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Existence and blow-up of solutions for a class of semilinear pseudo-parabolic equations with cone degenerate viscoelastic term

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Abstract

In this paper, we consider the semilinear pseudo-parabolic equation with cone degenerate viscoelastic term

$$u_t + \Delta_{\mathbb{B}}^2 u_t + \Delta_{\mathbb{B}}^2 u - \int_0^t g(t-s)\Delta_{\mathbb{B}}^2 u(s)ds = f(u), \text{ in int}\mathbb{B} \times (0, T),$$

with initial and boundary conditions, where $f(u) = |u|^{p-2}u - \frac{1}{|\mathbb{B}|}\int_{\mathbb{B}} |u|^{p-2}u \frac{dx_1}{x_1} dx'$. We construct several conditions for initial data which leads to global existence of the solutions or the solutions blowing up in finite time. Moreover, the asymptotic behavior and the bounds of blow-up time for the solutions are given.

Keywords Semilinear pseudo-parabolic equation · Cone degenerate viscoelastic term · Asymptotic behavior · Bounds for the blow-up time

Mathematics Subject Classification 35A01 · 35B44 · 35K58

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1 Introduction

In this paper, we consider the existence and blow-up of solutions for the following semilinear pseudo-parabolic equation with cone degenerate viscoelastic term

$$\begin{cases} u_t + \Delta_{\mathbb{B}}^2 u_t + \Delta_{\mathbb{B}}^2 u - \int_0^t g(t-s)\Delta_{\mathbb{B}}^2 u(s)ds = f(u), & (x,t) \in \text{int}\mathbb{B} \times (0,T), \\ u(x,t) = \Delta_{\mathbb{B}} u(x,t) = 0, & (x,t) \in \partial\mathbb{B} \times (0,T), \\ u(x,0) = u_0(x), & x \in \text{int}\mathbb{B}, \end{cases} \tag{1.1}$$

where $T \in (0, +\infty]$,

$$f(u) = |u|^{p-2}u - \frac{1}{|\mathbb{B}|}\int_{\mathbb{B}} |u|^{p-2}u \frac{dx_1}{x_1} dx',$$

$u_0 \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) \cap \mathcal{H}_2^{2,\frac{n}{2}}(\mathbb{B}) (n \geq 2)$ with spaces $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ and $\mathcal{H}_2^{2,\frac{n}{2}}(\mathbb{B})$ defined in Sect. 2. Here $\mathbb{B} = [0,1) \times X$, X is an (n-1)-dimensional closed compact manifold, which is regarded as the local model near the conical points, and $\partial\mathbb{B} = \{0\} \times X$. Moreover, the operator $\Delta_{\mathbb{B}}$ in (1.1) is defined by $(x_1\partial_{x_1})^2 + \partial_{x_2}^2 + \cdots + \partial_{x_n}^2$, which is an elliptic operator with conical degeneration on the boundary $x_1 = 0$, and $\Delta_{\mathbb{B}}^2 u := \Delta_{\mathbb{B}}(\Delta_{\mathbb{B}} u)$. Near $\partial\mathbb{B}$ we will often use coordinates $(x_1, x') = (x_1, x_2, \ldots, x_n)$ for $0 \leq x_1 < 1$, $x' \in X$. We assume that

(I) $g(x) : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 function satisfying

$$g(x) \geq 0, g'(s) \leq 0, 1 - \int_0^\infty g(s)ds = l > 0. \tag{1.2}$$

(II) p satisfies

$$2 < p < \infty \text{ if } n = 2;\ 2 < p < \frac{2n-2}{n-2} \text{ if } n \geq 3. \tag{1.3}$$

In this paper, we will study the behavior of solutions for pseudo-parabolic equations with conical singularity points. The theory of existence for such solutions plays an important role in fluid dynamics, aerodynamics and fracture mechanics [1]. Many scholars gave a lot of important results in operator algebra of quasi differential operator operations, singularity propagation of non-elliptic operators, spectral theories and so on. Here let us review the background related to singularity [2]. Space-time singularity, referred to as singularity, is a location in space-time where the gravitational field of a celestial body is predicted to become infinite by general relativity in a way that does not depend on the coordinate system. The laws of normal space-time cannot hold because such quantities become infinite within the singularity. Many kinds of mathematical singularities appear widely in physics theories. The ball of mass of some quantity becomes infinite or increases without limit is predicted by equations to these physical theories [3]. There are different types of singularities, each with different physical features, such as the different shape of the singularities, conical and curved. It has also been hypothesized that they occur without an event horizon, a structure that separates one part of space-time from another, in which the effects of events cannot exceed the horizon; these are called naked. Conial singularities occur when there exists a point where the limit of each heteromorphic invariant is finite, in which case space-time is not smooth at the limit point itself. Therefore, around this point, space-time looks like a cone, where the singularity is located at the tip of the cone. The metric can be finite everywhere and coordinate system is used. An example of such a conical singularity is a cosmic string and a Schwarzschild black hole [4]. In 2012, Chen et al. [5, 6] established the basic theories of weighted Sobolev spaces on manifolds with cone singularity, such as cone Sobolev inequality and Poincaré inequality. Based on these theories, they studied the following initial boundary value problem for a class of degenerate parabolic type equations

$$\partial_t u - \Delta_{\mathbb{B}} u = |u|^{p-1}u,\ x \in \mathrm{int}\mathbb{B},\ t > 0. \tag{1.4}$$

By using a family of potential wells, they obtained existence theorem of global solutions with exponential decay and showed the blow-up in finite time of solutions [7]. Especially, the relation between the above two phenomena was derived by them as a sharp condition. In 2017, Li et al. [8] studied the existence of solutions for

$$u_t - \Delta_{\mathbb{B}} u_t - \Delta_{\mathbb{B}} u = |u|^{p-1}u,\ x \in \mathrm{int}\mathbb{B},\ t > 0, \tag{1.5}$$

with initial and boundary conditions. They established the blow-up criterion for problem (1.5) by differential inequality.

As far as the present is concerned, there are few studies on the pseudo-parabolic equations with cone degenerate viscoelastic term and nonlocal source $|u|^{p-2}u - \frac{1}{|\mathbb{B}|}\int_{\mathbb{B}} |u|^{p-2}u\frac{dx_1}{x_1}dx'$. Thereupon, we will study the influence of viscoelastic term g on solutions, where Di and Shang [9] considered the case when $g = 0$ and the operator is $-\Delta_{\mathbb{B}}$.

To state our main results, we define the following "modified" energy functional and Nehari functional

$$\begin{aligned} J(u) &= \frac{1}{2}\int_0^t g(t-s)\|\Delta_{\mathbb{B}}u(t) - \Delta_{\mathbb{B}}u(s)\|_2^2 ds \\ &\quad + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\Delta_{\mathbb{B}}u(t)\|_2^2 - \frac{1}{p}\|u\|_p^p, \end{aligned} \tag{1.6}$$

$$\begin{aligned} I(u) &= \int_0^t g(t-s)\|\Delta_{\mathbb{B}}u(t) - \Delta_{\mathbb{B}}u(s)\|_2^2 ds \\ &\quad + \left(1 - \int_0^t g(s)ds\right)\|\Delta_{\mathbb{B}}u(t)\|_2^2 - \|u\|_p^p. \end{aligned} \tag{1.7}$$

It follows from (1.6) and (1.7) that

$$J(u) = \frac{1}{2}I(u) + \frac{p-2}{2p}\|u\|_p^p. \tag{1.8}$$

Here, $\|u\|_p$ is $\|u\|_{L_p^{\frac{n}{p}}(\mathbb{B})}$ and $\mathcal{H} := \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) \cap \mathcal{H}_2^{2,\frac{n}{2}}(\mathbb{B})$ defined in Sect. 2.

The main results of this paper are as follows.

Theorem 1.1 *Let p satisfy (1.3) and $u_0 \in \mathcal{H}$. If $J(u_0) \le d$, $I(u_0) > 0$ and $K(u_0) \ge 0$, then problem (1.1) has a global weak solution $u = u(x,t) \in L^\infty([0,+\infty);\mathcal{H})$ with $u_t \in L^2([0,+\infty);\mathcal{H})$. And for any $T > 0$, u satisfies*

$$\|u\|_{\mathcal{H}}^2 \le \left(2\alpha t + \|u_0\|_{\mathcal{H}}^2\right) e^{\beta t},\ 0 \le t \le T, \tag{1.9}$$

where $\alpha = \frac{(m^2+2)pd}{p-2}$, $\beta = (m+\frac{1}{m})^2(1-l) - m^2 - 2 > 0$, and $m \in \mathbb{R}$ satisfies $\frac{1}{(m^2+1)^2} > l$. Moreover, if $J(u_0) \le \min\left\{\frac{p-2}{q^2+2}\omega, d\right\}$, then u satisfies the following exponential decay

$$\|u\|_{\mathcal{H}}^2 \le \|u_0\|_{\mathcal{H}}^2 e^{1-\gamma t},\ t \ge 0, \tag{1.10}$$

where $\omega = \frac{K(u_0)}{p|\mathbb{B}|} \inf_{t>0} \|u\|_{p-1}^{p-1}$, $\gamma = \frac{q^2+2-(q+\frac{1}{q})^2(1-l)}{C_*^2+1} > 0$, C_* is mentioned in Remark 2.2 and $q \in \mathbb{R}$ satisfies $\frac{1}{(q^2+1)^2} < l$, $K(u_0) = \int_{\mathbb{B}} u_0 \frac{dx_1}{x_1} dx'$.

Theorem 1.2 *Let p satisfy (1.3) and $u_0 \in \mathcal{H}$. If $J(u_0) \le -\omega$, $I(u_0) < 0$, $K(u_0) < 0$ and*

$$\int_0^\infty g(s)ds < \frac{p-2}{p+\frac{1}{p}+2}, \tag{1.11}$$

then the weak solutions u of problem (1.1) blow up in finite time, where ω is the same as Theorem 1.1. Moreover, the maximal existence time T satisfies

$$\int_{\|u_0\|_{\mathcal{H}}^2}^\infty \frac{d\mu}{(p+2)C_1^p \mu^{\frac{p}{2}} - \frac{2K(u_0)}{|\mathbb{B}|} C_2^{p-1} \mu^{\frac{p-1}{2}}} \le T \le \frac{ab^2}{ab(p-2) - \|u_0\|_{\mathcal{H}}^2},$$

where positive parameters a and b are respectively given in (4.7) and (4.8), C_1 and C_2 are respectively the best constant of embedding $\mathcal{H} \hookrightarrow L_p^{\frac{n}{p}}(\mathbb{B})$ and $\mathcal{H} \hookrightarrow L_{p-1}^{\frac{n}{p-1}}(\mathbb{B})$.

This paper is organized as follows. First of all, we introduce several preliminaries relative to problem (1.1) in Sect. 2, including the definitions of cone Sobolev spaces, the weak solutions of problem (1.1) and several properties of potential wells and invariant sets. Next, we give the proof of Theorem 1.1 in Sect. 3 and the proof of Theorem 1.2 in Sect. 4.

2 Preliminaries

2.1 Relevant definitions and lemmas

We give some definitions and properties of the cone Sobolev spaces as follows [7].

Let X be a closed, compact, and C^∞ manifold. We set $X^\Delta = (\bar{\mathbb{R}}_+ \times X)/\{0\} \times X$, as a local model interpreted as a cone with the base X. Next, we Denote $X^\wedge = \mathbb{R}_+ \times X$ as the corresponding open stretched cone with the base X.

An n-dimensional manifold B with conical singularities is a topological space with a finite subset $B_0 = \{b_1, \dots, b_M\} \subset B$ of conical singularities, with the following two properties:

(i) $B \backslash B_0$ is a C^∞ manifold;
(ii) Every $b \in B_0$ has an open neighborhood U in B, such that there is a homeomorphism $\phi : U \to X^\Delta$ for some closed compact C^∞ manifold $X = X(b)$, and ϕ restricts to a diffeomorphism $\phi' : U \backslash \{b\} \to X^\wedge$.

For simplicity, we assume that the manifold B has only one conical point on the boundary. Thus, near the conical point, we have a stretched manifold $\mathbb{B}$, associated with B.

Definition 2.1 (*cf.* [7]) For $m \in \mathbb{N}$, $\gamma \in \mathbb{R}$, $p > 1$ and let $\mathbb{B} = [0, 1) \times X$ be a stretched manifold of the manifold B with conical singularity. Then the cone Sobolev space $\mathcal{H}_p^{m,\gamma}(\mathbb{B})$ is defined as

$$\mathcal{H}_p^{m,\gamma}(\mathbb{B}) := \left\{ u \in W_{loc}^{m,p}(\text{int}\mathbb{B}) : \omega u \in \mathcal{H}_p^{m,\gamma}(X^\wedge) \right\},$$

for any cut off function ω supported by a collar neighborhood of $(0, 1) \times \partial\mathbb{B}$. Moreover, the subspace $\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{B})$ of $\mathcal{H}_p^{m,\gamma}(\mathbb{B})$ is defined by

$$\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{B}) := [\omega]\mathcal{H}_{p,0}^{m,\gamma}(X^\wedge) + [1-\omega]W_0^{m,p}(\text{int}\mathbb{B}),$$

where $X^\wedge = \mathbb{R}_+ \times X$ denotes the open stretched cone with the base X, $W_0^{m,p}(\text{int}\mathbb{B})$ denotes the closure of $C_0^\infty(\text{int}\mathbb{B})$ in Sobolev spaces $W^{m,p}(\tilde{X})$ when $\tilde{X}$ is a closed compact C^∞ manifold of dimension n that containing $\mathbb{B}$ as a submanifold with boundary.

Definition 2.2 (*cf.* [7]) Let $\mathbb{B} = [0, 1) \times X$. We say $u(x) \in L_p^\gamma(\mathbb{B})$ with $1 < p < +\infty$ and $\gamma \in \mathbb{R}$, if

$$\|u\|_{L_p^\gamma(\mathbb{B})} = \left(\int_{\mathbb{B}} x_1^n |x_1^{-\gamma} u(x)|^p \frac{dx_1}{x_1} dx' \right)^{\frac{1}{p}} < +\infty.$$

Remark 2.1 (cf. [6]) We have the following properties:

(i) $\mathcal{H}_p^{m,\gamma}(\mathbb{B})$ is a Banach space for $1 \le p < \infty$, and is Hilbert space for $p = 2$;

(ii) $L_p^\gamma(\mathbb{B}) = \mathcal{H}_p^{0,\gamma}(\mathbb{B})$;

(iii) $L_p(\mathbb{B}) = \mathcal{H}_p^{0,0}(\mathbb{B})$;

(iv) $x_1^{\gamma_1}\mathcal{H}_p^{m,\gamma_2}(\mathbb{B}) = \mathcal{H}_p^{m,\gamma_1+\gamma_2}(\mathbb{B})$;

(v) The embedding $\mathcal{H}_p^{m,\gamma}(\mathbb{B}) \hookrightarrow \mathcal{H}_p^{m',\gamma'}(\mathbb{B})$ is continuous if $m \geq m'$, $\gamma \geq \gamma'$; and is compact embedding if $m > m'$, $\gamma > \gamma'$.

For simplicity, $\|u\|_{L_p^{\frac{n}{p}}(\mathbb{B})}$ is denoted by $\|u\|_p$ throughout the present paper, and $(\cdot,\cdot)$ represents the inner product in $L_2^{\frac{n}{2}}(\mathbb{B})$. Moreover, we also denote

$$\mathcal{H} := \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) \cap \mathcal{H}_2^{2,\frac{n}{2}}(\mathbb{B}) \text{ for } u = \Delta_{\mathbb{B}} u = 0 \text{ on } \partial\mathbb{B},$$

and

$$\|u\|_{\mathcal{H}}^2 = \|u\|_2^2 + \|\Delta_{\mathbb{B}} u\|_2^2.$$

Lemma 2.1 *If functions $u, v \in \mathcal{H}$, then*

$$\int_{\mathbb{B}} v \Delta_{\mathbb{B}}^2 u \frac{dx_1}{x_1} dx' = \int_{\mathbb{B}} \Delta_{\mathbb{B}} u \Delta_{\mathbb{B}} v \frac{dx_1}{x_1} dx'. \tag{2.1}$$

Lemma 2.2 (Cone Poincaré inequality, cf. [6]) Let $\mathbb{B} = [0, 1) \times X$ be a bounded subspace in $\mathbb{R}_+^n$ with $X \subset \mathbb{R}^{n-1}$. If $u(x) \in \mathcal{H}$, then

$$\|u(x)\|_2 \leq C \|\nabla_{\mathbb{B}} u(x)\|_2, \tag{2.2}$$

where the constant C depends only on $\mathbb{B}$.

Lemma 2.3 *Let $\mathbb{B} = [0, 1) \times X$ be a bounded subspace in $\mathbb{R}_+^n$ with $X \subset \mathbb{R}^{n-1}$. If $u(x) \in \mathcal{H}$, then*

$$\|\nabla_{\mathbb{B}} u(x)\|_2 \leq \frac{1}{\lambda_1} \|\Delta_{\mathbb{B}} u(x)\|_2, \tag{2.3}$$

where $\lambda_1 > 0$ (cf. [5]) is the first eigenvalue of the following equation

$$\begin{cases} -\Delta_{\mathbb{B}} u = \lambda u, & x \in int\mathbb{B}, \\ u = 0, & x \in \partial\mathbb{B}. \end{cases}$$

Proof For any $\varepsilon_0 > 0$ and $u(x) \in \mathcal{H}$, a simple calculation gives that

$$\begin{aligned} \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{dx_1}{x_1} dx' &= \int_{\partial\mathbb{B}} u \frac{\partial u}{\partial \nu} \frac{dx_1}{x_1} dx' - \int_{\mathbb{B}} u \Delta_{\mathbb{B}} u \frac{dx_1}{x_1} dx' \\ &\leq \frac{\varepsilon_0}{2} \int_{\mathbb{B}} |u|^2 \frac{dx_1}{x_1} dx' + \frac{1}{2\varepsilon_0} \int_{\mathbb{B}} |\Delta_{\mathbb{B}} u|^2 \frac{dx_1}{x_1} dx' \\ &\leq \frac{\varepsilon_0}{2\lambda_1} \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{dx_1}{x_1} dx' + \frac{1}{2\varepsilon_0} \int_{\mathbb{B}} |\Delta_{\mathbb{B}} u|^2 \frac{dx_1}{x_1} dx', \end{aligned}$$

where ν is the unit normal vector pointing toward the exterior of $\mathbb{B}$. Taking $\varepsilon_0 = \lambda_1$, we reach the conclusion of Lemma 2.3. □

Remark 2.2 It is easy to know that $\mathcal{H}$ is a Banach space with norm $\|\cdot\|_{\mathcal{H}}$, where the norm $\|\cdot\|_{\mathcal{H}}$ is equivalent to the norm $\|\Delta_{\mathbb{B}}\cdot\|_2$ by Lemmas 2.2 and 2.3. For simplicity, we denote $C_* = \frac{C}{\lambda_1}$, then $\|u\|^2_{\mathcal{H}} \le (C_*^2+1)\|\Delta_{\mathbb{B}}u\|_2^2$.

Lemma 2.4 (Cone Sobolev embedding, cf. [8]) For $1 < q < 2^* = \frac{2n}{n-2}$, the embedding $\mathcal{H} \hookrightarrow L_q^{\frac{n}{q}}(\mathbb{B})$ is continuous.

Lemma 2.5 (Hölder inequality, cf. [9]) For $p, q \in (1, +\infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$, if $u(x) \in L_p^{\frac{n}{p}}(\mathbb{B})$ and $v(x) \in L_q^{\frac{n}{q}}(\mathbb{B})$, then we have the following Hölder inequality

$$\int_{\mathbb{B}} |u(x)v(x)| \frac{dx_1}{x_1} dx' \le \left(\int_{\mathbb{B}} |u(x)|^p \frac{dx_1}{x_1} dx'\right)^{\frac{1}{p}} \left(\int_{\mathbb{B}} |v(x)|^q \frac{dx_1}{x_1} dx'\right)^{\frac{1}{q}}.$$

In view of the definitions and lemmas above, we give the definitions about the weak solutions below.

Definition 2.3 (*Weak solution*) A function $u = u(x,t)$ is called a weak solution of problem (1.1) on $\mathbb{B} \times [0, T)$, if $u \in L^\infty(0, T; \mathcal{H})$ with $u_t \in L^2(0, T; \mathcal{H})$ and satisfies (1.1) in the following distribution sense, namely

$$\begin{aligned} &(u_t, \phi) + (\Delta_{\mathbb{B}} u_t, \Delta_{\mathbb{B}}\phi) + (\Delta_{\mathbb{B}} u, \Delta_{\mathbb{B}}\phi) \\ &= \left(\int_0^t g(t-s)\Delta_{\mathbb{B}} u(s) ds, \Delta_{\mathbb{B}}\phi\right) + \left(|u|^{p-2}u - \frac{1}{|\mathbb{B}|}\int_{\mathbb{B}} |u|^{p-2}u \frac{dx_1}{x_1} dx', \phi\right), \end{aligned} \tag{2.4}$$

for any $\phi \in \mathcal{H}$, where $u_0 \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) \cap \mathcal{H}_2^{2,\frac{n}{2}}(\mathbb{B})$.

Definition 2.4 (*Maximal existence time*) Let $u(x,t)$ be a weak solution of (1.1). We define the maximal existence time T of $u(x,t)$ as follows:

(i) If $u(x,t)$ exists for all $0 \le t < \infty$, then $T = +\infty$;

(ii) If there exists $t_0 \in (0, \infty)$ such that $u(x,t)$ exists for $0 \le t < t_0$, but does not exist at $t = t_0$, then $T = t_0$.

Definition 2.5 (*Finite time blow-up*) Let $u(x,t)$ be a weak solution of (1.1). We say $u(x,t)$ blows up in finite time if the maximal existence time T is finite and

$$\lim_{t\to T^-} \|u\|^2_{\mathcal{H}} := \lim_{t\to T^-} \|u\|_2^2 + \lim_{t\to T^-} \|\Delta_{\mathbb{B}}u\|_2^2 = +\infty.$$

The following lemma motivates us to set up the initial value conditions in Theorems 1.1 and 1.2.

Lemma 2.6 *Let u be a weak solution of (1.1) and $u_0 \in \mathcal{H}\backslash\{0\}$, then $\int_{\mathbb{B}} u \frac{dx_1}{x_1} dx'$ is a constant for all $t \in [0, T)$, where T is the maximal existence time of u.*

Proof By the boundary condition $u = \Delta_{\mathbb{B}} u = 0$, it yields

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{B}} u\frac{dx_1}{x_1}dx' = \int_{\mathbb{B}} u_t\frac{dx_1}{x_1}dx' \\
&= \int_{\mathbb{B}} \left(-\Delta_{\mathbb{B}}^2 u_t - \Delta_{\mathbb{B}}^2 u + \int_0^t g(t-s)\Delta_{\mathbb{B}}^2 u(s)ds + |u|^{p-2}u \right.\\
&\left. -\frac{1}{|\mathbb{B}|}\int_{\mathbb{B}} |u|^{p-2}u\frac{dx_1}{x_1}dx'\right)\frac{dx_1}{x_1}dx' \\
&= -\frac{d}{dt}\int_{\partial\mathbb{B}} \nabla_{\mathbb{B}}(\Delta_{\mathbb{B}}u)\cdot \nu dS - \int_{\partial\mathbb{B}} \nabla_{\mathbb{B}}(\Delta_{\mathbb{B}}u)\cdot \nu dS \\
&\quad + \int_0^t g(t-s)\int_{\partial\mathbb{B}} \nabla_{\mathbb{B}}(\Delta_{\mathbb{B}}u(s))\cdot \nu dSds + \int_{\mathbb{B}} |u|^{p-2}u\frac{dx_1}{x_1}dx' \\
&\quad - \frac{1}{|\mathbb{B}|}\int_{\mathbb{B}} |u|^{p-2}u\frac{dx_1}{x_1}dx'\int_{\mathbb{B}}\frac{dx_1}{x_1}dx' = 0,
\end{aligned}
$$

where ν is the unit normal vector pointing toward the exterior of $\mathbb{B}$. □

Therefore, from Lemma 2.6, we can define

$$
K(u_0) = \int_{\mathbb{B}} u\frac{dx_1}{x_1}dx' = \int_{\mathbb{B}} u_0\frac{dx_1}{x_1}dx'.
$$

2.2 Potential wells and invariant sets

Lemma 2.7 *Let $u(x,t)$ be a weak solution of (1.1), then $J(u)$ is non-increasing about t, and*

$$
\begin{aligned}
J(u_0) = J(u) &- \frac{1}{2}\int_0^t\int_0^\tau g'(\tau-s)\|\Delta_{\mathbb{B}}u(\tau) - \Delta_{\mathbb{B}}u(s)\|_2^2 dsd\tau \\
&+ \frac{1}{2}\int_0^t g(\tau)\|\Delta_{\mathbb{B}}u(\tau)\|_2^2 d\tau + \int_0^t \|u_\tau\|_{\mathcal{H}}^2 d\tau,
\end{aligned} \tag{2.5}
$$

Proof Taking $\phi = u_t$ in (2.4), it follows from (1.2) and (1.6) that

$$
\frac{d}{dt}J(u) = \frac{1}{2}\int_0^t g'(t-s)\|\Delta_{\mathbb{B}}u(t) - \Delta_{\mathbb{B}}u(s)\|_2^2 ds - \frac{1}{2}g(t)\|\Delta_{\mathbb{B}}u\|_2^2 - \|u_t\|_{\mathcal{H}}^2 < 0.
$$

Integrating with respect to t over $(0,t)$, we accomplish the proof of (2.5). □

Lemma 2.8 *For any $u \in \mathcal{H}$ and $\|u\|_p \neq 0$, we have*

(i) $\lim_{\lambda\to 0^+} J(\lambda u) = 0$, $\lim_{\lambda\to+\infty} J(\lambda u) = -\infty$;

(ii) $J(\lambda u)$ *is increasing on* $0 \le \lambda \le \lambda^*$, *decreasing on* $\lambda^* \le \lambda < \infty$ *and takes the maximum at* $\lambda = \lambda^*$, *where*

$$\lambda^* = \left(\frac{\int_0^t g(t-s)\|\Delta_{\mathbb{B}}u(t) - \Delta_{\mathbb{B}}u(s)\|_2^2 ds + \left(1 - \int_0^t g(s)ds\right)\|\Delta_{\mathbb{B}}u(t)\|_2^2}{\|u\|_p^p} \right)^{\frac{1}{p-2}}.$$

Proof The proof process is similar to the proof of Lemma 2.1 in [10]. □

Then, we define the Nehari manifold

$$\mathcal{N} := \left\{ u \in \mathcal{H} : I(u) = 0, \int_{\mathbb{B}} |\Delta_{\mathbb{B}}u|^2 \frac{dx_1}{x_1} dx' \neq 0 \right\},$$

and

$$d := \inf \left\{ \sup_{\lambda \ge 0} J(\lambda u) : u \in \mathcal{H}\backslash\{0\} \right\}.$$

It follows from Lemma 2.8 that $0 < d = \inf_{u \in \mathcal{N}} J(u)$. The invariant sets are defined by

$$W := \{u \in \mathcal{H} : I(u) > 0, J(u) < d\} \cup \{0\},$$
$$V := \{u \in \mathcal{H} : I(u) < 0, J(u) < d\}.$$

The following properties of the invariant sets are important for us to get the main results of problem (1.1).

Lemma 2.9 *Let* $u(x,t)$ *be a weak solution of (1.1) with* $0 < J(u_0) < d$, *then*

(i) $u \in W$ *for any* $t \in [0, T)$ *provided* $I(u_0) > 0$;

(ii) $u \in V$ *for any* $t \in [0, T)$ *provided* $I(u_0) < 0$, *where* T *is the maximal existence time of* u.

Proof (i) If it is false, then there exists $t_0 \in (0, T)$ such that $u(x, t_0) \in \partial W$, namely

$$I(u(t_0)) = 0, \|\Delta_{\mathbb{B}}u(t_0)\| \neq 0, \text{ or } J(u(t_0)) = d.$$

By (2.5), $J(u(t_0)) = d$ is not true. If $I(u(t_0)) = 0$ but $\|\Delta_{\mathbb{B}}u(t_0)\| \neq 0$, we have $J(u(t_0)) \ge d$ by the definition of d, which is contradictive with (2.5).

(ii) By contradiction, then there exists $t_1 \in (0, T)$, such that $u \in V$ when $0 \le t < t_1$ but $u(t_1) \in \partial V$. Namely

$$I(u(t_1)) = 0, \text{ or } J(u(t_1)) = d.$$

(2.5) implies that $J(u(t_1)) = d$ is false. If $I(u(t_1)) = 0$ and for any $t \in (0, t_1)$, $I(u(t)) < 0$, we claim that there exists $r > 0$ such that $\|\Delta_{\mathbb{B}}u(t_1)\|_2 \ge r$. Indeed, (1.7) implies that

$$\begin{aligned} I(u) &\geq l\|\Delta_{\mathbb{B}}u\|_2^2 - \|u\|_p^p \\ &\geq l\|\Delta_{\mathbb{B}}u\|_2^2 - C_1^p\|u\|_{\mathcal{H}}^p \\ &\geq \frac{C_1^2}{C_*^2+1}\left(l - C_1^p(C_*^2+1)^{\frac{p}{2}}\|\Delta_{\mathbb{B}}u\|_2^{p-2}\right)\|u\|_p^2, \end{aligned}$$

where C_1 is the best constant of embedding $\mathcal{H} \hookrightarrow L_p^{\frac{n}{p}}(\mathbb{B})$, C_* is mentioned in Remark 2.2. Therefore,

$$r = \left(\frac{l}{C_1^p(C_*^2+1)^{\frac{p}{2}}}\right)^{\frac{1}{p-2}} > 0.$$

Hence, by the definition of d, we have $J(u(t_1)) \geq d$, which is contradictive with (2.5). □

Lemma 2.10 *Let $u(x,t)$ be a weak solution of (1.1) with $J(u_0) = d$, then $I(u(t)) > 0$ for any $t \in [0, T)$ provided $I(u_0) > 0$, where T is the maximal existence time of u.*

Proof If it is not true, there exists $t_* \in (0, T)$, such that for any $t \in (0, t_*)$, $I(u(t)) > 0$ but $I(u(t_*)) = 0$. If $\frac{d}{dt}\|u\|_p^p = 0$ for all $t \in (0, t_*]$, namely $\|u\|_p^p$ is a constant for all $t \in (0, t_*]$, it follows from (1.8) that

$$J(u(t_*)) = J(u_0) - \frac{1}{2}I(u_0) < d. \tag{2.6}$$

If there exists a $s \in (0, t_*]$, such that $\frac{d}{dt}\|u\|_p^p \neq 0$ at $t = s$, then $\|u_t\|_2 > 0$ by Hölder inequality, which means that $\int_0^{t_*}\|u_\tau\|_{\mathcal{H}}^2 d\tau$ is strictly positive(it is easy to verify that the solution here has some regularity by the standard regularity promotion process). Combining (2.5), we have

$$\begin{aligned} J(u(t_*)) = J(u_0) &+ \frac{1}{2}\int_0^{t_*}\int_0^{\tau} g'(\tau - s)\|\Delta_{\mathbb{B}}u(\tau) - \Delta_{\mathbb{B}}u(s)\|_2^2 ds d\tau \\ &- \frac{1}{2}\int_0^{t_*} g(\tau)\|\Delta_{\mathbb{B}}u(\tau)\|_2^2 d\tau - \int_0^{t_*}\|u_\tau\|_{\mathcal{H}}^2 d\tau < d. \end{aligned} \tag{2.7}$$

On the other hand, since $I(u(t_*)) = 0$ and for any $t \in (0, t_*)$, $I(u(t)) > 0$, then $\|\Delta_{\mathbb{B}}u(t_*)\|_2 \geq r > 0$, which can be proved as the proof of Lemma 2.9(ii). Consequently, $J(u(t_1)) \geq d$ by the definition of d, which is contradictive with (2.6) and (2.7). □

3 Global existence and asymptotic behaviors

If $u(x,t)$ is a solution of problem (1.1) with $J(u_0) \leq d$, $I(u_0) > 0$ and there exists $t_1 > 0$ such that $\|\Delta_{\mathbb{B}}u(t_1)\|_2 = 0$, then $\|\Delta_{\mathbb{B}}u(t)\|_2 = 0$ for all $t \geq t_1$. Therefore,

$u(x,t)$ is a global solution of problem (1.1) and satisfies the estimate (1.9) and (1.10). So in the following discussions, we do not consider this type of solutions.

Proof of Theorem 1.1 We divide the proof into four steps.
Step 1: The low initial energy $J(u_0) < d$.

By $J(u_0) < d$, $I(u_0) > 0$ and (1.8), we can get $J(u_0) > 0$. So we consider the case $0 < J(u_0) < d$ and $I(u_0) > 0$.

We construct an approximate weak solution of problem (1.1) by the Galerkin method. We choose $\{\omega_j(x)\}$ as the orthogonal basis of $\mathcal{H}$. Let

$$u_m(x,t) = \sum_{j=1}^{m} h_m^j(t)\omega_j(x), m = 1, 2, \cdots,$$

which satisfy

$$\begin{aligned}&(u_{mt}, \omega_k) + (\Delta_{\mathbb{B}} u_{mt}, \Delta_{\mathbb{B}}\omega_k) + (\Delta_{\mathbb{B}} u_m, \Delta_{\mathbb{B}}\omega_k)\\ &= \left(\int_0^t g(t-s)\Delta_{\mathbb{B}} u_m(s)ds, \Delta_{\mathbb{B}}\omega_k\right) + \left(|u_m|^{p-2}u_m - \frac{1}{|\mathbb{B}|}\int_{\mathbb{B}} |u_m|^{p-2}u_m \frac{dx_1}{x_1}dx', \omega_k\right),\end{aligned} \tag{3.1}$$

for $k = 1, 2, \cdots, m$, and

$$u_m(x,0) = \sum_{j=1}^{m} a_m^j(t)\omega_j(x) \to u_0 \text{ in } \mathcal{H}. \tag{3.2}$$

From (2.5), we obtain

$$\begin{aligned}J(u_m(0)) = J(u_m(t)) &- \frac{1}{2}\int_0^t \int_0^\tau g'(\tau - s)\|\Delta_{\mathbb{B}} u_m(\tau) - \Delta_{\mathbb{B}} u_m(s)\|_2^2 ds d\tau\\ &+ \frac{1}{2}\int_0^t g(\tau)\|\Delta_{\mathbb{B}} u_m(\tau)\|_2^2 ds d\tau + \int_0^t \|u_{m\tau}\|_{\mathcal{H}}^2 d\tau < d.\end{aligned} \tag{3.3}$$

for sufficiently large m and any $t \in [0, T)$, where T is the maximal existence time of u.

Similar to the proof of Lemma 2.9(i), it implies that $u_m(x,t) \in W$ for sufficiently large m and any $t \in [0, T)$. Letting $t_1 = \frac{1}{2}T$, then $u_m(x, t_1) \in W$, which means that $0 < J(u_m(t_1)) < d$ and $I(u_m(t_1)) > 0$. Taking $u_m(t_1)$ as initial value, similarly, when m is large enough and $t \in [t_1, t_1 + T) = [\frac{1}{2}T, \frac{3}{2}T)$, the corresponding formula (3.3) still holds. By the same way above, we obtain $u_m(x,t) \in W$ for sufficiently large m and any $t \in [\frac{1}{2}T, \frac{3}{2}T)$. Taking $t_2 = T, t_3 = \frac{3}{2}T, \cdots$ in sequence, we deduce that

$$u_m(x,t) \in W \text{ for sufficiently large m and any } 0 \le t < \infty.$$

From (3.3) and the discussion above, we obtain $I(u_m(t)) > 0$. Therefore, we have

$$J(u_m(t)) = \frac{1}{p} I(u_m(t)) + \frac{p-2}{2p} \int_0^t g(t-s) \|\Delta_{\mathbb{B}} u(t) - \Delta_{\mathbb{B}} u(s)\|_2^2 ds + \frac{p-2}{2p} \left(1 - \int_0^t g(s)ds\right) \|\Delta_{\mathbb{B}} u(t)\|_2^2 < d.$$

Hence, we deduce that

$$\|\Delta_{\mathbb{B}} u_m(t)\|_2^2 < \frac{2pd}{(p-2)l}, \tag{3.4}$$

$$\int_0^t \|u_{m\tau}\|_{\mathcal{H}}^2 d\tau < d, \tag{3.5}$$

and

$$\||u_m|^{p-2} u_m\|_{\frac{p}{p-1}} = \|u_m\|_p^{p-1} \le C_1^{p-1} \|u_m\|_{\mathcal{H}}^{p-1} < C_1^{p-1} \left(\frac{2pd(C_*^2+1)}{l(p-2)}\right)^{\frac{p-1}{2}}, \tag{3.6}$$

where C_1 is the best constant of embedding $\mathcal{H} \hookrightarrow L_p^{\frac{n}{p}}(\mathbb{B})$, C_* is mentioned in Remark 2.2.

Denote $\xrightarrow{\omega^*}$ as the weakly convergence. By (3.4)–(3.6), there exists u and subsequence $\{u_m\}$ (still denoted by $\{u_m\}$) such that as $m \to \infty$,

$$u_m \xrightarrow{\omega^*} u \text{ in } L^\infty([0,\infty); \mathcal{H}) \text{ and a.e. in } \mathbb{B} \times [0,\infty);$$
$$u_{mt} \xrightarrow{\omega^*} u_t \text{ in } L^2([0,\infty); \mathcal{H});$$
$$|u_m|^{p-2} u_m \xrightarrow{\omega^*} |u|^{p-2} u \text{ in } L^\infty([0,\infty); L_{\frac{p}{p-1}}^{\frac{(p-1)n}{p}}(\mathbb{B})).$$

Fixing k in (3.1) and letting $m \to \infty$, we have

$$(u_t, \omega_k) + (\Delta_{\mathbb{B}} u_t, \Delta_{\mathbb{B}} \omega_k) + (\Delta_{\mathbb{B}} u, \Delta_{\mathbb{B}} \omega_k) = \left(\int_0^t g(t-s) \Delta_{\mathbb{B}} u(s) ds, \Delta_{\mathbb{B}} \omega_k\right) + \left(|u|^{p-2} u - \frac{1}{|\mathbb{B}|} \int_{\mathbb{B}} |u|^{p-2} u \frac{dx_1}{x_1} dx', \omega_k\right),$$

and for any $\phi \in \mathcal{H}$ and $t > 0$,

$$(u_t, \phi) + (\Delta_{\mathbb{B}} u_t, \Delta_{\mathbb{B}} \phi) + (\Delta_{\mathbb{B}} u, \Delta_{\mathbb{B}} \phi) = \left(\int_0^t g(t-s) \Delta_{\mathbb{B}} u(s) ds, \Delta_{\mathbb{B}} \phi\right) + \left(|u|^{p-2} u - \frac{1}{|\mathbb{B}|} \int_{\mathbb{B}} |u|^{p-2} u \frac{dx_1}{x_1} dx', \phi\right).$$

On the other hand, (3.2) implies that $u(x, 0) = u_0(x) \in \mathcal{H}$. Therefore, u is a global weak solution of problem (1.1).

Step 2: The critical initial energy $J(u_0) = d$.

Consider the following problem

$$\begin{cases} u_t + \Delta_{\mathbb{B}}^2 u_t + \Delta_{\mathbb{B}}^2 u - \int_0^t g(t-s)\Delta_{\mathbb{B}}^2 u(s)ds = f(u), & (x,t) \in \text{int}\mathbb{B} \times (0,T), \\ u(x,t) = \Delta_{\mathbb{B}} u(x,t) = 0, & (x,t) \in \partial\mathbb{B} \times (0,T), \\ u(x,0) = u_{0m}(x), & x \in \text{int}\mathbb{B}, \end{cases} \tag{3.7}$$

where $u_{0m} = \mu_m u_0$, $\mu_m = 1 - \frac{1}{m}(m \geq 2)$. If $\|u_0\|_p = 0$, then $J(u_{0m}) = \mu_m^2 J(u_0) < d$ and $I(u_{0m}) = \mu_m^2 I(u_0) > 0$. If $\|u_0\|_p \neq 0$, it follows from $I(u_0) > 0$ and Lemma 2.8 that $\lambda^* = \lambda^*(u_0) \geq 1$. Then we can deduce that $J(u_{0m}) = J(\mu_m u_0) < J(u_0) = d$ and $I(u_{0m}) = \mu_m^2 I(u_0) + (\mu_m^2 - \mu_m^p)\|u_0\|_p^p > 0$. Similar to the proof in **step1**, for each m, problem (3.7) admits a global weak solution $u_m(t) \in L^\infty([0,+\infty);\mathcal{H})$ with $u_{mt} \in L^2([0,+\infty);\mathcal{H})$, which satisfies

$$\begin{aligned} &(u_{mt}, v) + (\Delta_{\mathbb{B}} u_{mt}, \Delta_{\mathbb{B}} v) + (\Delta_{\mathbb{B}} u_m, \Delta_{\mathbb{B}} v) \\ &= \left(\int_0^t g(t-s)\Delta_{\mathbb{B}} u_m(s)ds, \Delta_{\mathbb{B}} v\right) + \left(|u_m|^{p-2}u_m - \frac{1}{|\mathbb{B}|}\int_{\mathbb{B}} |u_m|^{p-2}u_m \frac{dx_1}{x_1}dx', v\right), \end{aligned}$$

for any $v \in \mathcal{H}$ and $t \in (0,\infty)$, and

$$\begin{aligned} J(u_{0m}) = J(u_m(t)) &- \frac{1}{2}\int_0^t \int_0^\tau g'(\tau-s)\|\Delta_{\mathbb{B}} u_m(\tau) - \Delta_{\mathbb{B}} u_m(s)\|_2^2 ds d\tau \\ &+ \frac{1}{2}\int_0^t g(\tau)\|\Delta_{\mathbb{B}} u_m(\tau)\|_2^2 ds d\tau + \int_0^t \|u_{m\tau}\|_{\mathcal{H}}^2 d\tau < d, \text{ for } 0 \leq t < \infty. \end{aligned}$$

Since $J(u_{0m}) < d$ and $I(u_{0m}) > 0$, it follows from Lemma 2.9(i) that for any $0 \leq t < \infty$, $I(u_m(t)) > 0$. Then for each m, (3.4)–(3.6) still hold. Therefore, there exists u and subsequence $\{u_m\}$ (still denoted by $\{u_m\}$) such that as $m \to \infty$,

$$\begin{aligned} &u_m \xrightarrow{\omega^*} u \text{ in } L^\infty([0,\infty);\mathcal{H}) \text{ and a.e. in } \mathbb{B} \times [0,\infty); \\ &u_{mt} \xrightarrow{\omega^*} u_t \text{ in } L^2([0,\infty);\mathcal{H}); \\ &|u_m|^{p-2}u_m \xrightarrow{\omega^*} |u|^{p-2}u \text{ in } L^\infty([0,\infty); L_{\frac{p}{p-1}}^{\frac{(p-1)n}{p}}(\mathbb{B})). \end{aligned}$$

Next, the proof is the same as that of the **Step 1** above, so we omit it here.

Step 3: Prove the global estimate (1.9).

We need the following lemma to obtain the result.

Lemma 3.1 (Gronwall Lemma, cf. [11]) *Assume that* $h(t) \in L^1(0,T)$ *is a non-negative function,* $g(t)$ *and* $\eta(t)$ *are the continuous function on* $[0,T]$. *If* $\eta(t)$ *satisfies*

$$\eta(t) \leq g(t) + \int_0^t h(\tau)\eta(\tau)d\tau \textit{ for all } t \in [0,T],$$

then

$$\eta(t) \leq g(t) + \int_0^t h(s)g(s)e^{\int_s^t h(\tau)d\tau} ds \text{ for all } t \in [0, T].$$

Moreover, if $g(t)$ is non-decreasing, one has

$$\eta(t) \leq g(t)e^{\int_0^t h(\tau)d\tau} \text{ for all } t \in [0, T].$$

Multiplying both sides of the first equation in (1.1) by u and then integrating the obtained results over $\mathbb{B} \times (0, t)$, for any $m \in \mathbb{R}$, it follows from (1.7), (1.8), (2.5), Lemma 2.9(i) and Lemma 2.10 that

$$\begin{aligned}
\|u\|_{\mathcal{H}}^2 - \|u_0\|_{\mathcal{H}}^2 &= -2\int_0^t \|\Delta_{\mathbb{B}} u(\tau)\|_2^2 d\tau + 2\int_0^t \int_0^\tau g(\tau - s) \int_{\mathbb{B}} \Delta_{\mathbb{B}} u(s) \Delta_{\mathbb{B}} u(\tau) \frac{dx_1}{x_1} dx' ds d\tau \\
&\quad + 2\int_0^t \|u(\tau)\|_p^p d\tau - \frac{2K(u_0)}{|\mathbb{B}|} \int_0^t \|u(\tau)\|_{p-1}^{p-1} d\tau \\
&\leq \left(-2 + \left(2 + \frac{1}{m^2}\right) \int_0^t g(s)ds\right) \int_0^t \|\Delta_{\mathbb{B}} u(\tau)\|_2^2 d\tau + 2\int_0^t \|u(\tau)\|_p^p d\tau \\
&\quad + m^2 \int_0^t \int_0^\tau g(\tau - s)\|\Delta_{\mathbb{B}} u(\tau) - \Delta_{\mathbb{B}} u(s)\|_2^2 ds d\tau \\
&\leq \left(-m^2 - 2 + \left(m^2 + \frac{1}{m^2} + 2\right) \int_0^t g(s)ds\right) \int_0^t \|\Delta_{\mathbb{B}} u(\tau)\|_2^2 d\tau \\
&\quad + \frac{2(m^2 + 2)pd}{p - 2} t.
\end{aligned}$$

Considering the arbitrariness of m, we choose m small enough such that $\frac{1}{(m^2+1)^2} > l$. Then we can deduce from (1.2) that

$$\|u\|_{\mathcal{H}}^2 \leq 2\alpha t + \|u_0\|_{\mathcal{H}}^2 + \beta \int_0^t \|u(\tau)\|_{\mathcal{H}}^2 d\tau.$$

It follows from Lemma 3.1 that (1.9) holds.

Step 4: Prove the exponential decay (1.10).

Lemma 3.2 (cf. [10]) *Let $y(t) : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function. Assume that there is a constant $A > 0$ such that*

$$\int_s^{+\infty} y(t)dt \leq Ay(s), \; 0 \leq s < +\infty.$$

Then $y(t) \leq y(0)e^{1-t/A}$ for all $t > 0$.

Let $Q(t)=-(u_t,u)-(\Delta_{\mathbb{B}}u_t,\Delta_{\mathbb{B}}u)$. Taking $\phi=u$ in (2.4), for any $q\in\mathbb{R}$, (1.7), (1.8), (2.5), Lemma 2.9(i) and Lemma 2.10 imply that

$$\begin{aligned} Q(t)&=\|\Delta_{\mathbb{B}}u\|_2^2-\int_0^t g(t-s)\int_{\mathbb{B}}\Delta_{\mathbb{B}}u(s)\Delta_{\mathbb{B}}u(t)\frac{dx_1}{x_1}dx'ds-\|u\|_p^p+\frac{K(u_0)}{|\mathbb{B}|}\|u\|_{p-1}^{p-1}\\ &\geq\left(1-\left(1+\frac{1}{2q^2}\right)\int_0^t g(s)ds\right)\|\Delta_{\mathbb{B}}u\|_2^2-\|u\|_p^p+\frac{K(u_0)}{|\mathbb{B}|}\|u\|_{p-1}^{p-1}\\ &\geq\left(\frac{q^2}{2}+1-\left(\frac{q^2}{2}+\frac{1}{2q^2}+1\right)(1-l)\right)\|\Delta_{\mathbb{B}}u\|_2^2-\frac{(q^2+2)p}{p-2}J(u_0)\\ &\quad+\frac{K(u_0)}{|\mathbb{B}|}\|u\|_{p-1}^{p-1}. \end{aligned}$$

Similar to the **Step 3**, we choose q large enough such that $\frac{1}{(q^2+1)^2}<l$. Then we can deduce from the condition $J(u_0)\leq\min\left\{\frac{p-2}{q^2+2}\omega,d\right\}$ that

$$Q(t)\geq\frac{\gamma}{2}\|u\|_{\mathcal{H}}^2, \tag{3.8}$$

where

$$\gamma=\frac{q^2+2-(q+\frac{1}{q})^2(1-l)}{C_*^2+1}>0.$$

On the other hand,

$$\int_t^T Q(\tau)d\tau=\frac{1}{2}\|u(t)\|_{\mathcal{H}}^2-\frac{1}{2}\|u(T)\|_{\mathcal{H}}^2\leq\frac{1}{2}\|u(t)\|_{\mathcal{H}}^2.$$

Combining (3.8), we have

$$\int_t^T\|u(\tau)\|_{\mathcal{H}}^2d\tau\leq\frac{1}{\gamma}\|u(t)\|_{\mathcal{H}}^2.$$

Letting $T\to+\infty$, it follows from Lemma 3.2 that

$$\|u\|_{\mathcal{H}}^2\leq\|u_0\|_{\mathcal{H}}^2e^{1-\gamma t},\ t\geq0.$$

This completes the proof of Theorem 1.1. □

4 Blow-up and bounds for the maximal existence time

In order to obtain the results of Theorem 1.2, we need the following lemma.

Lemma 4.1 (cf. [12]) Suppose that a positive and twice-differentiable function $\theta(t)$ satisfies the inequality

$$\theta''(t)\theta(t)-(1+\gamma)\theta'^2(t)\geq0,\ t>0,$$

where $\gamma > 0$. If $\theta(0) > 0$ and $\theta'(0) > 0$, then there exists a time $T^* \leq \frac{\theta(0)}{\gamma\theta'(0)}$ such that $\theta(t)$ tends to infinity as $t \to T^{*-}$.

Proof of Theorem 1.2 We divide the proof into two steps.

Step 1: Blowing up in finite time.

By contradiction, we assume that the maximal existence time $T = +\infty$. For any $\widehat{T} > 0$ and $t \in [0, \widehat{T})$, we let

$$G(t) = \int_0^t \|u\|_{\mathcal{H}}^2 d\tau + (\widehat{T} - t)\|u_0\|_{\mathcal{H}}^2 + a(t+b)^2, \tag{4.1}$$

where positive constant a and b are to be determined. It is easy to see that

$$\begin{aligned} G'(t) &= \|u\|_{\mathcal{H}}^2 - \|u_0\|_{\mathcal{H}}^2 + 2a(t+b) \\ &= 2\int_0^t (u, u_\tau) d\tau + 2\int_0^t (\Delta_{\mathbb{B}} u, \Delta_{\mathbb{B}} u_\tau) d\tau + 2a(t+b), \end{aligned} \tag{4.2}$$

and

$$G''(t) = 2\int_{\mathbb{B}} u u_t \frac{dx_1}{x_1} dx' + 2\int_{\mathbb{B}} \Delta_{\mathbb{B}} u \Delta_{\mathbb{B}} u_t \frac{dx_1}{x_1} dx' + 2a. \tag{4.3}$$

Furthermore, it follows from (4.1)–(4.3) that

$$\begin{aligned} &G(t)G''(t) - \frac{p}{2}G'^2(t) = G(t)G''(t) - 2p(C + a(t+b))^2 \\ &= G(t)G''(t) + 2p\Big\{\Big(A + a(t+b)^2\Big)\Big(B + a\Big) \\ &- \Big(G(t) - (\widehat{T} - t)\|u_0\|_{\mathcal{H}}^2\Big)\Big(\int_0^t \|u_\tau\|_{\mathcal{H}}^2 d\tau + a\Big) - \Big(C + a(t+b)\Big)^2\Big\}, \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} A &= \int_0^t \|u\|_2^2 d\tau + \int_0^t \|\Delta_{\mathbb{B}} u\|_2^2 d\tau, \\ B &= \int_0^t \|u_\tau\|_2^2 d\tau + \int_0^t \|\Delta_{\mathbb{B}} u_\tau\|_2^2 d\tau, \end{aligned}$$

and

$$C = \int_0^t (u, u_\tau) d\tau + \int_0^t (\Delta_{\mathbb{B}} u, \Delta_{\mathbb{B}} u_\tau) d\tau.$$

It is obvious that $AB \geq C^2$ by Schwarz inequality. Hence, we have

$$\Big(A + a(t+b)^2\Big)\Big(B + a\Big) \geq \Big(C + a(t+b)\Big)^2. \tag{4.5}$$

Combining (4.4) and (4.5), we calculate

$$G(t)G''(t) - \frac{p}{2}G'^2(t) \geq G(t)\left\{G''(t) - 2p\left(\int_0^t \|u_\tau\|_{\mathcal{H}}^2 d\tau + a\right)\right\} := G(t)\zeta(t),$$

where

$$\zeta(t) = 2\int_{\mathbb{B}} uu_t \frac{dx_1}{x_1}dx' + 2\int_{\mathbb{B}} \Delta_{\mathbb{B}}u\Delta_{\mathbb{B}}u_t \frac{dx_1}{x_1}dx' - 2p\int_0^t \|u_\tau\|_{\mathcal{H}}^2 d\tau - 2(p-1)a. \tag{4.6}$$

Taking $\phi = u$ in (2.4), it follows from (1.6), (2.5) and (4.6) that

$$\begin{aligned}
\zeta(t) &= -2\|\Delta_{\mathbb{B}}u\|_2^2 - 2\int_0^t g(t-s)\int_{\mathbb{B}} \Delta_{\mathbb{B}}u(s)\Delta_{\mathbb{B}}u(t)\frac{dx_1}{x_1}dx'ds + 2\|u\|_p^p \\
&\quad - \frac{2K(u_0)}{|\mathbb{B}|}\|u\|_{p-1}^{p-1} \\
&\quad + 2p\left\{\frac{1}{2}\int_0^t g(\tau)\|\Delta_{\mathbb{B}}u(\tau)\|_2^2 d\tau - \frac{1}{2}\int_0^t\int_0^\tau g'(\tau-s)\|\Delta_{\mathbb{B}}u(\tau) - \Delta_{\mathbb{B}}u(s)\|_2^2 dsd\tau \right. \\
&\quad \left. + J(u) - J(u_0)\right\} - 2(p-1)a \\
&\geq \left(p - 2 - p\int_0^t g(s)ds\right)\|\Delta_{\mathbb{B}}u\|_2^2 - 2\int_0^t g(t-s)\int_{\mathbb{B}} \Delta_{\mathbb{B}}u(s)\Delta_{\mathbb{B}}u(t)\frac{dx_1}{x_1}dx'ds \\
&\quad + p\int_0^t g(t-s)\|\Delta_{\mathbb{B}}u(t) - \Delta_{\mathbb{B}}u(s)\|_2^2 ds - 2(p-1)a \\
&\geq \left(p - 2 - \left(p + \frac{1}{p} + 2\right)\int_0^t g(s)ds\right)\|\Delta_{\mathbb{B}}u\|_2^2 - 2(p-1)a.
\end{aligned}$$

Taking a small enough such that

$$a \leq \frac{p - 2 - (p + \frac{1}{p} + 2)(1-l)}{2(p-1)}\|\Delta_{\mathbb{B}}u\|_2^2, \tag{4.7}$$

this implies that $\zeta(t) \geq 0$. From (4.1) and (4.2), we calculate $G(0) > 0$ and $G'(0) > 0$. We choose b large enough such that

$$ab(p-2) - \|u_0\|_{\mathcal{H}}^2 > 0. \tag{4.8}$$

Taking the arbitrariness of $\widehat{T}$ into consideration, let

$$\widehat{T} \geq \frac{ab^2}{ab(p-2) - \|u_0\|_{\mathcal{H}}^2}. \tag{4.9}$$

It follows from lemma 4.1 that there exists $T^* \in [0, \widehat{T}]$ such that $G(t) \to \infty$ as $t \to T^{*-}$, which means that

$$\|u\|_{\mathcal{H}}^2 \to \infty, \text{ as } t \to T^{*-}.$$

This is a contradiction with $T=+\infty$. Hence, $T<+\infty$, i.e. the solutions of problem (1.1) blow up in finite time.

Step 2: Bounds for the maximal existence time.

Lemma 4.1 and (4.9) imply that the maximal existence time T satisfies

$$T \leq \frac{ab^2}{ab(p-2)-\|u_0\|_{\mathcal{H}}^2},$$

where parameters a and b are respectively given in (4.7) and (4.8).

To state the estimate of the lower bound for the maximal existence time T, we define the function

$$\mathcal{L}(t)=\|u\|_{\mathcal{H}}^2.$$

Multiplying u on both sides of the first equation in (1.1) and integrating over $\mathbb{B}$ by parts, from (1.2), (1.7), (1.11) and Lemma 2.9(ii), we have

$$\begin{aligned}
\mathcal{L}'(t) &= -2\|\Delta_{\mathbb{B}}u\|_2^2+2\int_0^t g(t-s)\int_{\mathbb{B}}\Delta_{\mathbb{B}}u(s)\Delta_{\mathbb{B}}u(t)\frac{dx_1}{x_1}dx'ds+2\|u\|_p^p-\frac{2K(u_0)}{|\mathbb{B}|}\|u\|_{p-1}^{p-1}\\
&\leq -2\|\Delta_{\mathbb{B}}u\|_2^2+\left(2+\frac{1}{p}\right)\int_0^t g(s)ds\|\Delta_{\mathbb{B}}u\|_2^2+p\int_0^t g(t-s)\|\Delta_{\mathbb{B}}u(t)-\Delta_{\mathbb{B}}u(s)\|_2^2ds\\
&\quad +2\|u\|_p^p-\frac{2K(u_0)}{|\mathbb{B}|}\|u\|_{p-1}^{p-1}\\
&\leq \left(-p-2+\left(p+\frac{1}{p}+2\right)\int_0^t g(s)ds\right)\|\Delta_{\mathbb{B}}u\|_2^2+(p+2)\|u\|_p^p-\frac{2K(u_0)}{|\mathbb{B}|}\|u\|_{p-1}^{p-1}\\
&\leq (p+2)C_1^p\mathcal{L}^{\frac{p}{2}}(t)-\frac{2K(u_0)}{|\mathbb{B}|}C_2^{p-1}\mathcal{L}^{\frac{p-1}{2}}(t),
\end{aligned}$$

where C_1 and C_2 are respectively the best constant of embedding $\mathcal{H}\hookrightarrow L_p^{\frac{n}{p}}(\mathbb{B})$ and $\mathcal{H}\hookrightarrow L_{p-1}^{\frac{n}{p-1}}(\mathbb{B})$. Then, a simple calculation gives that

$$\int_{\|u_0\|_{\mathcal{H}}^2}^{\|u\|_{\mathcal{H}}^2}\frac{d\mu}{(p+2)C_1^p\mu^{\frac{p}{2}}-\frac{2K(u_0)}{|\mathbb{B}|}C_2^{p-1}\mu^{\frac{p-1}{2}}}\leq t.$$

From the proof in **Step 1** above, letting $t\to T^-$, we obtain

$$T\geq\int_{\|u_0\|_{\mathcal{H}}^2}^{\infty}\frac{d\mu}{(p+2)C_1^p\mu^{\frac{p}{2}}-\frac{2K(u_0)}{|\mathbb{B}|}C_2^{p-1}\mu^{\frac{p-1}{2}}}.$$

The proof of Theorem 1.2 is accomplished. □

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Declarations

Conflict of interest There are no conflicts of interest to this work.

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