J. Pseudo-Differ. Oper. Appl. (2023) 14:74
https://doi.org/10.1007/s11868-023-00569-9

Long time behavior of solutions for time-fractional pseudo-parabolic equations involving time-varying delays and superlinear nonlinearities

Do Lan[1] · **Tran Van Tuan**[2]

Received: 3 December 2022 / Revised: 11 October 2023 / Accepted: 13 October 2023 /
Published online: 9 November 2023

Abstract
We study the long time behavior of solutions for time-fractional pseudo-parabolic equations involving time-varying delays and nonlinear pertubations, where the non-linear term is allowed to have superlinear growth. Concerning the associated linear problem, we establish a variation-of-parameters formula of mild solutions and prove some regularity estimates of resolvent operators. In addition, thanks to local estimates on Hilbert scales, fixed point arguments and a new Halanay type inequality, we obtain some results on the global solvability, stability, dissipativity and the existence of decay solutions to our problem.

Keywords Decay solution · Dissipativity · Time-fractional pseudo-parabolic · Condensing map · Measure of non-compactness

Mathematics Subject Classification Primary 35B40 · 34K37; Secondary 35C15 · 45D05 · 45K05

1 Introduction

Over the past few years, fractional functional differential equations (FrFDEs) in both finite and infinite dimensional spaces have been widely investigated in the literature

✉ Do Lan
dolan@tlu.edu.vn

Tran Van Tuan
tranvantuan@hpu2.edu.vn

[1] Faculty of Computer Science and Engineering, Thuyloi University, 175 Tay Son, Dong Da, Hanoi, Vietnam

[2] Department of Mathematics, Hanoi Pedagogical University 2, Xuan Hoa, Phuc Yen, Vinh Phuc, Vietnam

by many researchers [1, 2, 13, 16, 24, 28–30, 33, 36, 37]. These studies are strongly motivated from the fact that FrFDEs are effective tools in the modeling various processes and phenomena arising from physics, biology, economics, engineering and other applied sciences, where the current state of such evolutionary processes depends on the state [3, 6, 34].

On the other hand, one of the important and interesting trends in qualitative treatments of FrFDEs is related to the long time behavior of solutions for such these equations. This topic has been studied extensively by many authors and many notable contributions have been established in the last few years. We mention recent results concerning to the qualitative analysis on the behavior of solutions such as the Mittag–Leffler stability, asymptotic stability, weakly asymptotic stability and existence of decay solutions governed by FrFDEs [1, 2, 4, 13, 16, 24, 28–30].

Motivated by the reasons above and the recent studies [25, 26], we investigate the global existence and long time behavior of solutions for time-fractional pseudo-parabolic equations (tFrPPEs) involving time-varying delays and superlinear nonlinearities

$$\partial_t^\alpha (u - \nu\Delta u) - \Delta u = f(t, u(t - \rho(t))) \text{ in } \Omega, t > 0, \tag{1.1}$$

$$u = 0 \ \text{ on } \partial\Omega,\ t \geq 0, \tag{1.2}$$

$$u(s, x) = \xi(s, x),\ \text{ in } \Omega, s \in [-q, 0], \tag{1.3}$$

where $\Omega \subset \mathbb{R}^d, d \geq 1$ be a bounded domain with smooth boundary $\partial\Omega$, Δ denotes the Laplacian, $\nu > 0$, $\partial_t^\alpha, \alpha \in (0, 1)$, stands for the Caputo fractional derivative of order α defined by

$$\partial_t^\alpha v(t, x) = \int_0^t g_{1-\alpha}(t - s)\partial_s v(s, x)ds, t > 0, x \in \Omega,$$

here $g_{1-\alpha}(t) = t^{-\alpha}/\Gamma(1 - \alpha), t > 0$. In the model problem, $\rho \in C(\mathbb{R}^+)$ be such that $-q \leq t - \rho(t) \leq t$, $\xi \in C([-q, 0]; L^2(\Omega))$ and $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a given nonlinear function which will be specified in Sect. 3.

Concerning the initial value problem (1.1)–(1.3) without delays, it should be noticed that the linear part of the Eq. (1.1) is used to describe many different phenomena in physics such as seepage of homogeneous liquids in fissured rocks [23] and the aggregation of populations [21]. The existence, stability and blow up in finite time of solutions governed by tFrPPEs and its nonlinear invariants were dealt with in a large number of published investigations; see, e.g., [8, 9, 17–19, 25–27, 32, 35]. However, to our knowledge, questions on the global existence and long time behavior of solutions for problem (1.1)–(1.3) have not yet been concerned in the literature. This is the main motivation of our study.

Regarding problem (1.1)–(1.3), our main goal is to find sufficient conditions on ρ and f to obtain the followings

- The global existence of solutions;
- The asymptotic stability and dissipativity of solutions;
- The existence of decay solutions.

In order to handle the first objective, we first construct an implicit representation formula of solutions via resolvent operators and formulate the question on the global solvability for the problem (1.1)–(1.3) as a fixed point problem of certain mapping related to f. Based on the smoothness of two resolvent families, compactness of the Cauchy operator (Lemma 2.3 and Remark 2.1) and fixed point arguments, we prove some results about the existence of solutions for the problem (1.1)–(1.3) under different situations of nonlinear terms including the sublinear and superlinear cases (Theorems 3.1, and 3.2). In addition, a new Halanay type inequality (Proposition 2.4) will be established to analyze the asymptotic stability and dissipativity of solutions. To deal with the third objective, we make use of the fixed point theorem for condensing mappings which is recently proposed in [8, 14]. With the aid of this technique, it is shown in Theorem 4.5 that if the nonlinearity f obeys a superlinear growth condition (see (**F**5) below) then the problem under consideration has a compact set of decay solutions.

The outline of the paper is as follows. In the next section, a variation of constants formula of mild solutions to the linear problem associated with problem (1.1)–(1.3) is presented. In addition, the smoothness of two resolvent operators, the compactness of the Cauchy operator and the Halanay type inequality are shown. The main goal in Sect. 3 is devoted to proving the global solvability. The last section shows the results on the asymptotic stability, the dissipativity and the existence of decay solutions.

2 Preliminaries

In this section, we aim to construct an integral representation of solutions to the linear problem associated with (1.1)–(1.3). For this purpose, let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions of $-\Delta$ subjected to the homogeneous boundary condition, i.e.,

$$-\Delta e_n = \lambda_n e_n \text{ in } \Omega,\ \ e_n = 0 \text{ on } \partial\Omega,$$

where one can assume that $0 < \lambda_1 \le \lambda_2 \le \ldots$ and $\lambda_n \to \infty$ as $n \to \infty$. Denote $(\cdot, \cdot)$, $\|\cdot\|$ the inner product and the standard norm in $L^2(\Omega)$. For $\gamma \in \mathbb{R}$, the fractional power operator $(-\Delta)^\gamma$ is defined as follows

$$(-\Delta)^\gamma v = \sum_{n=1}^{\infty} \lambda_n^\gamma (v, e_n) e_n,$$

$$D((-\Delta)^\gamma) = \left\{ v \in L^2(\Omega) : \sum_{n=1}^{\infty} \lambda_n^{2\gamma} (v, e_n)^2 < \infty \right\}.$$

Let $\mathbb{V}_\gamma = D((-\Delta)^\gamma)$. It should be noted that $\mathbb{V}_\gamma$ is a Banach space with the norm

$$\|z\|_{\mathbb{V}_\gamma} = \left(\sum_{n=1}^{\infty} \lambda_n^{2\gamma} \, |(z, e_n)|^2 \right)^{\frac{1}{2}}, \ z \in D((-\Delta)^\gamma).$$

Furthermore, for each $\gamma > 0$, we can identify $\mathbb{V}_{-\gamma} = D((-\Delta)^{-\gamma})$ with $\mathbb{V}_{\gamma}^{*}$, the dual space of $\mathbb{V}_{\gamma}$.

Consider the linear problem associated with (1.1)–(1.3) without delay of the form

$$\partial_t^{\alpha}(u - \nu\Delta u) - \Delta u = F \ \text{ in } \Omega, t > 0, \tag{2.1}$$

$$u = 0, \ \text{ on } \partial\Omega, \ t \geq 0, \tag{2.2}$$

$$u(0, \cdot) = \xi, \ \text{ in } \Omega, \tag{2.3}$$

where $F \in C(\mathbb{R}^+; L^2(\Omega))$.

Assume that

$$u(t, \cdot) = \sum_{n=1}^{\infty} u_n(t) e_n, \ F(t) = \sum_{n=1}^{\infty} F_n(t) e_n.$$

Inserting into (2.1)–(2.4) leads to

$$(g_{1-\alpha} * u_n')(t) + \frac{\lambda_n}{1 + \nu\lambda_n} u_n(t) = \frac{1}{1 + \nu\lambda_n} F_n(t), t > 0 \tag{2.4}$$

$$u_n(0) = \xi_n := (\xi, e_n). \tag{2.5}$$

In order to find u_n from (2.4)–(2.5), we consider the Mittag-Leffler function $E_{\alpha,\beta}$ defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, z \in \mathbb{C}, \alpha, \beta > 0.$$

Using the Laplace transform, it is useful to notice that the functions

$$s_\alpha(t, \mu) := E_{\alpha,1}(-\mu t^{\alpha}), r_\alpha(t, \mu) := t^{\alpha-1} E_{\alpha,\alpha}(-\mu t^{\alpha}), \tag{2.6}$$

where μ is a positive parameter, satisfy the following scalar Volterra integral equations

$$s_\alpha(t, \mu) + \mu\big(g_\alpha * s_\alpha(\cdot, \mu)\big)(t) = 1, \ t \geq 0, \tag{2.7}$$

$$r_\alpha(t, \mu) + \mu\big(g_\alpha * r_\alpha(\cdot, \mu)\big)(t) = g_\alpha(t), \ t > 0. \tag{2.8}$$

Let us remark that for each $\mu > 0$ the function $s_\alpha(t, \mu)$ is completely monotonic on $(0, \infty)$, that is,

$$(-1)^n \frac{\partial^n}{\partial t^n} s_\alpha(t, \mu) \geq 0, \ \text{ for all } n = 0, 1, 2, \ldots, t > 0,$$

thanks to [7, Proposition 3.23, p. 47] (see also [22]). Other useful properties of $s_\alpha(\cdot, \mu)$, $r_\alpha(\cdot, \mu)$ are gathered in the following proposition.

Proposition 2.1 *Let s_α, r_α are given by (2.6). Then the following assertions hold.*

(i) *For every* $\mu > 0$, *it holds that*

$$\frac{1}{1+\mu\Gamma(1-\alpha)t^\alpha} \le s_\alpha(t,\mu) \le \frac{1}{1+\frac{\mu t^\alpha}{\Gamma(1+\alpha)}}, \text{ for all } t \ge 0, \tag{2.9}$$

$$0 < -\frac{d}{dt}s_\alpha(t,\mu) < \mu t^{\alpha-1}, \text{ for almost all } t > 0. \tag{2.10}$$

(ii) $\mu\big(1 * r_\alpha(\cdot,\mu)\big)(t) = 1 - s_\alpha(t,\mu), t \ge 0$ *and* $\frac{d}{dt}s_\alpha(t,\mu) = -\mu r_\alpha(t,\mu)$ *for a.e.* $t > 0$.

(iii) *For every* $\mu > 0$, *the following bounds hold*

$$\mu r_\alpha(t,\mu) \le \frac{1}{t}, \text{ and } r_\alpha(t,\mu) \le g_\alpha(t), \text{ for all } t > 0. \tag{2.11}$$

(iv) *For each* $t > 0$, *the mappings*

$$\mu \mapsto s_\alpha(t,\mu),\ \ \mu \mapsto r_\alpha(t,\mu)$$

are nonincreasing.

(v) *For each* $t > 0$, *the mappings*

$$\mu \mapsto s_\alpha\left(t, \frac{\mu}{1+\nu\mu}\right),\ \ \mu \mapsto \frac{1}{1+\nu\mu} r_\alpha\left(t, \frac{\mu}{1+\nu\mu}\right)$$

are nonincreasing.

Proof The proof of the assertions (i)–(iv) can be found in [31, Proposition 2.1]. The assertion (vi) is followed by exploiting the chain rule, the claim (v) and the fact that the mapping $\mu \mapsto \frac{\mu}{1+\nu\mu}$ is increasing on $(0,\infty)$. □

Let us now consider the following initial value problem

$$(g_{1-\alpha} * v')(t) + \frac{a}{1+\nu a}v(t) = \frac{1}{1+\nu a}\omega(t), t > 0, \tag{2.12}$$

$$v(0) = v_0, \tag{2.13}$$

where $a > 0$ and $\omega \in C(\mathbb{R}^+)$. The following proposition gives a representation for the solution of (2.12)–(2.13).

Proposition 2.2 *The function*

$$v(t) = s_\alpha\left(t, \frac{a}{1+\nu a}\right)v_0 + \frac{1}{1+\nu a} r_\alpha\left(\cdot, \frac{a}{1+\nu a}\right) * \omega(t), t \ge 0, \tag{2.14}$$

be the unique solution of (2.12)–(2.13).

Proof Suppose that v is given by the formula (2.14). We now show that v be a solution to the problem (2.12)–(2.13). Indeed, by the formulation of v and by the fact that $s_\alpha(0, \frac{a}{1+\nu a}) = 1$, we have that $v(0) = s_\alpha(0, \frac{a}{1+\nu a})v_0 = v_0$. On the other hand, convolving the Eqs. (2.7), (2.8) by $g_{1-\alpha}$ and using $g_{1-\alpha} * g_\alpha = 1$, one obtains

$$g_{1-\alpha} * [s_\alpha(\cdot, \mu) - 1] + \mu(1 * s_\alpha(\cdot, \mu)) = 0, \text{ for all } \mu > 0, \tag{2.15}$$

and

$$g_{1-\alpha} * r_\alpha = s_\alpha, \tag{2.16}$$

thanks to Proposition 2.1(ii). Using the expression (2.16) and the formula of v, one knows that

$$\begin{aligned} g_{1-\alpha} * [v - v_0] &= g_{1-\alpha} * [s_\alpha - 1]v_0 + \frac{1}{1+\nu a} g_{1-\alpha} * r_\alpha * \omega \\ &= g_{1-\alpha} * [s_\alpha - 1]v_0 + \frac{1}{1+\nu a} s_\alpha * \omega \end{aligned}$$

Differentiating both sides of the later equality and owing to (2.15), Proposition 2.1(ii), we find that

$$\begin{aligned} &(g_{1-\alpha} * v')(t) + \frac{a}{1+\nu a} v(t) \\ &= \frac{d}{dt}\big(g_{1-\alpha} * [v - v_0]\big)(t) + \frac{a}{1+\nu a} v(t) \\ &= -\frac{a}{1+\nu a} s_\alpha(t, \frac{a}{1+\nu a})v_0 + \frac{1}{1+\nu a}\big(\omega(t) + s'_\alpha(\cdot, \frac{a}{1+\nu a}) * \omega(t)\big) + \frac{a}{1+\nu a} v(t) \\ &= -\frac{a}{1+\nu a} s_\alpha(t, \frac{a}{1+\nu a})v_0 + \frac{1}{1+\nu a}\frac{-a}{1+\nu a} r_\alpha(\cdot, \frac{a}{1+\nu a}) * \omega(t) + \frac{1}{1+\nu a}\omega(t) \\ &\quad + \frac{a}{1+\nu a} v(t) \\ &= \frac{1}{1+\nu a}\omega(t), \forall t > 0, \end{aligned}$$

which is the equality (2.12) as claimed.

Conversely, let v is a solution of (2.12)–(2.13). Taking the Laplace transform of both sides of the Eq. (2.12), we obtain

$$\lambda^{\alpha-1}(\lambda\widehat{v} - v_0) + \frac{a}{1+\nu a}\widehat{v} = \widehat{\omega},$$

or equivalently

$$\widehat{v}(\lambda) = \frac{v_0}{\lambda + \frac{a}{1+\nu a}\lambda^{1-\alpha}} + \frac{1}{1+\nu a}\frac{\widehat{\omega}}{\lambda^\alpha + \frac{a}{1+\nu a}}. \tag{2.17}$$

Using formulas (1.10.4), (1.10.9) in [15, p. 50], we know that

$$\widehat{s_\alpha(\cdot,\mu)}(\lambda) = \frac{\lambda^{\alpha-1}}{\lambda^\alpha+\mu}, \widehat{r_\alpha(\cdot,\mu)}(\lambda) = \frac{1}{\lambda^\alpha+\mu}, \Re(\lambda) > 0. \tag{2.18}$$

Combining (2.17), (2.18) and using the inverse Laplace transform, we find that

$$v(t) = s_\alpha\left(t, \frac{a}{1+\nu a}\right)v_0 + \frac{1}{1+\nu a}r_\alpha\left(\cdot, \frac{a}{1+\nu a}\right) * \omega(t),$$

which is (2.14). The proof is complete. □

Taking Proposition 2.2 into account, the solution of problem (2.4)–(2.5) is given by

$$u_n(t) = s_\alpha(t,\theta_n)\xi_n + \frac{1}{1+\nu\lambda_n}r_\alpha(\cdot,\theta_n) * F_n(t), t \ge 0,$$

where $\theta_n = \frac{\lambda_n}{1+\nu\lambda_n}$, $n = 1, 2, \ldots$ Thus

$$u(t,\cdot) = \mathcal{S}_\alpha(t)\xi + \int_0^t \mathcal{R}_\alpha(t-\tau)F(\tau)d\tau, \tag{2.19}$$

where

$$\mathcal{S}_\alpha(t)v = \sum_{n=1}^\infty s_\alpha(t,\theta_n)v_n e_n, v \in L^2(\Omega), t \ge 0, \tag{2.20}$$

$$\mathcal{R}_\alpha(t)v = \sum_{n=1}^\infty \frac{1}{1+\nu\lambda_n}r_\alpha(t,\theta_n)v_n e_n, v \in L^2(\Omega), t > 0. \tag{2.21}$$

Here and afterward, for any $T > 0$, the notations $\|\cdot\|_\infty$, $\|\cdot\|_0$ will stand for the norms in $C([0,T]; L^2(\Omega))$ and in $C([-q,0]; L^2(\Omega))$, and the symbol $\|\cdot\|_{op}$ will be employed for the operator norm of bounded linear operators on $L^2(\Omega)$. Furthermore, we make use of the notation $u(t)$ for $u(t,\cdot)$ and consider u as a function defined on $[0,T]$, taking values in $L^2(\Omega)$.

Clearly $\mathcal{S}_\alpha(t)$ and $\mathcal{R}_\alpha(t)$ defined by (2.20), (2.21) are bounded linear operators acting on $L^2(\Omega)$ for all $t \ge 0$. We collect in the following lemma some other interesting properties of these operators.

Lemma 2.3 *Let $\{\mathcal{S}_\alpha(t)\}_{t\ge0}$ and $\{\mathcal{R}_\alpha(t)\}_{t\ge0}$ be the families of linear operators defined by (2.20) and (2.21), respectively. Then*

(a) *For each $v \in L^2(\Omega)$ and $T > 0$, $\mathcal{S}_\alpha(\cdot)v \in C([0,T]; L^2(\Omega))$ and $\mathcal{S}_\alpha(\cdot)$ is differentiable on $(0,\infty)$. Furthermore, the following estimates hold*

$$\|\mathcal{S}_\alpha(t)v\| \le s_\alpha(t,\theta_1)\|v\|, \ t \in [0,T], \tag{2.22}$$

$$\|\mathcal{S}'_\alpha(t)v\| \le \frac{\|v\|}{t}, \forall v \in L^2(\Omega), \forall t > 0. \tag{2.23}$$

(b) *Let* $v \in L^2(\Omega), T > 0$ *and* $g \in C([0,T]; L^2(\Omega))$. *Then* $\mathcal{R}_\alpha(\cdot)v \in C([0,T]; L^2(\Omega))$ *and* $\mathcal{R}_\alpha * g \in C([0,T]; \mathbb{V}_\gamma)$, *for all* $\gamma \in (0,1)$. *Furthermore,*

$$\|\mathcal{R}_\alpha(t)v\| \leq \frac{1}{1+\nu\lambda_1} r_\alpha(t,\theta_1)\|v\|, \ t \in (0,T], \tag{2.24}$$

$$\|(\mathcal{R}_\alpha * g)(t)\| \leq \frac{1}{1+\nu\lambda_1}\int_0^t r_\alpha(t-\tau,\theta_1)\|g(\tau)\|d\tau, \ t \in [0,T], \tag{2.25}$$

$$\|(\mathcal{R}_\alpha * g)(t)\|_{\mathbb{V}_\gamma} \leq \nu^{-1}\lambda_1^{\gamma-1}\theta_1^{-1/2}\left(\int_0^t r_\alpha(t-\tau,\theta_1)\|g(\tau)\|^2 d\tau\right)^{\frac{1}{2}}, \ t \in [0,T]. \tag{2.26}$$

Proof (a) Obviously, the series given by (2.20) is uniformly convergent on $[0,T]$. Thus $\mathcal{S}_\alpha(\cdot)v \in C([0,T]; L^2(\Omega))$, for all $v \in L^2(\Omega)$. Moreover

$$\begin{aligned}\|\mathcal{S}_\alpha(t)v\|^2 &= \sum_{n=1}^\infty s_\alpha(t,\theta_n)^2 v_n^2\\ &\leq s_\alpha(t,\theta_1)^2\sum_{n=1}^\infty v_n^2\\ &= s_\alpha(t,\theta_1)^2\|v\|^2,\end{aligned}$$

thanks to Proposition 2.1(vi). Therefore, the bound (2.22) is followed. The proofs for the differentiability of $\mathcal{S}_\alpha$ and the bound (2.23) can be found in [31, Lemma 2. 3(c)].

(b) The proofs for $\mathcal{R}_\alpha(\cdot)v \in C([0,T]; L^2(\Omega))$ and $\mathcal{R}_\alpha * g \in C([0,T]; \mathbb{V}_\gamma)$ are done by using the same arguments as those proposed in the work of Ke et al. [12, Lemma 2.3]. It remains to show the bounds in (2.24), (2.25) and (2.26). Due to the nondecreasing of the sequence $\{\lambda_n\}_{n\geq 1}$, one has

$$\frac{1}{1+\nu\lambda_n} r_\alpha(t,\theta_n) \leq \frac{1}{1+\nu\lambda_1} r_\alpha(t,\theta_1), \forall n = 1,2,\ldots$$

thanks to Proposition 2.1(vi). Therefore, for all $t \in (0,T]$, we have that

$$\begin{aligned}\|\mathcal{R}_\alpha(t)v\| &= \left(\sum_{n=1}^\infty\left(\frac{1}{1+\nu\lambda_n} r_\alpha(t,\theta_n)\right)^2 v_n^2\right)^{1/2}\\ &\leq \frac{1}{1+\nu\lambda_1} r_\alpha(t,\theta_1)\|v\|.\end{aligned}$$

It is obvious that the bound in (2.25) follows immediately from (2.24). Noting that

$$(-\Delta)^\gamma\big(\mathcal{R}_\alpha * g\big)(t) = \sum_{n=1}^\infty \frac{\lambda_n^\gamma}{1+\nu\lambda_n}\int_0^t r_\alpha(t-\tau,\theta_n)g_n(\tau)d\tau,$$

and using the Hölder inequality, Proposition 2.1(ii)–(iv), we find that

$$
\begin{aligned}
&|\frac{\lambda_n^\gamma}{1+\nu\lambda_n}\int_0^t r_\alpha(t-\tau,\theta_n)g_n(\tau)d\tau| \\
&\quad\le \nu^{-1}\lambda_n^{\gamma-1}\Big(\int_0^t r_\alpha(t-\tau,\theta_n)\,d\tau\Big)^{1/2}\Big(\int_0^t r_\alpha(t-\tau,\theta_n)|g_n(\tau)|d\tau\Big)^{1/2} \\
&\quad\le \nu^{-1}\lambda_1^{\gamma-1}\theta_n^{-1/2}\Big(\int_0^t r_\alpha(t-\tau,\theta_n)|g_n(\tau)|d\tau\Big)^{1/2}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\|(-\Delta)^\gamma\big(\mathcal{R}_\alpha * g\big)(t)\|^2 &\le \sum_{n=1}^\infty \nu^{-2}\lambda_1^{2(\gamma-1)}\theta_n^{-1}\int_0^t r_\alpha(t-\tau,\theta_n)|g_n(\tau)|^2 d\tau \\
&\le \nu^{-2}\lambda_1^{2(\gamma-1)}\theta_1^{-1}\sum_{n=1}^\infty\int_0^t r_\alpha(t-\tau,\theta_n)|g_n(\tau)|^2 d\tau \\
&= \nu^{-2}\lambda_1^{2(\gamma-1)}\theta_1^{-1}\int_0^t r_\alpha(t-\tau,\theta_1)\|g(\tau)\|^2 d\tau.
\end{aligned}
$$

It follows that

$$
\|\big(\mathcal{R}_\alpha * g\big)(t)\|_{\mathbb{V}_\gamma} \le \nu^{-1}\lambda_1^{\gamma-1}\theta_1^{-1/2}\Big(\int_0^t r_\alpha(t-\tau,\theta_1)\|g(\tau)\|^2 d\tau\Big)^{1/2}.
$$

The proof is complete. □

Remark 2.1 Using Lemma 2.3(b) above, by similar arguments as in [24, Proposition 2.3], the Cauchy operator defined by

$$
\mathcal{Q}_\alpha : C([0,T];L^2(\Omega)) \to C([0,T];L^2(\Omega)),\ g \mapsto \mathcal{Q}_\alpha g(t) := (\mathcal{R}_\alpha * g)(t)
$$

is also compact.

To deal with the nonlinear problem (1.1)–(1.3), we consider the nonlinearity f as a map defined on $\mathbb{R}^+ \times L^2(\Omega)$ with values in $L^2(\Omega)$. Based on the representation (2.19), we introduce the following concept of mild solution to problem (1.1)–(1.3).

Definition 2.1 Let $\xi \in C([-q,0];L^2(\Omega))$ be given. A function $u \in C([-q,T];L^2(\Omega))$ is said to be a mild solution to (1.1)–(1.3) on the interval $[-q,T]$ iff $u(s)=\xi(s)$ for $s\in[-q,0]$ and

$$
u(t) = \mathcal{S}_\alpha(t)\xi(0) + \int_0^t \mathcal{R}_\alpha(t-\tau)f\big(\tau, u(\tau-\rho(\tau))\big)d\tau,\ t\in[0,T].
$$

For given $\xi \in C([-q,0];L^2(\Omega))$, denote $C_\xi([0,T];L^2(\Omega)) := \{u \in C([0,T];L^2(\Omega)) : u(0) = \xi(0)\}$. Then $C_\xi([0,T];L^2(\Omega))$ equipped with supremum norm $\|\cdot\|_\infty$ is a closed subset of $C([0,T];L^2(\Omega))$. For $u \in C_\xi([0,T];L^2(\Omega))$, we define $u[\xi] \in C([-q,T];L^2(\Omega))$ as follows

$$u[\xi](t) = \begin{cases} u(t) & \text{if } t \in [0,T] \\ \xi(t) & \text{if } t \in [-q,0]. \end{cases}$$

Hence

$$u[\xi]_\rho(t) = \begin{cases} u(t-\rho(t)) & \text{if } t-\rho(t) \in [0,T] \\ \xi(t-\rho(t)) & \text{if } t-\rho(t) \in [-q,0]. \end{cases}$$

Let $\mathcal{G} : C_\xi([0,T];L^2(\Omega)) \to C_\xi([0,T];L^2(\Omega))$ be the operator defined by

$$\mathcal{G}(u)(t) = \mathcal{S}_\alpha(t)\xi(0) + \int_0^t \mathcal{R}_\alpha(t-\tau) f\big(\tau, u[\xi]_\rho(\tau)\big) d\tau,$$

which will be referred to as *the solution operator*. This operator is continuous if f is a continuous map. Obviously, u is a fixed point of $\mathcal{G}$ iff $u[\xi]$ is a mild solution of (1.1)–(1.3).

The following proposition shows a Halanay type inequality which plays important role in our later analysis.

Proposition 2.4 *Let v be a continuous and nonnegative function satisfying*

$$v(t) \le s_\alpha(t, \frac{\mu}{1+\nu\mu})v_0 + \frac{1}{1+\nu\mu}\int_0^t r_\alpha(t-\tau, \frac{\mu}{1+\nu\mu})[a(\tau) + b \sup_{\theta\in[\tau-\rho(\tau),\tau]} v(\theta)]\, d\tau, t > 0, \tag{2.27}$$

$$v(s) = \psi(s), s \in [-q,0], \tag{2.28}$$

where $b \in \big(0,\mu\big)$, $\psi \in C([-q,0],\mathbb{R}^+)$ and $a \in L^1_{loc}(\mathbb{R}^+)$ which is nondecreasing. Then

$$v(t) \le \frac{\mu}{\mu-b}\Big[v_0 + \frac{1}{1+\nu\mu} r_\alpha(\cdot, \frac{\mu}{1+\nu\mu}) * a(t)\Big] + \frac{b}{\mu}\sup_{\theta\in[-q,0]}\psi(\theta), \forall t > 0. \tag{2.29}$$

In addition, if $r_\alpha(\cdot, \frac{\mu}{1+\nu\mu}) * a$ is bounded on $[0,\infty)$ and $\lim_{t\to\infty}(t-\rho(t)) = \infty$ then

$$\limsup_{t\to\infty} v(t) \le \frac{\mu}{\mu-b}\sup_{t\ge 0} r_\alpha(\cdot, \frac{\mu}{1+\nu\mu}) * a(t). \tag{2.30}$$

In particular, if $a = 0$ then $v(t) \to 0$ as $t \to \infty$.

Proof In order to prove this proposition, we need the following result [20, Lemma 2.3]: if $v \in C([-q,\infty);\mathbb{R}^+)$ is a nonnegative function satisfying

$$v(t) \le d(t) + \zeta \sup_{\theta\in[-q,t]} v(\theta), t > 0$$
$$v(s) = \psi(s), s \in [-q,0],$$

where $d(\cdot)$ is a nondecreasing function and $\zeta \in (0,1)$, then

$$v(t) \le (1-\zeta)^{-1}d(t) + \zeta \sup_{\theta\in[-q,0]} \psi(\theta), \forall t > 0. \tag{2.31}$$

Using Proposition 2.1, it follows from (2.27) that

$$\begin{aligned} v(t) &\le v_0 + \frac{1}{1+\nu\mu} r_\alpha\left(\cdot, \frac{\mu}{1+\nu\mu}\right) * a(t) + \frac{b}{1+\nu\mu} \sup_{\theta\in[-q,t]} v(\theta) \int_0^t r_\alpha\left(\tau, \frac{\mu}{1+\nu\mu}\right) d\tau \\ &= v_0 + \frac{1}{1+\nu\mu} r_\alpha\left(\cdot, \frac{\mu}{1+\nu\mu}\right) * a(t) + \frac{b}{\mu} \sup_{\theta\in[-q,t]} v(\theta) \left(1 - s_\alpha\left(t, \frac{\mu}{1+\nu\mu}\right)\right) \\ &\le v_0 + \frac{1}{1+\nu\mu} r_\alpha(\cdot,\mu) * a(t) + \frac{b}{\mu} \sup_{\theta\in[-q,t]} v(\theta). \end{aligned}$$

Since $a(\cdot)$ is nondecreasing, it implies that the function $t \mapsto v_0 + \frac{1}{1+\nu\mu} r_\alpha\left(\cdot, \frac{\mu}{1+\nu\mu}\right) * a(t)$ is nondecreasing as well. Using the inequality (2.31) with

$$d(t) = v_0 + \frac{1}{1+\nu\mu} r_\alpha\left(\cdot, \frac{\mu}{1+\nu\mu}\right) * a(t), \zeta = \frac{b}{\mu} < 1,$$

we get the inequality (2.29) as desired.

Now assume that $r_\alpha(\cdot, \frac{\mu}{1+\nu\mu}) * a$ is bounded on $[0,\infty)$. Then by (2.29), $v(\cdot)$ is bounded by

$$\bar{c} := \frac{\mu}{\mu - b}\left[v_0 + \frac{1}{1+\nu\mu} \sup_{t\ge 0} r_\alpha(\cdot,\mu) * a(t)\right] + \frac{b}{\mu} \sup_{\theta\in[-q,0]} \psi(\theta),$$

and thus the limit $\ell = \lim_{t\to\infty} \sup_{\zeta\in[t,\infty)} v(\zeta)$ exists. Due to $t - \rho(t) \to \infty$ as $t \to \infty$, then for any $\epsilon > 0$ one can find $T^* > 0$ such that

$$\sup_{\zeta\in[t-\rho(t),t]} v(\zeta) \le \sup_{\zeta\in[t-\rho(t),\infty]} v(\zeta) \le \ell + \epsilon, \forall t \ge T^*.$$

On the other hand, since for each $\mu > 0$, $s_\alpha(t, \frac{\mu}{1+\nu\mu}) \to 0$ as $t \to \infty$ and $r_\alpha(\cdot, \frac{\mu}{1+\nu\mu}) \in L^1(\mathbb{R}^+)$, then one can choose $t > T^*$ large enough such that

$$s_\alpha(t, \frac{\mu}{1+\nu\mu}) \le \epsilon, \int_{t-T^*}^t r_\alpha\left(\tau, \frac{\mu}{1+\nu\mu}\right) d\tau \le \epsilon.$$

From the these observations above and the inequality (2.27), for all $t > 0$, there holds that

$$
\begin{aligned}
v(t) &\le s_\alpha\left(t, \frac{\mu}{1+\nu\mu}\right) v_0 + \frac{1}{1+\nu\mu} r_\alpha(\cdot, \frac{\mu}{1+\nu\mu}) * a(t) \\
&\quad + \frac{b}{1+\nu\mu}\left(\int_0^{T^*} + \int_{T^*}^t\right) r_\alpha\left(t-\tau, \frac{\mu}{1+\nu\mu}\right) \sup_{\theta\in[\tau-\rho(\tau),\tau]} v(\theta) d\tau \\
&\le s_\alpha\left(t, \frac{\mu}{1+\nu\mu}\right) v_0 + \frac{1}{1+\nu\mu} r_\alpha\left(\cdot, \frac{\mu}{1+\nu\mu}\right) * a(t) \\
&\quad + \frac{b\bar{c}}{1+\nu\mu}\int_0^{T^*} r_\alpha\left(t-\tau, \frac{\mu}{1+\nu\mu}\right) ds + \frac{b(\ell+\epsilon)}{1+\nu\mu}\int_{T^*}^t r_\alpha\left(t-\tau, \frac{\mu}{1+\nu\mu}\right) ds \\
&\le \epsilon v_0 + \frac{1}{1+\nu\mu} r_\alpha\left(\cdot, \frac{\mu}{1+\nu\mu}\right) * a(t) \\
&\quad + \frac{b\bar{c}}{1+\nu\mu}\int_{t-T^*}^t r_\alpha\left(\tau, \frac{\mu}{1+\nu\mu}\right) ds + \frac{b(\ell+\epsilon)}{1+\nu\mu}\int_0^t r_\alpha\left(t-\tau, \frac{\mu}{1+\nu\mu}\right) ds \\
&\le \epsilon v_0 + \frac{1}{1+\nu\mu} r_\alpha\left(\cdot, \frac{\mu}{1+\nu\mu}\right) * a(t) + \frac{b\bar{c}\epsilon}{1+\nu\mu} + \frac{b(\ell+\epsilon)}{\mu}. \qquad (2.32)
\end{aligned}
$$

It follows from (2.32) that

$$
\ell = \lim_{t\to\infty} \sup_{\theta\in[t,\infty)} v(\theta) \le \frac{\ell b}{\mu} + \frac{1}{1+\nu\mu}\sup_{t\ge 0} r_\alpha\left(\cdot, \frac{\mu}{1+\mu}\right) * a(t) + \left(v_0 + \frac{b}{\mu}\right)\epsilon,
$$

which implies that

$$
\ell \le \frac{\mu}{\mu-b}\sup_{t\ge 0} r_\alpha\left(\cdot, \frac{\mu}{1+\nu\mu}\right) * a(t) + \frac{\mu}{\mu-b}\left(v_0 + \frac{b}{\mu}\right)\epsilon. \qquad (2.33)
$$

Since ϵ is an arbitrarily positive number, it follows from (2.33) that

$$
\limsup_{t\to\infty} v(t) \le \ell \le \frac{\mu}{\mu-b}\sup_{t\ge 0} r_\alpha\left(\cdot, \frac{\mu}{1+\nu\mu}\right) * a(t),
$$

from which we have the stated results in the lemma. □

We close this section by collecting some facts and basic results on measure of noncompactness, and fixed point theorem for condensing maps which are used to prove the existence of solutions in next sections.

Let E be a Banach space. Denote by $\mathcal{B}(E)$ the collection of nonempty bounded subsets of E. We will use the following definition of the measure of noncompactness (see, e.g. [11]).

Definition 2.2 A function $\psi : \mathcal{B}(E) \to \mathbb{R}^+$ is called a measure of noncompactness *(MNC)* on E if

$$\psi(\overline{\text{co}}\ D) = \psi(D) \text{ for every } D \in \mathcal{B}(E),$$

where $\overline{\text{co}}\ D$ is the closure of convex hull of D. An MNC ψ is said to be:

(i) monotone if for each $D_0, D_1 \in \mathcal{B}(E)$ such that $D_0 \subseteq D_1$, we have $\psi(D_0) \leq \psi(D_1)$;
(ii) nonsingular if $\psi(\{a\} \cup D) = \psi(D)$ for any $a \in E$, $D \in \mathcal{B}(E)$;
(iii) invariant with respect to the union with a compact set, if $\psi(K \cup D) = \psi(D)$ for every relatively compact set $K \subset E$ and $D \in \mathcal{B}(E)$;
(iv) algebraically semi-additive if $\psi(D_0 + D_1) \leq \psi(D_0) + \psi(D_1)$ for any $D_0, D_1 \in \mathcal{B}(E)$;
(v) regular if $\psi(D) = 0$ is equivalent to the relative compactness of D.

A typical example on MNC satisfying all properties stated in Definition 2.2 is the *Hausdorff* MNC $\chi(\cdot)$ defined by

$$\chi(D) = \inf\{\varepsilon > 0 : D \text{ has a finite } \varepsilon - \text{net}\}.$$

Definition 2.3 A continuous map $\mathcal{F} : Z \subseteq E \to E$ is said to be condensing with respect to an MNC ψ ($\psi-$condensing) if for any bounded set $D \subset Z$, the relation

$$\psi(D) \leq \psi(\mathcal{F}(D))$$

implies the relative compactness of D.

Let ψ be a monotone and nonsingular MNC in E. We have the following fixed point principle.

Theorem 2.5 [11, Corollary 3.3.1] *Let $\mathcal{M}$ be a bounded convex closed subset of E and let $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ be a ψ-condensing map. Then* $\text{Fix}(\mathcal{F}) := \{x \in E : x = \mathcal{F}(x)\}$ *is a nonempty and compact set.*

3 Solvability results

This section deals with the existence of global in time solutions to the problem (1.1)–(1.3) on a finite time interval $[-q, T]$ for every $T > 0$. Our first result about the global existence of solutions to problem (1.1)–(1.3) reads as the following.

Theorem 3.1 *Assume that the nonlinearity $f : [0, T] \times L^2(\Omega) \to L^2(\Omega)$ satisfies*

(F1) *f is continuous such that*

$$\|f(t, v)\| \leq \psi_f(\|v\|), \ \forall t \in [0, T], v \in L^2(\Omega),$$

where $\psi_f \in C(\mathbb{R}^+)$ is a nonnegative and nondecreasing function such that

$$\limsup_{r\to 0} \frac{\psi_f(r)}{r} < \lambda_1. \tag{3.1}$$

Then there exists $\delta > 0$ such that the problem (1.1)–(1.3) has a compact set of mild solutions on $[-q, T]$, provided $\|\xi\|_0 \le \delta$. Furthermore, if f obeys

- ***(F2)*** *$f(\cdot, 0) = 0$ and is locally Lipschitz continuous with respect to the second variable, i.e., for each $r > 0$, there exists a nonnegative constant $\kappa(r)$ such that*

$$\|f(t, v_1) - f(t, v_2)\| \le \kappa(r)\|v_1 - v_2\|, \tag{3.2}$$

for all $t \in [0, T]$, $v_i \in L^2(\Omega)$ with $\|v_i\| \le r$, $i \in \{1, 2\}$ and $\limsup_{r\to 0} \kappa(r) < \lambda_1$, then the mild solution to (1.1)–(1.3) is unique.

Proof To prove this theorem, we make use of Theorem 2.5. We first show that $\mathcal{G}(\mathsf{B}_\varrho) \subset \mathsf{B}_\varrho$ for some $\varrho > 0$, where B_ϱ be the closed ball in $C_\xi([0, T]; L^2(\Omega))$ centered at origin with radius ϱ. Let $\alpha_f = \limsup\limits_{r\to 0} \frac{\psi_f(r)}{r}$. Then by assumption (3.1), for $\epsilon \in (0, \lambda_1 - \ell)$, one can find $\varrho > 0$ such that

$$\frac{\psi_f(r)}{r} \le \alpha_f + \epsilon, \forall r \in (0, 2\varrho].$$

Choosing $\delta = \frac{(\lambda_1 - \alpha_f - \epsilon)\varrho}{\lambda_1 + \alpha_f + \epsilon}$, it is obvious that $0 < \delta \le \varrho$. Considering $\mathcal{G}$ on B_ϱ with $\|\xi\|_0 \le \delta$, one first sees that

$$\|u[\xi]_\rho(t)\| \le \|\xi\|_0 + \sup_{s\in[0,t]} \|u(s)\| \le \delta + \varrho, \forall t \ge 0.$$

Therefore, according to Lemma 2.3 and the formulas of $\mathcal{G}$, δ, we find that

$$\begin{aligned}
\|\mathcal{G}(u)(t)\| &\le s_\alpha(t, \theta_1)\|\xi(0)\| + \frac{1}{1+\nu\lambda_1}\int_0^t r_\alpha(t-\tau, \theta_1)\|f(\tau, u[\xi]_\rho(\tau))\|d\tau \\
&\le \|\xi(0)\| + \frac{1}{1+\nu\lambda_1}\int_0^t r_\alpha(t-\tau, \theta_1)\psi_f(\|u[\xi]_\rho(\tau)\|)d\tau \\
&\le \|\xi\|_0 + \frac{\alpha_f+\epsilon}{1+\nu\lambda_1}\int_0^t r_\alpha(t-\tau, \theta_1)\|u[\xi]_\rho(\tau)\|d\tau \\
&\le \delta + \frac{(\alpha_f+\epsilon)(\delta+\varrho)}{1+\nu\lambda_1}\int_0^t r_\alpha(t-\tau, \theta_1)d\tau \\
&\le \delta + \frac{(\alpha_f+\epsilon)(\delta+\varrho)}{1+\nu\lambda_1}\frac{1}{\theta_1} \\
&\le \varrho, \forall t \in [0, T],
\end{aligned} \tag{3.3}$$

thanks to the facts $s_\alpha(t,\mu) \le 1$, $\int_0^t r_\alpha(\tau,\mu)d\tau \le \mu^{-1}$, $\forall t \ge 0, \mu > 0$. The inequality (3.3) ensures that $\mathcal{G}(\mathsf{B}_\varrho) \subset \mathsf{B}_\varrho$, provided $\|\xi\|_0 \le \delta$. We now consider $\mathcal{G} : \mathsf{B}_\varrho \to \mathsf{B}_\varrho$. Since f is continuous, it is easily seen that $\mathcal{G}$ is continuous as well.

Now let χ be the Hausdorff MNC in $C([0,T]; L^2(\Omega))$. According to the decomposition

$$\begin{aligned}\mathcal{G}(u) &= \mathcal{S}_\alpha(\cdot)\xi(0) + \mathcal{Q}_\alpha \circ N_f(u),\\ N_f(u)(t) &= f\big(t, u[\xi]_\rho(t)\big),\end{aligned}$$

we get $\mathcal{G}$ is a compact operator, thanks to the compactness of $\mathcal{Q}_\alpha$ stated in Remark 2.1. Consequently, for any bounded set $D \subset \mathsf{B}_\varrho$, we get that $\mathcal{G}(D)$ is a relatively compact in $C([0,T]; L^2(\Omega))$. It follows that $\mathcal{G}$ is χ-condensing. Thus, the existence of global solutions of Theorem 3.1 follows by applying Theorem 2.5.

We now suppose that the nonlinearity f satisfies the locally Lipschitz condition (**F2**). In this case, the assumption (**F1**) is also fulfilled for $\psi_f(r) = r\kappa(r)$. Let us fix ϱ, δ, and ϵ as above. We now testify the uniqueness of solutions. Assume that $u, v \in C([-q,T]; L^2(\Omega))$ are two solutions of (1.1)–(1.3) with the initial condition ξ, then one can assume that $u, v \in \mathsf{B}_{\bar{R}}$ for some $\bar{R} > 0$. Due to the representation formula of u, v, one has

$$\begin{aligned}\|u(t) - v(t)\| &\le \int_0^t r_\alpha(t-\tau,\theta_1)\kappa(\bar{R})\|u[\xi]_\rho(\tau) - v[\xi]_\rho(\tau)\|d\tau\\ &\le \kappa(\bar{R})\int_0^t r_\alpha(t-\tau,\theta_1)\sup_{\theta\in[0,\tau]}\|u(\theta)-v(\theta)\|d\tau,\ \forall t\in[0,T],\end{aligned}$$

thanks to the fact that $u(t) = v(t) = \xi(t), \forall t \in [-q,0]$. Since the last inequality is nondecreasing in t, we thus obtain

$$\begin{aligned}\sup_{\tau\in[0,t]}\|u(\tau)-v(\tau)\| &\le \kappa(\bar{R})\int_0^t r_\alpha(t-\tau,\theta_1)\sup_{\theta\in[0,\tau]}\|u(\theta)-v(\theta)\|d\tau\\ &\le \frac{\kappa(\bar{R})}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\sup_{\theta\in[0,\tau]}\|u(\theta)-v(\theta)\|d\tau,\end{aligned}$$

thanks to Proposition 2.1(iii). Utilizing the Gronwall type inequality [10, Theorem 4.3, p. 99], we get $\sup\limits_{\tau\in[0,t]}\|u(\tau)-v(\tau)\| = 0$ for all $t \in [0,T]$, which implies $u = v$. The proof is complete. □

In the next theorem, we will show that the smallness condition on initial data and coefficients of f can be relaxed if the nonlinear function f fulfills a sublinear growth or a global Lipschitz condition. More precisely, we prove

Theorem 3.2 *Assume that f satisfies the condition*

*(**F3**) f is continuous such that $\|f(t,v)\| \le a(t) + b(t)\|v\|$, for all $t \in [0,T]$, $v \in L^2(\Omega)$, where $a \in L^1_{loc}(\mathbb{R}^+)$, $b \in L^\infty(0,T)$ are nonnegative functions.*

Then the problem (1.1)–(1.3) *has a compact set of mild solutions on* $[-q, T]$. *Moreover, if the nonlinearity function* f *satisfies*

(F4) $\|f(t, v_1)-f(t, v_2)\| \le c(t)\|v_1-v_2\|$, *for all* $t \in [0, T]$, $v_i \in L^2(\Omega)$, $i \in \{1, 2\}$, *where* $c \in L^1_{loc}(\mathbb{R}^+)$ *is a nonnegative function,*

then the problem (1.1)–(1.3) *has a unique mild solution on* $[-q, T]$.

Proof Assume that the assumption (**F**3) holds. Let

$$\mathsf{D} = \{u \in C_\xi([0, T]; L^2(\Omega)) : \sup_{\tau\in[0,t]} \|u(\tau)\| \le \vartheta(t),\ \forall t \in [0, T]\},$$

where ϑ is the unique solution of the integral equation

$$\vartheta(t) = \big(1 + \lambda_1^{-1}\|b\|_{L^\infty(0,T)}\big)\|\xi\|_0 + \frac{1}{1+\nu\lambda_1}\sup_{t\in[0,T]} r_\alpha(\cdot, \theta_1) * a(t) + \frac{\|a\|_{L^\infty(0,T)}}{1+\nu\lambda_1}\int_0^t r_\alpha(t-\tau, \theta_1)\vartheta(\tau)d\tau,\ t \in [0, T].$$

Then D is a closed, bounded and convex set in $C_\xi([0, T]; L^2(\Omega))$. Considering the solution operator $\mathcal{G}$ on D, we see that

$$\begin{aligned}
\|\mathcal{G}(u)(t)\| &\le \|\mathcal{S}_\alpha(t)\xi(0)\| + \int_0^t \|\mathcal{R}_\alpha(t-\tau)\|_{op}\|f(\tau, u[\xi]_\tau)\|d\tau \\
&\le s_\alpha(t, \theta_1)\|\xi(0)\| + \frac{1}{1+\nu\lambda_1}\int_0^t r_\alpha(t-\tau, \theta_1)[a(\tau)+b(\tau)\|u[\xi]_\rho(\tau)\|]d\tau \\
&\le s_\alpha(t, \theta_1)\|\xi\|_0 \\
&\quad + \frac{1}{1+\nu\lambda_1}\int_0^t r_\alpha(t-\tau, \theta_1)\big[a(\tau) + b(\tau)\big(\|\xi\|_0 + \sup_{\theta\in[0,\tau]}\|u(\theta)\|\big)\big]d\tau \\
&\le \|\xi\|_0 + \frac{1}{1+\nu\lambda_1}\int_0^t r_\alpha(t-\tau, \theta_1)a(\tau)d\tau + \frac{\|b\|_{L^\infty(0,T)}\|\xi\|_0}{1+\nu\lambda_1}\int_0^t r_\alpha(\tau, \theta_1)d\tau \\
&\quad + \frac{\|b\|_{L^\infty(0,T)}}{1+\nu\lambda_1}\int_0^t r_\alpha(t-\tau, \theta_1)\sup_{\theta\in[0,\tau]}\|u(\theta)\|d\tau \\
&\le \|\xi\|_0 + \frac{\|b\|_{L^\infty(0,T)}\|\xi\|_0}{\lambda_1} + \frac{1}{1+\nu\lambda_1}\sup_{t\in[0,T]} r_\alpha(\cdot, \theta_1) * a(t) \\
&\quad + \frac{\|b\|_{L^\infty(0,T)}}{1+\nu\lambda_1}\int_0^t r_\alpha(t-\tau, \theta_1)\sup_{\theta\in[0,\tau]}\|u(\theta)\|d\tau \\
&= \big(1 + \lambda_1^{-1}\|b\|_{L^\infty(0,T)}\big)\|\xi\|_0 + \frac{1}{1+\nu\lambda_1}\sup_{t\in[0,T]} r_\alpha(\cdot, \theta_1) * a(t) \\
&\quad + \frac{\|b\|_{L^\infty(0,T)}}{1+\nu\lambda_1}\int_0^t r_\alpha(t-\tau, \theta_1)\sup_{\theta\in[0,\tau]}\|u(\theta)\|d\tau, \qquad (3.4)
\end{aligned}$$

for any $u \in \mathsf{D}$, thanks to Lemma 2.3. Since the function $t \mapsto \sup_{\tau\in[0,t]} \|u(\tau)\|$ is nondecreasing, the integral term in the right hand side of (3.4) is nondecreasing in t as well. Thus

$$\begin{aligned}\sup_{\tau\in[0,t]} \|\mathcal{G}(u)(\tau)\| &\le \left(1+\lambda_1^{-1}\|b\|_{L^\infty(0,T)}\right)\|\xi\|_0 + \frac{1}{1+\nu\lambda_1}\sup_{t\in[0,T]} r_\alpha(\cdot,\theta_1)*a(t)\\ &\quad + \frac{\|b\|_{L^\infty(0,T)}}{1+\nu\lambda_1}\int_0^t r_\alpha(t-\tau,\theta_1)\sup_{\theta\in[0,\tau]}\|u(\theta)\|d\tau\\ &\le \left(1+\lambda_1^{-1}\|b\|_{L^\infty(0,T)}\right)\|\xi\|_0 + \frac{1}{1+\nu\lambda_1}\sup_{t\in[0,T]} r_\alpha(\cdot,\theta_1)*a(t)\\ &\quad + \frac{\|b\|_{L^\infty(0,T)}}{1+\nu\lambda_1}\int_0^t r_\alpha(t-\tau,\theta_1)\vartheta(\tau)d\tau,\ \text{ for all } t\in[0,T]. \end{aligned} \tag{3.5}$$

From the inequality (3.5) yields $\mathcal{G}(\mathsf{D}) \subset \mathsf{D}$. The existence of global mild solutions is followed by using the same lines as the proof of Theorem 3.1 for the case of (**F1**). Besides, in the case of (**F4**), one can consider a suitable weighted norm in $C_\xi([0,T];L^2(\Omega))$ and prove that the solution operator $\mathcal{G}$ is a contraction operator. Thus, the proof of the theorem is now complete. □

4 Long time behavior of solutions

This section is devoted to analyzing the long time behavior of solutions to the problem (1.1)–(1.3). Firstly, by using the Halanay type inequality, we establish results about stability and dissipativity of solutions, whose definitions are given in the following definitions.

Definition 4.1 Let $u(\cdot,\xi) \in C([-q,\infty);L^2(\Omega))$ be the solution of the problem (1.1)–(1.3) with the initial datum ξ. The solution $u(\cdot,\xi)$ is said to be asymptotic stable if it is the following:

(i) stable: for all $\epsilon > 0$ there exists $\delta > 0$ such that if $\psi \in C([-q,0];L^2(\Omega))$ is obeying $\|\xi-\psi\|_0 < \delta$, then $\|u_t - v_t\|_\infty < \epsilon$, for all $t > 0$.
(ii) attractive: there exists $r > 0$ such that for every $\psi \in C([-q,0];L^2(\Omega))$ satisfying $\|\xi-\psi\|_0 < r$ then $\lim_{t\to\infty}\|u_t - v_t\|_\infty = 0$.

Definition 4.2 The problem (1.1)–(1.3) is said to be dissipativity with an absorbing set B_R if one can find a positive constant R: such that for each $\xi \in C([-q,0];L^2(\Omega))$ there exists $T > 0$ such that the solution $u(\cdot,\xi)$ satisfying

$$\|u_t\|_\infty \le R,\ \text{ for all } t > T.$$

We are now ready to present results about stability and dissipativity of solutions of our problem.

Theorem 4.1 *Assume that the hypothesis (**F2**) of Theorem 3.1 holds for any $T > 0$. Then the zero solution of (1.1) is asymptotically stable.*

Proof Take ϱ, δ, and ϵ as in the proof of Theorem 3.1. Then for every $\|\xi\|_0 \le \delta$, there exists a unique mild solution to (1.1)–(1.3) such that $\|u(t)\| \le \varrho$ for all $t \ge 0$. Note that

$$\|u[\xi]_\rho(t)\| \le \|\xi\|_0 + \|u\|_\infty \le 2\varrho, \text{ for all } t \in [0, T],$$

and it holds that

$$\begin{aligned}\|u(t)\| &\le \|\mathcal{S}_\alpha(t)\xi(0)\| + \int_0^t \|\mathcal{R}_\alpha(t-\tau)\|_{op}\|f(\tau, u[\xi]_\rho(\tau)) - f(\tau, 0)\|d\tau \\ &\le s_\alpha(t,\theta_1)\|\xi(0)\| + \frac{1}{1+\nu\lambda_1}\int_0^t r_\alpha(t-\tau,\theta_1)\kappa(2\varrho)\|u[\xi]_\rho(\tau)\|d\tau \\ &\le s_\alpha(t,\theta_1)\|\xi(0)\| + \frac{1}{1+\nu\lambda_1}\int_0^t r_\alpha(t-\tau,\theta_1)(\alpha_f+\epsilon)\sup_{\theta\in[\tau-\rho(\tau),\tau]}\|u(\theta)\|d\tau.\end{aligned}$$

Employing the Halanay type inequality in Proposition 2.4 with $v(t) = \|u(t)\|, t \ge -q$, $\mu = \lambda_1$, we obtain

$$\|u(t)\| \le \frac{\lambda_1}{\lambda_1 - \alpha_f - \epsilon}\|\xi(0)\| + \frac{\alpha_f + \epsilon}{\lambda_1}\|\xi\|_0, \forall t \ge 0, \tag{4.1}$$

$$\lim_{t\to\infty}\|u(t)\| = 0. \tag{4.2}$$

The inequalities (4.1), (4.2) guarantee the stability and attractivity of the zero solution, respectively. We thus finish the proof of this theorem. □

Considering the case when f is globally Lipschitzian, we have a stronger result.

Theorem 4.2 *Assume that the hypothesis **(F4)** holds for any $T > 0$ and for $c \in L^\infty(\mathbb{R}^+;\mathbb{R}^+)$. If $\|c\|_{L^\infty(\mathbb{R}^+)} < \lambda_1$, then every mild solution of* (1.1)–(1.3) *is asymptotically stable.*

Proof Let u and v be solutions of (1.1)–(1.3). Then, by the formula of solutions and by Lemma 2.3, we get that

$$\begin{aligned}\|u(t) - v(t)\| &\le \|\mathcal{S}_\alpha(t)[u(0) - v(0)]\| \\ &\quad + \frac{1}{1+\nu\lambda_1}\int_0^t \mathcal{R}_\alpha(t-\tau)\|_{op}\|f(\tau, u[\xi]_\rho(\tau)) - f(\tau, v[\xi]_\rho(\tau))\|d\tau \\ &\le s_\alpha(t,\theta_1)\|u(0) - v(0)\| \\ &\quad + \frac{1}{1+\nu\lambda_1}\int_0^t r_\alpha(t-\tau,\theta_1)c(\tau)\|u[\xi]_\rho(\tau) - v[\xi]_\rho(\tau)\|d\tau \\ &\le s_\alpha(t,\theta_1)\|u(0) - v(0)\| \\ &\quad + \frac{1}{1+\nu\lambda_1}\int_0^t r_\alpha(t-\tau,\theta_1)\|c\|_{L^\infty(\mathbb{R}^+)}\sup_{[\tau-\rho(\tau),\tau]}\|u(\theta) - v(\theta)\|d\tau.\end{aligned}$$

Applying Proposition 2.4 leads to

$$\|u(t)-v(t)\| \leq \frac{\lambda_1\|u(0)-v(0)\|}{\lambda_1-\|c\|_{L^\infty(\mathbb{R}^+)}} + \frac{\|c\|_{L^\infty(\mathbb{R}^+)}}{\lambda_1}\|u(0)-v(0)\|_0, \forall t \geq 0,$$
$$\lim_{t\to\infty}\|u(t)-v(t)\| = 0,$$

from which we obtain the conclusion of this theorem. □

In the next theorem, we establish a result on the dissipativity of solutions of our system.

Theorem 4.3 *Let the assumption (**F3**) of Theorem 3.2 hold for any $T > 0$ with $b \in L^\infty(\mathbb{R}^+;\mathbb{R}^+)$ satisfying $\|b\|_{L^\infty(\mathbb{R}^+)} < \lambda_1$ and $a \in L^1_{loc}(\mathbb{R}^+;\mathbb{R}^+)$ is nondecreasing such that $r_\alpha(\cdot,\theta_1) * a$ is a bounded function on $\mathbb{R}^+$. Then there exists an absorbing set for solutions of (1.1)–(1.3) with arbitrary initial data. Moreover, if $a = 0$, then the zero solution of (1.1) is asymptotically stable.*

Proof Let u be a solution of (1.1)–(1.3). Using Lemma 2.3 and the estimate of f, we obtain

$$\|u(t)\| \leq s_\alpha(t,\theta_1)\|\xi(0)\| + \frac{1}{1+\nu\lambda_1}\int_0^t r_\alpha(t-\tau,\theta_1)[a(\tau)+b(\tau)\|u[\xi]_\rho(\tau)\|]d\tau$$
$$\leq s_\alpha(t,\theta_1)\|\xi(0)\| + \frac{1}{1+\nu\lambda_1}\int_0^t r_\alpha(t-\tau,\theta_1)[a(\tau)+\|b\|_{L^\infty(\mathbb{R}^+)}\|u[\xi]_\rho(\tau)\|]\,d\tau.$$

Using Proposition 2.4 again, we arrive at

$$\limsup_{t\to\infty}\|u(t)\| \leq \frac{\lambda_1}{\lambda_1-\|b\|_{L^\infty(\mathbb{R}^+)}}\sup_{t\geq 0} r_\alpha(\cdot,\theta_1) * a(t).$$

Put

$$R = \epsilon + \frac{\lambda_1}{\lambda_1-\|b\|_{L^\infty(\mathbb{R}^+)}}\sup_{t\geq 0} r_\alpha(\cdot,\theta_1) * a(t)$$

for some $\epsilon > 0$, then the ball B_R is an absorbing set for solutions of (1.1)–(1.3). Finally, if $a = 0$, then (1.1) admits the zero solution and it holds that

$$\|u(t)\| \leq \frac{\lambda_1}{\lambda_1-\|b\|_{L^\infty(\mathbb{R}^+)}}\|\xi(0)\| + \frac{\|b\|_{L^\infty(\mathbb{R}^+)}}{\lambda_1}\|\xi\|_0, \forall t \geq 0,$$
$$\lim_{t\to\infty}\|u(t)\| = 0,$$

thanks to Proposition 2.4 again, which ensures the asymptotic stability of the zero solution. □

Remark 4.1 Results about the asymptotic stability of solutions to the problem (1.1)–(1.3) obtained in Theorems 4.2 and 4.3 under the assumption that the coefficient of the nonlinearity f is small. Our approach is based on the Halanay type inequality. It is interesting to ask that are these results still valid or invalid in the different cases (that is, $\|c\|_{L^\infty(\mathbb{R}^+)} \geq \lambda_1$ in Theorem 4.2 and $\|b\|_{L^\infty(\mathbb{R}^+)} \geq \lambda_1$ in Theorem 4.3)? These issues will be a topic of future works.

In the remainder of this section, we deal with the existence of decay solutions to (1.1)–(1.3). In particular, we assume that, the nonlinearity f is non-Lipschitzian and possibly superlinear. More precisely,

(**F**5) $f : \mathbb{R}^+ \times L^2(\Omega) \to L^2(\Omega)$ is a continuous mapping such that

$$\|f(t, v)\| \leq p(t)\Phi(\|v\|),\ \forall t \in \mathbb{R}^+, v \in L^2(\Omega),$$

where $p \in L^1_{loc}(\mathbb{R}^+)$ is a nonnegative function and $\Phi \in C(\mathbb{R}^+)$ is a nonnegative and nondecreasing function such that

$$\limsup_{r\to 0} \frac{\Phi(r)}{r} \cdot \sup_{t\geq 0} \int_0^t r_\alpha(t-\tau, \theta_1)p(\tau)d\tau < 1 + \nu\lambda_1, \tag{4.3}$$

and

$$\lim_{T\to\infty} \sup_{t\geq T} \int_0^{\beta t} r_\alpha(t-\tau, \theta_1)p(\tau)d\tau = 0. \tag{4.4}$$

for some $\beta \in (0, 1)$.

We start with considering the solution operator $\mathcal{G}$ on $BC_0(\mathbb{R}^+; L^2(\Omega))$, the space of continuous functions on $\mathbb{R}^+$, taking values in $L^2(\Omega)$ and decaying as $t \to \infty$. Given $\xi \in C([-q, 0]; L^2(\Omega))$, put $\mathcal{BC}_0^\xi = \{u \in BC_0(\mathbb{R}^+; L^2(\Omega)) : u(0) = \xi(0)\}$. Then $\mathcal{BC}_0^\xi$ is a closed subset of $BC_0(\mathbb{R}^+; L^2(\Omega))$ furnished by the sup norm $\|\cdot\|_\infty$.

Let D be a bounded set in $\mathcal{BC}_0^\xi$ and $\pi_T : \mathcal{BC}_0^\xi \to C([0, T]; L^2(\Omega))$ the restriction operator on $\mathcal{BC}_0^\xi$, i.e. $\pi_T(u)$ is the restriction of $u \in \mathcal{BC}_0^\xi$ to the interval $[0, T]$. Define

$$d_\infty(D) = \lim_{T\to\infty} \sup_{u\in D} \sup_{t\geq T} \|u(t)\|,$$
$$\chi_\infty(D) = \sup_{T>0} \chi_T(\pi_T(D)),$$

where $\chi_T(\cdot)$ is the Hausdorff MNC in $C([0, T]; L^2(\Omega))$. Then the following MNC defined in [1],

$$\chi^*(D) = d_\infty(D) + \chi_\infty(D), \tag{4.5}$$

possesses all properties stated in Definition 2.2. In addition, if $\chi^*(D) = 0$ then D is relatively compact in $BC_0(\mathbb{R}^+; L^2(\Omega))$. Especially, $d_\infty(\{u\}) = 0$ iff $u \in BC_0(\mathbb{R}^+; L^2(\Omega))$.

Lemma 4.4 *Let (**F**5) hold. Then there exist positive numbers δ and R such that $\mathcal{G}(\mathsf{B}_R) \subset \mathsf{B}_R$ for $\|\xi\|_0 \le \delta$, where B_R is the closed ball in $\mathcal{BC}_0^\xi$ centered at origin with radius R.*

Proof Let

$$\alpha_0 = \limsup_{r\to 0} \frac{\Phi(r)}{r}, \mathsf{M}_\infty = \sup_{t\ge 0} \int_0^t r_\alpha(t-\tau,\theta_1)p(\tau)d\tau.$$

By (4.3), one can take $\eta > 0$ such that

$$(\alpha_0 + \eta)\mathsf{M}_\infty < 1 + \nu\lambda_1, \tag{4.6}$$

and we can find $R > 0$ satisfying $\frac{\Phi(r)}{r} \le \alpha_0 + \eta$ for all $r \in (0, 2R]$ thanks to the definition of lim sup. Recalling that the solution operator $\mathcal{G}$ is defined by

$$\mathcal{G}(u)(t) = \mathcal{S}_\alpha(t)\xi(0) + \int_0^t \mathcal{R}_\alpha(t-\tau) f(\tau, u[\xi]_\rho(\tau))d\tau, u \in \mathcal{BC}_0^\xi.$$

Considering $\mathcal{G}$ on B_R with $\|\xi\|_0 \le R$, we have

$$\|\mathcal{G}(u)(t)\| \le s_\alpha(t,\theta_1)\|\xi(0)\| + \frac{1}{1+\nu\lambda_1}\int_0^t r_\alpha(t-\tau,\theta_1)p(\tau)\Phi\big(\|u[\xi]_\rho(\tau)\|\big)d\tau. \tag{4.7}$$

We first check that $\mathcal{BC}_0^\xi$ is invariant under $\mathcal{G}$, i.e., $\mathcal{G}\big(\mathcal{BC}_0^\xi\big) \subset \mathcal{BC}_0^\xi$. In view of (4.7) and the fact $s_\alpha(t,\theta_1)\|\xi(0)\| \to 0$ as $t \to \infty$, we need to show that

$$H(t) = \int_0^t r_\alpha(t-\tau,\theta_1)p(\tau)\Phi\big(\|u[\xi]_\rho(\tau)\|\big)d\tau \to 0 \text{ as } t \to \infty.$$

Since $t - \rho(t) \to \infty$ as $t \to \infty$, we have both $\|u(t)\|$ and $\|u[\xi]_\rho(t)\|$ go to zero as $t \to \infty$. This means that, for any $\epsilon > 0$, there exists $T > 0$ such that $\Phi\big(\|u[\xi]_\rho(t)\|\big) \le \epsilon$ for all $t \ge T$, thanks to the fact that Φ is continuous and $\Phi(0) = 0$. Therefore, for $t > T$, we find that

$$\begin{aligned}
H(t) &= \left(\int_0^T + \int_T^t\right) r_\alpha(t-\tau,\theta_1)p(\tau)\Phi\big(\|u[\xi]_\rho(\tau)\|\big)d\tau \\
&\le \Phi(2R)\int_0^T r_\alpha(t-\tau,\theta_1)p(\tau)d\tau + \epsilon\int_T^t r_\alpha(t-\tau,\theta_1)p(\tau)d\tau \\
&\le \Phi(2R) r_\alpha(t-T,\theta_1)\int_0^T p(\tau)d\tau + \epsilon\mathsf{M}_\infty \\
&\le [\Phi(2R) + \mathsf{M}_\infty]\epsilon,
\end{aligned}$$

because of the fact that $r_\alpha(t-T,\theta_1)\int_0^T p(\tau)d\tau \to 0$ as $t \to \infty$ (see, e.g. [5, Proposition 2.4(i)]). We have shown that $\mathcal{G}(u) \in \mathcal{BC}_0^\xi$, provided $u \in \mathcal{BC}_0^\xi$.

Now let

$$\delta = \frac{R}{1+\nu\lambda_1+(\alpha_0+\epsilon)\mathsf{M}_\infty} \inf_{t\geq 0}\Big(1+\nu\lambda_1-(\alpha_0+\eta)\int_0^t r_\alpha(t-\tau,\theta_1)p(\tau)d\tau\Big), \tag{4.8}$$

then $\delta > 0$ and $\delta \leq R$ due to (4.6). We now show that $\mathcal{G}(u) \in \mathsf{B}_R$, provided $u \in \mathsf{B}_R$ and $\|\xi\|_0 \leq \delta$. Indeed, by the formula of $\mathcal{G}$ and by Lemma 2.3, we have

$$\begin{aligned}
\|\mathcal{G}(u)(t)\| &\leq s_\alpha(t,\theta_1)\|\xi(0)\| + \frac{1}{1+\nu\lambda_1}\int_0^t r_\alpha(t-\tau,\theta_1)p(\tau)\Phi\big(\|u[\xi]_\rho(\tau)\|\big)d\tau \\
&\leq \|\xi\|_0 + \frac{1}{1+\nu\lambda_1}\int_0^t r_\alpha(t-\tau,\theta_1)p(\tau)(\alpha_0+\eta)\|u[\xi]_\rho(\tau)\|d\tau \\
&\leq \|\xi\|_0 + \frac{1}{1+\nu\lambda_1}\int_0^t r_\alpha(t-\tau,\theta_1)p(\tau)(\alpha_0+\eta)\big(\|\xi\|_0 + \sup_{\theta\in[0,\tau]}\|u(\theta)\|\big)d\tau \\
&\leq \delta + \frac{(\alpha_0+\eta)(\delta+R)}{1+\nu\lambda_1}\int_0^t r_\alpha(t-\tau,\theta_1)p(\tau)d\tau \\
&\leq \left[1+\frac{(\alpha_0+\epsilon)\mathsf{M}_\infty}{1+\nu\lambda_1}\right]\delta + \frac{(\alpha_0+\eta)R}{1+\nu\lambda_1}\int_0^t r_\alpha(t-\tau,\theta_1)p(\tau)d\tau \\
&\leq R, \forall t \geq 0,
\end{aligned}$$

thanks to the formulation of δ given by (4.8). The proof is complete. □

The following theorem gives a result on the existence of decay solutions to the problem (1.1)–(1.3).

Theorem 4.5 *Let the hypothesis (**F**5) hold. Then there exists $\delta > 0$ such that the problem* (1.1)–(1.3) *has a compact set of decay solutions, provided that $\|\xi\|_0 \leq \delta$.*

Proof Taking δ and B_R from Lemma 4.4, we consider the solution map $\mathcal{G} : \mathsf{B}_R \to \mathsf{B}_R$. By standard reasoning, we get that $\mathcal{G}$ is continuous. We will show that $\mathcal{G}$ is χ^*-condensing. Using the same arguments as in the proof of Theorem 3.1, one has $\pi_T \circ \mathcal{G}$ is compact. This implies $\chi_T(\pi_T(\mathcal{G}(D))) = 0$ for $D \subset \mathsf{B}_R$ and then $\chi_\infty(\mathcal{G}(D)) = 0$. We are now in a position to estimate $d_\infty(\mathcal{G}(D))$.

Let $z \in \mathcal{G}(D)$ and $u \in D$ be such that $z = \mathcal{G}(u)$. Since $t-\rho(t) \to \infty$ as $t \to \infty$, for given $T > 0$, there exists $T_1 > T$ such that $t-\rho(t) \geq T$ for all $t \geq T_1$. Thus, for $t > \beta^{-1}T_1$, we find that

$$\begin{aligned}
\|z(t)\| &\leq s_\alpha(t,\theta_1)\|\xi(0)\| + \frac{1}{1+\nu\lambda_1}\int_0^t r_\alpha(t-\tau,\theta_1)p(\tau)\Phi(\|u[\xi]_\rho(\tau)\|)d\tau \\
&\leq s_\alpha(t,\theta_1)\|\xi(0)\| + \frac{(\alpha_0+\eta)}{1+\nu\lambda_1}\int_0^t r_\alpha(t-\tau,\theta_1)p(\tau)\|u[\xi]_\rho(\tau)\|d\tau
\end{aligned}$$

$$
\begin{aligned}
&= s_\alpha(t,\theta_1)\|\xi(0)\| + \frac{(\alpha_0+\eta)}{1+\nu\lambda_1}\left(\int_0^{\beta t} + \int_{\beta t}^{t}\right) r_\alpha(t-\tau,\theta_1)p(\tau)\|u[\xi]_\rho(\tau)\|\,d\tau \\
&\le s_\alpha(t,\theta_1)\|\xi(0)\| + \frac{(\alpha_0+\eta)(\delta+R)}{1+\nu\lambda_1}\int_0^{\beta t} r_\alpha(t-\tau,\theta_1)p(\tau)d\tau \\
&\quad + \frac{(\alpha_0+\eta)}{1+\nu\lambda_1}\sup_{\theta\ge\beta t}\|u(\theta-\rho(\theta))\|\int_{\beta t}^{t} r_\alpha(t-\tau,\theta_1)p(\tau)d\tau \\
&\le s_\alpha(t,\theta_1)\|\xi(0)\| + \frac{(\alpha_0+\eta)(\delta+R)}{1+\nu\lambda_1}\int_0^{\beta t} r_\alpha(t-\tau,\theta_1)p(\tau)d\tau \\
&\quad + \frac{(\alpha_0+\eta)}{1+\nu\lambda_1}\sup_{u\in D}\sup_{s\ge T}\|u(s)\|\int_0^{t} r_\alpha(t-\tau,\theta_1)p(\tau)d\tau.
\end{aligned}
$$

Since $z \in \mathcal{G}(D)$ is taken arbitrarily, we thus obtain

$$
\begin{aligned}
\sup_{z\in\mathcal{G}(D)}\sup_{t\ge\beta^{-1}T_1}\|z(t)\| &\le \sup_{t\ge\beta^{-1}T_1} s_\alpha(t,\theta_1)\|\xi(0)\| \\
&\quad + \frac{(\alpha_0+\eta)(\delta+R)}{1+\nu\lambda_1}\sup_{t\ge\mu^{-1}T_1}\int_0^{\beta t} r_\alpha(t-\tau,\theta_1)p(\tau)d\tau \\
&\quad + \frac{(\alpha_0+\eta)}{1+\nu\lambda_1}\sup_{u\in D}\sup_{s\ge T}\|u(s)\|\sup_{t\ge\mu^{-1}T_1}\int_0^{t} r_\alpha(t-\tau,\theta_1)p(\tau)d\tau \\
&\le \sup_{t\ge T} s_\alpha(t,\theta_1)\|\xi(0)\| \\
&\quad + \frac{(\alpha_0+\eta)(\delta+R)}{1+\nu\lambda_1}\sup_{t\ge T}\int_0^{\beta t} r_\alpha(t-\tau,\theta_1)p(\tau)d\tau \\
&\quad + \frac{(\alpha_0+\eta)\mathsf{M}_\infty}{1+\nu\lambda_1}\sup_{u\in D}\sup_{s\ge T}\|u(s)\|.
\end{aligned}
$$

Letting $T \to \infty$ then $\mu^{-1}T_1 \to \infty$ and using the fact that

$$
\sup_{t\ge T} s_\alpha(t,\theta_1)\|\xi(0)\| = s_\alpha(T,\theta_1)\|\xi(0)\| \to 0 \text{ as } T \to \infty,
$$

we find that

$$
d_\infty(\mathcal{G}(D)) \le \frac{(\alpha_0+\eta)\mathsf{M}_\infty}{1+\nu\lambda_1} d_\infty(D),
$$

thanks to (4.4). Therefore,

$$
\begin{aligned}
\chi^*(\mathcal{G}(D)) = \chi_\infty(\mathcal{G}(D)) + d_\infty(\mathcal{G}(D)) = d_\infty(\mathcal{G}(D)) &\le \frac{(\alpha_0+\eta)\mathsf{M}_\infty}{1+\nu\lambda_1} d_\infty(D) \\
&\le \frac{(\alpha_0+\eta)\mathsf{M}_\infty}{1+\nu\lambda_1}[d_\infty(D)+\chi_\infty(D)] = \frac{(\alpha_0+\eta)\mathsf{M}_\infty}{1+\nu\lambda_1}\chi^*(D).
\end{aligned}
$$

Now if $\chi^*(D) \le \chi^*(\mathcal{G}(D))$ then $\chi^*(D) \le \frac{(\alpha_0+\eta)\mathsf{M}_\infty}{1+\nu\lambda_1}\chi^*(D)$ which implies $\chi^*(D) = 0$, thanks to (4.6). Thus $\mathcal{G}$ is χ^*-condensing and it admits a fixed point, according to Theorem 2.5. Denote by $\mathcal{D}$ the fixed point set of $\mathcal{G}$ in B_R. Then $\mathcal{D}$ is closed and $\mathcal{D} \subset \mathcal{G}(\mathcal{D})$. Hence,

$$\chi^*(\mathcal{D}) \le \chi^*(\mathcal{G}(\mathcal{D})) \le \frac{(\alpha_0+\eta)\mathsf{M}_\infty}{1+\nu\lambda_1}\chi^*(\mathcal{D}),$$

which ensures $\chi^*(\mathcal{D}) = 0$ and $\mathcal{D}$ is a compact set. The proof is complete. □

Remark 4.2 It should be remarked that when $p \in L^\infty(\mathbb{R}^+;\mathbb{R}^+)$ then the conditions (4.3)–(4.4) can be simplified. In this case, it is testified that

$$\begin{aligned}\sup_{t\ge 0}\int_0^t r_\alpha(t-\tau,\theta_1)p(\tau)d\tau &\le \|p\|_{L^\infty(\mathbb{R}^+)}\sup_{t\ge 0}\int_0^t r_\alpha(t-\tau,\theta_1)d\tau\\ &\le \|p\|_{L^\infty(\mathbb{R}^+)}\theta_1^{-1}\sup_{t\ge 0}(1-s_\alpha(t,\theta_1))\\ &= \|p\|_{L^\infty(\mathbb{R}^+)}\theta_1^{-1}.\end{aligned}$$

Therefore condition (4.3) is replaced by

$$\|p\|_{L^\infty(\mathbb{R}^+)}\limsup_{r\to 0}\frac{\Phi(r)}{r} < \lambda_1.$$

On the other hand, we observe that

$$\begin{aligned}\int_0^{\beta t} r_\alpha(t-\tau,\theta_1)p(\tau)d\tau &\le \|p\|_{L^\infty(\mathbb{R}^+)}\int_0^{\beta t} r_\alpha(t-\tau,\theta_1)d\tau\\ &= \|p\|_{L^\infty(\mathbb{R}^+)}\int_{(1-\beta)t}^{t} r_\alpha(\tau,\theta_1)d\tau.\end{aligned}$$

Then

$$\begin{aligned}\sup_{t\ge T}\int_0^{\beta t} r_\alpha(t-\tau,\theta_1)p(\tau)d\tau &\le \|p\|_{L^\infty(\mathbb{R}^+)}\int_{(1-\beta)T}^{\infty} r_\alpha(\tau,\theta_1)d\tau\\ &\to 0 \text{ as } T\to\infty,\end{aligned}$$

due to the fact that $r_\alpha(\cdot,\theta_1) \in L^1(\mathbb{R}^+)$. Thus condition (4.4) is satisfied as claimed.

Let us finish our work with an example of the nonlinear function f. Let

$$f(t,u_\rho)(x) = h(t)\tilde{f}\left(\int_\Omega |u(rt-q,x)|^2dx\right)u(rt-q,x),\ t\ge 0, \tag{4.9}$$

where $\tilde{f}:\mathbb{R}^+\to\mathbb{R}$ is a continuous function and $\rho(t) = (1-r)t+q$, $r\in(0,1)$. The nonlinear function of this type is contructed to depend on both the history state and its energy. Concerning $\tilde{f}$, h, we first assume that

- h is continuous and bounded function on $\mathbb{R}^+$;
- $\tilde{f}:\mathbb{R}\to\mathbb{R}$ is a function such that $|\tilde{f}(z)|\le a|z|^\sigma$ for some $a>0,\sigma>0$.

By these assumptions, we testify that f is locally Lipschitzian. For $v_1, v_2\in L^2(\Omega)$, $\|v_1\|, \|v_2\|\le r$, one has

$$\begin{aligned}\|f(t,v_1)-f(t,v_2)\| &\le |p(t)|\big[\big|\tilde{f}(\|v_1\|^2)-\tilde{f}(\|v_2\|^2)\big|\|v_2\|+\big|\tilde{f}(\|v_1\|^2)\big|\|v_1-v_2\|\big]\\ &\le \|p\|_\infty\big[(\|v_1\|+\|v_2\|)\|v_2\|\big|\tilde{f}'\big((1-\zeta)\|v_2\|^2+\zeta\|v_1\|^2\big)\big|\\ &\quad + a\|v_1\|^{2\sigma}\big]\|v_1-v_2\|,\end{aligned}$$

where $\zeta\in[0,1]$, thanks to the mean value formula. Thus

$$\|f(t,v_1)-f(t,v_2)\|\le\|p\|_\infty\big[2r^2\sup_{s\in[0,r^2]}|\tilde{f}'(s)|+ar^{2\sigma}\big]\|v_1-v_2\|,$$

which means that f obeys (**F**3) with

$$\kappa(r)=\|p\|_\infty\big[2r^2\sup_{s\in[0,r^2]}|\tilde{f}'(s)|+ar^{2\sigma}\big]\to 0 \text{ as } r\to 0.$$

Applying Theorem 4.1, we conclude that the zero solution of (1.1) is asymptotically stable.

Now if we drop the assumption that $\tilde{f}\in C^1(\mathbb{R}^+)$, then the Lipschitz property for f is unavailable. Assuming $\tilde{f}\in C(\mathbb{R}^+)$ such that $|\tilde{f}(s)|\le a|s|^\sigma$ for $a,\sigma>0$, we get the estimate

$$\|f(t,v)\|\le|p(t)|\big|\tilde{f}(\|v\|^2)\big|\|v\|\le|p(t)|a\|v\|^{2\sigma+1}.$$

As pointed out in Remark 4.2, f in this case fulfills (**F**5) with $\Phi(r)=ar^{2\sigma+1}$ satisfying that $\frac{\Phi(r)}{r}=ar^{2\sigma}\to 0$ as $r\to 0$. Therefore, Theorem 4.5 ensures the existence of a compact set of decay solutions to the problem (1.1)–(1.3).

Author Contributions DL and TVT wrote the main manuscript text. All authors reviewed the manuscript.

Data availability All data used to support the current study are included within the article.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Anh, N.T., Ke, T.D.: Decay integral solutions for neutral fractional differential equations with infinite delays. Math. Methods Appl. Sci. **38**, 1601–1622 (2015)
2. Anh, N.T., Ke, T.D., Quan, N.N.: Weak stability for integro-differential inclusions of diffusion-wave type involving infinite delays. Discrete Contin. Dyn. Syst. Ser. B **21**, 3637–3654 (2016)

3. Burlică, M.-D., Necula, M., Roşu, D., Vrabie, I.I.: Delay Differential Evolutions Subjected to Nonlocal Initial Conditions. Monographs and Research Notes in Mathematics. Chapman and Hall/CRC Press, Boca Raton (2016)
4. Cong, N.D., Tuan, H.T., Trinh, H.: On asymptotic properties of solutions to fractional differential equations. J. Math. Anal. Appl. **484**, 123759 (2020)
5. Cong, N.D., Son, D.T., Siegmund, S., Tuan, H.T.: Linearized asymptotic stability for fractional differential equations. Electron. J. Qual. Theory Differ. Equ. **39**, 1–13 (2016)
6. Erneux, T.: Applied Delay Differential Equations, Surveys and Tutorials in the Applied Mathematical Sciences, vol. 3. Springer, New York (2009)
7. Gorenflo, R., Kilbas, A.A., Mainardi, F., Rogosin, S.V.: Mittag–Leffler Functions, Related Topics and Applications. Springer Monographs in Mathematics. Springer, Heidelberg (2014)
8. Hien, L.V., Ke, T.D., Kinh, C.T.: Globally attracting solutions to impulsive fractional differential inclusions of Sobolev type. Acta Math. Sci. Ser. B (Engl. Ed.) **37**, 1295–1318 (2017)
9. Jin, L., Li, L., Fang, S.: The global existence and time-decay for the solutions of the fractional pseudo-parabolic equation. Comput. Math. Appl. **73**, 2221–2232 (2017)
10. Jin, B.: Fractional Differential Equations: An Approach via Fractional Derivatives. Springer, Berlin (2021)
11. Kamenskii, M., Obukhovskii, V., Zecca, P.: Condensing multivalued maps and semilinear differential inclusions in Banach spaces. In: de Gruyter Series in Nonlinear Analysis and Applications, vol. 7, Walter de Gruyter, Berlin (2001)
12. Ke, T.D., Thang, N.N., Thuy, L.T.P.: Regularity and stability analysis for a class of semilinear nonlocal differential equations in Hilbert spaces. J. Math. Anal. Appl. **483**, 123655 (2020)
13. Ke, T.D., Thuy, L.T.P.: Dissipativity and stability for semilinear anomalous diffusion equations with delay. Math. Methods Appl. Sci. **43**, 1–17 (2020)
14. Ke, T.D., Lan, D.: Fixed point approach for weakly asymptotic stability of fractional differential inclusions involving impulsive effects. J. Fixed Point Theory Appl. **19**, 2185–2208 (2017)
15. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations, vol. 204. Elsevier, Amsterdam (2006)
16. Lan, D., Phong, V.N.: Decay solutions to retarded fractional evolution inclusions with superlinear perturbations. Fixed Point Theory **23**, 293–309 (2022)
17. Lian, W., Wang, J., Xu, R.: Global existence and blow up of solutions for pseudo-parabolic equation with singular potential. J. Differ. Equ. **269**, 4914–4959 (2020)
18. Lyubanova, A.S., Velisevich, A.V.: Inverse problems for the stationary and pseudoparabolic equations of diffusion. Appl. Anal. **98**, 1997–2010 (2019)
19. Lyubanova, A.S., Tani, A.: An inverse problem for pseudoparabolic equation of filtration: the stabilization. Appl. Anal. **92**, 573–585 (2013)
20. Phong, V.N., Lan, D.: Finite-time attractivity of solutions for a class of fractional differential inclusions with finite delay. J. Pseudo-Differ. Oper. Appl. **12**, 5 (2021)
21. Padron, V.: Effect of aggregation on population recovery modeled by a forward–backward pseudoparabolic equation. Trans. Am. Math. Soc. **356**, 2739–2756 (2004)
22. Pollard, H.: The completely monotonic character of the Mittag–Leffler function $E_\alpha(-x)$. Bull. Am. Math. Soc. **54**, 1115–1116 (1948)
23. Showalter, R.E., Ting, T.W.: Pseudoparabolic partial differential equations. SIAM J. Math. Anal. **1**, 1–26 (1970)
24. Thang, N.N., Ke, T.D., Dac, N.V.: Stability analysis for nonlocal evolution equations involving infinite delays. J. Fixed Point Theory Appl. **25**(1), 22 (2023)
25. Tuan, N.H., Tuan, N.A., Yang, C.: Global well-posedness for fractional Sobolev–Galpern type equations. Discrete Contin. Dyn. Syst. (2022)
26. Tuan, N.H., Au, V.V., Xu, R.: Semilinear Caputo time-fractional pseudo-parabolic equations. Commun. Pure Appl. Anal. **20**, 583–621 (2021)
27. Tuan, N.H., Au, V.V., Tri, V.V., O'Regan, D.: On the well-posedness of a nonlinear pseudo-parabolic equation. J. Fix. Point Theory Appl. **22**, 77 (2020)
28. Tuan, H.T., Siegmund, S.: Stability of scalar nonlinear fractional differential equations with linearly dominated delay. Fract. Calc. Appl. Anal. **23**, 250–267 (2020)
29. Tuan, H.T., Trinh, H.: A qualitative theory of time delay nonlinear fractional-order systems. SIAM J. Control. Optim. **58**, 1491–1518 (2020)

30. Tuan, H.T., Thai, H.D., Garrappa, R.: An analysis of solutions to fractional neutral differential equations with delay. Commun. Nonlinear Sci. Numer. Simul. **100**, 105854 (2021)
31. Tuan, T.V.: Existence and regularity in inverse source problem for fractional reaction–subdiffusion equation perturbed by locally Lipschitz sources. Evol. Equ. Control Theory **12**(1), 336–361 (2023)
32. Ting, T.W.: Parabolic and pseudo-parabolic partial differential equations. J. Math. Soc. Jpn. **21**, 440–453 (1969)
33. Vergara, V., Zacher, R.: Optimal decay estimates for time-fractional and other nonlocal subdiffusion equations via energy methods. SIAM J. Math. Anal. **47**(1), 210–239 (2015)
34. Wu, J.: Theory and Applications of Partial Functional Differential Equations. Applied Mathematical Sciences, vol. 119. Springer, Berlin (1996)
35. Xu, R., Su, J.: Global existence and finite time blow-up for a class of semilinear pseudo-parabolic equations. J. Funct. Anal. **264**, 2732–2763 (2013)
36. Zhou, Y.: Basic Theory of Fractional Differential Equations. World Scientific, Singapore (2014)
37. Zhou, Y., Wang, R.N., Peng, L.: Topological Structure of the Solution Set for Evolution Inclusions. Springer, Singapore (2017)