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Mixed local and nonlocal equation with singular nonlinearity having variable exponent

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Abstract

In this paper, we consider the mixed local-nonlocal quasilinear elliptic problem, with singular nonlinearity having a variable exponent,

$$(P)\begin{cases} -\Delta_p u + (-\Delta)^s_p u = \dfrac{f}{u^{\gamma(x)}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded regular domain in $\mathbb{R}^N$ with $0 < s < 1 < p < N$, γ is positive continuous function, having a convenient behavior near $\partial\Omega$ and f be a nonnegative function belonging to a suitable Lebesgue space. By using approximation methods, we get the existence and regularity of positive solutions for such problems. The summability of the finite energy solutions to Problem (P) in the case $\gamma(x) \equiv 0$ in Ω is also studied.

Keywords Mixed local and nonlocal p-Laplace operators · Singular problem · Existence · Positive solutions · Variable exponent · Regularity

Mathematics Subject Classification 35R11 · 35J92 · 35J75 · 35B65

1 Introduction

In this work, we study the following mixed local-nonlocal quasilinear elliptic problem, with singular nonlinearity having a variable exponent,

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$$
\begin{cases}
-\Delta_p u + (-\Delta)^s_p u = \dfrac{f}{u^{\gamma(x)}} & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases} \tag{1.1}
$$

where $\Omega \subset \mathbb{R}^N$ be a bounded regular domain with $0 < s < 1 < p < N$, $\gamma \in C^1(\overline{\Omega})$ is positive function, having a convenient behavior near $\partial\Omega$ and f be a nonnegative function belonging to a suitable Lebesgue space.

Here $\Delta_p = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is standard p-Laplace operator and $(-\Delta)^s_p$ denotes the so-called *fractional* $p-$*Laplacian* operator, is defined as,

$$
(-\Delta)^s_p u(x) := P.V \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}}\, dy,
$$

where $P.V$ denotes the Cauchy principal value P.V, as usual denoted the principal-value of the integral.

Recently, several studies and great attention have concentrated to the mixed local and non-local operator from different points of view, including the regularity theory, existence and non-existence results and eigenvalue problems, we refer readers to [1, 18–22, 24, 25, 29, 49, 52] and the references therein.

Before stating our main results, we begin by recalling some well known results related to the singular term.

• *Local case* $(s = 1)$. In this case, we consider the following problem,

$$
\begin{cases}
-\Delta_p u = \dfrac{f}{u^{\gamma(x)}} & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{in } \partial\Omega,
\end{cases} \tag{1.2}
$$

In the case of semilinear problem corresponding to $p = 2$ and $\gamma(x) = \gamma \in \mathbb{R}^+_*$, the study of singular elliptic equations was initiated in pioneering work [16] which constitutes the starting point of a large literature, see for instance [4, 9–12] and the references therein. The quasilinear case that is for $p \in (1, \infty)$ with $\gamma(x) = \gamma \in \mathbb{R}^+$, authors in [39] answered the question of existence, multiplicity and regularity of weak solutions in the case $\gamma^+ \in (0, 1)$, which was further extended to the case of $\gamma^+ \geq 1$ in [7] where the authors have showed the multiplicity of weak solutions (see also [13, 26]). Recently for the weighted p-Laplace operator with Muckenhoupt class of weights with $\gamma(x) = \gamma \in \mathbb{R}^+$ the existence and multiplicity is proved in [32, 43]. In [14], the authors consider a singular semilinear elliptic problem with variable exponent $\gamma(x)$, they obtained existence and regularity of the solution, under some conditions on the behavior of the function $\gamma(x)$ near the boundary of Ω. Other related works can be found [28, 37, 38, 46, 47] and the references therein.

Notice that Problem (1.2) has been treated by another type of operator, notably an anisotropic operator see [30], (see also [31, 44]).

• *Non-local case* $(0 < s < 1)$. In this case, Problem (1.1) reduces to,

$$\begin{cases} (-\Delta_p)^s u = \dfrac{f}{u^{\gamma(x)}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.3}$$

This kind of nonlocal problems with nonlinear singular term has been widely studied in recent years.

Problem (1.3) has been studied in [8] when $p = 2$ and $\gamma(x) = \gamma^+ \in \mathbb{R}_*^+$, the authors proved the existence and uniqueness of positive solutions, according to the range of γ^+ and to the summability of f, (see also [54]). In [15], the authors generalize the results of [8] to delicate case of the $p-$ fractional Laplace operator. Notice that, the quasilinear nonlocal elliptic problem (1.3) with variable singular exponent have been considered in [35], where the authors established the existence results by using the approximations arguments. Readers may refer to the related work in [2, 34, 36, 40, 46] and the references therein.

Needless to say, the references mentioned above do not exhaust the rich literature on the subject.

• *Mixed local and non-local case* Problem (1.1) with $p = 2$ and $\gamma(x) = \gamma^+ \in \mathbb{R}_*^+$, it was considered recently in [6]. The authors have showed the existence, uniqueness and regularities properties of the weak solution by deriving uniform a priori estimates and using the method of approximation. In the very recent work [33], the authors have obtained the existence and regularity of solution to problem (1.1) for $p > 1$, $\gamma(x) = \gamma^+ \in \mathbb{R}_*^+$ and under some conditions of f.

The main goal of this paper is to look the natural conditions on f and $\gamma(x)$ which allow us to establish the existence and uniqueness of solution to problem (1.1). As far as we aware, our main results are new even in the semilinear case $p = 2$.

Notice that, the summability of solutions of the following problem,

$$\begin{cases} -\Delta_p u + (-\Delta)_p^s u = f & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.4}$$

is also studied according to the summability of f.

The paper is organized as follows. In the next section we recall some preliminaries dealing with the functional setting associated to our problem, like the concepts of solutions, some functional inequalities and useful lemmas are included that will be needed along of the paper. Section 3 is devoted to proving the summability of the solution of (1.4) in terms of the summability of the right-hand side of this problem. In Sect. 4, we study approximating regular problems, where the singular nonlinearity is replaced by a regular one and we prove that the sequence of solutions to such approximating problems converges to the solution of problem (1.1). Finnaly, in last section, we establish some useful results that will be used in this paper.

2 The functional setting and tools

In this section we present some basic results for fractional Sobolev spaces that will be used in the proofs of our theorems. We refer to Section 2.2 in [51] for the proofs and for other useful estimates and properties of the fractional Sobolev spaces.

Let $\Omega \subset \mathbb{R}^N$ with $N \geq 2$ and $0 < s < 1 < p < \infty$ be the real numbers. The fractional Sobolev space is defined by,

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) : \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dxdy < +\infty \right\}.$$

$W^{s,p}(\Omega)$ is Banach space endowed with the norm,

$$\|u\|_{W^{s,p}(\Omega)} = \left(||u||_{L^p(\Omega)} + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dxdy \right)^{\frac{1}{p}}.$$

The space $W^{s,p}(\mathbb{R}^N)$ is defined analogously. The space $W_0^{s,p}(\Omega)$ is the set of functions defined as,

$$W_0^{s,p}(\Omega) = \left\{ u \in W^{s,p}(\mathbb{R}^N) : u = 0 \quad \text{in } \mathbb{R}^N \backslash \Omega \right\}.$$

Both $W^{s,p}(\Omega)$ and $W_0^{s,p}(\Omega)$ are reflexive Banach spaces, see [5, 27] for more details. The need the next Sobolev inequality, a simple proof can be seen in [48].

Theorem 2.1 (fractional Sobolev inequality)*: Assume that $0 < s < 1$ and $p > 1$ are such that $ps < N$, then there exists a positive constant $S(N, s)$ such that for all $v \in C_0^\infty(\mathbb{R}^N)$, we have that*

$$S(N, s)\left(\int_{\mathbb{R}^N} |v(x)|^{p_s^*} dx \right)^{\frac{p}{p_s^*}} \leq \iint_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^p}{|x - y|^{N+ps}} dx\, dy, \tag{2.1}$$

where $p_s^* = \frac{PN}{N-ps}$ is Sobolev critical exponent.

Now we define, for $1 < p < \infty$, the Sobolev space

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) : |\nabla u| \in L^p(\Omega) \right\},$$

endowed with classical norm,

$$||u||_{W^{1,p}(\Omega)} = ||u||_{L^p(\Omega)} + ||\nabla u||_{L^p(\Omega)}.$$

Notice that $(W^{1,p}(\Omega), ||.||_{W^{1,p}(\Omega)})$ is Banach reflexive space.

The space $W_0^{1,p}(\Omega)$ is defined as the closure of the space $C_0^\infty(\Omega)$ of smooth functions with compact support in the norm of the Sobolev space $W^{1,p}(\Omega)$.

The next result asserts that Sobolev space $W^{1,p}(\Omega)$ is continuously embedded in the fractional Sobolev space, see [27] for more details.

Lemma 2.2 *Let $\Omega \subset \mathbb{R}^N$ be bounded Lipschitz domain and $0 < s < 1 < p < \infty$, then, there exists a constant $C = C(N, p, s)$ such that,*

$$||u||_{W^{s,p}(\Omega)} \leq C||u||_{W^{1,p}(\Omega)}, \quad \forall u \in W^{1,p}(\Omega). \tag{2.2}$$

We need also the following result, where the proof can be found in [21],

Lemma 2.3 *Under the same hypothesis of the previous lemma, then, there exists a constant $C = C(N, p, s, \Omega)$ such that,*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \leq C \int\limits_{\Omega} |\nabla u|^p dx, \quad \forall u \in W_0^{1,p}(\Omega). \tag{2.3}$$

Remark 2.4 It clear that from previous inequality, the following norm on the space $W_0^{1,p}(\Omega)$ defined by

$$||u||_{W_0^{1,p}(\Omega)} = \left(\int\limits_{\Omega} |\nabla u|^p dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \right)^{\frac{1}{p}} \tag{2.4}$$

is equivalent to

$$||u||_{W_0^{1,p}(\Omega)} = \left(\int\limits_{\Omega} |\nabla u|^p dx \right)^{\frac{1}{p}}. \tag{2.5}$$

We define the notion of zero of Dirichlet boundary condition as follows,

Definition 2.5 We say that $u \leq 0$ on $\partial\Omega$, if $u = 0$ in $\mathbb{R}^N \backslash \Omega$ and for every $\varepsilon > 0$, we have

$$(u-\varepsilon)_+ \in W_0^{1,p}(\Omega).$$

We say that $u = 0$ on $\partial\Omega$, if u in nonnegative and $u \leq 0$ on $\partial\Omega$.

Now, we need to precise the sense of the weak solution for the problem (1.1).

Definition 2.6 Assume that $u \in W_{loc}^{1,p}(\Omega) \cap L^{p-1}(\Omega)$. We say that u is a weak solution to problem (1.1), if $u > 0$ in Ω, $\quad u = 0$ on $\partial\Omega$ in the sense of Definition 2.5 with $\frac{f}{u^{\gamma(.)}} \in L_{loc}^1(\Omega)$ and for every $\phi \in C_c^1(\Omega)$, we have

$$\int\limits_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \phi + \int\limits_{\Omega} \phi((-\Delta)_p^s u) dx = \int\limits_{\Omega} \frac{f}{u^{\gamma(x)}} \phi dx. \tag{2.6}$$

Remark 2.7 Let $u \in W^{1,p}_{loc}(\Omega)$ be a nonnegative function in Ω, satisfy $u^{\alpha} \in W^{1,p}_0(\Omega)$, for some $\alpha \geq 1$, then, we have $u = 0$ on $\partial\Omega$ according to Definition 2.5, (see [33] for more details.)

In order to give the summability of solutions to problem (1.3), we need to clarify the sense of the energy solution to (1.3),

Definition 2.8 Let $f \in W^{-1,p'}(\Omega)$ be nonnegative function where $W^{-1,p'}(\Omega)$ is dual space of $W^{1,p}_0(\Omega)$. We say that $u \in W^{1,p}_0(\Omega)$ is a energy solution to (1.3), if

$$u > 0 \quad \text{in } \Omega, u = 0 \text{ on } \mathbb{R}^N\backslash\Omega, \text{ and for every } \phi \in C^{\infty}_c(\Omega), \text{ we have that}$$

$$\int\int_{D_\Omega} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y)}{|x-y|^{N+ps}}dxdy + \int_\Omega |\nabla u|^{p-2}\nabla u \nabla\phi dx = \int_\Omega f\phi dx, \tag{2.7}$$

where $D_\Omega = \mathbb{R}^{2N}\backslash(\Omega^c \times \Omega^c)$.

We need also the following result, the proof of which can be found in [17].

Theorem 2.9 *Suppose that $s \in (0,1)$, $f \in L^m(\Omega)$ for some $m \geq 1$ and define $w \in W^{s,p}_0(\Omega)$ to be the unique solution to*

$$\begin{cases} (-\Delta)^s_p w = f & in\ \Omega, \\ w > 0 & in\ \Omega, \\ w = 0 & in\ \mathbb{R}^N\backslash\Omega, \end{cases} \tag{2.8}$$

then, we have that

(i) *If $m > \frac{N}{ps}$, then, there exists a constant $C > 0$ depending on $N, s, p, \Omega, \|f\|_{L^m(\Omega)}$ such that,*

$$\|w\|_{L^\infty(\Omega)} \leq C. \tag{2.9}$$

(ii) *If $f \in L^{\frac{N}{ps}}(\Omega)$, then, there exists, $\alpha > 0$ such that,*

$$\int_\Omega e^{\alpha|u|} < +\infty. \tag{2.10}$$

(iii) *If $\frac{pN}{(p-1)N+ps} = (p^*_s)' \leq m < \frac{N}{ps}$, then, there exists a constant, $C = C(N,m,s) > 0$ such that,*

$$\|w\|_{L^{m^{**}_s}(\Omega)} \leq C\|f\|_{L^m(\Omega)} \tag{2.11}$$

where $m^{**}_s = \frac{(p-1)mN}{N-pms}$.

Definition 2.10 We define for every $k \geq 0$ and $\sigma \in \mathbb{R}$ the functions,

$$T_k(\sigma) := max\{-k, \min\{k, \sigma\}\} \quad \text{and } G_k(\sigma) := \sigma - T_k(\sigma). \tag{2.12}$$

For the following result, see [17].

Proposition 2.11 *Assume that $v \in W_0^{s,p}(\Omega)$. Then, we have that*

(i) *If $\psi \in Lip(\mathbb{R})$ is such that $\psi(0) = 0$, then $\psi(v) \in W_0^{s,p}(\Omega)$. In particular, for any $k \geq 0$, $T_k(v)$, $G_k(v) \in W_0^{s,p}(\Omega)$;*

(ii) *For any $k \geq 0$,*

$$\|G_k(v)\|^2_{W_0^{s,p}(\Omega)} \leq \int_\Omega G_k(v)\,(-\Delta)^s_p v\,dx; \tag{2.13}$$

(iii) *For any $k \geq 0$,*

$$\|T_k(v)\|^2_{W_0^{s,p}(\Omega)} \leq \int_\Omega T_k(v)\,(-\Delta)^s_p v\,dx. \tag{2.14}$$

The next elementary algebraic inequality from [3] will be used in some arguments.

Lemma 2.12 *Let $a, b \geq 0$, $p \geq 1$ and $\alpha > 0$. Then, there exists a positive constant $C > 0$ such that*

$$|a-b|^{p-2}(a-b)(a^\alpha - b^\alpha) \geq C|a^{\frac{p+\alpha-1}{p}} - b^{\frac{p+\alpha-1}{p}}|^p. \tag{2.15}$$

Next, we state also the following algebraic inequality, the proof of which can be found in [23].

Lemma 2.13 *Let $1 < p < \infty$. Then for any $\xi_1, \xi_2 \in \mathbb{R}^N$, there exists a constant positive $C := C(p)$ such that*

$$\langle |\xi_1|^{p-2}\xi_1 - |\xi_2|^{p-2}\xi_2, \xi_1 - \xi_2\rangle \geq C\frac{|\xi_1 - \xi_2|^2}{(|\xi_2| + |\xi_1|)^{2-p}}. \tag{2.16}$$

Finally we state the following classical numerical iteration result proved in [53] and that we will use later for some boundedness results.

Lemma 2.14 *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function such that*

$$\psi(h) \leq \frac{M\,\psi(k)^\delta}{(h-k)^\gamma}, \quad \forall h > k > 0,$$

where $M > 0$, $\delta > 1$ and $\gamma > 0$. Then $\psi(d) = 0$, where $d^\gamma = M\,\psi(0)^{\delta-1}\,2^{\frac{\delta\gamma}{\delta-1}}$.

3 Summability of solution when $f \in L^m(\Omega)$, $m \geq (p_s^*)' := \frac{pN}{(p-1)N+ps}$

In this section we study the summability of solutions to problem (1.4) when we vary the regularity of source term $f \in L^m(\Omega)$ with $m \geq (p_s^*)' := \frac{pN}{(p-1)N+ps}$. As we commented in the introduction of this work, we will adapt the technique used in [42] for the case $p = 2$ to the general case $1 < p < \infty$.

3.1 Boundedness of solution when $m > \frac{N}{p-1+s}$

In this subsection, we will show the boundedness of any solutions to problem (1.4) if $f \in L^m(\Omega)$ with
$m > \frac{N}{p-1+s}$. Notice that the following result can be seen as a generalization of proposition 9 of [51] to the case of mixed operators.

Theorem 3.1 *Let $0 \lneq f \in L^m(\Omega)$ with $m > \frac{N}{p-1+s}$. Let $u \in W_0^{1,p}(\Omega)$ be the unique energy solution to problem (1.4). Then, there exists a positive constant C depending on $N, \Omega, ||f||_{L^m(\Omega)}, s, ||u||_{W_0^{1,p}(\Omega)}$ such that*

$$||u||_{L^\infty(\Omega)} \leq C.$$

Proof Notice that the existence and uniqueness of solution $u \in W_0^{1,p}(\Omega)$ follows by arguing exactly as in the proof of Lemma 3.1 in [33]. Let be $k > 0$ and consider the function $G_k(u)$ defined in (2.12) as test function in(1.4), we get,

$$\int\int_{D_\Omega} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(G_k(u(x))-G_k(u(y))}{|x-y|^{N+ps}} dxdy. + \int_\Omega |\nabla u|^{p-2}\nabla u \nabla G_k(u)dx = \int_\Omega f G_k(u)dx, \tag{3.1}$$

On the other hand, we know that, $u(x) = T_k(u(x)) + G_k(u(x))$, then, by applying Proposition 2.11, it holds that,

$$||G_k(u)||^p_{W_0^{s,p}(\Omega)} \leq \int_\Omega f G_k(u)dx \leq \int_{A_k} f G_k(u)dx,$$

where $A_k = \{x \in \Omega : u(x) \geq k\}$.

Hence, by using Sobolev inequality (2.1) and Hölder inequality, we get,

$$S||G_k(u)||^p_{L^{p_s^*}(\Omega)} \leq ||f||_{L^m(\Omega)}||G_k(u)||_{L^{p_s^*}(\Omega)}|A_k|^{1-\frac{1}{m}-\frac{1}{p_s^*}}.$$

Thus, we conclude that,

$$||G_k(u)||_{L^{p_s^*}(\Omega)} \leq C||f||_{L^m(\Omega)}^{\frac{1}{p-1}}|A_k|^{\frac{1}{p-1}\left(1-\frac{1}{m}-\frac{1}{p_s^*}\right)}. \tag{3.2}$$

Since $\nabla u(x) = \nabla G_k(u(x))$ for every $x \in A_k$, it follows that,

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla G_k(u))dx = \int_\Omega |\nabla G_k(u)|^p dx. \tag{3.3}$$

So, combining with (3.1), (3.3), Proposition 2.11 and Sobolev embedding, we derive that,

$$\begin{aligned} S||G_k(u)||_{L^{p^*}(\Omega)}^p &\leq \int_{A_k} |\nabla G_k(u)|^p dx \\ &\leq \int_{A_k} f G_k(u) dx \\ &\leq ||f||_{L^m(\Omega)}||G_k(u)||_{L^{p_s^*}(\Omega)}|A_k|^{1-\frac{1}{m}-\frac{1}{p_s^*}} \end{aligned} \tag{3.4}$$

Thus, combining with (3.2) and (3.4), it holds that,

$$\begin{aligned} ||G_k(u)||_{L^{p^*}(\Omega)}^p &\leq C||f||_{L^m(\Omega)}||G_k(u)||_{L^{p_s^*}(\Omega)}|A_k|^{1-\frac{1}{m}-\frac{1}{p_s^*}} \\ &\leq C||f||_{L^m(\Omega)}^{\frac{p}{p-1}}|A_k|^{\frac{p}{p-1}\left(1-\frac{1}{m}-\frac{1}{p_s^*}\right)}. \end{aligned}$$

Since for every $h > k$, we know that $A_h \subset A_k$ and $|G_k(u(x)|\chi_{A_h}(x) \geq (h-k)$ in Ω, we have that

$$\begin{aligned} (h-k)|A_h|^{\frac{1}{p^*}} &\leq ||G_k(u)||_{L^{p^*}(\Omega)} \\ &\leq C||f||_{L^m(\Omega)}^{\frac{1}{p-1}}|A_k|^{\frac{1}{p-1}\left(1-\frac{1}{m}-\frac{1}{p_s^*}\right)}. \end{aligned}$$

So,

$$|A_h| \leq C\frac{||f||_{L^m(\Omega)}^{\frac{p^*}{p-1}}|A_k|^{\frac{p^*}{p-1}\left(1-\frac{1}{m}-\frac{1}{p_s^*}\right)}}{(h-k)^{p^*}}.$$

Now we observe that,

$$\frac{p^*}{p-1}\left(1-\frac{1}{m}-\frac{1}{p_s^*}\right) > 1,$$

if $m > \frac{N}{p-1+s}$. Hence, we can apply Lemma 2.14, with

$$0 < M = C||f||_{L^m(\Omega)}^{\frac{p^*}{p-1}}, \quad \delta = \frac{p^*}{p-1}\left(1 - \frac{1}{m} - \frac{1}{p_s^*}\right) > 1, \quad \gamma = p^* > 0$$
$$\text{and } \psi(\sigma) = |A_\sigma|,$$

to derive that, there exists, $k_0 > 0$ such that $|A_k| = 0$ for every $k > k_0$ and thus,

$$\text{esssup}_\Omega |u| \leq k_0.$$

□

3.2 Summability of solution when $\frac{pN}{(p-1)N+ps} \leq m < \frac{N}{p-1+s}$

The main result of this section is the following result,

Theorem 3.2 *Assume that $f \in L^m(\Omega)$ with*

$$\frac{pN}{(p-1)N+ps} \leq m < \frac{N}{p-1+s}, \tag{3.5}$$

and let $u \in W_0^{1,p}(\Omega)$ be the unique energy solution to problem (1.4). Then, there exists a positive constant $C = C(N, m, s)$ such that,

$$||u||_{L^{m^{**}}(\Omega)} \leq C||f||_{L^m(\Omega)}^{\frac{1}{p-1}} \tag{3.6}$$

where

$$m^{**} = \frac{(p-1)Nm(N-ps)}{(N-p)(N-psm)}. \tag{3.7}$$

Proof For $T > 0$ big to be precise later, we define the following function,

$$\psi(\sigma) = \begin{cases} \sigma^\beta & \sigma \leq T, \\ \beta T^{\beta-1}(\sigma - T) + T^\beta, & \sigma > T, \end{cases} \tag{3.8}$$

where $\beta = \frac{m_s^{**}}{m'}$ and $m_s^{**} = \frac{(p-1)mN}{N-pms}$.

Since $\frac{pN}{(p-1)N+ps} \leq m < \frac{N}{p-1+s}$, so, we deduce that, $\beta \geq 1$.

Using $\psi(u)$ as test function in problem (1.4), we get,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\psi(u(x)) - \psi(u(y))}{|x-y|^{N+ps}} dxdy$$

$$+\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \psi(u)dx = \int_\Omega f\psi(u)dx, \tag{3.9}$$

Observe that,

$$\begin{aligned}\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\psi(u)dx &= \int_{\Omega\cap\{u>T\}} |\nabla u|^{p-2}\nabla u\nabla\psi(u)dx \\ &\quad + \int_{\Omega\cap\{0\le u\le T\}} |\nabla u|^{p-2}\nabla u\nabla\psi(u)dx \\ &= \beta T^{\beta-1}\int_{\Omega\cap\{u>T\}} |\nabla u|^p dx \\ &\quad + \int_{\Omega\cap\{0\le u\le T\}} |\nabla u|^{p-2}\nabla u\nabla u^\beta dx \\ &= \int_{\Omega\cap\{u>T\}} |\nabla u|^p dx \\ &\quad +\beta\left(\frac{p}{p+\beta-1}\right)^p \int_{\Omega\cap\{0\le u\le T\}} |\nabla u^{\frac{p-1+\beta}{p}}|^p dx \ge 0.\end{aligned} \tag{3.10}$$

Combing with (3.9) and (3.10), we get,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\psi(u(x))-\psi(u(y))}{|x-y|^{N+ps}}dxdy \le f\psi(u)dx. \tag{3.11}$$

The same reasoning of the proof of Theorem 2.9, we get that,

$$||u||_{L^{m_s^{**}}(\Omega)} \le C||f||_{L^m(\Omega)}, \tag{3.12}$$

where $m_s^{**} = \frac{(p-1)mN}{N-pms}$ and C is positive constant depends on s, N, m.

Now, we claim that,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\psi(u(x))-\psi(u(y))}{|x-y|^{N+ps}}dxdy \ge 0. \tag{3.13}$$

Indeed, in first we decompose $\mathbb{R}^N$ as follows,

$$\mathbb{R}^N = \Big\{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N : u(x) > T\Big\} \cup \Big\{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N : 0\le u(x)\le T\Big\}$$

Now, we define the following sets,

$$\begin{aligned}\Omega_1 &= \Big\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : u(x) > T, u(y) > T\Big\}\\ \Omega_2 &= \Big\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : u(x) > T, 0 \le u(y) \le T\Big\},\\ \Omega_3 &= \Big\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : 0 \le u(x) \le T, u(y) > T\Big\},\\ \Omega_4 &= \Big\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : 0 \le u(x) \le T, 0 \le u(y) \le T\Big\}.\end{aligned}$$

Hence,

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\psi(u(x))-\psi(u(y))}{|x-y|^{N+ps}}dxdy\\ &\quad= \sum_{i=1}^{4}\int\int_{\Omega_i} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\psi(u(x))-\psi(u(y))}{|x-y|^{N+ps}}dxdy.\end{aligned} \tag{3.14}$$

Let us start by I_1, where

$$I_1 = \int\int_{\Omega_1} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\psi(u(x))-\psi(u(y))}{|x-y|^{N+ps}}dxdy.$$

Therefore, by using the definition of ψ and for $(x, y) \in \Omega_1$, it follows that,

$$\psi(u(x)) - \psi(u(y)) = \beta T^{\beta-1}(u(x) - u(y)),$$

which implies,

$$I_1 = \beta T^{\beta-1}\int\int_{\Omega_1} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy \ge 0. \tag{3.15}$$

Now, we treat I_2. Let $(x, y) \in \Omega_2$, then, we have that

$$\psi(u(x)) - \psi(u(y)) = \beta T^{\beta-1}(u(x) - T) + T^\beta - u^\beta(y) \ge 0.$$

Since $u(x) \ge u(y)$ for all $(x, y) \in \Omega_2$, therefore,

$$\begin{aligned}I_2 &= \int\int_{\Omega_2} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\psi(u(x))-\psi(u(y))}{|x-y|^{N+ps}}dxdy\\ &= \int\int_{\Omega_2} \frac{(u(x)-u(y))^{p-1}(\beta T^{\beta-1}(u(x)-T)+T^\beta-u^\beta(y)}{|x-y|^{N+ps}}dxdy \ge 0.\end{aligned} \tag{3.16}$$

Respect I_3. Let $(x, y) \in \Omega_3$, then, by using the definition of ψ, we get,

$$\psi(u(x)) - \psi(u(y)) = u^\beta(x) - T^\beta - \beta T^{\beta-1}(u(y) - T)) \le 0.$$

Obviously $u(x) \le u(y)$ for every $(x, y) \in \Omega_3$, which leads to,

$$I_3 = \int\int_{\Omega_3} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\psi(u(x))-\psi(u(y))}{|x-y|^{N+ps}}dxdy$$
$$= \int\int_{\Omega_3} \frac{(u(y)-u(x))^{p-2}(u(x)-u(y))((u^\beta(x)-T^\beta)-\beta T^{\beta-1}(u(y)-T))}{|x-y|^{N+ps}}dxdy \ge 0. \tag{3.17}$$

Finally, we consider I_4. It clear that, for $(x, y) \in \Omega_4$, we have that

$$\psi(u(x)) - \psi(u(y)) = u^\beta(x) - u^\beta(y).$$

Hence, by using Lemma 2.12, we get the existence a positive constant C, such that

$$I_4 = \int\int_{\Omega_4} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\psi(u(x))-\psi(u(y))}{|x-y|^{N+ps}}dxdy$$
$$= \int\int_{\Omega_4} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(u^\beta(x)-u^\beta(y))}{|x-y|^{N+ps}}dxdy$$
$$\ge C\int\int_{\Omega_4} \frac{|u^{\frac{p+\beta-1}{p}}(x)-u^{\frac{p+\beta-1}{p}}(y)|^p}{|x-y|^{N+ps}}dxdy \ge 0. \tag{3.18}$$

Combining with (3.14), (3.15), (3.16), (3.17) and (3.18) and claim follows.

Going back to (3.9) and using (3.13), it follows that,

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\psi(u)dx \le \int_\Omega f\psi(u)dx. \tag{3.19}$$

On the other hand, by taking T large enough in the definition of ψ, it holds that, $\psi(u) = u^\beta$ if $0 \le u \le T$.
Now, by using Hölder inequality in (3.19), we get,

$$\beta\left(\frac{p}{p+\beta-1}\right)^p \int_\Omega |\nabla u^{\frac{p+\beta-1}{p}}|^p dx \le ||f||_{L^m(\Omega)}||u||^\beta_{L^{m_s^{**}}(\Omega)},$$

here, we have used the facts,

$$\frac{1}{m} + \frac{\beta}{m_s^{**}} = 1, \quad and \quad \frac{p+\beta-1}{p} \ge 1.$$

Since $\frac{p^*}{p}(p+\beta-1) > 1$, then, by using (2.9), we obtain that,

$$||u^{\frac{p+\beta-1}{p}}||^p_{L^{p^*}(\Omega)} \le C||f||^{\frac{p-1+\beta}{p-1}}_{L^m(\Omega)},$$

which implies that,

$$||u||_{L^{m^{**}}(\Omega)} \le C||f||^{\frac{1}{p-1}}_{L^m(\Omega)}, \tag{3.20}$$

with

$$m^{**} = \frac{(p-1)Nm(N-ps)}{(N-p)(N-psm)}.$$

Hence the result follows. □

Remark 3.3 Observe that, m^{**} is increasing in s and

$$\lim_{s\to 1^-} \frac{(p-1)Nm(N-ps)}{(N-p)(N-psm)} = \frac{(p-1)Nm}{(N-pm)}.$$

Thus, we have that

$$\frac{(p-1)mN}{N-pms} < m^{**} < \frac{(p-1)Nm}{(N-pm)}, \tag{3.21}$$

which shows that the exponent defined in (3.7) is better than the one coming from the $p-$Laplacian fractional only, but it less than the one coming from the $p-$Laplacian only.

Hence, the previous result clarify that the mixed local and nonlocal p-Laplace operators has it own features and we can not consider the fractional $p-$Laplacian as a lower order perturbation only of the classical elliptic problem.

4 Existence results

In this section we study the existence and uniqueness of positive solution to problem (1.1) under some extra hypothesis on f and γ.

4.1 Approximation problems

In order to deal with Problem (1.1), we follow closely the approximate scheme of [33] (see also [6] in case $p=2$).

For $n \in I\!N$, let us consider the following approximating problems,

$$\begin{cases} -\Delta_p u_n + (-\Delta)^s_p u_n = \dfrac{f_n}{(u_n + \frac{1}{n})^{\gamma(x)}} & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ u_n = 0 & \text{in } \mathbb{R}^N \backslash \Omega. \end{cases} \tag{4.1}$$

where $f_n = T_n(f)$.

Let us start by proving that problem (4.1) has a positive solution $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, more precisely we have the following result,

Lemma 4.1 *Problem* (4.1) *has a nonnegative positive solution* $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Proof Let $n \in I\!N$ be fixed and $v \in L^p(\Omega)$, then, by using Lemma 3.1 in [33], there exists a unique positive solution $w \in W_0^{1,p}(\Omega)$ to the following problem,

$$\begin{cases} -\Delta_p w + (-\Delta)^s_p w = \dfrac{f_n}{(v^+ + \frac{1}{n})^{\gamma(x)}} & \text{in } \Omega, \\ w > 0 & \text{in } \Omega, \\ w = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{4.2}$$

where $v^+ = \max\{v, 0\}$.

So, we can define the operator $v \in L^p(\Omega) \mapsto w := S(v) \in W_0^{1,p}(\Omega) \subset L^p(\Omega)$, where w is the unique solution to problem (4.2).

Since $\gamma \in C^1(\overline{\Omega})$, we can define $\gamma^* = ||\gamma||_{L^\infty(\Omega)}$. Choosing w as test function in (4.2) and using Poincaréand Hölder's inequalities, we obtain

$$\int_\Omega |\nabla w|^p dx \le n^{\gamma^*+1} |\Omega|^{\frac{p-1}{p}} \left(\int_\Omega |w|^p dx \right)^{\frac{1}{p}} \le C n^{\gamma^*+1} |\Omega|^{\frac{p-1}{p}} \left(\int_\Omega |\nabla w|^p dx \right)^{\frac{1}{p}},$$

where C is positive constant.

So,

$$\left(\int_\Omega |\nabla w|^p dx \right)^{\frac{1}{p}} \le C^{\frac{1}{p-1}} n^{\frac{\gamma^*+1}{p-1}} |\Omega|^{\frac{1}{p}} := R,$$

which means that the ball of the radius R in $L^p(\Omega)$ is invariant by S. Now, by using the same arguments as in the proof of Proposition 2.3 in [15], (see also Lemma 3.2 in [33]), it follows that the mapping S is continuous and compact. Therefore, by applying Schauder's fixed point Theorem, the operator T admits at least one fixed point $u_n \in W_0^{1,p}(\Omega)$ such that $S(u_n) = u_n$. Hence, we get the existence u_n which is the solution to Problem (4.1).

Since the r.h.s of (4.1) belongs to $L^\infty(\Omega)$, then, by applying Theorem 3.1, we get $u_n \in L^\infty(\Omega)$.

Now, keeping in mind that, $\frac{f_n}{(u_n^+ + \frac{1}{n})^{\gamma(x)}} \geq 0$, thus, using u_n^- as test function in problem (4.1), we get

$$0 \leq \int_\Omega |\nabla u_n^-|^p dx \leq \int_\Omega \frac{f_n}{(u_n^+ + \frac{1}{n})^{\gamma(x)}} u_n^- \leq 0.$$

Hence $u_n \geq 0$, and result follows. □

Lemma 4.2 *The sequence $\{u_n\}_n$ obtained in the previous lemma is increasing with respect to n and*

$$u_n(x) \geq c(\omega) > 0, \quad \textit{for a.e} \quad x \in \omega \subset\subset \Omega.$$

Proof Let u_n and u_{n+1} are two positive solutions to (4.1), then, by taking $\varphi^+ = (u_n - u_{n+1})^+$ as test function in (4.1), we get

$$\int\int_{D_\Omega} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))\left(\varphi^+(x) - \varphi^+(y)\right)}{|x-y|^{N+ps}} dxdy + \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi^+(x) dx = \int_\Omega \frac{f_n}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} \varphi^+(x) dx, \tag{4.3}$$

and

$$\int\int_{D_\Omega} \frac{|u_{n+1}(x) - u_{n+1}(y)|^{p-2}(u_{n+1}(x) - u_{n+1}(y))\left(\varphi^+(x) - \varphi^+(y)\right)}{|x-y|^{N+ps}} dxdy + \int_\Omega |\nabla u_{n+1}|^{p-2} \nabla u_{n+1} \nabla \varphi^+(x) dx = \int_\Omega \frac{f_{n+1}}{\left(u_{n+1} + \frac{1}{n+1}\right)^{\gamma(x)}} \varphi^+(x) dx, \tag{4.4}$$

where $\varphi^+ = (u_n - u_{n+1})^+$.

Since, $\{f_n\}_n$ is increasing with respect to n, it follows that,

$$\int_\Omega \left[\frac{f_n}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} - \frac{f_{n+1}}{\left(u_{n+1} + \frac{1}{n+1}\right)^{\gamma(x)}} \right] \varphi^+(x) dx \leq 0. \tag{4.5}$$

Now, subtracting (4.3) with (4.4) and by taking into consideration (4.5), we obtain,

$$\int\int_{D_\Omega} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\varphi^+(x) - \varphi^+(y))}{|x-y|^{N+ps}} dxdy - \int\int_{D_\Omega} \frac{|u_{n+1}(x) - u_{n+1}(y)|^{p-2}(u_{n+1}(x) - u_{n+1}(y))(\varphi^+(x) - \varphi^+(y))}{|x-y|^{N+ps}} dxdy$$

$$+\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_{n+1}|^{p-2}\nabla u_{n+1}\right)\nabla\varphi^+(x)dx \le 0. \tag{4.6}$$

Hence, arguing exactly as in the proof of Lemma 9 in [41], it holds that,

$$\begin{aligned}&\int\int_{D_\Omega} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi^+(x)-\varphi^+(y))}{|x-y|^{N+ps}}dxdy\\ &-\int\int_{D_\Omega} \frac{|u_{n+1}(x)-u_{n+1}(y)|^{p-2}(u_{n+1}(x)-u_{n+1}(y))(\varphi^+(x)-\varphi^+(y))}{|x-y|^{N+ps}}dxdy \ge 0. \end{aligned}\tag{4.7}$$

Therefore, combining (4.6) and (4.7), we get

$$\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_{n+1}|^{p-2}\nabla u_{n+1}\right)\nabla\varphi^+(x)dx \le 0. \tag{4.8}$$

Thus, by applying Lemma 2.13, we derive that, $u_{n+1} \ge u_n$. This concludes the proof of the first assertion.

Now, we will show the second assertion. Observe that $u_1 \in L^\infty(\Omega)$ solves,

$$-\Delta_p u_1 + (-\Delta)_p^s u_1 = \frac{f_1}{(u_1+1)^{\gamma(x)}} \in L^\infty(\Omega).$$

Thus, by using Theorem 8.3 in [45], we have for every $\omega \subset\subset \Omega$, there exists a constant $c(\omega) > 0$ such that $u_1 \ge c(\omega) > 0$ in Ω.
Hence by using monotonicity of $\{u_n\}_n$, the second assertion follows. □

4.2 Passing to the limit

For fixed $\delta > 0$, we define the following set,

$$\Omega_\delta = \{x \in \Omega : dist(x, \partial\Omega) < \delta\}.$$

Theorem 4.3 *Let $f \in L^{\sigma_1}(\Omega)^+$ with $\sigma_1 = \dfrac{pN}{(p-1)N+p}$. Suppose that, there exists a $\delta > 0$ such that $\gamma(x) \le 1$ in Ω_δ. Then, there exists a solution $u \in W_0^{1,p}(\Omega)$ to problem (1.1).*

Proof Let us denote $\omega_\delta = \Omega\backslash\Omega_\delta$, by previous Lemma, we know that, $u_n(x) \ge C_{\omega_\delta} > 0$. Now taking u_n as test function in (4.1), we obtain

$$\int_\Omega |\nabla u_n|^p dx + \int\int_{D_\Omega} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}dxdy$$

$$
\begin{aligned}
&= \int\limits_{\Omega} \frac{f_n}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} u_n dx, \\
&= \int_{\Omega_\delta} \frac{f_n}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} u_n dx + \int_{\omega_\delta} \frac{f_n}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} u_n dx \\
&\leq \int_{\Omega_\delta} f u_n^{1-\gamma(x)} dx + \int_{\omega_\delta} \frac{f}{C_{\omega_\delta}^{\gamma(x)}} u_n dx \\
&\leq \int_{\Omega_\delta \cap \{u_n \leq 1\}} f u_n^{1-\gamma(x)} dx + \int_{\Omega_\delta \cap \{u_n \geq 1\}} f u_n^{1-\gamma(x)} dx + \int_{\omega_\delta} \frac{f}{C_{\omega_\delta}^{\gamma(x)}} u_n dx \\
&\leq ||f||_{L^1(\Omega)} + \left(1 + ||C_{\omega_\delta}^{-\gamma(.)}||_{L^\infty(\Omega)}\right) \int\limits_{\Omega} f u_n dx. \qquad (4.9)
\end{aligned}
$$

Therefore, by using Hölder's and Sobolev's inequalities, it holds that,

$$
||u_n||^p_{W_0^{1,p}(\Omega)} \leq ||f||_{L^1(\Omega)} + C\left(1 + ||C^{-\gamma(.)}||_{L^\infty(\Omega)}\right) ||f||_{L^{\sigma_1}(\Omega)} ||u_n||_{W_0^{1,p}(\Omega)},
$$

hence, we conclude

$$
||u_n||_{W_0^{1,p}(\Omega)} \leq C, \quad \text{for all } n \in I\!N,
$$

where C is positive constant independent of n.

Since $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, then (up to a subsequence) $\{u_n\}$ such that, $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$, $u_n \to u$ strongly in $L^r(\Omega)$ for every $r < p^*$ and $u_n(x) \to u(x)$ a.e in Ω. Hence, the pointwise limit u belong also to $L^{p-1}(\Omega)$.

So, by applying Theorem 5.2, we get

$$
\nabla u_n \to \nabla u \quad \text{pointwise almost everywhere in } \Omega.
$$

Hence, we conclude that

$$
\lim_{n\to\infty} \int\limits_{\Omega} |\nabla u_n|^{p-2} \nabla u_n dx \varphi = \int\limits_{\Omega} |\nabla u|^{p-2} \nabla u \varphi dx, \qquad (4.10)
$$

for every $\varphi \in C_0^1(\Omega)$.

On the other hand, $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$ and $\varphi \in C_0^1(\Omega)$, then by using Lemma 2.3, it follows that

$$
\frac{|u_n(x) - u_n(y)|^{p-2} \left(u_n(x) - u_n(y)\right)}{|x-y|^{\frac{N+ps}{p'}}} \in L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)
$$

is uniformly bounded and

$$\frac{\varphi(x)-\varphi(y)}{|x-y|^{\frac{N+ps}{p}}} \in L^p(\mathbb{R}^N \times \mathbb{R}^N).$$

Whence, by using the weak convergence, we get

$$\lim_{n\to\infty} \int\int_{D_\Omega} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}} dxdy$$
$$= \int\int_{D_\Omega} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}} dxdy. \tag{4.11}$$

Using now Lemma 4.2, we obtain

$$0 \le \left| \frac{f_n(x)\varphi}{\left(u_n+\frac{1}{n}\right)^{\gamma(x)}} \right| \le ||C_\omega^{-\gamma(x)}\varphi||_{L^\infty(\Omega)} f(x),$$

for every $\varphi \in C_0^1(\Omega)$ whenever $\varphi \neq 0$ and on the set $\{u_n \ge C_\omega\}$, ω being the support of φ.
So, by applying Lebesgue dominated convergence theorem, we derive that

$$\lim_{n\to\infty} \int_\Omega \frac{f_n}{\left(u_n+\frac{1}{n}\right)^{\gamma(x)}} \varphi dx = \int_\Omega \frac{f}{u^{\gamma(x)}} \varphi dx. \tag{4.12}$$

Using (4.10), (4.11), (4.12) and by taking into consideration remark 2.7, we get the desired result. □

Theorem 4.4 *Assume that for some $\gamma^* > 1$ and $\delta > 0$ we have that, $||\gamma||_{L^\infty(\Omega)} \le \gamma^*$. Suppose that, $f \in L^{\sigma_2}(\Omega)^+$ with $\sigma_2 = \dfrac{N(p-1+\gamma^*)}{N(p-1)+p\gamma^*}$, then, there exists a unique weak positive $u \in W^{1,p}_{loc}(\Omega) \cap L^{p-1}(\Omega)$ to problem (1.1) such that $u^{\frac{p-1+\gamma^*}{p}} \in W_0^{1,p}(\Omega)$.*

Proof We proceed as in the proof of the previous result. Let u_n be the unique positive solution to (4.2).

Since $\gamma^* > 1$, then, by using Lemma 5.1, we can chose $u_n^{\gamma^*} \in W_0^{1,p}(\Omega)$ as test function in (4.1), we get

$$\int\int_{D_\Omega} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))\left(u_n^{\gamma^*}(x)-u_n^{\gamma^*}(y)\right)}{|x-y|^{N+ps}} dxdy$$
$$+ \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla u_n^{\gamma^*}(x)dx = \int_\Omega \frac{f_n}{\left(u_n+\frac{1}{n}\right)^{\gamma(x)}} u_n^{\gamma^*}(x)dx, \tag{4.13}$$

Therefore, by applying Lemma 2.12, for all most $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$, we obtain that

$$|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))\left(u_n^{\gamma^*}(x) - u_n^{\gamma^*}(y)\right) \geq 0. \tag{4.14}$$

Hence, from (4.13) and (4.14), it follows that

$$\begin{aligned}
\gamma^*\left(\frac{p}{p-1+\gamma^*}\right)^p \int_\Omega |\nabla u_n^{\frac{p-1+\gamma^*}{p}}|^p dx &= \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla u_n^{\gamma^*}(x)dx \\
&\leq \int_\Omega \frac{f_n}{\left(u_n+\frac{1}{n}\right)^{\gamma(x)}} u_n^{\gamma^*}(x)dx, \\
&\leq \int_{\Omega_\delta} \frac{f_n}{\left(u_n+\frac{1}{n}\right)^{\gamma(x)}} u_n^{\gamma^*}(x)dx + \int_{\omega_\delta} \frac{f_n}{\left(u_n+\frac{1}{n}\right)^{\gamma(x)}} u_n^{\gamma^*}(x)dx \\
&\leq \int_{\Omega_\delta} f u_n^{\gamma^*-\gamma(x)}dx + \int_{\omega_\delta} \frac{f}{C_{\omega_\delta}^{\gamma(x)}} u_n dx \\
&\leq ||f||_{L^1(\Omega)} + \left(1 + ||C_{\omega_\delta}^{-\gamma(.)}||_{L^\infty(\Omega)}\right) \int_\Omega f u_n^{\gamma^*} dx \\
&\leq ||f||_{L^1(\Omega)} + \left(1 + ||C_{\omega_\delta}^{-\gamma(.)}||_{L^\infty(\Omega)}\right) \left(\int_\Omega f^{\sigma_2}\right)^{\frac{1}{\sigma}2} \left(\int_\Omega u_n^{\gamma^*\sigma_2'}\right)^{\frac{1}{\sigma}'2}.
\end{aligned} \tag{4.15}$$

Since $\sigma_2 = \dfrac{N(p-1+\gamma^*)}{N(p-1)+p\gamma^*}$, so, $\gamma^*\sigma_2' = \dfrac{N(p-1+\gamma^*)}{N-p}$ and therefore,

$$\begin{aligned}
&\int_\Omega |\nabla u_n^{\frac{p-1+\gamma^*}{p}}|^p dx \\
&\quad \leq C||f||_{L^1(\Omega)} + C\left(1 + ||C^{-\gamma(x)}||_{L^\infty(\Omega)}\right)\left(\int_\Omega f^{\sigma_2}\right)^{\frac{1}{\sigma}2} \left(\int_\Omega |\nabla u_n^{\frac{p-1+\gamma^*}{p}}|^p\right)^{\frac{p^*}{p\sigma_2'}},
\end{aligned} \tag{4.16}$$

where in the rhs of (4.16) we have used the Sobolev's inequality.

By using the fact, $\frac{p^*}{p\sigma_2'} < 1$ in the last inequality, we get that $\left\{u_n^{\frac{p-1+\gamma^*}{p}}\right\}$ in bounded in $W_0^{1,p}(\Omega)$.

Since $\gamma^* > 1$, then, by applying Lemma 4.2, we have that

$$\int_\omega |\nabla u_n|^p dx = \left(\frac{p}{p-1+\gamma^*}\right)^p \int_\omega u_n^{1-\gamma^*} |\nabla u_n^{\frac{p-1+\gamma^*}{p}}|^p dx$$

$$\leq \left(\frac{p}{p-1+\gamma^*}\right)^p C(\omega)^{1-\gamma^*} C,$$

which implies that, $\{u_n\}$ in bounded in $W^{1,p}_{loc}(\Omega)$.

Hence $u^{\frac{p-1+\gamma^*}{p}} \in W_0^{1,p}(\Omega)$, and therefore $u \in L^{p-1}(\Omega)$.
To complete the proof, we follow the same steps as in previous result. □

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Conflict of interest The author declare that there is no conflict of interest regarding the publication of this paper.

5 Appendix

In this section we give some useful Lemmas that we have used in the proof of Theorems 4.3–4.4.

First, let us start by the following result, which allow us to use the test functions from the space $W_0^{1,p}(\Omega)$ in the Eq. (2.6). Notice that in the case $\gamma(x) = costant$ it was proved in [33].

Lemma 5.1 *Let $\gamma \in C^1(\overline{\Omega})$ and $f \in L^\sigma(\Omega)^+$ where $\sigma = \sigma_i$ with $i \in \{1, 2\}$ (where σ_1 and σ_2 are defined in Theorems 4.3–4.4 respectively). Assume that u be a weak solution of the problem (1.1), then, the Eq. (2.6) holds for every $\varphi \in W_0^{1,p}(\Omega)$.*

Proof The result follows by using the same reasoning as in the proof of Lemma 5.1 of [33]. □

Now, by adapting the same strategy as in the proof of Theorem A.1 of the Appendix in [33], we can show also, in the case $\gamma \in C^1(\overline{\Omega})$, the pointwise convergence of the gradient of the approximate solutions $\{u_n\}$ founded in Lemma 4.1, more precisely we have the following result,

Lemma 5.2 *Let $p > 1$ and $\gamma \in C^1(\overline{\Omega})$. Assume that, $\{u_n\}_n$ be the sequence of approximate solutions to problem (4.1) given by Lemma 4.1 and u is the pointwise limit of $\{u_n\}_n$. For $\gamma(x) \leq 1$ in Ω_δ, let $f \in L^{\sigma_1}(\Omega)^+$ where σ_1 is defined in Theorem 4.3 and for $\gamma^* > 1$ where $||\gamma||_{L^\infty(\Omega)} \leq \gamma^*$, let $f \in L^{\sigma_2}(\Omega)^+$ where σ_2 is defined in Theorem 4.4. Then, up to a subsequence, $\nabla u_n(x) \to \nabla u(x)$ a.e in Ω.*

Proof Let $f \in L^{\sigma_1}(\Omega)^+$ and $\gamma \in C^1(\overline{\Omega})$ such that, $\gamma(x) \leq 1$, then, by Theorem 4.3, we have the sequence $\{u_n\}_n$ is uniformly bounded in $W_0^{1,p}(\Omega)$.

Now, if we take $f \in L^{\sigma_2}(\Omega)^+$ with $\gamma(x) > 1$, therefore, by Theorem 4.4, the sequences $\{u_n\}_n$ and $\left\{u_n^{\frac{p-1+\gamma^*}{p}}\right\}_n$ are bounded in $W^{1,p}_{loc}(\Omega)$ and in $W^{1,p}_0(\Omega)$ respectively.

Thus, we have

$$u_n \rightharpoonup u \quad \text{weakly in } W^{1,p}_{loc}(\Omega),$$

and

$$u_n \to u \quad \text{strongly in } L^p_{loc}(\Omega).$$

On the other hand, by Lemma 4.2, for all $n \in I\!N$, we have

$$u_n \le u \quad \text{in } \mathbb{R}^N.$$

Let K be a compact set and consider a function $\phi_K \in C^1_c(\Omega)$ such that supp$\phi_K = \omega$, $0 \le \phi_K \le 1$ in Ω and $\phi_K \equiv 1$ in K. For $\mu > 0$, we chose $v_n = \phi_K T_\mu(u_n - u) \in W^{1,p}_0(\Omega)$ as test function in (4.1) and by arguing exactly as in the step 1 and step 2 of the proof Theorem A.1 of [33], we can show that, (up to a subsequence) $\nabla u_n(x) \to \nabla u(x)$ a.e in Ω. As desired. □

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