TT^*\right\|_{ber} + \frac{1}{|\alpha|}\mathbf{ber}\left(T^2\right).$$

Proof Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}(\Omega)$. Then, by using Lemma 1.5 we have

$$\begin{aligned}
\left|\left\langle T\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 &= \left|\left\langle T\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\left|\left\langle \hat{k}_\lambda, T^*\hat{k}_\lambda\right\rangle\right| \\
&\leq \frac{1}{|\alpha|}\left(\max\{1, |1-\alpha|\}\left\|T\hat{k}_\lambda\right\|\left\|T^*\hat{k}_\lambda\right\| + \left|\left\langle T\hat{k}_\lambda, T^*\hat{k}_\lambda\right\rangle\right|\right) \\
&\leq \frac{1}{|\alpha|}\left(\max\{1, |1-\alpha|\}\left\|T\hat{k}_\lambda\right\|\left\|T^*\hat{k}_\lambda\right\| + \left|\left\langle T^2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\right) \\
&\leq \frac{1}{|\alpha|}\left(\frac{\max\{1, |1-\alpha|\}}{2}\left(\left\|T\hat{k}_\lambda\right\|^2 + \left\|T^*\hat{k}_\lambda\right\|^2\right) + \left|\left\langle T^2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\right) \\
&= \frac{1}{|\alpha|}\left(\frac{\max\{1, |1-\alpha|\}}{2}\left\langle\left(T^*T + TT^*\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle + \left|\left\langle T^2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\right) \\
&= \frac{1}{2|\alpha|}\max\{1, |1-\alpha|\}\left\langle\left(T^*T + TT^*\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle + \frac{1}{|\alpha|}\left|\left\langle T^2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right| \\
&\leq \frac{1}{2|\alpha|}\max\{1, |1-\alpha|\}\left\|T^*T + TT^*\right\|_{ber} + \frac{1}{|\alpha|}\mathbf{ber}\left(T^2\right).
\end{aligned}$$

Taking the supremum over $\lambda \in \Omega$, we get

$$\mathbf{ber}^2(T) \leq \frac{1}{2|\alpha|}\max\{1, |1-\alpha|\}\left\|T^*T + TT^*\right\|_{ber} + \frac{1}{|\alpha|}\mathbf{ber}\left(T^2\right).$$

□

Considering $\alpha = n \in \mathbb{N}$ in Theorem 2.13 we get the following corollary.

Corollary 2.14 *Let $T \in \mathcal{B}(\mathcal{H}(\Omega))$. Then*

$$\mathbf{ber}^2(T) \leq \frac{n-1}{2n}\left\|T^*T + TT^*\right\|_{ber} + \frac{1}{n}\mathbf{ber}\left(T^2\right),$$

for all $n \in \mathbb{N}$.

For $n = 2$ in Corollary 2.14, we get the inequality (1.3).

Corollary 2.15 *Let $T \in \mathcal{B}(\mathcal{H}(\Omega))$. Then*

$$\mathbf{ber}^2(T) \leq \frac{1}{4}\left\|T^*T + TT^*\right\|_{ber} + \frac{1}{2}\mathbf{ber}\left(T^2\right).$$

Remark 2.16 If we take $n \to \infty$ in Corollary 2.14, then we obtain

$$\mathbf{ber}^2(T) \leq \frac{1}{2}\left\|T^*T + TT^*\right\|_{ber}.$$

Theorem 2.17 *Let $T, S \in \mathcal{B}(\mathcal{H}(\Omega))$. Then*

$$\mathbf{ber}^2\left(S^*T\right) \leq \frac{1}{2|\alpha|}\max\{1, |1-\alpha|\}\left\||T|^4 + |S|^4\right\|_{ber} + \frac{1}{|\alpha|}\mathbf{ber}\left(|S|^2|T|^2\right).$$

Proof Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}(\Omega)$. Then

$$\begin{aligned}
\left|\left\langle S^*T\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 &= \left|\left\langle T\hat{k}_\lambda, S\hat{k}_\lambda\right\rangle\right|^2 \\
&\leq \left\|T\hat{k}_\lambda\right\|^2\left\|S\hat{k}_\lambda\right\|^2 \\
&= \left|\left\langle |T|^2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\left|\left\langle |S|^2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right| \\
&= \left|\left\langle |T|^2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\left|\left\langle \hat{k}_\lambda, |S|^2\hat{k}_\lambda\right\rangle\right| \\
&\leq \frac{1}{|\alpha|}\left(\max\{1, |1-\alpha|\}\left\||T|^2\hat{k}_\lambda\right\|\left\||S|^2\hat{k}_\lambda\right\| + \left|\left\langle |T|^2\hat{k}_\lambda, |S|^2\hat{k}_\lambda\right\rangle\right|\right) \\
&\leq \frac{1}{|\alpha|}\left(\max\{1, |1-\alpha|\}\left\||T|^2\hat{k}_\lambda\right\|\left\||S|^2\hat{k}_\lambda\right\|\right. \\
&\qquad \left. + \left|\left\langle |S|^2|T|^2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\right) \\
&\leq \frac{1}{|\alpha|}\left(\frac{\max\{1, |1-\alpha|\}}{2}\left(\left\||T|^2\hat{k}_\lambda\right\|^2 + \left\||S|^2\hat{k}_\lambda\right\|^2\right)\right. \\
&\qquad \left. + \left|\left\langle |S|^2|T|^2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\right) \\
&= \frac{1}{|\alpha|}\left(\frac{\max\{1, |1-\alpha|\}}{2}\left\langle\left(|T|^4 + |S|^4\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle + \left|\left\langle |S|^2|T|^2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\right) \\
&= \frac{1}{2|\alpha|}\max\{1, |1-\alpha|\}\left\langle\left(|T|^4 + |S|^4\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle + \frac{1}{|\alpha|}\left|\left\langle |S|^2|T|^2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|
\end{aligned}$$

$$\leq \frac{1}{2|\alpha|}\max\{1,|1-\alpha|\}\left\||T|^4+|S|^4\right\|_{ber}+\frac{1}{|\alpha|}\mathbf{ber}\left(|S|^2|T|^2\right).$$

□

Considering $\alpha = n \in \mathbb{N}$ in Theorem 2.17 we get the following corollary.

Corollary 2.18 *Let $T, S \in \mathcal{B}(\mathcal{H}(\Omega))$). Then*

$$\mathbf{ber}^2\left(S^*T\right) \leq \frac{n-1}{2n}\left\||T|^4+|S|^4\right\|_{ber}+\frac{1}{n}\mathbf{ber}\left(|S|^2|T|^2\right),$$

for all $n \in \mathbb{N}$.

Remark 2.19 If we take $n \to \infty$ in Corollary 2.18, then we obtain

$$\mathbf{ber}^2\left(S^*T\right) \leq \frac{1}{2}\left\||T|^4+|S|^4\right\|_{ber}.$$

In [24] the authors proved that

$$\mathbf{ber}^r\left(S^*T\right) \leq \left\|\alpha|T|^{\frac{r}{\alpha}}+(1-\alpha)|S|^{\frac{r}{1-\alpha}}\right\|_{ber}, \tag{2.2}$$

where $0 < \alpha < 1$ and $r \geq 1$.

Next, we improve the inequality (2.2) in the following theorem.

Theorem 2.20 *Let $T, S \in \mathcal{B}(\mathcal{H}(\Omega))$ and let $0 < \alpha < 1$, $r \geq 2$. Then*

$$\mathbf{ber}^r\left(S^*T\right) \leq \left\|\alpha|T|^{\frac{r}{\alpha}}+(1-\alpha)|S|^{\frac{r}{1-\alpha}}\right\|_{ber}-\inf_{\lambda\in\Omega}\delta\left(\hat{k}_\lambda\right),$$

where $\delta\left(\hat{k}_\lambda\right)=\min\{\alpha,1-\alpha\}\left(\sqrt{\left\langle|T|^{\frac{r}{\alpha}}\hat{k}_\lambda,\hat{k}_\lambda\right\rangle}-\sqrt{\left\langle|S|^{\frac{r}{1-\alpha}}\hat{k}_\lambda,\hat{k}_\lambda\right\rangle}\right)^2.$

Proof Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}(\Omega)$. Then

$$\begin{aligned}
\left|\left\langle S^*T\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^r &= \left|\left\langle T\hat{k}_\lambda,S\hat{k}_\lambda\right\rangle\right|^r \\
&\leq \left\|T\hat{k}_\lambda\right\|^r\left\|S\hat{k}_\lambda\right\|^r \\
&= \left\langle|T|^2\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^{\frac{r}{2}}\left\langle|S|^2\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^{\frac{r}{2}} \\
&= \left\langle\left(|T|^{\frac{2}{\alpha}}\right)^{\alpha}\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^{\frac{r}{2}}\left\langle\left(|S|^{\frac{2}{1-\alpha}}\right)^{1-\alpha}\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^{\frac{r}{2}} \\
&\leq \left(\left\langle|T|^{\frac{2}{\alpha}}\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^{\frac{r}{2}}\right)^{\alpha}\left(\left\langle|S|^{\frac{2}{1-\alpha}}\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^{\frac{r}{2}}\right)^{1-\alpha} \\
&\quad \text{(by Lemma 1.2)}
\end{aligned}$$

$$\leq \left(\left\langle |T|^{\frac{r}{\alpha}} \hat{k}_\lambda, \hat{k}_\lambda \right\rangle\right)^{\alpha} \left(\left\langle |S|^{\frac{r}{1-\alpha}} \hat{k}_\lambda, \hat{k}_\lambda \right\rangle\right)^{1-\alpha}$$

(by Lemma 1.2)

$$\leq \left(\alpha \left\langle |T|^{\frac{r}{\alpha}} \hat{k}_\lambda, \hat{k}_\lambda \right\rangle + (1-\alpha) \left\langle |S|^{\frac{r}{1-\alpha}} \hat{k}_\lambda, \hat{k}_\lambda \right\rangle\right)$$

$$- \min\{\alpha, 1-\alpha\} \left(\sqrt{\left\langle |T|^{\frac{r}{\alpha}} \hat{k}_\lambda, \hat{k}_\lambda \right\rangle} - \sqrt{\left\langle |S|^{\frac{r}{1-\alpha}} \hat{k}_\lambda, \hat{k}_\lambda \right\rangle}\right)^2$$

(by Lemma 1.4)

$$\leq \left\| \alpha\, |T|^{\frac{r}{\alpha}} + (1-\alpha)\, |S|^{\frac{r}{1-\alpha}} \right\|_{ber} - \inf_{\lambda \in \Omega} \delta\left(\hat{k}_\lambda\right),$$

where $\delta\left(\hat{k}_\lambda\right) = \min\{\alpha, 1-\alpha\} \left(\sqrt{\left\langle |T|^{\frac{r}{\alpha}} \hat{k}_\lambda, \hat{k}_\lambda \right\rangle} - \sqrt{\left\langle |S|^{\frac{r}{1-\alpha}} \hat{k}_\lambda, \hat{k}_\lambda \right\rangle}\right)^2.$

Taking the supremum over $\lambda \in \Omega$, we get the desired result. □

Corollary 2.21 *If $T, S \in \mathcal{B}(\mathcal{H}(\Omega))$ and $r \geq 2$, then*

$$\mathbf{ber}^r\left(S^*T\right) \leq \left\| |T|^{2r} + |S|^{2r} \right\|_{ber} - \inf_{\lambda \in \Omega} \zeta\left(\hat{k}_\lambda\right)$$
$$\leq \frac{1}{2} \left\| |T|^{2r} + |S|^{2r} \right\|_{ber},$$

where $\zeta\left(\hat{k}_\lambda\right) = \frac{1}{2}\left(\sqrt{\left\langle |T|^{2r} \hat{k}_\lambda, \hat{k}_\lambda \right\rangle} - \sqrt{\left\langle |S|^{2r} \hat{k}_\lambda, \hat{k}_\lambda \right\rangle}\right)^2.$

Remark 2.22 We note that the inequality in Corollary 2.21 refines the inequality (1.2).

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Author Contributions ..

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Declarations

Conflict of interest The authors declare that there is no Conflict of interest.

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