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Semilinear degenerate elliptic equation in the presence of singular nonlinearity

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Abstract

Given $\Omega (\subseteq \mathbb{R}^{1+m})$, a smooth bounded domain and a nonnegative measurable function f defined on Ω with suitable summability. In this paper, we will study the existence and regularity of solutions to the quasilinear degenerate elliptic equation with a singular nonlinearity given by:

$$
\begin{aligned}
-\Delta_\lambda u &= \frac{f}{u^\nu} \text{ in } \Omega \\
u &> 0 \text{ in } \Omega \\
u &= 0 \text{ on } \partial\Omega
\end{aligned}
$$

where the operator Δ_λ is given by

$$\Delta_\lambda u = u_{xx} + |x|^{2\lambda} \Delta_y u;\ (x, y) \in \mathbb{R} \times \mathbb{R}^m$$

is known as the Grushin operator.

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1 Introduction

In this paper, we are interested in the semilinear elliptic problem, whose model is given by

$$-\Delta_\lambda u = \frac{f}{u^\nu} \text{ in } \Omega \tag{1}$$

$$u > 0 \text{ in } \Omega \tag{2}$$

$$u = 0 \text{ on } \partial\Omega \tag{3}$$

where the operator Δ_λ is given by

$$\Delta_\lambda u = u_{xx} + |x|^{2\lambda}\Delta_y u;\ \ \lambda \geq 0$$

is known as the Grushin operator. Δ_y denotes the Laplacian operator w.r.t y variable. $\Omega \subseteq \mathbb{R}^{1+m}$ is a $\Lambda-$connected bounded open set (definition provided in the next section) and $X = (x, y) \in \Omega$, $x \in \mathbb{R}$, $y = (y_1, y_2, ..., y_m) \in \mathbb{R}^m$, $m \geq 1$. Here $\nu > 0$ is a positive real number, and f is a nonnegative measurable function lying in some Lebesgue space.

To understand the context of our study, we start by looking at available literature concerning (1). Starting with the classical work by Crandall et al. [7] where the case $\lambda = 0$ was considered and showed to have a unique solution in $C^2(\Omega) \cap C(\bar{\Omega})$ such that the solution behaves like some power of the distance function near the boundary, a plethora of work followed provided $f \in C^\alpha(\Omega)$. Of particular significance is the work of Lazer–Mckenna, where the solution was shown to exist in $H_0^1(\Omega)$ if and only if $0 < \delta < 3$. When $f \in L^1(\Omega)$, Boccardo and Orsina [5] proved if $0 < \nu \leq 1$ then there exist a solution of (1) in $H_0^1(\Omega)$ and for $\nu > 1$ there exist a solution $u \in H^1_{loc}(\Omega)$ such that $u^{\frac{\nu+1}{2}} \in H_0^1(\Omega)$ among other regularity results. The p-laplacian case was settled in [6], where existence, uniqueness, and some regularity results were proved.

In this paper, we would like to relook at the equation (1) by replacing the Laplacian with a degenerate elliptic equation whose prototype is given by Grushin Laplacian Δ_λ. We will prove the existence and regularity results analog to [5]. It is worth pointing out that there are several issues when degeneracy is introduced. If the distance between the domain Ω and the plane $x = 0$ is positive, then the Grushin operator will become uniformly elliptic in Ω, and in this case, the problem is settled in [5]. We assume the domain Ω intersects the $x = 0$ plane, thus degenerating the operator in Ω. To handle this kind of degeneracy, assuming that Δ_λ admits a uniformly elliptic direction, we discuss the solvability of (1) in the weighted degenerate Sobolev space $H^{1,\lambda}(\Omega)$ which is defined in [8, 10]. We would also need to have a notion of convergence of sequence in the space $H^{1,\lambda}(\Omega)$ for which Monticelli-Payne [18] introduced the concept of a quasi-gradient, hence providing a proper representation of elements of $H^{1,\lambda}(\Omega)$. Another issue is the lack of availability of the Strong Maximum Principle, which we showed to hold using weak Harnack inequality of Franchi-Lanconelli [11, Theorem 4.3] valid for $d-$metric on Ω provided $\lambda \geq 1$ and assuming that Ω is $\Lambda-$connected (definition is provided in the next section). We conclude our study with a brief discussion of how singular variable exponent for Grushin Laplacian may be handled, whose Laplacian counterpart can be found in Garain-Mukherjee [13]. For further reading into the topic, one may look at the papers [1–4, 19] and the references therein.

Notation 1.1 *Throughout the paper, if not explicitly stated, C will denote a positive real number depending only on Ω and N, whose value may change from line to line. We denote by $\langle .,. \rangle$ the Euclidean inner product on $\mathbb{R}^n$ and denote by $|A| := \sup_{|\xi|=1}\langle A\xi, \xi\rangle$ the norm of a real, symmetric $N \times N$ matrix A. The Lebesgue measure of $S \subset \mathbb{R}^N$ is denoted by $|S|$. The Hölder conjugate of $r \geq 1$ is denoted by r'.*

This paper is organized into seven sections. Section 2 discusses functional, analytical settings related to our problem and a few related results. We state our main results in Sect. 3. Sections 4 and 5 are devoted to proving a few auxiliary results. We prove our main results in Sect. 6. Finally, in Sect. 7, we consider the variable singular exponent case.

2 Preliminaries and few useful results

We define a few crucial notions, and the metric introduced in Franchi-Lanconelli [11].

Definition 2.1 An open subset $\Omega(\subset \mathbb{R}^N)$ is said to be $\Lambda-$connected if for every $X, Y \in \Omega$, there exists a continuous curve lying in Ω which is piecewise an integral curve of the vector fields $\pm\partial_x, \pm|x|^\lambda\partial_{y_1}, ..., \pm|x|^\lambda\partial_{y_m}$ connecting X and Y.

Note that every $\Lambda-$connected open set in $\mathbb{R}^N$ is connected. We denote by $P(\Lambda)$ the set of all continuous curves which are piecewise integral curves of the vector fields $\pm\partial_x, \pm|x|^\lambda\partial_{y_1}, ..., \pm|x|^\lambda\partial_{y_m}$. Let $\gamma : [0, T] \to \Omega$ is an element in $P(\Lambda)$ and define $l(\gamma) = T$.

Definition 2.2 Let $X, Y \in \Omega$, we define a new metric d on Ω by $d(X, Y) = \inf\{l(\gamma) : \gamma \in P(\Lambda)$ connecting X and $Y\}$.

The $d-$ball around $X \in \Omega$ with radius $r > 0$ is denoted by $S_d(X, r)$ and is given by $S_d(X, r) = \{Y \in \Omega : d(X, Y) < r\}$. ([10, Proposition 2.9]) ensures that the usual metric is equivalent to the d in Ω.

Let $N = k + m$ and $\Omega \subseteq \mathbb{R}^N$ be a bounded domain. Let $A = \begin{pmatrix} I_k & O \\ O & |x|^{2\lambda} I_m \end{pmatrix}$ and define the set

$$V_A(\Omega) = \{u \in C^1(\Omega) | \int_\Omega |u|^p \, dX + \int_\Omega \langle A\nabla u, \nabla u\rangle^{\frac{p}{2}} \, dX < \infty\}$$

Consider the normed linear spaces $(V_A(\Omega), \|.\|)$ and $(C_0^1(\Omega), \|.\|_0)$ where

$$\|u\| = \left(\int_\Omega |u|^p \, dX + \int_\Omega \langle A\nabla u, \nabla u\rangle^{\frac{p}{2}} \, dX\right)^{\frac{1}{p}}$$

and

$$\|u\|_0 = \left(\int_\Omega \langle A\nabla u, \nabla u\rangle^{\frac{p}{2}} \, dX\right)^{\frac{1}{p}}$$

Now $W^{1,\lambda,p}(\Omega)$ and $W_0^{1,\lambda,p}(\Omega)$ is defined as the completion of $(V_A(\Omega), \|.\|)$ and $(C_0^1(\Omega), \|.\|_0)$ respectively. Each element $[\{u_n\}]$, of the Banach space $W^{1,\lambda,p}(\Omega)$ is a class of Cauchy sequence in $(V_A(\Omega), \|.\|)$ and $\|[\{u_n\}]\| = \lim_{n\to\infty} \|u_n\|$. A function u is said to be in $W_{loc}^{1,\lambda,p}(\Omega)$ if and only if $u \in W^{1,\lambda,p}(\Omega')$ for every $\Omega' \Subset \Omega$. For more information, one can look into Monticelli-Payne [18].

The following theorem proves that $\|.\|_0$ and $\|.\|$ are equivalent norm on $W_0^{1,\lambda,p}(\Omega)$.

Theorem 2.1 (Poincaré Inequality) (Monticelli–Payne [18, Theorem 2.1]) *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, and A is given as above. Then for any $1 \le p < \infty$ there exists a constant $C_p = C(N, p, \|A\|_\infty, d(\Omega)) > 0$ such that*

$$\|u\|_{L^p(\Omega)}^p \le C_p \int_\Omega \langle A\nabla u, \nabla u\rangle^{\frac{p}{2}} \, dX \ \textit{ for all } u \in C_0^1(\Omega)$$

where $d(\Omega)$ denotes the diameter of Ω.

Now the suitable representation of an element of $W^{1,\lambda,p}(\Omega)$ and $W_0^{1,\lambda,p}(\Omega)$ is given by the following theorem, whose proof follows exactly that of Monticelli-Payne where it is done for $p = 2$.

Theorem 2.2 (Monticelli–Payne [18, Theorem 2.1])) *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, and A is given as above. Then for every $[\{u_n\}] \in W^{1,\lambda,p}(\Omega)$ there exists unique $u \in L^p(\Omega)$ and $U \in (L^p(\Omega))^N$ such that the following properties hold*

(i) $u_n \to u$ *in* $L^p(\Omega)$ *and* $\sqrt{A}\nabla u_n \to U$ *in* $(L^p(\Omega))^N$.
(ii) $\sqrt{A}^{-1} U$ *is the weak gradient of u in each of the component of* $\Omega \setminus \Sigma$
(iii) *If* $|[\sqrt{A}]^{-1}| \in L^{p'}(\Omega)$ *then* $[\sqrt{A}]^{-1} U$ *is the weak gradient of u in* Ω.
(iv) *One has*

$$\|[u_n]\|^p = \|u\|_{L^p(\Omega)}^p + \|U\|_{(L^p(\Omega))^N}^p$$

where $\Sigma = \{X \in \Omega : det[A(X)] = 0\}$, $p' = \frac{p}{p-1}$.

Proof Let $[\{u_n\}] \in W^{1,\lambda,p}$. So $[\{u_n\}]$ is a Cauchy sequence in $(V_A, \|.\|)$. Clearly $\{u_n\}$ and $\{\sqrt{A}\nabla u_n\}$ are Cauchy in $L^p(\Omega)$ and $L^p(\Omega)^N$ respectively. Hence there exists $u \in L^p(\Omega)$ and $U \in L^p(\Omega)^N$ such that $u_n \to u$ in $L^p(\Omega)$ and $\{\sqrt{A}\nabla u_n\} \to U$ in

$L^p(\Omega)^N$ as $n \to \infty$. If $[\{u_n\}] = [\{v_n\}]$ and $\{\sqrt{A}\nabla u_n\} \to U$, $\{\sqrt{A}\nabla v_n\} \to V$ in $L^p(\Omega)^N$ as $n \to \infty$. Then

$$\|U - V\|_{L^p(\Omega)^N} \leq \|\sqrt{A}\nabla u_n - U\|_{L^p(\Omega)^N} + \|\sqrt{A}\nabla u_n \\ - \sqrt{A}\nabla v_n\|_{L^p(\Omega)^N} + \|\sqrt{A}\nabla v_n - V\|_{L^p(\Omega)^N} \\ \to 0 \text{ as } n \to \infty$$

which implies $U = V$ a.e in Ω. So U does not depend on the representative of the class $[\{u_n\}]$. Let $\phi \in C_0^\infty(\Omega)$. Since $u_n \to u$ in $L^p(\Omega)$ so u_n converges to u in the distributional sense as well. As $u_n \in C^1(\Omega)$ so

$$\int_\Omega u_n \nabla\phi dx = -\int_\Omega \phi\nabla u_n dx$$

Taking limit $n \to \infty$ we have

$$\int_\Omega u\nabla\phi dx = -\lim_{n\to\infty}\int_\Omega \phi\nabla u_n dx = -\lim_{n\to\infty}\int_\Omega \phi\sqrt{A}^{-1}\sqrt{A}\nabla u_n dx$$

Hence if $|\phi\sqrt{A}^{-1}| \in L^{p'}(\Omega)$ then

$$\int_\Omega u\nabla\phi dx = -\int_\Omega \phi\sqrt{A}^{-1}U dx \tag{4}$$

If support of ϕ is contained in a component of $\Omega \setminus \Sigma$ then $|\phi\sqrt{A}^{-1}| \in L^{p'}(\Omega)$. By using (4) we can conclude that $\sqrt{A}^{-1}U$ is the weak gradient of u in that component of $\Omega \setminus \Sigma$. Hence (ii) is proved. Also, if $|\sqrt{A}^{-1}| \in L^{p'}(\Omega)$ then (4) is true for every $\phi \in C_0^\infty(\Omega)$. So $\sqrt{A}^{-1}U$ is the weak gradient of u in Ω. Which proves (iii). For $[\{u_n\}] \in W^{1,\lambda,p}(\Omega)$,

$$\|[\{u_n\}]\|^p = \lim_{n\to\infty}(\|u_n\|^p_{L^p(\Omega)} + \|\sqrt{A}\nabla u_n\|^p_{L^p(\Omega)^N}) = (\|u\|^p_{L^p(\Omega)} + \|U\|^p_{L^p(\Omega)^N})$$

Hence (iv) is proved. □

Using the above theorem, we have the following embedding theorem.

Corollary 2.3 *The space $W^{1,\lambda,p}(\Omega)$ is continuously embedded into $L^p(\Omega)$.*

Proof Define the map $T : W^{1,\lambda,p}(\Omega) \to L^p(\Omega)$ by $T([\{u_n\}]) = u$. T is a bounded linear map.
Claim: $T$is injective. Let $u = 0$. If we can prove $U = 0$, then we are done. Since Σ has measure zero, we can prove that $U = 0$ a.e in each component of $\Omega \setminus \Sigma$. Let Ω' be a component of $\Omega \setminus \Sigma$. By the above theorem for every $\phi \in C_0^\infty(\Omega')$

$$\int_{\Omega'} \phi\sqrt{A}^{-1}U dx = -\int_{\Omega'} u\nabla\phi dx = 0$$

which ensures us $\sqrt{A}^{-1}U = 0$ a.e in Ω'. So $U = 0$ a.e in Ω'. □

Henceforth we use the notation u for the element $[\{u_n\}] \in W^{1,\lambda,p}(\Omega)$ or $[\{u_n\}] \in W_0^{1,\lambda,p}(\Omega)$ which is determined in Theorem (2.2). Using the properties of $U \in (L^p(\Omega))^N$ in the theorem we introduce the following definition:

Definition 2.3 For $u \in W^{1,\lambda,p}(\Omega)$ we denote the weak quasi gradient of u by $\nabla^* u$ and defined by

$$\nabla^* u := (\sqrt{A})^{-1} U$$

which is a vector-valued function defined almost everywhere in Ω.

Also for $u \in W^{1,\lambda,p}(\Omega)$,

$$\begin{aligned}\|u\|^p &= \|u\|^p_{L^p(\Omega)} + \|\sqrt{A}\nabla^* u\|^p_{L^p(\Omega)} \\ &= \int_\Omega |u|^p dx + \int_\Omega \langle A\nabla^* u, \nabla^* u\rangle^{\frac{p}{2}}.\end{aligned}$$

We define $H^{1,\lambda}(\Omega) := W^{1,\lambda,2}(\Omega)$ and $H_0^{1,\lambda}(\Omega) := W_0^{1,\lambda,2}(\Omega)$. $(H^{1,\lambda}(\Omega), \|.\|)$ and $(H_0^{1,\lambda}(\Omega), \|.\|_0)$ are Hilbert spaces.

Theorem 2.4 (Embedding Theorem) ([12, Theorem 2.6] and [16, Proposition 3.2]) *Let $\Omega \subset \mathbb{R}^{k+m}$ be an open set. The embedding*

$$H_0^{1,\lambda}(\Omega) \hookrightarrow L^q(\Omega)$$

is continuous for every $q \in [1, 2^*_\lambda]$ and compact for $q \in [1, 2^*_\lambda)$, where $2^*_\lambda = \frac{2Q}{Q-2}$, $Q = k + (\lambda + 1)m$.

Theorem 2.5 (Stampacchia–Kinderlehrer [15, lemma B.1]) *Let $\phi : [k_0, \infty) \to \mathbb{R}$ be a nonnegative and nonincreasing such that for $k_0 \le k \le h$,*

$$\phi(h) \le [C/(h-k)^\alpha]|\phi(k)|^\beta$$

where C, α, β are positive constant with $\beta > 1$. Then

$$\phi(k_0 + d) = 0$$

where $d^\alpha = C2^{\frac{\alpha\beta}{\beta-1}}|\phi(k_0)|^{(\beta-1)}$

Now we will prove the Strong Maximum Principle for super-solutions of $-\Delta_\lambda u = 0$. In this proof, we denote ρ and S_ρ, which are defined in [10, Definition 2.6]. The constants a, c_1 are introduced in [10, Theorem 4.3]. Also, c and ϵ_0 are defined in [10, Proposition 2.9].

Theorem 2.6 (Strong Maximum Principle) *Let $\Omega \subset \mathbb{R}^{1+m}$ be a $\Lambda-$connected, bounded open set and $\lambda \geq 1$. Let u be a nonnegative (not identically zero) function in $H_0^{1,\lambda}(\Omega)$ such that u is a super solution of $-\Delta_\lambda u = 0$, i.e., for every nonnegative $v \in H_0^{1,\lambda}(\Omega)$,*

$$\int_\Omega \langle A\nabla^* u, \nabla^* v\rangle dX \geq 0.$$

If there exist a ball $B_r(x_0) \Subset \Omega$ with $\inf_{B_r(x_0)} u = 0$ then u is identically zero in Ω.

Proof Let n_0 be a natural number such that $n_0^{\epsilon_0} > 2c_1$. We can choose $r > 0$ such that $B(X_0, n_0 r) \Subset \Omega$, $\inf_{B_r(X_0)} u = 0$ and $S_\rho(X, ac(n_0 r)^{\epsilon_0}) \subset \Omega$. By using ([10, Proposition 2.9]) and ([10, Theorem 2.7]) we have

$$B(X_0, r) \subset B(X_0, n_0 r) \subset S_d(X_0, c(n_0 r)^{\epsilon_0}) \subset S_\rho(X_0, ac(n_0 r)^{\epsilon_0}) \subset \Omega$$

Put $R = \frac{ac(n_0 r)^{\epsilon_0}}{c_1}$ and by [10, Theorem 4.3] with $p = 1$, we have

$$\inf_{S_\rho(X_0, \frac{R}{2})} u \geq M|S_\rho(X_0, R)|^{-1} \int_{S_\rho(X_0, R)} |u|\, dX. \tag{5}$$

By using ([10, Proposition 2.9]) and ([10, Theorem 2.7]) we easily can show that $B(X_0, r) \subset S_\rho(X_0, \frac{R}{2})$. Hence, $\inf_{S_\rho(X_0, \frac{R}{2})} u = 0$. By (5) we have $u = 0$ a.e. in $S_\rho(X_0, R)$ and hence, in $B(X_0, r)$. Let $Y \in \Omega$ and $r_0 = r$. Since Ω is a bounded domain, we can find a finite collection of balls $\{B(X_i, r_i)\}_{i=0}^{i=k}$ such that $B(X_i, n_0 r_i) \Subset \Omega$, $S_\rho(X_i, ac(n_0 r_i)^{\epsilon_0}) \subset \Omega$, $B(X_{i-1}, r_{i-1}) \cap B(X_i, r_i) \neq \emptyset$ for $i = 1, 2...k$ and $Y \in B(X_k, r_k)$. We can use the previous process to show that $u = 0$ a.e. in $B(X_1, r_1)$. Iterating we have $u = 0$ a.e. in $B(X_k, r_k)$. Hence, $u = 0$ a.e.in Ω. □

Now we are ready to define the notion of solution of (1).

Definition 2.4 A function $u \in H_{loc}^{1,\lambda}(\Omega)$ is said to be a weak solution of (1) if for every $\Omega' \Subset \Omega$, there exists a positive constant $C(\Omega')$ such that

$$u \geq C(\Omega') > 0 \text{ a.e in } \Omega',$$
$$\int_\Omega \langle A\nabla^* u, \nabla v\rangle\, dX = \int_\Omega \frac{fv}{u^\nu}\, dX \quad \text{for all } v \in C_0^1(\Omega)$$

and

- if $\nu \leq 1$ then $u \in H_0^{1,\lambda}(\Omega)$.
- if $\nu > 1$ then $u^{\frac{\nu+1}{2}} \in H_0^{1,\lambda}(\Omega)$.

3 Existence and regularity results

Henceforth, we will assume $N = 1 + m$, and $\Omega \subset \mathbb{R}^N$ is a $\Lambda-$connected, bounded open set. We will also assume f is a nonnegative (not identically zero) function and $\lambda \geq 1$. Our main results are the following:

3.1 The case $\nu = 1$

Theorem 3.1 *Let $\nu = 1$ and $f \in L^1(\Omega)$. Then the Dirichlet boundary value problem (1) has a unique solution in the sense of definition (2.4).*

Theorem 3.2 *Let $\nu = 1$ and $f \in L^r(\Omega), r \geq 1$. Then the solution given by Theorem 3.1 satisfies the following*

(i) If $r > \frac{Q}{2}$ then $u \in L^\infty(\Omega)$.
(ii) If $1 \leq r < \frac{Q}{2}$ then $u \in L^s(\Omega)$.

where $Q = (m + 1) + \lambda m$ and $s = \frac{2Qr}{Q-2r}$.

3.2 The case $\nu > 1$

Theorem 3.3 *Let $\nu > 1$ and $f \in L^1(\Omega)$. Then there exists $u \in H^{1,\lambda}_{loc}(\Omega)$ which satisfies Eq. (1) in sense of definition (2.4).*

Theorem 3.4 *Let $\nu > 1$ and $f \in L^r(\Omega),\ r \geq 1$. Then the solution u of (1) given by the above theorem is such that*

(i) If $r > \frac{Q}{2}$ then $u \in L^\infty(\Omega)$.
(ii) If $1 \leq r < \frac{Q}{2}$ then $u \in L^s(\Omega)$.

where $s = \frac{Qr(\nu+1)}{(Q-2r)}$ and $Q = (m + 1) + \lambda m$.

3.3 The case $\nu < 1$

Theorem 3.5 *Let $\nu < 1$ and $f \in L^r(\Omega), r = (\frac{2^*_\lambda}{1-\lambda})'$. Then (1) has a unique solution in $H^{1,\lambda}_0(\Omega)$.*

Theorem 3.6 *Let $\nu < 1$ and $f \in L^r(\Omega),\ r \geq (\frac{2^*_\lambda}{1-\nu})'$. Then the solution u of (1) given by the above theorem is such that*

(i) If $r > \frac{Q}{2}$ then $u \in L^\infty(\Omega)$.
(ii) If $(\frac{2^*_\lambda}{1-\nu})' \leq r < \frac{Q}{2}$ then $u \in L^s(\Omega)$.

where $s = \frac{Qr(\nu+1)}{(Q-2r)},\ Q = (m + 1) + \lambda m$ and r' denotes the Hölder conjugate of r.

Theorem 3.7 *Let $\nu < 1$ and $f \in L^r(\Omega)$ for some $1 \le r < \frac{2Q}{(Q+2)+\nu(Q-2)}$. Then there exists $u \in W_0^{1,\lambda,q}(\Omega)$ which is a solution of (1) in the sense*

$$\int_\Omega \langle A\nabla^* u, \nabla v\rangle dX = \int_\Omega \frac{fv}{u^\nu}\, dX \text{ for all } v \in C_0^1(\Omega)$$

where $q = \frac{Qr(\nu+1)}{Q-r(1-\nu)}$.

4 Approximation of the Equation (1)

Let f be a nonnegative (not identically zero) measurable function and $n \in N$. Let us consider the equation

$$\begin{aligned} -\Delta_\lambda u_n &= \frac{f_n}{(u_n + \frac{1}{n})^\nu} \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned} \tag{6}$$

where $f_n := \min\{f, n\}$.

Lemma 4.1 *Equation (6) has a unique solution $u_n \in H_0^{1,\lambda}(\Omega) \cap L^\infty(\Omega)$.*

Proof Let $w \in L^2(\Omega)$ be a fixed element. Now consider the equation

$$\begin{aligned} -\Delta_\lambda u &= g_n \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned} \tag{7}$$

where $g_n = \frac{f_n}{(|w|+\frac{1}{n})^\nu}$. Since $|g_n(x)| \le n^{\nu+1}$ one has $g_n \in L^2(\Omega)$. By [18, Theorem 4.4], we can say equation (7) has a unique solution $u_w \in H_0^{1,\lambda}(\Omega)$ and the map $T : L^2(\Omega) \to H_0^{1,\lambda}(\Omega)$ such that $T(w) = u_w$ is continuous. By Theorem 2.4, we have the compact embedding

$$H_0^{1,\lambda}(\Omega) \hookrightarrow L^2(\Omega).$$

Hence, the $T : L^2(\Omega) \to L^2(\Omega)$ is continuous as well as compact.

Let $S = \{w \in L^2(\Omega) : w = \lambda T w \text{ for some } 0 \le \lambda \le 1\}$.

Claim: The set S is bounded.

Let $w \in S$. By the Poincaré inequality (see [18, Theorem 2.1]), there exists a constant $C > 0$ such that,

$$\begin{aligned} \|u_w\|^2_{L^2(\Omega)} &\le C \int_\Omega \langle A\nabla^* u_w, \nabla^* u_w\rangle\, dX \\ &= C \int_\Omega g_n(x) u_w\, dX \le C n^{\nu+1} \int_\Omega u_w\, dX \le C n^{\nu+1} |\Omega|^{\frac{1}{2}} \|u_w\|_{L^2(\Omega)} \end{aligned}$$

Hence, we have

$$\|u_w\|_{L^2(\Omega)} \leq Cn^{\nu+1}|\Omega|^{\frac{1}{2}}$$

where $C > 0$ is a independent of w. This proves S is bounded. Hence by Schaefer's fixed point theorem, there exists $u_n \in H_0^{1,\lambda}(\Omega)$ such that

$$-\Delta_\lambda u_n = \frac{f_n}{(|u_n| + \frac{1}{n})^\nu} \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega \tag{8}$$

By Weak Maximum Principle (see [18, Theorem 4.4]), we have $u_n \geq 0$ in Ω. So u_n is a solution of (6). Hence,

$$\int_\Omega \langle A\nabla^* u_n, \nabla v\rangle dX = \int_\Omega \frac{f_n v}{(u_n + \frac{1}{n})^\nu} dX \text{ for every } v \in C_0^1(\Omega) \tag{9}$$

Now, we want to prove $u_n \in L^\infty(\Omega)$.

Let $k > 1$ and define $S(k) = \{x \in \Omega : u_n(x) \geq k\}$. We can treat the function

$$v(x) = \begin{cases} u_n(x) - k & x \in S(k) \\ o & \text{otherwise} \end{cases}$$

as a function in $C_0^1(\Omega)$. By putting v in (9), we obtain

$$\int_{S(k)} \langle A\nabla^* v, \nabla^* v\rangle\, dX = \int_{S(k)} \frac{f_n v}{(v + k + \frac{1}{n})^\nu}\, dX \leq n^{\nu+1} \int_{S(k)} v\, dX$$
$$\leq n^{\nu+1} \|v\|_{L^{2^*_\lambda}(\Omega)} |S(k)|^{1-\frac{1}{2^*_\lambda}}$$

Here, $2^*_\lambda = \frac{2Q}{Q-2}$ and $Q = (m+1) + \lambda m$. Now, by Theorem 2.4 there exists $C > 0$ such that

$$\|v\|^2_{L^{2^*_\lambda}(\Omega)} \leq C \int_\Omega \langle A\nabla^* v, \nabla^* v\rangle\, dX = C \int_{S(k)} \langle A\nabla^* v, \nabla^* v\rangle\, dX$$
$$\leq Cn^{\nu+1} \|v\|_{L^{2^*_\lambda}(\Omega)} |S(k)|^{1-\frac{1}{2^*_\lambda}}.$$

We have

$$\|v\|_{L^{2^*_\lambda}(\Omega)} \leq Cn^{\nu+1} |S(k)|^{1-\frac{1}{2^*_\lambda}} \tag{10}$$

Assume $1 < k < h$ and using Inequality (10) we get

$$|S(h)|^{\frac{1}{2^*_\lambda}}(h-k) = \left(\int_{S(h)} (h-k)^{2^*_\lambda}\, dX\right)^{\frac{1}{2^*_\lambda}}$$
$$\leq \left(\int_{S(k)} (v(x))^{2^*_\lambda}\, dX\right)^{\frac{1}{2^*_\lambda}} \leq \|v\|_{L^{2^*_\lambda}(\Omega)} \leq Cn^{\nu+1}|S(k)|^{1-\frac{1}{2^*_\lambda}}$$

The above two inequalities implies

$$|S(h)| \leq \left(\frac{Cn^{\nu+1}}{(h-k)})^{2^*_\lambda}\right)|S(k)|^{2^*_\lambda-1}$$

Let $d^{2^*_\lambda} = (Cn^{\nu+1})^{2^*_\lambda})2^{\frac{2^*_\lambda(2^*_\lambda-1)}{2^*_\lambda-2}}|S(1)|^{2^*_\lambda-2}$ then by the Theorem 2.5, we get $|S(1+d)| = 0$. Hence, $u_n(x) \leq 1+d$ a.e in Ω. We get a positive constant $C(n)$ such that $u_n \leq C(n)$ a.e in Ω. Consequently, $u_n \in L^\infty(\Omega)$.

Let u_n and v_n be two solutions of (6). The function $w = (u_n - v_n)^+ \in H_0^{1,\lambda}(\Omega)$ can be considered as a test function. It is clear that

$$\left[\left(v_n + \frac{1}{n}\right)^\nu - \left(u_n + \frac{1}{n}\right)^\nu\right] w \leq 0 \tag{11}$$

Since u_n and v_n are two solutions of (6) so by putting w in (9) we get

$$\int_\Omega \langle A\nabla^* u_n, \nabla^* w\rangle dX = \int_\Omega \frac{f_n w}{(u_n+\frac{1}{n})^\nu} dX$$
$$\text{and} \int_\Omega \langle A\nabla^* v_n, \nabla^* w\rangle dX = \int_\Omega \frac{f_n w}{(v_n+\frac{1}{n})^\nu} dX$$

Therefore,

$$\int_\Omega \langle A\nabla^*(u_n - v_n), \nabla^* w\rangle\, dX = \int_\Omega \frac{f_n[(v_n+\frac{1}{n})^\nu - (u_n+\frac{1}{n})^\nu]}{(u_n+\frac{1}{n})^\nu (v_n+\frac{1}{n})^\nu} w\, dX$$

Using (11) we have

$$\int_\Omega \langle A\nabla^* w, \nabla^* w\rangle\, dX \leq 0$$

Hence, $w = 0$ and so $(u_n - v_n) \leq 0$. By a similar argument, we can prove that $(v_n - u_n) \leq 0$. Consequently, $u_n = v_n$ a.e in Ω. □

Lemma 4.2 *Let for each $n \in \mathbb{N}$, u_n be the solution of (6). Then the sequence $\{u_n\}$ is an increasing sequence and for each $\Omega' \Subset \Omega$, there exists a constant $C(\Omega') > 0$ such that*

$$u_n(x) \geq C(\Omega') > 0 \ \ a.e\ x \in \Omega' \ \ and\ for\ all\ n \in \mathbb{N}$$

Proof Let $n \in \mathbb{N}$ be fixed. Define $w = (u_n - u_{n+1})^+$. It is clear that

$$\left[\left(u_{n+1} + \frac{1}{n+1}\right)^{\nu} - \left(u_n + \frac{1}{n}\right)^{\nu}\right] w \leq 0.$$

w can be considered as a test function. Arguing as in the proof of the previous theorem, we obtain $w = 0$. Hence, $u_n - u_{n+1} \leq 0 \implies u_n \leq u_{n+1}$ a.e in Ω and for all $n \in \mathbb{N}$. Since f is not identically zero so f_i is not identically zero for some $i \in N$. Without loss of generality, we may assume that f_1 is not identically zero.

Consider the equation

$$\begin{aligned} -\Delta_\lambda u_1 &= \frac{f_1}{(u_1+1)^{\nu}} \text{ in } \Omega \\ u_1 &= 0 \text{ on } \partial\Omega \end{aligned} \tag{12}$$

Since f_1 is not identically zero so u_1 is not identically zero. So by Theorem 2.6, we have $u_1 > 0$ in Ω. Hence, for every compact set $\Omega' \Subset \Omega$, there exists a constant $C(\Omega') > 0$ such that $u_1 \geq C(\Omega')$ a.e. in Ω'. Monotonicity of the sequence implies that for every $n \in N$,

$$u_n \geq C(\Omega').$$

□

5 A few auxiliary results

We start this section with the proof of a priori estimates on u_n.

Lemma 5.1 *Let u_n be the solution of equation (6) with $\nu = 1$ and assume $f \in L^1(\Omega)$ is a nonnegative function (not identically zero). Then the sequence $\{u_n\}$ is bounded in $H_0^{1,\lambda}(\Omega)$.*

Proof Since $u_n \in H_0^{1,\lambda}(\Omega)$ is a solution of (6) so from (9) we obtain

$$\int_\Omega \langle A\nabla^* u_n, \nabla^* u_n\rangle \, dX = \int_\Omega \frac{f_n u_n}{(u_n + \frac{1}{n})} dX \leq \int_\Omega f \, dX = \|f\|_{L^1(\Omega)}$$

Hence, $\{u_n\}$ is bounded in $H_0^{1,\lambda}(\Omega)$. □

Lemma 5.2 *Let u_n be the solution of the Eq. (6) with $\nu > 1$ and $f \in L^1(\Omega)$ is a nonnegative function (not identically zero). Then $\{u_n^{\frac{\nu+1}{2}}\}$ is bounded in $H_0^{1,\lambda}(\Omega)$ and $\{u_n\}$ is bounded in $H_{loc}^{1,\lambda}(\Omega)$ and in $L^s(\Omega)$, where $s = \frac{(\nu+1)Q}{(Q-2)}$.*

Proof Since $\nu > 1$ and $u_n \in H_0^{1,\lambda}(\Omega)$ so by putting $v = u_n^\nu$ in (9) we have,

$$\int_\Omega \langle A\nabla^* u_n, \nabla^* u_n^\nu \rangle dX = \int_\Omega \frac{f_n u_n^\nu}{(u_n + \frac{1}{n})^\nu} dX \leq \int_\Omega f dX.$$

Now,

$$\int_\Omega \langle A\nabla^* u_n^{\frac{\nu+1}{2}}, \nabla^* u_n^{\frac{\nu+1}{2}} \rangle dX = \frac{(\nu+1)^2}{4\nu} \int_\Omega \nu u_n^{\nu-1} \langle A\nabla^* u_n, \nabla^* u_n \rangle dX$$
$$= \frac{(\nu+1)^2}{4\nu} \int_\Omega \langle A\nabla^* u_n, \nabla^* u_n^\nu \rangle dX \leq \frac{(\nu+1)^2}{4\nu} \int_\Omega f dX. \tag{13}$$

Hence, $\{u_n^{\frac{\nu+1}{2}}\}$ is bounded in $H_0^{1,\lambda}(\Omega)$. By Theorem 2.4, there exists a constant $C > 0$ such that

$$\|u_n^{\frac{\nu+1}{2}}\|_{L^{2^*_\lambda}(\Omega)} \leq C \|u_n^{\frac{\nu+1}{2}}\|_{H_0^{1,\lambda}(\Omega)}$$

By using (13), we have

$$\left(\int_\Omega u_n^{2^*_\lambda \frac{(\nu+1)}{2}} dX \right)^{\frac{2}{2^*_\lambda}} \leq C \frac{(\nu+1)^2}{4\nu} \|f\|_{L^1(\Omega)}$$

Since $s = 2^*_\lambda \frac{(\nu+1)}{2}$ so

$$\int_\Omega u_n^s dX \leq \left(C \frac{(\nu+1)^2}{4\nu} \|f\|_{L^1(\Omega)} \right)^{\frac{2^*_\lambda}{2}}$$

Hence, $\{u_n\}$ is bounded in $L^s(\Omega)$. To prove $\{u_n\}$ is bounded in $H_{\text{loc}}^{1,\lambda}(\Omega)$, let $\Omega' \Subset \Omega$ and $\eta \in C_0^\infty(\Omega)$ such that $0 \leq \eta \leq 1$ and $\eta = 1$ in Ω'. It is a test function as $u_n \eta^2 \in H_0^{1,\lambda}(\Omega)$. By Lemma 4.2, there exists a constant $C > 0$ such that $u_n \geq C$ a.e in supp(η). Put $v = u_n \eta^2$ in (9) we have

$$\int_\Omega \langle A\nabla^* u_n, \nabla^* (u_n \eta^2) \rangle dX = \int_\Omega \frac{f_n u_n \eta^2}{(u_n + \frac{1}{n})^\nu} dX \tag{14}$$

Also,

$$\int_\Omega \langle A\nabla^* u_n, \nabla^* (u_n \eta^2) \rangle dX = \int_\Omega \{\eta^2 \langle A\nabla^* u_n, \nabla^* u_n \rangle + 2\eta u_n \langle A\nabla^* u_n, \nabla \eta \rangle\} \tag{15}$$

From (14) and (15) we get

$$\int_\Omega \eta^2 \langle A\nabla^* u_n, \nabla^* u_n\rangle dX = \int_\Omega \frac{f_n \eta^2}{C^{(\nu-1)}} dX - \int_\Omega 2\eta u_n \langle A\nabla^* u_n, \nabla \eta\rangle dX \tag{16}$$

Choose $\epsilon > 0$ and use Young's inequality; one has

$$\begin{aligned} |\int_\Omega 2\eta u_n \langle A\nabla^* u_n, \nabla\eta\rangle dX| &\leq \int_\Omega 2|\langle \eta\sqrt{A}\nabla^* u_n, u_n\sqrt{A}\nabla\eta\rangle| dX \\ &\leq \frac{1}{\epsilon}\int_\Omega \eta^2 |\sqrt{A}\nabla^* u_n|^2 dX + \epsilon \int_\Omega u_n^2 |\sqrt{A}\nabla\eta|^2 dX, \end{aligned} \tag{17}$$

Put $\epsilon = 2$ then we get

$$\begin{aligned} |\int_\Omega 2\eta u_n \langle A\nabla^* u_n, \nabla\eta\rangle dX| &\leq \frac{1}{2}\int_\Omega \eta^2 |\sqrt{A}\nabla^* u_n|^2 dX + 2\int_\Omega u_n^2 |\sqrt{A}\nabla\eta|^2 dX \\ &= \frac{1}{2}\int_\Omega \eta^2 \langle A\nabla^* u_n, \nabla^* u_n\rangle dX + 2\int_\Omega u_n^2 \langle A\nabla\eta, \nabla\eta\rangle dX \end{aligned} \tag{18}$$

Using (16) and (18), we have

$$\begin{aligned} \int_\Omega \eta^2 \langle A\nabla^* u_n, \nabla^* u_n\rangle dX &\leq 2\int_\Omega \frac{f\eta^2}{C^{(\nu-1)}} dX + 4\int_\Omega u_n^2 \langle A\nabla\eta, \nabla\eta\rangle dX \\ &\leq \frac{2\|\eta\|_\infty^2 \|f\|_{L^1(\Omega)}}{C^{\nu-1}} + 4\|\langle A\nabla\eta, \nabla\eta\rangle\|_\infty \int_\Omega u_n^2 dX \end{aligned}$$

Since $\{u_n\}$ is bounded in $L^s(\Omega)$ and $s > 2$ So $\{u_n\}$ is bounded in $L^2(\Omega)$.

$$\begin{aligned} \int_\Omega \eta^2 \langle A\nabla^* u_n, \nabla^* u_n\rangle dX &\leq \frac{2\|\eta\|_\infty^2 \|f\|_{L^1(\Omega)}}{C^{\nu-1}} + 4\|\langle A\nabla\eta, \nabla\eta\rangle\|_\infty \int_\Omega u_n^2 dX \\ &\leq C(f, \eta) \end{aligned}$$

Now,

$$\int_{\Omega'} \langle A\nabla^* u_n, \nabla^* u_n\rangle dX \leq \int_\Omega \eta^2 \langle A\nabla^* u_n, \nabla^* u_n\rangle dX \leq C(f, \eta)$$

Hence, $\{u_n\}$ is bounded in $H^{1,\lambda}_{\mathrm{loc}}(\Omega)$. □

Lemma 5.3 *Let u_n be the solution of (6) with $\nu < 1$ and $f \in L^r$, $r = (\frac{2^*_\lambda}{1-\nu})'$ is a nonnegative (not identically zero) function. Then $\{u_n\}$ is bounded in $H^{1,\lambda}_0(\Omega)$.*

Proof Since $r = (\frac{2^*_\lambda}{1-\nu})'$, we can choose $v = u_n$ in (9) and using Hölder inequality, one has

$$\int_\Omega \langle A\nabla^* u_n, \nabla^* u_n\rangle dX = \int_\Omega \frac{f_n u_n}{(u_n + \frac{1}{n})^\nu} \le \int_\Omega f u_n^{1-\nu} dX \le \|f\|_{L^r(\Omega)} \left(\int_\Omega u_n^{(1-\nu)r'} dX\right)^{\frac{1}{r'}}$$
$$\le \|f\|_{L^r(\Omega)} \left(\int_\Omega u_n^{2^*_\lambda} dX\right)^{\frac{1-\nu}{2^*_\lambda}}. \tag{19}$$

By Theorem 2.4 and using the above inequality, we get

$$\int_\Omega u_n^{2^*_\lambda} dX \le C\left(\int_\Omega \langle A\nabla^* u_n, \nabla^* u_n\rangle dX\right)^{\frac{2^*_\lambda}{2}} \le C(\|f\|_{L^r(\Omega)} \left(\int_\Omega u_n^{2^*_\lambda} dX\right)^{\frac{1-\nu}{2^*_\lambda}})^{\frac{2^*_\lambda}{2}}. \tag{20}$$

So we have

$$\int_\Omega u_n^{2^*_\lambda} dX \le C\|f\|_{L^r(\Omega)}^{\frac{2^*_\lambda}{1+\nu}}. \tag{21}$$

Hence, $\{u_n\}$ is bounded $L^{2^*_\lambda}(\Omega)$. Using (19) and (21), we can conclude $\|u_n\|_{H_0^{1,\lambda}(\Omega)} \le C\|f\|_{L^r(\Omega)}^{\frac{1}{1+\nu}}$ where C is independent of n. Hence, $\{u_n\}$ is bounded in $H_0^{1,\lambda}(\Omega)$. □

6 Proof of main results

6.1 The case $\nu = 1$

Proof of Theorem 3.1: Consider the above sequence $\{u_n\}$ and define u as the pointwise limit of the sequence $\{u_n\}$. Since $H_0^{1,\lambda}(\Omega)$ is Hilbert space and $\{u_n\}$ is bounded in $H_0^{1,\lambda}(\Omega)$ so it admits a weakly convergent subsequence. Assume u_n weakly converges to v in $H_0^{1,\lambda}(\Omega)$ and hence u_n converges to v in $L^2(\Omega)$. So $\{u_n\}$ has a subsequence that converges to v pointwise. Consequently, $u = v$. So we may assume that the sequence $\{u_n\}$ weakly converges to u in $H_0^{1,\lambda}(\Omega)$. Choose $v' \in C_0^1(\Omega)$. By Lemma 4.2, there exists $C > 0$ such that $u \ge u_n \ge C$ a.e in supp(v') and for all $n \in \mathbb{N}$. So

$$|\frac{f_n v'}{(u_n + \frac{1}{n})}| \le \frac{\|v'\|_\infty |f|}{C} \quad \text{for all } n \in \mathbb{N}$$

By Dominated Convergence Theorem, we have

$$\lim_{n\to\infty} \int_\Omega \frac{f_n v'}{(u_n + \frac{1}{n})} dX = \int_\Omega \lim_{n\to\infty} \frac{f_n v'}{(u_n + \frac{1}{n})} dX = \int_\Omega \frac{f v'}{u} dX. \tag{22}$$

As u_n is a solution of (6) so from (9) we get,

$$\int_\Omega \langle A\nabla^* u_n, \nabla v'\rangle dX = \int_\Omega \frac{f_n v'}{(u_n + \frac{1}{n})} dX$$

Take $n \to \infty$ and use (22) we obtain,

$$\int_\Omega \langle A\nabla^* u, \nabla v'\rangle dX = \int_\Omega \frac{f v'}{u} dX$$

Hence, $u \in H_0^{1,\lambda}(\Omega)$ is a solution of (1).

Let u and v be two solutions of (1). The function $w = (u - v)^+ \in H_0^{1,\lambda}(\Omega)$ can be considered as a test function. Since u_n and v_n are two solutions of (1) so we have

$$\int_\Omega \langle A\nabla^* u, \nabla^* w\rangle dX = \int_\Omega \frac{f w}{u} dX$$
and
$$\int_\Omega \langle A\nabla^* v, \nabla^* w\rangle dX = \int_\Omega \frac{f w}{v} dX$$

By subtracting one from the other, we get

$$\int_\Omega \langle A\nabla^* (u - v), \nabla^* w\rangle \, dX = \int_\Omega \frac{f(v-u)}{uv} w dX \le 0.$$

Which ensures us

$$\int_\Omega \langle A\nabla^* w, \nabla^* w\rangle \, dX \le 0.$$

Hence, $w = 0$ and so $(u - v) \le 0$. By interchanging the role of u and v, we get $(v - u) \le 0$. Consequently, $u = v$ a.e in Ω. □

Proof of Theorem 3.2: (i) Let $k > 1$ and define $S(k) = \{x \in \Omega : u_n(x) \ge k\}$. We can treat the function

$$v(x) = \begin{cases} u_n(x) - k & x \in S(k) \\ o & \text{otherwise} \end{cases}$$

as a function in $C_0^1(\Omega)$. So by (5) we have

$$\begin{aligned} \int_{S(k)} \langle A\nabla^* v, \nabla^* v\rangle \, dX &= \int_{S(k)} \frac{f_n v}{(v + k + \frac{1}{n})} dX \\ &\le \int_{S(k)} f v \, dX \le \|f\|_{L^r(\Omega)} \|v\|_{L^{2^*_\lambda}(\Omega)} |S(k)|^{1 - \frac{1}{2^*_\lambda} - \frac{1}{r}} \end{aligned} \tag{23}$$

where $2^*_\lambda = \frac{2Q}{Q-2}$. By Theorem 2.4, there exists $C > 0$ such that

$$\|v\|^2_{L^{2^*_\lambda}(\Omega)} \leq C \int_\Omega \langle A\nabla^* v, \nabla^* v\rangle dX$$
$$= C \int_{S(k)} \langle A\nabla^* v, \nabla^* v\rangle \, dX \leq C\|f\|_{L^r(\Omega)}\|v\|_{L^{2^*_\lambda}(\Omega)}|S(k)|^{1-\frac{1}{2^*_\lambda}-\frac{1}{r}} \tag{24}$$

The last inequality follows from (23). Inequality (24) ensures us

$$\|v\|_{L^{2^*_\lambda}(\Omega)} \leq C\|f\|_{L^r(\Omega)}|S(k)|^{1-\frac{1}{2^*_\lambda}-\frac{1}{r}}$$

Assume $1 < k < h$. Using last inequality, we obtain

$$|S(h)|^{\frac{1}{2^*_\lambda}}(h-k) = \left(\int_{S(h)} (h-k)^{2^*_\lambda} \, dX\right)^{\frac{1}{2^*_\lambda}} \leq \left(\int_{S(k)} (v(x))^{2^*_\lambda} \, dX\right)^{\frac{1}{2^*_\lambda}}$$
$$\leq \|v\|_{L^{2^*_\lambda}(\Omega)} \leq C\|f\|_{L^r(\Omega)}|S(k)|^{1-\frac{1}{2^*_\lambda}-\frac{1}{r}}$$

So,

$$|S(h)| \leq \left(\frac{C\|f\|_{L^r(\Omega)}}{(h-k)}\right)^{2^*_\lambda} |S(k)|^{2^*_\lambda(1-\frac{1}{2^*_\lambda}-\frac{1}{r})}$$

As $r > \frac{Q}{2}$ we have, $2^*_\lambda(1 - \frac{1}{2^*_\lambda} - \frac{1}{r}) > 1$. Let

$$d^{2^*_\lambda} = (C\|f\|_{L^r(\Omega)})^{2^*_\lambda} 2^{\frac{(2^*_\lambda)^2(1-\frac{1}{2^*_\lambda}-\frac{1}{r})}{[2^*_\lambda(1-\frac{1}{(2^*_\lambda}-\frac{1}{r})-1]}} |S(1)|^{2^*_\lambda(1-\frac{1}{2^*_\lambda}-\frac{1}{r})-2}$$

By Theorem 2.5 we have $|S(1+d)| = 0$. Hence, $u_n(x) \leq 1+d$ a.e in Ω. We get a positive constant C independent of n such that $u_n \leq C\|f\|_{L^r(\Omega)}$ a.e in Ω for all $n \in \mathbb{N}$. Hence, $\|u\|_{L^\infty(\Omega)} \leq C\|f\|_{L^r(\Omega)}$

(ii) If $r = 1$ then $s = 2^*_\lambda$. Since $u \in H_0^{1,\lambda}(\Omega)$ so by Theorem 2.4, we have $u \in L^s(\Omega)$. If $1 < r < \frac{Q}{2}$. Choose $\delta > 1$ (to be determined later). Consider the function $w = u^{2\delta-1}$. By the density argument, w can be treated as a test function. Put w in (9), we have

$$\int_\Omega (2\delta-1)u_n^{(2\delta-2)}\langle A\nabla^* u_n, \nabla^* u_n\rangle dX = \int_\Omega \frac{f_n w}{u_n + \frac{1}{n}} dX \leq \int_\Omega f u_n^{2\delta-2} dX$$

By using Hölder inequality on the RHS of the above inequality, we get

$$\int_\Omega \langle A\nabla^* u_n^\delta, \nabla^* u_n^\delta\rangle dX = \int_\Omega \delta^2 u_n^{(2\delta-2)}\langle A\nabla^* u_n, \nabla^* u_n\rangle dX$$

$$\leq \tfrac{\delta^2}{(2\delta-1)}\|f\|_{L^r(\Omega)}\left(\int_\Omega u_n^{(2\delta-2)r'}dX\right)^{\frac{1}{r'}} \tag{25}$$

where $\frac{1}{r}+\frac{1}{r'}=1$. By Theorem 2.4, we have

$$\begin{aligned}\int_\Omega u_n^{2^*_\lambda\delta} &\leq C\left(\int_\Omega \langle A\nabla^* u_n^\delta, \nabla^* u_n^\delta\rangle dX\right)^{\frac{2^*_\lambda}{2}}\\ &\leq C\left\{\frac{\delta^2}{(2\delta-1)}\|f\|_{L^r(\Omega)}\left(\int_\Omega u_n^{(2\delta-2)r'}dX\right)^{\frac{1}{r'}}\right\}^{\frac{2^*_\lambda}{2}}, \text{ [by (25)]}\end{aligned} \tag{26}$$

We choose δ such that $2^*_\lambda\delta = (2\delta-2)r'$ so $\delta = \frac{r(Q-2)}{(Q-2r)}$. Clearly, $\delta > 1$ and $2^*_\lambda\delta = s$. By using (26), we have

$$\left(\int_\Omega u_n^s dX\right)^{(1-\frac{2^*_\lambda}{2r'})} \leq C$$

Also, $(1-\frac{2^*_\lambda}{2r'}) > 0$ as $r < \frac{Q}{2}$. So we get

$$\int_\Omega u_n^s dX \leq C, C > 0 \text{ is independent of } n.$$

By Dominated Convergence Theorem, we have

$$\int_\Omega u^s dX \leq C.$$

Hence we are done.

□

6.2 The Case $\nu > 1$

Proof of Theorem 3.3: Define u as the pointwise limit of $\{u_n\}$. By Lemma 5.2, $\{u_n\}$ and $\{u_n^{\frac{\nu+1}{2}}\}$ are bounded in $H^{1,\lambda}_{loc}(\Omega)$ and $H^{1,\lambda}_0(\Omega)$ respectively. So by the similar argument as the proof of Theorem 3.1 we can prove $u \in H^{1,\lambda}_{loc}(\Omega)$ and $u^{\frac{\nu+1}{2}} \in H^{1,\lambda}_0(\Omega)$.

Let $v \in C^1_0(\Omega)$ and $\Omega' = \text{supp}(v)$. Without loss of generality we can assume u_n weakly converges to u in $H^{1,\lambda}(\Omega')$. By Lemma 4.2, there exists $C > 0$ such that $u_n(x) \geq C$ a.e $x \in \Omega'$ and for all $n \in \mathbb{N}$. So, $u \geq C > 0$ a.e in Ω'. Also,

$$\left|\frac{f_n v}{(u_n+\frac{1}{n})^\nu}\right| \leq \frac{\|v\|_\infty |f|}{C^\nu}, \text{ for all } n \in \mathbb{N}$$

By the Dominated Convergence Theorem, we have

$$\lim_{n\to\infty}\int_{\Omega'}\frac{f_n v}{(u_n+\frac{1}{n})^{\nu}}dX=\int_{\Omega'}\lim_{k\to\infty}\frac{f_n v}{(u_n+\frac{1}{n})^{\nu}}dX=\int_{\Omega'}\frac{fv}{u^{\nu}}dX. \tag{27}$$

As u_n is a solution of (6) so

$$\int_{\Omega'}\langle A\nabla^* u_n,\nabla v\rangle dX=\int_{\Omega'}\frac{f_n v}{(u_n+\frac{1}{n})^{\nu}}dX$$

Take $n\to\infty$ and use (27), we get

$$\int_{\Omega}\langle A\nabla^* u,\nabla v\rangle dX=\int_{\Omega}\frac{fv}{u^{\nu}}dX$$

Hence, $u\in H^{1,\lambda}_{loc}(\Omega)$ is a solution of (1). □

Proof of Theorem 3.4: (i) The same proof of Theorem (3.2) will work.

(ii) If $r=1$ then $s=\frac{2^*_\lambda(\nu+1)}{2}$. Also, $u^{\frac{\nu+1}{2}}\in H^{1,\lambda}_0(\Omega)$. By Theorem 2.4, we have $u\in L^s(\Omega)$. If $1<r<\frac{Q}{2}$. Choose $\delta>\frac{\nu+1}{2}$. By the density argument, $v=u_n^{2\delta-1}$ can be considered a test function. From (9), we have

$$\int_{\Omega}\langle A\nabla^* u_n,\nabla^* u_n^{2\delta-1}\rangle\, dX=\int_{\Omega}\frac{f_n u_n^{2\delta-1}}{(u_n+\frac{1}{n})^{\nu}}\, dX$$

which gives us

$$\begin{aligned}\int_{\Omega}(2\delta-1)u_n^{2\delta-2}\langle A\nabla^* u_n,\nabla^* u_n\rangle dX &\le \int_{\Omega} f u_n^{2\delta-\nu-1}dX\\ &\le \|f\|_{L^r(\Omega)}\left(\int_{\Omega}u_n^{(2\delta-\nu-1)r'}dX\right)^{\frac{1}{r'}}\end{aligned} \tag{28}$$

By Theorem 2.4, there exists $C>0$ such that

$$\int_{\Omega}u_n^{\delta 2^*_\lambda}dX\le C\left(\int_{\Omega}\langle A\nabla^* u_n^{\delta},\nabla^* u_n^{\delta}\rangle dX\right)^{\frac{2^*_\lambda}{2}}\le C\left(\int_{\Omega}\delta^2 u_n^{2\delta-2}\langle A\nabla^* u_n,\nabla^* u_n\rangle dX\right)^{\frac{2^*_\lambda}{2}} \tag{29}$$

By using (28) and (29), we get

$$\int_{\Omega}u_n^{\delta 2^*_\lambda}dX\le C\{\frac{\delta^2}{(2\delta-1)}\|f\|^r_L(\Omega)\}^{\frac{2^*_\lambda}{2}}\left(\int_{\Omega}u_n^{(2\delta-\nu-1)r'}dX\right)^{\frac{2^*_\lambda}{2r'}}$$

Choose δ such that $\delta 2^*_\lambda = (2\delta - \nu - 1)r'$ then $2^*_\lambda \delta = s$. As $r < \frac{Q}{2}$ so $1 - \frac{2^*_\lambda}{2r'} > 0$. we have $\int_\Omega u_n^s dX \leq C$. Hence, by Dominated Convergence Theorem we get $u \in L^s(\Omega)$.

□

6.3 The Case $\nu < 1$

Proof of Theorem 3.5: Since $\{u_n\}$ is bounded in $H_0^{1,\lambda}(\Omega)$ so it has a subsequence which converges to u weakly in $H_0^{1,\lambda}(\Omega)$. Without loss of generality we can assume $u_n \rightharpoonup u$in $H_0^{1,\lambda}(\Omega)$. Let $v \in C_0^1(\Omega)$. By the Lemma 4.2, there exists $C > 0$ such that $u_n(x) \geq C$ a.e $x \in \text{supp}(v)$ and for all $n \in \mathbb{N}$. So

$$|\frac{f_n v}{(u_n + \frac{1}{n})^\nu}| \leq \frac{\|v\|_\infty |f|}{C^\nu} \text{ for all } n \in \mathbb{N}$$

By the Dominated Convergence Theorem, we have

$$\lim_{n\to\infty} \int_\Omega \frac{f_n v}{(u_n + \frac{1}{n})^\nu} dX = \int_\Omega \lim_{k\to\infty} \frac{f_n v}{(u_n + \frac{1}{n})^\nu} dX = \int_\Omega \frac{f v}{u^\nu} dX. \tag{30}$$

As u_n is a solution of (6) so,

$$\int_\Omega \langle A\nabla^* u_n, \nabla v\rangle dX = \int_\Omega \frac{f_n v}{(u_n + \frac{1}{n})^\nu} dX$$

Take $n \to \infty$ and (30) we get

$$\int_\Omega \langle A\nabla^* u, \nabla v\rangle dX = \int_\Omega \frac{f v}{u^\nu} dX$$

Hence, $u \in H_0^{1,\lambda}(\Omega)$ is a solution of (1) with $\nu < 1$. The proof of uniqueness is similar to Theorem 3.1. □

Proof of Theorem 3.6: (i) The proof is similar to the proof of Theorem 3.2.
(ii) If $r = (\frac{2^*_\lambda}{1-\nu})'$ then $s = 2^*_\lambda$. By the embedding theorem and (9), we have

$$\left(\int_\Omega u_n^{2^*_\lambda} dX\right)^{\frac{1}{2^*_\lambda}} \leq C\left(\int_\Omega \langle A\nabla^* u_n, \nabla^* u_n\rangle dX\right)^{\frac{1}{2}}$$
$$= C\left(\int_\Omega \frac{f_n u_n}{(u_n + \frac{1}{n})^\nu} dX\right)^{\frac{1}{2}} \leq C\left(\int_\Omega f u_n^{1-\nu} dX\right)^{\frac{1}{2}}$$
$$\leq C\|f\|_{L^r(\Omega)}^{\frac{1}{2}} \left(\int_\Omega u_n^{(1-\nu)r'} dX\right)^{\frac{1}{2r'}}$$

Since $r' = \frac{2^*_\lambda}{1-\nu}$ so using the above inequality we get

$$\int_\Omega u_n^{2^*_\lambda} dX \le C\|f\|_{L^r(\Omega)}^{\frac{2^*_\lambda}{1+\nu}}$$

By Dominated Convergence Theorem we have $u \in L^{2^*_\lambda}(\Omega)$.

Let $(\frac{2^*_\lambda}{1-\nu})' < r < \frac{Q}{2}$. Choose $\delta > 1$ (to be determined later). We can treat the function $v = u_n^{2\delta-1}$ as a test function and put it in (9), we obtain

$$\begin{aligned}\int_\Omega \langle A\nabla^* u_n, \nabla^* u_n^{2\delta-1}\rangle dX &= \int_\Omega \frac{f_n u_n^{2\delta-1}}{(u_n+\frac{1}{n})^\nu} dX \\ &\le \int_\Omega f u_n^{2\delta-\nu-1} dX \le \|f\|_{L^r(\Omega)} \left(\int_\Omega u_n^{(2\delta-\nu-1)r'} dX\right)^{\frac{1}{r'}}\end{aligned} \tag{31}$$

Also,

$$\begin{aligned}\int_\Omega \langle A\nabla^* u_n, \nabla^* u_n^{2\delta-1}\rangle dX &= \int_\Omega (2\delta-1) u_n^{2\delta-2}\langle A\nabla^* u_n, \nabla^* u_n\rangle dX \\ &= \int_\Omega \frac{(2\delta-1)}{\delta^2}\langle A\nabla^* u_n^\delta, \nabla^* u_n^\delta\rangle dX\end{aligned} \tag{32}$$

Using (31) and (32) we have

$$\int_\Omega \langle A\nabla^* u_n^\delta, \nabla^* u_n^\delta\rangle dX) \le \frac{\delta^2}{(2\delta-1)}\|f\|_{L^r(\Omega)} \left(\int_\Omega u_n^{(2\delta-\nu-1)r'} dX\right)^{\frac{1}{r'}}$$

By Theorem 2.4, there exists $C > 0$ such that

$$\begin{aligned}\int_\Omega u_n^{\delta 2^*_\lambda} dX &\le C\left(\int_\Omega \langle A\nabla^* u_n^\delta, \nabla^* u_n^\delta\rangle dX\right)^{\frac{2^*_\lambda}{2}} \\ &\le C\{\frac{\delta^2}{(2\delta-1)}\|f\|_{L^r(\Omega)\}}^{\frac{2^*_\lambda}{2}} \left(\int_\Omega u_n^{(2\delta-\nu-1)r'} dX\right)^{\frac{2^*_\lambda}{2r'}}\end{aligned}$$

Choose δ such that $\delta 2^*_\lambda = (2\delta-\nu-1)r'$ then $2^*_\lambda \delta = s$. As $(\frac{2^*_\lambda}{1-\nu})' < r < \frac{Q}{2}$ so $\delta > 1$ and $\frac{2^*_\lambda}{2r'} < 1$. Hence, we have $\int_\Omega u_n^s dX \le C$. Hence, by Dominated Convergence Theorem, we get $u \in L^s(\Omega)$.

□

Proof of Theorem 3.7: Let $\epsilon < \frac{1}{n}$ and $v = (u_n+\epsilon)^{2\delta-1} - \epsilon^{2\delta-1}$ with $\frac{1+\nu}{2} \le \delta < 1$. We can treat v as a function in $C_0^1(\Omega)$. Put v in (9) and we obtain

$$\int_\Omega \langle A\nabla^* u_n, \nabla^* u_n\rangle (u_n+\epsilon)^{2\delta-2} dX \le \frac{1}{(2\delta-1)}\int_\Omega \frac{fv}{(u_n+\frac{1}{n})^\nu}$$

As $\epsilon < \frac{1}{n}$ so we have

$$\int_\Omega \langle A\nabla^* u_n, \nabla^* u_n\rangle (u_n+\epsilon)^{2\delta-2} dX \leq \frac{1}{(2\delta-1)} \int_\Omega f(u_n+\epsilon)^{2\delta-1-\nu}\, dX \quad (33)$$

By some simple calculation, we get

$$\int_\Omega \langle A\nabla^* v, \nabla^* v\rangle dX \leq \frac{\delta^2}{(2\delta-1)} \int_\Omega f(u_n+\epsilon)^{2\delta-1-\nu} dX$$

By Theorem 2.4, we have

$$\left(\int_\Omega v^{2^*_\lambda} dX\right)^{\frac{2}{2^*_\lambda}} \leq \frac{C\delta^2}{(2\delta-1)} \int_\Omega f(u_n+\epsilon)^{2\delta-1-\nu}$$

Take $\epsilon \to 0$ and use Dominated convergence Theorem we have,

$$\left(\int_\Omega u_n^{2^*_\lambda \delta}\right)^{\frac{2}{2^*_\lambda}} \leq \frac{C\delta^2}{(2\delta-1)} \int_\Omega f u_n^{2\delta-1-\nu} \quad (34)$$

If $r = 1$ then choose $\delta = \frac{\nu+1}{2}$ and from the previous inequality we have $\{u_n\}$ is bounded in $L^s(\Omega)$ with $s = \frac{Q(\nu+1)}{(Q-2)}$.

If $r > 1$ then choose δ in such a way that $(2\delta - 1 - \nu)r' = 2^*_\lambda \delta$. Now, applying Hölder inequality on RHS of (34) we have,

$$\begin{aligned}\left(\int_\Omega u_n^{2^*_\lambda \delta}\right)^{\frac{2}{2^*_\lambda}} &\leq \frac{C\delta^2}{(2\delta-1)} \|f\|_{L^r(\Omega)} \left(\int_\Omega u_n^{(2\delta-1-\nu)r'}\right)^{\frac{1}{r'}} \\ &= \frac{C\delta^2}{(2\delta-1)} \|f\|_{L^r(\Omega)} \left(\int_\Omega u_n^{2^*_\lambda \delta}\right)^{\frac{1}{r'}}\end{aligned}$$

As $1 \leq r < \frac{2Q}{(Q+2)+\nu(Q-2)} < \frac{Q}{2}$ so $\frac{2}{2^*_\lambda} > \frac{1}{r'}$. Hence, $\{u_n\}$ is bounded in $L^s(\Omega)$ with $s = 2^*_\lambda \delta = \frac{Qr(\nu+1)}{(Q-2r)}$. Using Hölder inequality in (33), we have

$$\int_\Omega \langle A\nabla^* u_n, \nabla^* u_n\rangle (u_n+\epsilon)^{2\delta-2} dX \leq \frac{1}{(2\delta-1)} \|f\|_{L^r(\Omega)} \left(\int_\Omega (u_n+\epsilon)^{2^*_\lambda \delta}\right)^{\frac{1}{r'}}$$

Since u_n is bounded in $L^s(\Omega)$ so

$$\int_\Omega \langle A\nabla^* u_n, \nabla^* u_n\rangle (u_n+\epsilon)^{2\delta-2} dX \leq C.$$

For $q = \frac{Qr(\nu+1)}{Q-r(1-\nu)}$ and above chosen δ satisfies the condition $(2-2\delta)q = (2-q)s$.

So,

$$\int_\Omega \langle A\nabla^* u_n, \nabla^* u_n\rangle^{\frac{q}{2}} dX = \int_\Omega \frac{|\sqrt{A}\nabla^* u_n|^q}{(u_n+\epsilon)^{q-q\delta}}(u_n+\epsilon)^{q-\delta q} dX$$
$$\leq \left(\int_\Omega \frac{|\sqrt{A}\nabla^* u_n|^2}{(u_n+\epsilon)^{2-2\delta}} dX\right)\left(\int_\Omega (u_n+\epsilon)^s dX\right)^{1-\frac{q}{2}}$$

since $\{u_n\}$ is bounded in $L^s(\Omega)$ and $\epsilon < \frac{1}{n}$ so $\{u_n+\epsilon\}$ is bounded in $L^s(\Omega)$. Consequently, $\{u_n\}$ is bounded in $W_0^{1,\lambda,q}(\Omega)$. Hence $u \in W_0^{1,\lambda,q}(\Omega)$. □

7 Variable singular exponent

Consider the equation

$$\begin{aligned} -\Delta_\lambda u &= \frac{f}{u^{\nu(x)}} \text{ in } \Omega \\ u &> 0 \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned} \tag{35}$$

where $\nu \in C^1(\overline{\Omega})$ is a positive function.

Theorem 7.1 *Let $f \in L^{(2^*_\lambda)'}(\Omega)$ be a function. If there exists $K \Subset \Omega$ such that $0 < \nu(x) \leq 1$ in K^c (complement of K) then (35) has an unique solution in $H_0^{1,\lambda}(\Omega)$ provided $\lambda \geq 1$.*

Proof The same approximation used in the earlier section yields the existence of a strictly positive function u, which is the increased limit of the sequence $\{u_n\} \subset H_0^{1,\lambda}(\Omega) \cap L^\infty(\Omega)$. Also, Lemma 4.2 is satisfied. As $K \Subset \Omega$ so by Lemma 4.2, there exists $C > 0$ such that $u_n(x) \geq C$ for a.e $x \in K$ and for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, u_n solves

$$\begin{aligned} -\Delta_\lambda u_n &= \frac{f_n}{(u_n+\frac{1}{n})^{\nu(x)}} \quad \text{in } \Omega \\ u &> 0 \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned} \tag{36}$$

By using Hölder inequality and the embedding theorem, we have

$$\int_\Omega \langle A\nabla^* u_n, \nabla^* u_n\rangle dx = \int_\Omega \frac{f_n u_n}{(u_n+\frac{1}{n})^{\nu(x)}} dx$$
$$= \int_K \frac{f_n u_n}{(u_n+\frac{1}{n})^{\nu(x)}} dx + \int_{\{K^c\cap\Omega\}} \frac{f_n u_n}{(u_n+\frac{1}{n})^{\nu(x)}} dx$$

$$
\begin{aligned}
&\le ||\frac{1}{C^{\nu(x)}}||_\infty \int_K f u_n dx + \int_{\{x\in K^c\cap\Omega:u_n(x)\le 1\}} f u_n^{1-\nu(x)} dx \\
&\quad + \int_{x\in K^c\cap\Omega:u_n(x)\ge 1} f u_n^{1-\nu(x)} dx \\
&\le ||\frac{1}{C^{\nu(x)}}||_\infty \int_K f u_n dx + \int_{\{x\in K^c\cap\Omega:u_n(x)\le 1\}} f dx \\
&\quad + \int_{x\in K^c\cap\Omega:u_n(x)\ge 1} f u_n dx \\
&\le ||\frac{1}{C^{\nu(x)}}||_\infty ||f||_{L^{(2^*_\lambda)'}(\Omega)} ||u_n||_{L^{2^*_\lambda}} + ||f||_{L^1(\Omega)} \\
&\quad + ||f||_{L^{(2^*_\lambda)'}(\Omega)} ||u_n||_{L^{2^*_\lambda}(\Omega)} \\
&\le C||f||_{L^{(2^*_\lambda)'}(\Omega)} ||u_n||_{H_0^{1,\lambda}(\Omega)} + ||f||_{L^1(\Omega)}
\end{aligned}
$$

We obtain

$$
||u_n||^2_{H_0^{1,\lambda}(\Omega)} \le C||f||_{L^{(2^*_\lambda)'}(\Omega)} ||u_n||_{H_0^{1,\lambda}(\Omega)} + ||f||_{L^1(\Omega)}.
$$

Hence, u_n is bounded in $H_0^{1,\lambda}(\Omega)$. Without loss of generality we can assume that u_n weakly converges to u in $H_0^{1,\lambda}(\Omega)$. Let $w \in C_c^1(\Omega)$. Using Lemma 4.2, there exists $c > 0$ such that $u_n \ge c$ for a.e x in $\text{supp}(w)$. Since u_n solves (36) so

$$
\int_\Omega \langle A\nabla^* u_n, \nabla w\rangle dx = \int_\Omega \frac{f_n w}{(u_n + \frac{1}{n})^{\nu(x)}} dx
$$

Taking $n \to \infty$ and using the dominated convergence theorem, we get

$$
\int_\Omega \langle A\nabla^* u, \nabla w\rangle dx = \int_\Omega \frac{f w}{u^{\nu(x)}} dx
$$

Hence, u is a solution of (35). The proof of the uniqueness part is identical to the one given in Theorem 3.1. □

Theorem 7.2 *Let u be the solution of Eq. (35) with $f \in L^r(\Omega)$, $r > \frac{Q}{2}$. Then $u \in L^\infty(\Omega)$, where $Q = (m+1) + \lambda m$.*

Proof The proof is similar to that of the Theorem 3.2 and is omitted here. □

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Declarations

Conflict of interest The authors declare no competing interests.

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