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Time–frequency analysis associated with the *k*-Hankel Gabor transform on $\mathbb{R}^d$

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Abstract
In this paper, we investigate the k-Hankel Gabor transform on $\mathbb{R}^d$ in some problems of time–frequency analysis. Firstly, we present the main theorems of Harmonic analysis as Plancherel's, Lieb's and inversion formulas for this transform. Next, we formulate some novel uncertainty principles including the Heisenberg and logarithmic uncertainty principles, Benedick–Amrein–Berthier's uncertainty principle, local uncertainty principles and Shapiro's uncertainty principle. In sequel, we introduce the localization operators associated with the k-Hankel Gabor transform on $\mathbb{R}^d$ and we develop corresponding theory. In particular we study their trace class properties and we prove that they are in the Schatten–von Neumann.

Keywords k-Hankel transform on $\mathbb{R}^d$ · k-Hankel Gabor transform on $\mathbb{R}^d$ · Heisenberg's uncertainty principles · Logarithmic uncertainty principles · Shapiro's uncertainty principles · Time–frequency concentration

Mathematics Subject Classification Primary 47G10; Secondary 42B10 · 47G30

1 Introduction

The classical Fourier transform $\mathcal{F}$, initially defined on $L^1(\mathbb{R}^d)$, extends to an isometry of $L^2(\mathbb{R}^d)$ and it commutes with *the rotation group*. Recently, Ben Said et al. [5] gave in a foundation of the deformation theory of the classical situation, by constructing a *generalization* $\mathcal{F}_{k,a}$ *of the Fourier transform*, commuting with *finite Coxeter groups*. The deformation parameters consists of a real parameter $a > 0$ coming from the

This paper is dedicated to Emeritus Professor Khalifa Trimèche on the occasion of his 75 birthday.

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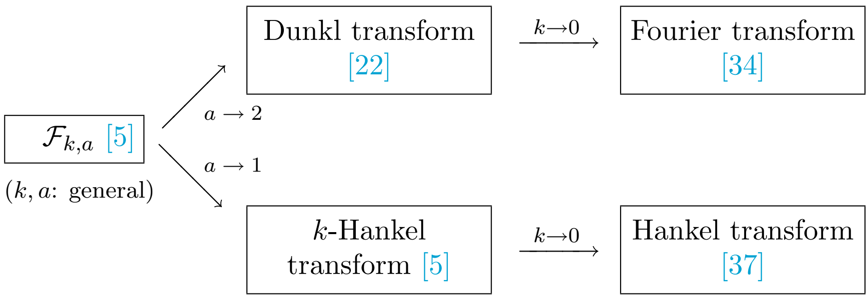
interpolation of the minimal unitary representations of two different reductive groups and a parameter k coming from Dunkl's theory of differential difference operators [21]. As it turned out, the unitary operator $\mathcal{F}_{k,a}$ includes some known integral transforms as special cases:



We pin down that $\mathcal{F}_{k,1}$ has a rich structure, as much as the Dunkl transform, and recently has been gaining a lot of attention (see, e.g., [6,7,11,16,29,35,44–46,48,49]). We shall call the generalized Fourier transform $\mathcal{F}_{k,1}$ the *k-Hankel transform* and we will simply denote it by $\mathcal{F}_k$.

The Heisenberg's uncertainty principle is one of the premier inequalities in quantum mechanics governing the uncertainty in knowing the position and the momentum of a moving particle simultaneously. That is, a precise knowledge of the position (momentum) leads to a diluted knowledge of the momentum (position). Motivated by 'quantum mechanics', in 1946 the physicist Gabor defined elementary time–frequency atoms as waveforms that have a minimal spread in a time–frequency plane [26]. To measure time–frequency 'information' content, he proposed decomposing signals over these elementary atomic waveforms. By showing that such decompositions are closely related to our sensitivity to sounds, and that they exhibit important structures in speech and music recordings, Gabor demonstrated the importance of localized time–frequency signal processing. The Gabor transformation has been found to be very useful in many physical and engineering applications, including wave propagation, signal processing and quantum optics [10]. For more details on the Gabor transform and its basic properties, we refer the reader to [17]. We may also refer to [30] where the author extends Gabor theory to the setup of locally compact abelian groups, and to [62] for the Gabor transform on Gelfand pairs. We note also that the notion of the Gabor transform for strong hypergroups was first introduced by Czaja and Gigante [13].

Motivated by the previous works, in [49] we have extend the Gabor transform to the setup of the minimal unitary representation of the conformal group $\mathrm{O}(d+1, 1)$, and then we have investigate for this transform the general theory of reproducing kernels theory.

The purpose of the present paper is twofold. On one hand, we want to study many versions of quantitative uncertainty principles for the k-Hankel Gabor transform. On the other hand we want to study the localization operators associated with this transform.

Roughly speaking, the uncertainty principle states that a non-zero integrable function f and its Fourier transform $\mathcal{F}(f)$, cannot both be sharply localized. To make such a principle concrete, many classical qualitative uncertainty principles (Hardy, Cowling-Price, Morgan, Beurling and Miyachi, etc), state that f and $\mathcal{F}(f)$ cannot both have arbitrarily rapid Gaussian decay, unless f is identically zero.

It is worth mentioning that quantitative uncertainty principles have a long and rich history; we refer the reader to the survey [25], the book [32] and the references [2,4,20,36,42,43,56,57,59,60,66] for numerous versions of uncertainty principles for the Fourier transform in different settings.

In the Euclidean case, the notion of the quantitative uncertainty principles for the Gabor transform was first introduced by Wilczok [64]. Later on, similar results appeared for several extended Gabor transforms in different setups (see, e.g., [3,8,23,24,40,41]).

Time–frequency localization operators are a mathematical tool to define a restriction of functions to a region in time–frequency plane that is compatible with the uncertainty principle and to extract time–frequency features. In the classical setting, this notion have been introduced and studied by Daubechies [14,15], Ramanathan and Topiwala [54], developed in the paper [33] by He and Wong, and detailed in the book [65] by Wong. Recently, the localization operators have found many applications to time–frequency analysis, the theory of differential equations, quantum mechanics and they are now extensively investigated as an important mathematical tool in signal analysis and other applications [9,12,18,19,31,63,65]. Next, this subject has been extended for the generalized integral transforms (see [1,28,45,46,48] and others).

Keeping in view the fact that the theory of localization operators associated with the k-Hankel Gabor transforms is yet to be investigated exclusively, our second endeavour is to introduce the localization operators associated with the k-Hankel Gabor transform on $\mathbb{R}^d$ and to develop the corresponding theory.

The objectives of this study are mentioned below:

- To prove a new inversion formula for the k-Hankel Gabor transform on $\mathbb{R}^d$.
- To derive several versions of the Heisenberg uncertainty principle via different techniques including generalized entropy, the contraction semigroup method of the homogeneous integral transform and others.
- To study the concentration-based uncertainty principles, including the Benedick–Amrein–Berthier, Shapiro's and the local-type uncertainty principles for the k-Hankel Gabor transform on $\mathbb{R}^d$.
- To study some weighted uncertainty, including Pitt's and Beckner's inequalities, pertaining to the k-Hankel Gabor transform on $\mathbb{R}^d$.
- To investigate the theory of localization operators in the setting of k-Hankel Gabor transform on $\mathbb{R}^d$.

The remaining part of the paper is organized as follows. In Sect. 2, we recall the main results about the harmonic analysis associated with the k-Hankel transform on $\mathbb{R}^d$. Section 3 deals with the k-Hankel Gabor transform on $\mathbb{R}^d$. More precisely we review some properties as the Plancherel's and Lieb's formulas and we prove a new inversion formula for this transform. Section 4 deals to derive many variants of Heisenberg's inequalities for the proposed transform. In Sect. 5, we present two concentration uncertainty principles for the k-Hankel Gabor transform on $\mathbb{R}^d$ such as Benedick–Amrein–Berthier's uncertainty principle and local uncertainty principles. Section 6 is devoted to prove the Shapiro uncertainty principle for the k-Hankel Gabor transform on $\mathbb{R}^d$. In Sect. 7, we derive two weighted uncertainty principles for the k-Hankel Gabor transform on $\mathbb{R}^d$. Towards the culmination, in last Section, we study the localization operators theory in the setting of k-Hankel Gabor transform on $\mathbb{R}^d$. In particular the boundedness and compactness of proposed operators are investigated in the Schatten classes.

2 Preliminaries

This section gives an introduction to the theory of k-Hankel transform on $\mathbb{R}^d$, the generalized translation operators and and Schatten–von Neumann classes. Main references are [5–7,65].

2.1 The *k*-Hankel transform

Let $\mathbb{R}^d$ denotes the Euclidean space with $\{e_i,\ i = 1, \dots, d\}$ as the Hamel basis and $\langle , \rangle$ as the scalar product. For any non-trivial vector $\alpha \in \mathbb{R}^d$, let σ_α denotes be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α. That is,

$$\sigma_\alpha(x) = x - 2\frac{\langle \alpha, x\rangle}{||\alpha||^2}\alpha. \tag{2.1}$$

A finite set $R \subset \mathbb{R}^d \backslash \{0\}$ is called a root system if $R \bigcap \mathbb{R}\alpha = \{\pm\alpha\}$ and $\sigma_\alpha(R) = R$ for all $\alpha \in R$. For a given root system R the reflections σ_α, $\alpha \in R$, generate a finite group $W \subset O(d)$, called the reflection group associated with R. In what follows, we define a positive root system $R_+ = \Big\{\alpha \in R\ : \langle \alpha, \beta\rangle > 0\Big\}$ for some β belongs to $\mathbb{R}^d \backslash \bigcup_{\alpha \in R} H_\alpha$. Also, we assume that $\langle \alpha, \alpha\rangle = 2$ for all $\alpha \in R_+$.

A function $k : \mathcal{R} \longrightarrow \mathbb{C}$ is called a multiplicity function if it is invariant under the action of the associated reflection group W.

For typographical convenience, we fix some notations as under:

- For $\alpha \in R_+$, the sum over $k(\alpha)$ is denoted by $\langle k\rangle$. That is,

$$\langle k\rangle = \sum_{\alpha \in R_+} k(\alpha). \tag{2.2}$$

- Let λ_k denotes the weight function

$$\lambda_k(x) = ||x||^{-1} \prod_{\alpha \in R_+} |\langle \alpha, x\rangle|^{2k(\alpha)}, \tag{2.3}$$

which is W-invariant and homogeneous of degree $2\langle k\rangle - 1$.

- Let d_k denotes the constant given by

$$d_k := \Big(\int_{S^{d-1}} \lambda_k(x) d\sigma(x) \Big)^{-1},$$

where $d\sigma$ denotes the Lebesgue surface measure on the unit sphere S^{d-1}.
- Denote $d\gamma_k(x) := \lambda_k(x)dx$.
- For $p \in [1, \infty]$, let p' denotes the conjugate exponent of p.
- $L^p_k(\mathbb{R}^d)$, $1 \leq p \leq \infty$, the space of measurable functions on $\mathbb{R}^d$ such that

$$||f||_{L^p_k(\mathbb{R}^d)} := \Big(\int_{\mathbb{R}^d} |f(x)|^p d\gamma_k(x) \Big)^{\frac{1}{p}} < \infty, \quad \text{if} \quad 1 \leq p < \infty,$$
$$||f||_{L^\infty_k(\mathbb{R}^d)} := \text{ess} \sup_{x \in \mathbb{R}^d} |f(x)| < \infty.$$

For $p = 2$, we provide this space with the scalar product

$$\langle f, g \rangle_{L^2_k(\mathbb{R}^d)} := \int_{\mathbb{R}^d} f(x) \overline{g(x)} d\gamma_k(x).$$

In this paper we assume that k is a non-negative multiplicity function satisfying

$$2\langle k \rangle + d > 2.$$

For $f \in L_k(\mathbb{R}^d)$, the k-Hankel transform $\mathcal{F}_k$ is defined by

$$\mathcal{F}_k(f)(\lambda) = \frac{1}{c_k} \int_{\mathbb{R}^d} f(x) \mathcal{B}_k(x, \lambda) d\gamma_k(x), \quad \text{for all } \lambda \in \mathbb{R}^d, \tag{2.4}$$

where

$$c_k := \int_{\mathbb{R}^d} e^{-||x||} d\gamma_k(x) = \frac{\Gamma(2\langle k \rangle + d - 1)}{d_k}, \tag{2.5}$$

and the kernel $\mathcal{B}_k(x, y)$ is obtained via the following proposition.

Proposition 2.1 (i) *Suppose that $d = 1$ and $k > \frac{1}{2}$. We have*

$$\mathcal{B}_k(\lambda, x) = j_{2k-1}\big(2\sqrt{|\lambda x|}\big) - \frac{\lambda x}{2k(2k+1)} j_{2k+1}\big(2\sqrt{|\lambda x|}\big). \tag{2.6}$$

Here

$$j_\alpha(u) := \Gamma(\alpha + 1) \Big(\frac{u}{2}\Big)^{-\alpha} J_\alpha(u) = \Gamma(\alpha + 1) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(\alpha + m + 1)} \Big(\frac{u}{2}\Big)^{2m} \tag{2.7}$$

denotes the normalized Bessel function of index α.

(ii) *Suppose that $d \geq 2$. In the polar coordinates $x = r\omega$ and $y = s\eta$, the kernel $\mathcal{B}_k(x, y)$ is given by*

$$\mathcal{B}_k(x, y) = \Gamma(\langle k\rangle + \frac{d-1}{2})V\Big(j_{\langle k\rangle+\frac{d-3}{2}}(\sqrt{2rs(1+\langle\omega, .\rangle)})\Big)(\eta),$$

here V_k is the Dunkl intertwining operator defined as follow, if h is a continuous function on $\mathbb{R}^d$,

$$V_k h(x) = \int_{\mathbb{R}^d} h(z)d\mu_x(z), \quad x \in \mathbb{R}^d \tag{2.8}$$

where μ_x is a positive probability measure on $\mathbb{R}^d$, with support in the closed ball $\overline{B}_d(0, ||x||)$ of center 0 and radius $||x||$. (See [55]).

Some of the basic properties of the kernel $\mathcal{B}_k$ are assembled in the following proposition.

Proposition 2.2 (i) *For $z, t \in \mathbb{C}^d$ and for all $\lambda > 0$, we have*

$$\mathcal{B}_k(z, t) = \mathcal{B}_k(t, z);\;\; \mathcal{B}_k(z, 0) = 1 \text{ and } \mathcal{B}_k(\lambda z, t) = \mathcal{B}_k(z, \lambda t).$$

(ii) *For all $x, y \in \mathbb{R}^d$ we have*

$$|\mathcal{B}_k(x, y)| \leq 1. \tag{2.9}$$

We note that the previous inequality implies that the k-Hankel transform is bounded on the space $L^1_k(\mathbb{R}^d)$, and for all f in $L^1_k(\mathbb{R}^d)$ we have

$$||\mathcal{F}_k(f)||_{L^\infty_k(\mathbb{R}^d)} \leq \frac{1}{c_k}||f||_{L^1_k(\mathbb{R}^d)}. \tag{2.10}$$

Remark 2.3 When $f(x) = F(||x||)$ is a radial function on $\mathbb{R}^d$ and belongs to $L^1_k(\mathbb{R}^d)$, we have

$$\forall\lambda \in \mathbb{R}^d,\;\; \mathcal{F}_k(f)(\lambda) = \mathcal{F}_B^{2\langle k\rangle+d-2}(F)(\|\xi\|). \tag{2.11}$$

where $\mathcal{F}^\nu_B$ is the deformed Hankel transform of one variable defined by

$$\mathcal{F}^\nu_B(\psi)(s) := \frac{1}{\Gamma(\nu+1)}\int_0^\infty \psi(r)j_\nu\Big(2(rs)^{\frac{1}{2}}\Big)r^{(\nu+1)-1}dr, \tag{2.12}$$

for a function ψ defined on $\mathbb{R}_+$. Here, j_ν is the normalized Bessel function given by (2.7).

Some fundamental properties of the k-Hankel transform are given in the following proposition, whose proof can be found in [5].

Proposition 2.4 (i) (Plancherel's theorem for $\mathcal{F}_k$). *The k-Hankel transform $f \mapsto \mathcal{F}_k(f)$ is an isometric isomorphism on $L^2_k(\mathbb{R}^d)$ and we have*

$$\int_{\mathbb{R}^d} |f(x)|^2 d\gamma_k(x) = \int_{\mathbb{R}^d} |\mathcal{F}_k(f)(\lambda)|^2 d\gamma_k(\lambda). \tag{2.13}$$

(ii) (Parseval's formula for $\mathcal{F}_k$). *For all f, g in $L^2_k(\mathbb{R}^d)$ we have*

$$\int_{\mathbb{R}^d} f(x)\overline{g(x)} d\gamma_k(x) = \int_{\mathbb{R}^d} \mathcal{F}_k(f)(\lambda)\overline{\mathcal{F}_k(g)(\lambda)} d\gamma_k(\lambda). \tag{2.14}$$

(iii) *Inversion formula. We have*

$$\mathcal{F}_k^{-1} = \mathcal{F}_k. \tag{2.15}$$

2.2 Generalized translation operator

Definition 2.5 ([7]) Let $x \in \mathbb{R}^d$. We define the generalized translation operator $f \mapsto \tau_x f$ on $L^2_k(\mathbb{R}^d)$ by

$$\mathcal{F}_k(\tau_x f) = \mathcal{B}_k(., x)\mathcal{F}_k(f). \tag{2.16}$$

It is useful to have a class of functions in which (2.16) holds pointwise. One such class is given by the generalized Wigner space $\mathcal{W}_k(\mathbb{R}^d)$ given by

$$\mathcal{W}_k(\mathbb{R}^d) := \Big\{ f \in L^1_k(\mathbb{R}^d) : \mathcal{F}_k(f) \in L^1_k(\mathbb{R}^d) \Big\}.$$

Proposition 2.6 ([7]). *The following statements hold true.*
(i) *Let f be in $L^2_k(\mathbb{R}^d)$, we have*

$$\|\tau_x f\|_{L^2_k(\mathbb{R}^d)} \le \|f\|_{L^2_k(\mathbb{R}^d)}, \quad \forall x \in \mathbb{R}^d.$$

(ii) *For all f in $\mathcal{W}_k(\mathbb{R}^d)$ we have*

$$\tau_x f(y) = \frac{1}{c_k} \int_{\mathbb{R}^d} \mathcal{B}_k(x,\xi)\mathcal{B}_k(y,\xi)\mathcal{F}_k(f)(\xi) d\gamma_k(\xi), \quad \forall x, y \in \mathbb{R}^d.$$

(iii) *For all f in $L^2_k(\mathbb{R}^d)$ and for all $x, y \in \mathbb{R}^d$, we have*

$$\tau_x f(y) = \tau_y(f)(x). \tag{2.17}$$

At the moment an explicit formula for the generalized translation operators is known only in the following two cases.
1^{st} *case*([6]): $d = 1$ and $W = \mathbb{Z}_2$. For all $f \in C(\mathbb{R})$ we have

$$\tau^k_x f(y) = \int_{\mathbb{R}} f(z) d\zeta^k_{x,y}(z), \tag{2.18}$$

here

$$d\zeta^k_{x,y}(z) = \begin{cases} \mathcal{K}_k(x,y,z)|z|^{2k-1}dz, & \text{if } xy \neq 0, \\ d\delta_x(z), & \text{if } y = 0, \\ d\delta_y(z), & \text{if } x = 0, \end{cases}$$

where $\mathcal{K}_k(x,y,z)$ is supported on

$$(\sqrt{|x|} - \sqrt{|y|})^2 < |z| < (\sqrt{|x|} + \sqrt{|y|})^2$$

and is given by

$$\mathcal{K}_k(x,y,z) = K_{\mathrm{B}}^{2k-1}(\sqrt{|x|}, \sqrt{|y|}, \sqrt{|z|})\, \nabla_k(x,y,z), \tag{2.19}$$

where

$$\begin{aligned} \nabla_k(x,y,z) := \frac{1}{4}\Big\{1 &+ \frac{\operatorname{sgn}(xy)}{(4\langle k\rangle + 2d - 2)}\Big[4k\Delta(|x|,|y|,|z|)^2 - 1\Big] \\ &+ \frac{\operatorname{sgn}(xz)}{(4\langle k\rangle + 2d - 2)}\Big[4k\Delta(|z|,|x|,|y|)^2 - 1\Big] \\ &+ \frac{\operatorname{sgn}(yz)}{(4\langle k\rangle + 2d - 2)}\Big[4k\Delta(|z|,|y|,|x|)^2 - 1\Big]\Big\}, \end{aligned} \tag{2.20}$$

$$\Delta(u,v,w) := \frac{1}{2\sqrt{uv}}(u+v-w), \qquad \text{for } u,v,w \in \mathbb{R}^*_+ \tag{2.21}$$

and K_{B}^{2k-1} is the positive kernel given by

$$K_{\mathrm{B}}^{2k-1}(u,v,w) = \frac{\Gamma(2k)}{2^{4\langle k\rangle+2d-2}\Gamma(2k-\frac{1}{2})\Gamma(\frac{1}{2})} \frac{\Big\{\big[(u+v)^2 - w^2\big]\big[w^2 - (u-v)^2\big]\Big\}^{2k-\frac{3}{2}}}{(uvw)^{4k-2}} \tag{2.22}$$

for $|u-v| < w < u+v$ and $K_{\mathrm{B}}^{2k-1}(u,v,w) = 0$ elsewhere.

The previous explicit formula implies the L^p-boundedness of $\tau^k_y f$. More precisely, we have.

Proposition 2.7 ([6]) *For all $f \in L^p_k(\mathbb{R})$, $1 \leq p \leq \infty$, there exists a positive constant A_k such that*

$$\forall\, y \in \mathbb{R}, \quad ||\tau^k_y f||_{L^p_k(\mathbb{R})} \leq A_k ||f||_{L^p_k(\mathbb{R})}. \tag{2.23}$$

<u>2^{nd}*case*</u>: ([7]) For all radial function f in $\mathcal{W}_k(\mathbb{R}^d)$ and for all $x, y \in \mathbb{R}^d$, we have

$$\tau_y f(x) = \frac{\Gamma(\frac{d-1}{2} + \langle k\rangle)}{\sqrt{\pi}\,\Gamma(\frac{d-2}{2} + \langle k\rangle)}$$

$$V_k\Big[\int_{-1}^{1} f_0\Big(||x|| + ||y|| - \sqrt{2||x||\,||y||\big(1 + \langle \frac{x}{||x||}, .\rangle\big)u}\Big)(1-u^2)^{\frac{d+2\langle k\rangle-4}{2}} du\Big]\Big(\frac{y}{||y||}\Big),$$

with f_0 the function on $[0, \infty)$ given by $f(x) = f_0(||x||)$ and V_k is the Dunkl intertwining given by (2.8).

Several essential properties of $\tau_y f$ is established for f being radial functions. This is collected in the following proposition ([7]). Let $L^p_{k,rad}(\mathbb{R}^d)$ stands for the subspace of radial functions in $L^p_k(\mathbb{R}^d)$.

Proposition 2.8 (i) *Let f be in $L^1_{k,rad}(\mathbb{R}^d)$ and nonnegative. Then we have*

$$\forall\, y \in \mathbb{R}^d, \quad \tau_y f \geq 0, \quad \tau_y f \in L^1_k(\mathbb{R}^d)$$

and

$$\int_{\mathbb{R}^d} \tau_y f(x) d\gamma_k(x) = \int_{\mathbb{R}^d} f(x) d\gamma_k(x). \tag{2.24}$$

(ii) *Let f be in $L^p_{k,rad}(\mathbb{R}^d)$, $1 \leq p \leq \infty$, we have*

$$\forall\, y \in \mathbb{R}^d, \quad ||\tau_y f||_{L^p_k(\mathbb{R}^d)} \leq ||f||_{L^p_k(\mathbb{R}^d)}. \tag{2.25}$$

By means of the generalized translation operator, the generalized convolution product is defined on the space $L^2_k(\mathbb{R}^d)$ by:

$$\forall\, x \in \mathbb{R}^d, \quad f *_k g(x) = \frac{1}{c_k} \int_{\mathbb{R}^d} \tau_x f(y) g(y) d\gamma_k(y). \tag{2.26}$$

We close the notion of the generalized convolution product by giving the following results which play a significant role in the next sections.

Proposition 2.9 ([46]) (i) *For $f \in L^2_k(\mathbb{R}^d)$ and $g \in L^1_k(\mathbb{R}^d)$ we have*

$$\mathcal{F}_k(f *_k g) = \mathcal{F}_k(f)\mathcal{F}_k(f). \tag{2.27}$$

(ii) *Let $f, g \in L^2_k(\mathbb{R}^d)$. Then $f *_k g \in L^2_k(\mathbb{R}^d)$ if and only if $\mathcal{F}_k(f)\mathcal{F}_k(g)$ belongs to $L^2_k(\mathbb{R}^d)$, and in this case we have*

$$\mathcal{F}_k(f *_k g) = \mathcal{F}_k(f)\mathcal{F}_k(g). \tag{2.28}$$

An immediate consequence of Proposition 2.9 ii) and Plancherel's formula (2.13) that will be used in the next section is the following.

Proposition 2.10 ([46]) *Let f and g be in $L^2_k(\mathbb{R}^d)$. Then, we have*

$$\int_{\mathbb{R}^d} |f *_k g(x)|^2 d\gamma_k(x) = \int_{\mathbb{R}^d} |\mathcal{F}_k(f)(\xi)|^2 |\mathcal{F}_k(g)(\xi)|^2 d\gamma_k(\xi) \tag{2.29}$$

where both sides are finite or infinite.

2.3 Schatten–von Neumann classes

We denote by $B(L^2_k(\mathbb{R}^d))$ the space of bounded operators from $L^2_k(\mathbb{R}^d)$ into itself.

Definition 2.11 (1) The singular values $(s_n(A))_{n\in\mathbb{N}}$ of a compact operator A in $B(L^2_k(\mathbb{R}^d))$ are by definition the eigenvalues of the positive self-adjoint operator $|A| = \sqrt{A^*A}$.

(2) For $1 \leq p < \infty$, the Schatten class S_p is defined as the space of all compact operators whose singular values lie in $l^p(\mathbb{N})$. The space S_p is equipped with the norm

$$||A||_{S_p} := \Big(\sum_{n=1}^{\infty}(s_n(A))^p\Big)^{\frac{1}{p}}. \tag{2.30}$$

Remark 2.12 We note that S_2 is the space of Hilbert–Schmidt operators, while S_1 is the space of trace class operators.

Definition 2.13 The trace of an operator A in S_1 is defined by

$$\mathrm{tr}(A) = \sum_{n=1}^{\infty}\langle Av_n, v_n\rangle_{L^2_k(\mathbb{R}^d)}, \tag{2.31}$$

where $(v_n)_n$ is an orthonormal basis of $L^2_k(\mathbb{R}^d)$.

Remark 2.14 If A is positive, then

$$\mathrm{tr}(A) = ||A||_{S_1}. \tag{2.32}$$

Moreover, a compact operator A acting on $L^2_k(\mathbb{R}^d)$ is of Hilbert–Schmidt if the positive operator A^*A is in the space of trace class S_1. In this case,

$$||A||^2_{HS} := ||A||^2_{S_2} = ||A^*A||_{S_1} = \mathrm{tr}(A^*A) = \sum_{n=1}^{\infty}||Av_n||^2_{L^2_k(\mathbb{R}^d)}, \tag{2.33}$$

where $(v_n)_n$ is an orthonormal basis of $L^2_k(\mathbb{R}^d)$.

Definition 2.15 Define $S_\infty := B(L^2_k(\mathbb{R}^d))$ equipped with the norm,

$$||A||_{S_\infty} := \sup_{v\in L^2_k(\mathbb{R}^d):||v||_{L^2_k(\mathbb{R}^d)}=1}||Av||_{L^2_k(\mathbb{R}^d)}. \tag{2.34}$$

3 *k*-Hankel Gabor transform

The aim of this section is to survey and revisit some results for the k-Hankel Gabor transform on $\mathbb{R}^d$ studied in [49].

For $1 \leq p \leq \infty$, let $L^p_{\mu_k}(\mathbb{R}^{2d})$ be the space of measurable functions f on $\mathbb{R}^{2d}$ such that

$$\|f\|_{L^p_{\mu_k}(\mathbb{R}^{2d})} := \left(\int_{\mathbb{R}^{2d}} |f(x,y)|^p d\mu_k(x,y)\right)^{\frac{1}{p}} < \infty, \qquad 1 \leq p < \infty$$
$$\|f\|_{L^\infty_{\mu_k}(\mathbb{R}^{2d})} := \operatorname*{ess\,sup}_{(x,y)\in\mathbb{R}^{2d}} |f(x,y)| < \infty,$$

where $d\mu_k(x,y) := d\gamma_k(x)d\gamma_k(y)$.

Definition 3.1 For any function h in $L^2_{k,rad}(\mathbb{R}^d)$ and any $\nu \in \mathbb{R}^d$, we define the modulation of h by ν as :

$$h_\nu := \mathcal{F}_k(\sqrt{\tau^k_\nu(|h|^2)}), \tag{3.1}$$

where τ^k_ν, $\nu \in \mathbb{R}^d$, are the k-Hankel translation operators.

Remark 3.2 (i) Using the positivity of the generalized translation operator on radial functions given by Proposition 2.8, we see that the formula (3.1) is well defined.

(ii) Using Plancherel's formula (2.13) and relation (2.24), we get for all h in $L^2_{k,rad}(\mathbb{R}^d)$

$$\|h_\nu\|_{L^2_k(\mathbb{R}^d)} = \|h\|_{L^2_k(\mathbb{R}^d)}. \tag{3.2}$$

We consider the family $h_{y,\nu}(x)$, $\nu, y \in \mathbb{R}^d$ defined by

$$h_{y,\nu}(x) = \tau^k_y h_\nu(x), \;\; x \in \mathbb{R}^d.$$

We note that we have

$$\forall\, y, \nu \in \mathbb{R}^d, \quad ||h_{y,\nu}||_{L^2_k(\mathbb{R}^d)} \leq ||h||_{L^2_k(\mathbb{R}^d)}. \tag{3.3}$$

Definition 3.3 Let h be in $L^2_{k,rad}(\mathbb{R}^d)$. For a function f in $L^2_k(\mathbb{R}^d)$ we define its k-Hankel Gabor transform by

$$\mathcal{G}^k_h(f)(y,\nu) := \frac{1}{c_k}\int_{\mathbb{R}^d} f(x)h_{y,\nu}(x)d\gamma_k(x), \tag{3.4}$$

which can also be written in the form

$$\mathcal{G}^k_h(f)(y,\nu) := f *_k h_\nu(y). \tag{3.5}$$

Remark 3.4 By a standard computation it is easy to see that, for every $f \in L^2_k(\mathbb{R}^d)$ and h in $L^2_{k,rad}(\mathbb{R}^d)$, for all $\lambda > 0$ and for all $(y,\nu) \in \mathbb{R}^{2d}$, we have

$$\mathcal{G}^k_{h_{\frac{1}{\lambda}}}(f_\lambda)(y,\nu) = \mathcal{G}^k_h(f)\left(\frac{y}{\lambda}, \lambda\nu\right), \tag{3.6}$$

where

$$\forall\, t > 0,\ \forall\, x \in \mathbb{R}^d,\ g_t(x) := \frac{1}{t^{\frac{2\langle k\rangle+d-1}{2}}} g\left(\frac{x}{t}\right).$$

Proposition 3.5 *For f in $L^2_k(\mathbb{R}^d)$ and h in $L^2_{k,rad}(\mathbb{R}^d)$ we have*

$$\|\mathcal{G}^k_h(f)\|_{L^\infty_{\mu_k}(\mathbb{R}^{2d})} \leq \frac{1}{c_k}\|f\|_{L^2_k(\mathbb{R}^d)}\|h\|_{L^2_k(\mathbb{R}^d)}. \tag{3.7}$$

Proposition 3.6 (Plancherel's formula) *Let h be in $L^2_{k,rad}(\mathbb{R}^d)$. Then, for all f in $L^2_k(\mathbb{R}^d)$, we have*

$$||\mathcal{G}^k_h(f)||_{L^2_{\mu_k}(\mathbb{R}^{2d})} = \|h\|_{L^2_k(\mathbb{R}^d)}\|f\|_{L^2_k(\mathbb{R}^d)}. \tag{3.8}$$

As in the classical case, the continuous k-Hankel Gabor transform preserves the orthogonality relation. However, we have the following result.

Corollary 3.7 *Let h be in $L^2_{k,rad}(\mathbb{R}^d)$. Then, for all f, g in $L^2_k(\mathbb{R}^d)$, we have*

$$\int_{\mathbb{R}^{2d}} \mathcal{G}^k_h(f)(y,\nu)\overline{\mathcal{G}^k_h(g)(y,\nu)}d\mu_k(y,\nu) = ||h||^2_{L^2_k(\mathbb{R}^d)}\int_{\mathbb{R}^d} f(x)\overline{g(x)}d\gamma_k(x). \tag{3.9}$$

Proposition 3.8 *Let h be in $L^2_{k,rad}(\mathbb{R}^d)$. Then for any f be in $L^2_k(\mathbb{R}^d)$ and any p belongs to $[2,\infty)$, we have*

$$\|\mathcal{G}^k_h(f)\|_{L^p_{\mu_k}(\mathbb{R}^{2d})} \leq c_k^{\frac{2-p}{p}}\|f\|_{L^2_k(\mathbb{R}^d)}\|h\|_{L^2_k(\mathbb{R}^d)}. \tag{3.10}$$

Proof Using Propositions 3.5 and 3.6 the result follows by applying the Riesz–Thorin interpolation theorem. □

By simple calculations we prove the following:

Lemma 3.9 *Let $h \in L^2_{k,rad}(\mathbb{R}^d)\bigcap L^\infty_k(\mathbb{R}^d)$, then for any $f \in L^2_k(\mathbb{R}^d)$, we have*

$$\mathcal{F}_k\Big(\mathcal{G}^k_h(f)(.,\nu)\Big)(\xi) = \mathcal{F}_k(f)(\xi)\sqrt{\tau^k_\nu|h|^2(\xi)}. \tag{3.11}$$

Henceforth, the function h will denote an arbitrary nonzero element in $L^2_{k,rad}(\mathbb{R}^d)$.

Now, we will prove a new inversion formula for the k-Hankel Gabor transform on $\mathbb{R}^d$.

Theorem 3.10 (L^2_k inversion formula). *Let h be in $(L^2_{k,rad}(\mathbb{R}^d)\cap L^\infty_k(\mathbb{R}^d))$ such that $||h||_{L^2_k(\mathbb{R}^d)} = 1$. Then, for any function f in $L^2_k(\mathbb{R}^d)$, we have*

$$f_n(x) = \frac{1}{c_k}\int_{B_d(0,n)}\int_{\mathbb{R}^d} \mathcal{G}^k_h(f)(y,\nu)\tau^k_y h_\nu(x)d\mu_k(\nu,y) \tag{3.12}$$

in $L_k^2(\mathbb{R}^d)$ *and satisfies*

$$\lim_{n\to\infty} ||f - f_n||_{L_k^2(\mathbb{R}^d)} = 0.$$

Here $B_d(0,n)$ *is the open ball of* $\mathbb{R}^d$ *of center* 0 *and radius* n*.*

For proof this theorem we need the following Lemmas.

Lemma 3.11 *Let* h *be as above. For any positive integer* n *define the two functions*

$$G_n(x) := \frac{1}{c_k}\int_{B_d(0,n)}\int_{\mathbb{R}^d} \mathcal{B}_k(\xi,x)|\mathcal{F}_k(h_\nu)(\xi)|^2 d\gamma_k(\nu)d\gamma_k(\xi), \text{ for } x \in \mathbb{R}^d,$$

and

$$H_n(\xi) := \int_{B_d(0,n)} |\mathcal{F}_k(h_\nu)(\xi)|^2 d\gamma_k(\nu), \text{ for } \xi \in \mathbb{R}^d.$$

Then we have

$$G_n \in L_k^2(\mathbb{R}^d), \quad H_n \in L_k^1(\mathbb{R}^d)\cap L_k^\infty(\mathbb{R}^d), \quad \textit{and } \mathcal{F}_k(G_n) = H_n.$$

Proof Using the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}\forall\, x \in \mathbb{R}^d,\quad |G_n(x)|^2 &\le \tfrac{1}{c_k^2}\Big(\int_{B_d(0,n)} d\gamma_k(\nu)\Big)\int_{B_d(0,n)}\Big|\int_{\mathbb{R}^d}\mathcal{B}_k(\xi,x)|\mathcal{F}_k(h_\nu)(\xi)|^2 d\gamma_k(\xi)\Big|^2 d\gamma_k(\nu)\\ &\le C\int_{B_d(0,n)}\Big|\int_{\mathbb{R}^d}\mathcal{B}_k(\xi,x)|\mathcal{F}_k(h_\nu)(\xi)|^2 d\gamma_k(\xi)\Big|^2 d\gamma_k(\nu).\end{aligned}$$

Therefore by Fubini's theorem, the relations (2.9), (2.13), (2.15), (3.1) and Proposition 2.9

$$\begin{aligned}\int_{\mathbb{R}^d}|G_n(x)|^2 d\gamma_k(x) &\le C\int_{B_d(0,n)}\int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d}\mathcal{B}_k(\xi,x)|\mathcal{F}_k(h_\nu)(\xi)|^2 d\gamma_k(\xi)\Big|^2 d\gamma_k(\nu)d\gamma_k(x)\\ &\le C\int_{B_d(0,n)}\int_{\mathbb{R}^d}|\mathcal{F}_k^{-1}(|\mathcal{F}_k(h_\nu)|^2)(x)|^2 d\gamma_k(x)d\gamma_k(\nu)\\ &\le C\int_{B_d(0,n)}\int_{\mathbb{R}^d}|\tau_\nu^k|h|^2(\xi)|^2 d\gamma_k(\nu)d\gamma_k(\xi)\\ &\le C\int_{B_d(0,n)}\Big|\Big|\tau_\nu^k|h|^2\Big|\Big|_{L_k^1(\mathbb{R}^d)}\Big|\Big|\tau_\nu^k|h|^2\Big|\Big|_{L_k^\infty(\mathbb{R}^d)} d\gamma_k(\nu)\\ &\le C\int_{B_d(0,n)}||\tau_\nu^k|h|^2||_{L_k^\infty(\mathbb{R}^d)} d\gamma_k(\nu) < \infty.\end{aligned}$$

Now we will prove that $H_n \in L^1_k(\mathbb{R}^d) \cap L^\infty_k(\mathbb{R}^d)$. Indeed, from (3.1) we have

$$\begin{aligned}
\forall\, \xi \in \mathbb{R}^d, \quad |H_n(\xi)| &= \Big| \int_{B_d(0,n)} |\mathcal{F}_k(h_\nu)(\xi)|^2 d\gamma_k(\nu) \Big| \\
&= \int_{B_d(0,n)} \tau^k_\nu |h|^2(\xi) d\gamma_k(\nu) \\
&\leq \int_{\mathbb{R}^d} \tau^k_\nu |h|^2(\xi) d\gamma_k(\nu) \\
&= \int_{\mathbb{R}^d} \tau^k_\xi |h|^2(\nu) d\gamma_k(\nu) = ||h||^2_{L^2_k(\mathbb{R}^d)} < \infty.
\end{aligned}$$

Thus H_n belongs to $L^\infty_k(\mathbb{R}^d)$.

On the other hand, by Fubini's theorem and the relation (2.24), we have

$$\begin{aligned}
||H_n||_{L^1_k(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |H_n(\xi)| d\gamma_k(\xi) &= \int_{\mathbb{R}^d} \Big| \int_{B_d(0,n)} |\mathcal{F}_k(h_\nu)(\xi)|^2 d\gamma_k(\nu) \Big| d\gamma_k(\xi) \\
&= \int_{B_d(0,n)} \Big(\int_{\mathbb{R}^d} \tau^k_\nu |h|^2(\xi) d\gamma_k(\xi) \Big) d\gamma_k(\nu) \\
&\leq ||h||^2_{L^2_k(\mathbb{R}^d)} \int_{B_d(0,n)} d\gamma_k(\nu) < \infty.
\end{aligned}$$

Hence H_n belongs to $L^1_k(\mathbb{R}^d)$. Finally, using Fubini's theorem we obtain

$$\begin{aligned}
\forall\, y \in \mathbb{R}^d, \quad \mathcal{F}^{-1}_k(H_n)(y) &= \tfrac{1}{c_k} \int_{\mathbb{R}^d} H_n(\xi) \mathcal{B}_k(\xi, y) d\gamma_k(\xi) \\
&= \tfrac{1}{c_k} \int_{\mathbb{R}^d} \mathcal{B}_k(\xi, y) \int_{B_d(0,n)} |\mathcal{F}_k(h_\nu)(\xi)|^2 d\gamma_k(\nu) d\gamma_k(\xi) \\
&= \tfrac{1}{c_k} \int_{B_d(0,n)} \int_{\mathbb{R}^d} \mathcal{B}_k(\xi, y) |\mathcal{F}_k(h_\nu)(\xi)|^2 d\gamma_k(\nu) d\gamma_k(\xi) = G_n(y).
\end{aligned}$$

□

Lemma 3.12 *Let h be as above. For any positive integer n the function*

$$G_n(x) := \frac{1}{c_k} \int_{B_d(0,n)} \int_{\mathbb{R}^d} \mathcal{B}_k(\xi, x) |\mathcal{F}_k(h_\nu)(\xi)|^2 d\gamma_k(\xi) d\gamma_k(\nu), \quad x \in \mathbb{R}^d$$

can be written

$$G_n(x) = \int_{B_d(0,n)} h_\nu *_k h_\nu(x) d\gamma_k(\nu), \quad x \in \mathbb{R}^d.$$

Proof From Proposition 2.9 we have

$$\begin{aligned}\forall\, x \in \mathbb{R}^d, \quad G_n(x) &= \frac{1}{c_k}\int_{B_d(0,n)}\int_{\mathbb{R}^d} \mathcal{B}_k(\xi, x)|\mathcal{F}_k(h_\nu)(\xi)|^2 d\gamma_k(\nu)d\gamma_k(\xi)\\ &= \int_{B_d(0,n)} \mathcal{F}_k^{-1}(|\mathcal{F}_k(h_\nu)|^2)(x)d\gamma_k(\nu)\\ &= \int_{B_d(0,n)} h_\nu *_k h_\nu(x)d\gamma_k(\nu).\end{aligned}$$

□

Lemma 3.13 *Let h be in $L^2_{k,rad}(\mathbb{R}^d) \cap L^\infty_k(\mathbb{R}^d)$. Then, for any function f in $L^2_k(\mathbb{R}^d)$, we have*

$$f_n = G_n *_k f. \tag{3.13}$$

Proof We have

$$\begin{aligned}\forall\, x \in \mathbb{R}^d, \quad f_n(x) &= \frac{1}{c_k}\int_{B_d(0,n)}\int_{\mathbb{R}^d} \mathcal{G}_h^k(f)(y,\nu)\tau_y^k h_\nu(x)d\mu_k(\nu, y)\\ &= \int_{B_d(0,n)} \Big(\mathcal{G}_h^k(f)(.,\nu) *_k h_\nu\Big)(x)d\gamma_k(\nu)\\ &= \int_{B_d(0,n)} f *_k h_\nu *_k h_\nu(x)d\gamma_k(\nu)\\ &= \frac{1}{c_k}\int_{B_d(0,n)}\int_{\mathbb{R}^d} \tau_x^k f(y)h_\nu *_k h_\nu(y)d\mu_k(\nu, y)\\ &= \frac{1}{c_k}\int_{\mathbb{R}^d} \tau_x^k f(y)\Big(\int_{B_d(0,n)} h_\nu *_k h_\nu(y)d\gamma_k(\nu)\Big)d\gamma_k(y)\\ &= \frac{1}{c_k}\int_{\mathbb{R}^d} \tau_x^k f(y)G_n(y)d\gamma_k(y)\\ &= f *_k G_n(x).\end{aligned}$$

On the follow we justify the use of Fubini's theorem in the last sequence of equalities observe that

$$\frac{1}{c_k}\Big|\int_{B_d(0,n)}\int_{\mathbb{R}^d} \tau_x^k f(y)h_\nu *_k h_\nu(y)d\mu_k(\nu, y)\Big| \le \int_{B_d(0,n)} |f *_k h_\nu *_k h_\nu(x)|d\gamma_k(\nu).$$

Now, using Proposition 2.9 and hypothesis on h we see that $h_\nu *_k h_\nu \in L^2_k(\mathbb{R}^d)$. Next using Young's inequality and Parseval's theorem we obtain

$$\begin{aligned}||f *_k h_\nu *_k h_\nu||_{L^\infty_k(\mathbb{R}^d)} &\le ||f||_{L^2_k(\mathbb{R}^d)}||h_\nu *_k h_\nu||_{L^2_k(\mathbb{R}^d)}\\ &\le C||f||_{L^2_k(\mathbb{R}^d)}||h||_{L^2_k(\mathbb{R}^d)}||h||_{L^\infty_k(\mathbb{R}^d)}\end{aligned}$$

and

$$\int_{B_d(0,n)} |f *_k h_\nu *_k h_\nu(x)| d\gamma_k(\nu)$$
$$\leq C\Big(\int_{B_d(0,n)} d\gamma_k(\nu)\Big) ||f||_{L^2_k(\mathbb{R}^d)} ||h||_{L^2_k(\mathbb{R}^d)} ||h||_{L^\infty_k(\mathbb{R}^d)}.$$

The proof is complete. □

Proof of Theorem 3.10 It follows from Proposition 2.10, Lemmas 3.11 and 3.13 that $f_n \in L^2_k(\mathbb{R}^d)$ and

$$\forall \xi \in \mathbb{R}^d, \quad \mathcal{F}_k(f_n)(\xi) = H_n(\xi)\mathcal{F}_k(f)(\xi).$$

By this, the Plancherel formula (2.13), the fact that $H_n \to 1$ pointwise as $n \to \infty$, and the dominated convergence theorem, it follows that

$$||f - f_n||^2_{L^2_k(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\mathcal{F}_k(f)(\xi) - H_n(\xi)\mathcal{F}_k(f)(\xi)|^2 d\gamma_k(\xi)$$
$$= \int_{\mathbb{R}^d} |\mathcal{F}_k(f)(\xi)(1 - H_n(\xi))|^2 d\gamma_k(\xi) \to 0$$

as $n \to \infty$ which achieves the proof. □

Remark 3.14 Let h be in $L^2_{k,rad}(\mathbb{R}^d)$. We proceed as in [13], we define the modulation of h by ν otherwise, as follow:

$$\mathcal{M}_\nu(h) := \mathcal{F}_k\big(\sqrt{\tau^k_\nu(|\mathcal{F}_k(h)|^2)}\big). \tag{3.14}$$

Subsequently, we define the generalized Gabor transform $\mathcal{V}^k_h$ as follow:

$$\forall (y, \nu) \in \mathbb{R}^{2d}, \ \mathcal{V}^k_h(f)(y, \nu) := \frac{1}{c_k}\int_{\mathbb{R}^d} f(x)\tau^k_y\big(\mathcal{M}_\nu(h)\big)(y) d\gamma_k(x) = f *_k \mathcal{M}_\nu(h)(y). \tag{3.15}$$

It is clear that

$$\mathcal{V}^k_h = \mathcal{G}^k_{\mathcal{F}_k(h)}. \tag{3.16}$$

Thus, by involving Plancherel's formula (2.13), we derive that the two integral transforms are equivalent and then all results proved for one are valuables for the second. So, I reclame that all results proved in [49] and in this paper for the k-Hankel Gabor transform $\mathcal{G}^k_h$ are valuables for the integral transform $\mathcal{V}^k_h$ and it is suffice to replace h by $\mathcal{F}_k(h)$ to derive the analogues results. Finally, I note and I insist that any adaptation of results proved for the k-Hankel Gabor transform $\mathcal{G}^k_h$ in the context of the transformation $\mathcal{V}^k_h$ is a plagiarism (in particular results proved in [49] and in the current paper), since I mentioned that the two transformations coincide modulo the formulas (3.16) and (2.13).

4 Heisenberg type uncertainty principles

Recall that the window function h in $\mathcal{G}_h^k$ is a non trivial radial function in $L_k^2(\mathbb{R}^d)$.

4.1 A generalized Heisenberg uncertainty principle

Let us recall the Heisenberg uncertainty principle for the k-Hankel transform $\mathcal{F}_k$.

Proposition 4.1 (See [5,35]) *For $s, t > 0$, there exists a positive constant $C_k(s,t)$, such that for every $f \in L_k^2(\mathbb{R}^d)$, the following inequality holds*

$$\Big\| \, ||\xi||^s \mathcal{F}_k(f) \Big\|_{L_k^2(\mathbb{R}^d)}^{\frac{t}{s+t}} \Big\| \, ||x||^t f \Big\|_{L_k^2(\mathbb{R}^d)}^{\frac{s}{s+t}} \geq C_k(s,t) \|f\|_{L_k^2(\mathbb{R}^d)}. \tag{4.1}$$

For $s, t \geq \frac{1}{2}$, $C_k(s,t) = (\frac{2\langle k\rangle + d - 1}{2})^{\frac{2st}{s+t}}$.

Theorem 4.2 (Heisenberg's uncertainty principle for $\mathcal{G}_h^k$) *Let $s, t > 0$. For every f belongs to $L_k^2(\mathbb{R}^d)$, we have*

$$\begin{aligned} &\left(\int_{\mathbb{R}^{2d}} ||y||^{2t} |\mathcal{G}_h^k(f)(y,\nu)|^2 d\mu_k(y,\nu)\right)^{\frac{s}{s+t}} \left(\int_{\mathbb{R}^d} ||\xi||^{2s} |\mathcal{F}_k(f)(\xi)|^2 d\gamma_k(\xi)\right)^{\frac{t}{s+t}} \\ &\quad \geq \big(C_k(s,t)\big)^2 \|h\|_{L_k^2(\mathbb{R}^d)}^{\frac{2s}{s+t}} \|f\|_{L_k^2(\mathbb{R}^d)}^2. \end{aligned} \tag{4.2}$$

Here $C_k(s,t)$ is the same constant as in Proposition 4.1.

Proof Let us consider the non-trivial case where both integrals on the left hand side of (4.2) are finite. Fixing ν arbitrary, Heisenberg's inequality (4.1) gives

$$\begin{aligned} &\left(\int_{\mathbb{R}^d} ||\xi||^{2s} |\mathcal{F}_k(\mathcal{G}_h^k(f)(.,\nu))(\xi)|^2 d\gamma_k(\xi)\right)^{\frac{t}{s+t}} \left(\int_{\mathbb{R}^d} ||y||^{2t} |\mathcal{G}_h^k(f)(y,\nu)|^2 d\gamma_k(y)\right)^{\frac{s}{s+t}} \\ &\quad \geq \big(C_k(s,t)\big)^2 \int_{\mathbb{R}^d} |\mathcal{G}_h^k(f)(y,\nu)|^2 d\gamma_k(y). \end{aligned}$$

Integrating over ν with respect to the measure $d\gamma_k(\nu)$, and using Cauchy–Schwarz's inequality, we obtain

$$\begin{aligned} &\left(\int_{\mathbb{R}^{2d}} ||\xi||^{2s} |\mathcal{F}_k(\mathcal{G}_h^k(f)(.,\nu))(\xi)|^2 d\mu_k(\xi,\nu)\right)^{\frac{t}{s+t}} \left(\int_{\mathbb{R}^2} ||y||^{2t} |\mathcal{G}_h^k(f)(y,\nu)|^2 d\mu_k(y,\nu)\right)^{\frac{s}{s+t}} \\ &\quad \geq \big(C_k(s,t)\big)^2 \int_{\mathbb{R}^2} |\mathcal{G}_h^k(f)(y,\nu)|^2 d\mu_k(y,\nu). \end{aligned}$$

Further, using the fact that

$$\int_{\mathbb{R}^{2d}} ||\xi||^{2s} |\mathcal{F}_k(\mathcal{G}_h^k(f)(.,\nu))(\xi)|^2 d\mu_k(\xi,\nu) = \|h\|_{L_k^2(\mathbb{R}^d)}^2 \int_{\mathbb{R}^d} ||\xi||^{2s} |\mathcal{F}_k(f)(\xi)|^2 d\gamma_k(\xi),$$

we deduce that

$$\|h\|^{\frac{2t}{s+t}}_{L^2_k(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} ||\xi||^{2s} |\mathcal{F}_k(f)(\xi)|^2 d\mu_k(\xi)\right)^{\frac{t}{t+s}} \left(\int_{\mathbb{R}^{2d}} ||y||^{2t} |\mathcal{G}^k_h(f)(y,\nu)|^2 d\mu_k(y,\nu)\right)^{\frac{s}{s+t}}$$
$$\geq \big(C_k(s,t)\big)^2 \int_{\mathbb{R}^{2d}} |\mathcal{G}_h f(y,\nu)|^2 d\mu_k(y,\nu) = \big(C_k(s,t)\big)^2 \|h\|^2_{L^2_k(\mathbb{R}^d)} \|f\|^2_{L^2_k(\mathbb{R}^d)}.$$

This proves the result. □

Proposition 4.3 (Nash's uncertainty principle for $\mathcal{G}^k_h$) *For every $s > 0$, there exists a positive constant $\mathcal{C}(k,s)$ such that, for all $f \in L^2_k(\mathbb{R}^d)$, we have*

$$||h||_{L^2_k(\mathbb{R}^d)} ||f||_{L^2_k(\mathbb{R}^d)} \leq \mathcal{C}(k,s) \Big\| \, ||(y,\nu)||^s \mathcal{G}^k_h(f) \Big\|_{L^2_{\mu_k}(\mathbb{R}^{2d})}. \tag{4.3}$$

Proof It is clear that the relation (4.3) holds if $f = 0$. Assume that $0 \neq f \in L^2_k(\mathbb{R}^d)$ and let $R > 0$. From Plancherel's formula (3.8) we have

$$\|h\|^2_{L^2_k(\mathbb{R}^d)} \|f\|^2_{L^2_k(\mathbb{R}^d)} = \|\mathcal{G}^k_h(f)\|^2_{L^2_{\mu_k}(\mathbb{R}^{2d})}$$
$$= ||\mathbb{1}_{B_{2d}(0,R)} \mathcal{G}^k_h(f)||^2_{L^2_{\mu_k}(\mathbb{R}^{2d})} + ||(1 - \mathbb{1}_{B_{2d}(0,R)}) \mathcal{G}^k_h(f)||^2_{L^2_{\mu_k}(\mathbb{R}^{2d})},$$

where

$$B_{2d}(0,R) := \Big\{(y,\nu) \in \mathbb{R}^{2d} : ||(y,\nu)|| \leq R\Big\}.$$

By (3.7), we have

$$||\mathbb{1}_{B_{2d}(0,R)} \mathcal{G}^k_h(f)||^2_{L^2_{\mu_k}(\mathbb{R}^{2d})} \leq \frac{1}{c_k^2} \|h\|^2_{L^2_k(\mathbb{R}^d)} \|f\|^2_{L^2_k(\mathbb{R}^d)} \int_{\mathbb{R}^{2d}} \mathbb{1}_{B_{2d}(0,R)} d\mu_k(y,\nu)$$
$$\leq C R^{4\langle k\rangle + 2d - 2} ||h||^2_{L^2_k(\mathbb{R}^d)} \|f\|^2_{L^2_k(\mathbb{R}^d)}.$$

On the other hand,

$$||(1 - \mathbb{1}_{B_{2d}(0,R)}) \mathcal{G}^k_h(f)||^2_{L^2_{\mu_k}(\mathbb{R}^{2d})} \leq R^{-2s} \Big\| (1 - \mathbb{1}_{B_{2d}(0,R)}) \, ||(y,\nu)||^s \mathcal{G}^k_h(f) \Big\|^2_{L^2_{\mu_k}(\mathbb{R}^{2d})}$$
$$\leq R^{-2s} \Big\| \, ||(y,\nu)||^s \mathcal{G}^k_h(f) \Big\|^2_{L^2_{\mu_k}(\mathbb{R}^{2d})}.$$

It follows then

$$\|h\|^2_{L^2_k(\mathbb{R}^d)} \|f\|^2_{L^2_k(\mathbb{R}^d)} \leq C R^{4\langle k\rangle + 2d - 2} ||h||^2_{L^2_k(\mathbb{R}^d)} \|f\|^2_{L^2_k(\mathbb{R}^d)}$$
$$+ R^{-2s} \Big\| \, ||(y,\nu)||^s \mathcal{G}^k_h(f) \Big\|^2_{L^2_{\mu_k}(\mathbb{R}^{2d})}.$$

Minimizing over $R > 0$ the right hand side of the above inequality gives

$$\|h\|^2_{L^2_k(\mathbb{R}^d)}||f||^2_{L^2_k(\mathbb{R}^d)} \leq C(k,s)||h||^{\frac{4s}{4\langle k\rangle+2d+2s-2}}_{L^2_k(\mathbb{R}^d)}||f||^{\frac{4s}{4\langle k\rangle+2d+2s-2}}_{L^2_k(\mathbb{R}^d)} \left\| ||(y,\nu)||^s \mathcal{G}^k_h(f)\right\|^{\frac{8\langle k\rangle+4d-4}{4\langle k\rangle+2d+2s-2}}_{L^2_{\mu_k}(\mathbb{R}^{2d})}. \tag{4.4}$$

The desired result follows immediately from (4.4). □

4.2 Heisenberg uncertainty principles via the *k*-entropy

Let ρ be a probability density function on $\mathbb{R}^{2d}$, i.e. a nonnegative measurable function on $\mathbb{R}^{2d}$ satisfying

$$\int_{\mathbb{R}^{2d}} \rho(y,\nu)d\mu_k(y,\nu) = 1.$$

Following Shannon [58], the k-entropy of a probability density function ρ on $\mathbb{R}^{2d}$ is defined by

$$E_k(\rho) := -\int_{\mathbb{R}^{2d}} \ln(\rho(y,\nu))\rho(y,\nu)d\mu_k(y,\nu).$$

Henceforth, we extend the definition of the k-entropy of a nonnegative measurable function ρ on $\mathbb{R}^{2d}$ whenever the previous integral on the right hand side is well defined.

The aim of this part is to study the localization of the k-entropy of the k-Hankel Gabor transform. Indeed, we have the following result.

Proposition 4.4 *For all $f \in L^2_k(\mathbb{R}^d)$, we have*

$$E_k(|\mathcal{G}^k_h(f)|^2) \geq -2\|f\|^2_{L^2_k(\mathbb{R}^d)}\|h\|^2_{L^2_k(\mathbb{R}^d)} \ln\left(\frac{\|f\|_{L^2_k(\mathbb{R}^d)}\|h\|_{L^2_k(\mathbb{R}^d)}}{c_k}\right). \tag{4.5}$$

Proof Assume that $\|f\|_{L^2_k(\mathbb{R}^d)}\|h\|_{L^2_k(\mathbb{R}^d)} = c_k$. By (3.7),

$$|\mathcal{G}^k_h(f)(y,\nu)| \leq \frac{1}{c_k}\|f\|_{L^2_k(\mathbb{R}^d)}\|h\|_{L^2_k(\mathbb{R}^d)} = 1. \tag{4.6}$$

In particular $E_k(|\mathcal{G}^k_h(f)|^2) \geq 0$. Next, let us drop the above assumption, and let

$$\phi := \frac{c_k f}{\|f\|_{L^2_k(\mathbb{R}^d)}} \quad \text{and} \quad \psi := \frac{h}{\|h\|_{L^2_k(\mathbb{R}^d)}}.$$

Then, $\phi, \psi \in L^2_k(\mathbb{R}^d)$ and $\|\phi\|_{L^2_k(\mathbb{R}^d)}\|\psi\|_{L^2_k(\mathbb{R}^d)} = c_k$.

Therefore, $E_k(|\mathcal{G}_\psi^k(\phi)|^2) \geq 0$. Moreover,

$$\mathcal{G}_\psi^k(\phi) = \frac{c_k}{\|f\|_{L_k^2(\mathbb{R}^d)}\|h\|_{L_k^2(\mathbb{R}^d)}}\mathcal{G}_h^k(f),$$

which implies

$$E_k(|\mathcal{G}_\psi^k(\phi)|^2) = \frac{c_k^2}{\|f\|^2_{L_k^2(\mathbb{R}^d)}\|h\|^2_{L_k^2(\mathbb{R}^d)}}E_k(|\mathcal{G}_h^k(f)|^2) + 2c_k^2\ln(\frac{\|f\|_{L_k^2(\mathbb{R}^d)}\|h\|_{L_k^2(\mathbb{R}^d)}}{c_k}).$$

Using the fact that $E_k(|\mathcal{G}_\psi^k(\phi)|^2) \geq 0$, we deduce that

$$E_k(|\mathcal{G}_h^k(f)|^2) \geq -2\|f\|^2_{L_k^2(\mathbb{R}^d)}\|h\|^2_{L_k^2(\mathbb{R}^d)}\ln\left(\frac{\|f\|_{L_k^2(\mathbb{R}^d)}\|h\|_{L_k^2(\mathbb{R}^d)}}{c_k}\right).$$

□

Using the k-entropy of the k-Hankel Gabor transform, we can obtain another version of the Heisenberg uncertainty principle for $\mathcal{G}_h^k$.

Theorem 4.5 *Let $p, q > 0$. Then for every $f \in L_k^2(\mathbb{R}^d)$ we have*

$$\left(\int_{\mathbb{R}^{2d}} ||y||^p|\mathcal{G}_h^k(f)(y,\nu)|^2 d\mu_k(y,\nu)\right)^{\frac{q}{p+q}}\left(\int_{\mathbb{R}^{2d}} ||\nu||^q|\mathcal{G}_h^k(f)(y,\nu)|^2 d\mu_k(y,\nu)\right)^{\frac{p}{p+q}}$$
$$\geq M_{p,q}(k)\|f\|^2_{L_k^2(\mathbb{R}^d)}\|h\|^2_{L_k^2(\mathbb{R}^d)},$$

where

$$M_{p,q}(k) = \frac{2\langle k\rangle + d - 1}{p^{\frac{q}{p+q}}q^{\frac{p}{p+q}}}\exp\left(\frac{pq}{(2\langle k\rangle + d - 1)(p+q)}\ln\left(\frac{pq(d_k)^2}{\Gamma(\frac{2\langle k\rangle+d-1}{p})\Gamma(\frac{2\langle k\rangle+d-1}{q})}\right) - 1\right).$$

Proof For every positive real numbers $t,\ p,\ q$, let $\eta^k_{t,p,q}$ be the function defined on $\mathbb{R}^{2d}$ by

$$\eta^k_{t,p,q}(y,\nu) := \frac{pq(d_k)^2}{\Gamma(\frac{2\langle k\rangle+d-1}{p})\Gamma(\frac{2\langle k\rangle+d-1}{q})}\frac{\exp\left(-\frac{||y||^p+||\nu||^q}{t}\right)}{t^{\frac{(2\langle k\rangle+d-1)(p+q)}{pq}}}.$$

By simple computation, we see that

$$\int_{\mathbb{R}^{2d}}\eta^k_{t,p,q}(y,\nu)d\mu_k(y,\nu) = 1.$$

In particular, the measure $d\sigma^k_{t,p,q}(y,\nu) := \eta^k_{t,p,q}(y,\nu)d\mu_k(y,\nu)$ is a probability measure on $\mathbb{R}^{2d}$. Since the function $\varphi(t) = t\ln(t)$ is convex over $(0,\infty)$, then by using Jensen's inequality for convex functions we get

$$\int_{\mathbb{R}^{2d}} \frac{|\mathcal{G}^k_h(f)(y,\nu)|^2}{\eta^k_{t,p,q}(y,\nu)} \ln\left(\frac{|\mathcal{G}^k_h(f)(y,\nu)|^2}{\eta^k_{t,p,q}(y,\nu)}\right) d\sigma^k_{t,p,q}(y,\nu) \geq 0,$$

which implies in terms of k-entropy that

$$\begin{aligned}
&E_k(|\mathcal{G}^k_h(f)|^2) + \ln\left(\frac{pq(d_k)^2}{\Gamma(\frac{2\langle k\rangle+d-1}{p})\Gamma(\frac{2\langle k\rangle+d-1}{q})}\right)\|f\|^2_{L^2_k(\mathbb{R}^d)}\|h\|^2_{L^2_k(\mathbb{R}^d)} \\
&\leq \ln\left(t^{\frac{(2\langle k\rangle+d-1)(p+q)}{pq}}\right)\|f\|^2_{L^2_k(\mathbb{R}^d)}\|h\|^2_{L^2_k(\mathbb{R}^d)} \\
&\quad + \frac{1}{t}\int_{\mathbb{R}^{2d}} (||y||^p + ||\nu||^q)|\mathcal{G}^k_h(f)(y,\nu)|^2 d\mu_k(y,\nu).
\end{aligned}$$

Assume that $\|f\|_{L^2_k(\mathbb{R}^d)}\|h\|_{L^2_k(\mathbb{R}^d)} = c_k$. Then, by Proposition 4.4 we get

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}} (||y||^p + ||\nu||^q)|\mathcal{G}^k_h(f)(y,\nu)|^2 d\mu_k(y,\nu) \\
&\geq t\left(\ln\left(\frac{pq(d_k)^2}{\Gamma(\frac{2\langle k\rangle+d-1}{p})\Gamma(\frac{2\langle k\rangle+d-1}{q})}\right) - \ln\left(t^{\frac{(2\langle k\rangle+d-1)(p+q)}{pq}}\right)\right)c_k^2.
\end{aligned}$$

However, the expression

$$t\left(\ln\left(\frac{pq(d_k)^2}{\Gamma(\frac{2\langle k\rangle+d-1}{p})\Gamma(\frac{2\langle k\rangle+d-1}{q})}\right) - \ln\left(t^{\frac{(2\langle k\rangle+d-1)(p+q)}{pq}}\right)\right)c_k^2$$

attains its upper bound at

$$t_0 = \exp\left(\frac{pq}{(2\langle k\rangle+d-1)(p+q)}\ln\left(\frac{pq(d_k)^2}{\Gamma(\frac{2\langle k\rangle+d-1}{p})\Gamma(\frac{2\langle k\rangle+d-1}{q})}\right) - 1\right),$$

and consequently

$$\int_{\mathbb{R}^{2d}} (||y||^p + ||\nu||^q)|\mathcal{G}^k_h(f)(y,\nu)|^2 d\mu_k(y,\nu) \geq C_{p,q}(k)c_k^2,$$

where

$$C_{p,q}(k) = \frac{(2\langle k\rangle + d - 1)(p+q)}{pq} \exp\left(\frac{pq}{(2\langle k\rangle + d - 1)(p+q)} \ln\left(\frac{pq(d_k)^2}{\Gamma(\frac{2\langle k\rangle + d-1}{p})\Gamma(\frac{2\langle k\rangle + d-1}{q})}\right) - 1\right).$$

Therefore, for every $f \in L_k^2(\mathbb{R}^d)$ and $h \in L_{k,rad}^2(\mathbb{R}^d)$ such that $\|f\|_{L_k^2(\mathbb{R}^d)}\|h\|_{L_k^2(\mathbb{R}^d)} = c_k$, we get

$$\int_{\mathbb{R}^{2d}} ||y||^p |\mathcal{G}_h^k(f)(y,\nu)|^2 d\mu_k(y,\nu) + \int_{\mathbb{R}^{2d}} ||\nu||^q |\mathcal{G}_h^k(f)(y,\nu)|^2 d\mu_k(y,\nu) \geq C_{p,q}(k) c_k^2.$$

Now, for every $\lambda > 0$, the dilates f_λ and $h_{\frac{1}{\lambda}}$ belong to $L_k^2(\mathbb{R}^d)$. Then, by substituting f by f_λ and h by $h_{\frac{1}{\lambda}}$ and using the fact that

$$\|f_\lambda\|_{L_k^2(\mathbb{R}^d)}\|h_{\frac{1}{\lambda}}\|_{L_k^2(\mathbb{R}^d)} = \|f\|_{L_k^2(\mathbb{R}^d)}\|h\|_{L_k^2(\mathbb{R}^d)} = c_k,$$

the above inequality gives

$$\int_{\mathbb{R}^{2d}} ||y||^p |\mathcal{G}_{h_{\frac{1}{\lambda}}}^k(f_\lambda)(y,\nu)|^2 d\mu_k(y,\nu) + \int_{\mathbb{R}^{2d}} ||\nu||^q |\mathcal{G}_{h_{\frac{1}{\lambda}}}^k(f_\lambda)(y,\nu)|^2 d\mu_k(y,\nu) \geq C_{p,q}(k) c_k^2.$$

Using (3.6), we deduce that

$$\lambda^p \int_{\mathbb{R}^{2d}} ||y||^p |\mathcal{G}_h^k(f)(y,\nu)|^2 d\mu_k(y,\nu) + \lambda^{-q} \int_{\mathbb{R}^{2d}} ||\nu||^q |\mathcal{G}_h^k(f)(y,\nu)|^2 d\mu_k(y,\nu) \geq C_{p,q}(k) c_k^2.$$

In particular, the inequality holds at the point

$$\lambda = \left(\frac{p\int_{\mathbb{R}^{2d}} ||y||^p |\mathcal{G}_h^k(f)(y,\nu)|^2 d\mu_k(y,\nu)}{q\int_{\mathbb{R}^{2d}} ||\nu||^q |\mathcal{G}_h^k(f)(y,\nu)|^2 d\mu_k(y,\nu)}\right)^{\frac{-1}{p+q}},$$

which implies that

$$\left(\int_{\mathbb{R}^{2d}} ||y||^p |\mathcal{G}_h^k(f)(y,\nu)|^2 d\mu_k(y,\nu)\right)^{\frac{q}{p+q}} \left(\int_{\mathbb{R}^{2d}} ||\nu||^q |\mathcal{G}_h^k(f)(y,\nu)|^2 d\mu_k(y,\nu)\right)^{\frac{p}{p+q}} \geq M_{p,q}(k) c_k^2,$$

where

$$M_{p,q}(k) = C_{p,q}(k)\frac{p^{\frac{p}{p+q}}q^{\frac{q}{p+q}}}{p+q}$$
$$= \frac{2\langle k\rangle + d - 1}{p^{\frac{q}{p+q}}q^{\frac{p}{p+q}}}\exp\left(\frac{pq}{(2\langle k\rangle + d - 1)(p+q)}\ln\Big(\frac{pq(d_k)^2}{\Gamma(\frac{2\langle k\rangle+d-1}{p})\Gamma(\frac{2\langle k\rangle+d-1}{q})}\Big) - 1\right).$$

Now, the general formula follows from above by substituting f by $c_k f/\{\|f\|_{L^2_k(\mathbb{R}^d)}\}$ and h by $h/\|h\|_{L^2_k(\mathbb{R}^d)}$. □

Remark 4.6 When $p = q = 2$, we get

$$\Big\| \, ||y|| \, \mathcal{G}_h^k(f)\Big\|_{L^2_{\mu_k}(\mathbb{R}^{2d})} \Big\| \, ||\nu|| \, \mathcal{G}_h^k(f)\Big\|_{L^2_{\mu_k}(\mathbb{R}^{2d})}$$
$$\geq \frac{2\langle k\rangle + d - 1}{2e}\left(\frac{2d_k}{\Gamma(\frac{2\langle k\rangle+d-1}{2})}\right)^{\frac{2}{2\langle k\rangle+d-1}}\|f\|^2_{L^2_k(\mathbb{R}^d)}\|h\|^2_{L^2_k(\mathbb{R}^d)}.$$

4.3 L^p-Heisenberg's uncertainty principle

In this subsection we will establish a general from of L^p-Heisenberg's uncertainty principle.

For $t > 0$, we set

$$\Gamma_t(y,\nu) := e^{-t||(y,\nu)||^2}, \qquad (y,\nu) \in \mathbb{R}^{2d}.$$

By simple calculations it is easy to check that for every $1 \leq q < \infty$, there exists a positive constant C such that

$$||\Gamma_t||_{L^q_{\mu_k}(\mathbb{R}^{2d})} = Ct^{-\frac{2\langle k\rangle+d-1}{q}}. \tag{4.7}$$

Lemma 4.7 *Let $1 < p \leq 2$ and $0 < a < \frac{2\langle k\rangle+d-1}{2p'}$, where p' denotes the conjugate exponent of p. Then, there exists a positive constant C such that, for all $f \in L^2_k(\mathbb{R}^d)$ and $t > 0$,*

$$\Big\|\Gamma_t \, \mathcal{G}_h^k(f)\Big\|_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})}$$
$$\leq C\|h\|_{L^2_k(\mathbb{R}^d)}t^{-2a}\Big[\Big\| \, ||y||^a f\Big\|_{L^2_k(\mathbb{R}^d)} + \Big\| \, ||y||^a f\Big\|_{L^{2p}_k(\mathbb{R}^d)}\Big]. \tag{4.8}$$

Proof Inequality (4.8) holds whenever $\| \, ||y||^a f\|_{L^2_k(\mathbb{R}^d)} + \| \, ||y||^a f\|_{L^{2p}_k(\mathbb{R}^d)} = \infty$. Let us assume that

$$\Big\| \, ||y||^a f\Big\|_{L^2_k(\mathbb{R}^d)} + \Big\| \, ||y||^a f\Big\|_{L^{2p}_k(\mathbb{R}^d)} < \infty.$$

For $s > 0$, let $f_s = \mathbb{1}_{B_d(0,s)} f$ and $f^s = f - f_s$. Since

$$|f^s(y)| \leq s^{-a} | \, ||y||^a f(y)|,$$

we deduce from Proposition 3.8 that

$$\begin{aligned}\left\|\Gamma_t \, \mathcal{G}_h^k(\mathbb{1}_{B_d^c(0,s)} f)\right\|_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})} &\leq \|\Gamma_t\|_{L^\infty_{\mu_k}(\mathbb{R}^{2d})} \left\|\mathcal{G}_h^k(\mathbb{1}_{B_d^c(0,s)} f)\right\|_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})} \\ &\leq c_k^{\frac{2-p'}{p'}} \|h\|_{L^2_k(\mathbb{R}^d)} \left\|\mathbb{1}_{B_d^c(0,s)} f\right\|_{L^2_k(\mathbb{R}^d)} \\ &\leq c_k^{\frac{2-p'}{p'}} s^{-a} \|h\|_{L^2_k(\mathbb{R}^d)} \| \, ||y||^a f\|_{L^2_k(\mathbb{R}^d)} .\end{aligned}$$

On the other hand, by (3.7) and Hölder's inequality

$$\begin{aligned}\left\|\Gamma_t \, \mathcal{G}_h^k(\mathbb{1}_{B_d(0,s)} f)\right\|_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})} &\leq \|\Gamma_t\|_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})} \left\|\mathcal{G}_h^k(\mathbb{1}_{B_d(0,s)} f)\right\|_{L^\infty_{\mu_k}(\mathbb{R}^{2d})} \\ &\leq \tfrac{1}{c_k} \|h\|_{L^2_k(\mathbb{R}^d)} \|\Gamma_t\|_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})} \left\|\mathbb{1}_{B_d(0,s)} f\right\|_{L^2_k(\mathbb{R}^d)} \\ &\leq \tfrac{1}{c_k} \|h\|_{L^2_k(\mathbb{R}^d)} \|\Gamma_t\|_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})} \left\| \, ||y||^{-a} \mathbb{1}_{B_d(0,s)}\right\|_{L^{2p'}_k(\mathbb{R}^d)} \| \, ||y||^a f\|_{L^{2p}_k(\mathbb{R}^d)} .\end{aligned}$$

A simple calculation shows that there exists a positive constant C such that

$$\left\| \, ||y||^{-a} \mathbb{1}_{B_d(0,s)}\right\|_{L^{2p'}_k(\mathbb{R}^d)} = C s^{-a + \frac{2\langle k\rangle + d - 1}{2p'}} .$$

Therefore,

$$\begin{aligned}\|\Gamma_t \, \mathcal{G}_h^k(f)\|_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})} &\leq \|\Gamma_t \, \mathcal{G}_h^k(f_s)\|_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})} + \|\Gamma_t \, \mathcal{G}_h^k(f^s)\|_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})} \\ &\leq C s^{-a} \|h\|_{L^2_k(\mathbb{R}^d)} \left[c_k^{\frac{2-p'}{p'}} \| \, ||y||^a f\|_{L^2_k(\mathbb{R}^d)} + \tfrac{1}{c_k} s^{\frac{2\langle k\rangle+d-1}{2p'}} ||\Gamma_t||_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})} \| \, ||y||^a f\|_{L^{2p}_k(\mathbb{R}^d)} \right].\end{aligned}$$

Choosing $s = (c_k)^{\frac{4}{2\langle k\rangle + d - 1}} t^2$ and using (4.7), we obtain the desired inequality. □

Theorem 4.8 *Let* $1 < p \leq 2$, $0 < a < \frac{2\langle k\rangle + d - 1}{2p'}$ *and* $b > 0$. *Then, there exists a positive constant C such that for all* $f \in L^2_k(\mathbb{R}^d)$, *we have*

$$\begin{aligned}\left\|\mathcal{G}_h^k(f)\right\|_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})} &\leq C \|h\|_{L^2_k(\mathbb{R}^d)}^{\frac{b}{a+b}} \left[\| \, ||y||^a f\|_{L^2_k(\mathbb{R}^d)} + \| \, ||y||^a f\|_{L^{2p}_k(\mathbb{R}^d)} \right]^{\frac{b}{a+b}} \\ &\quad \left\| \, ||(y,\nu)||^{4b} \mathcal{G}_h^k(f)\right\|_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})}^{\frac{a}{a+b}} .\end{aligned} \tag{4.9}$$

Proof Inequality (4.9) holds whenever $\mathcal{G}_h^k(f) = 0$. Assume that $\mathcal{G}_h^k(f) \neq 0$. Let $1 < p \leq 2$ and $0 < a < \frac{2\langle k\rangle + d - 1}{2p'}$. Let us assume that $b \leq \frac{1}{2}$. From the previous lemma, for all $t > 0$, we have

$$\begin{aligned}\left\|\mathcal{G}_h^k(f)\right\|_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})} &\leq \left\|\Gamma_t\,\mathcal{G}_h^k(f)\right\|_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})} + \left\|(1-\Gamma_t)\mathcal{G}_h^k(f)\right\|_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})}\\ &\leq C\|h\|_{L^2_k(\mathbb{R}^d)}t^{-2a}\Big[\big\|\,||y||^a f\big\|_{L^2_k(\mathbb{R}^d)}\\ &\quad + \big\|\,||y||^a f\big\|_{L^{2p}_k(\mathbb{R}^d)}\Big] + \left\|(1-\Gamma_t)\mathcal{G}_h^k(f)\right\|_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})}.\end{aligned}$$

On the other hand,

$$\left\|(1-\Gamma_t)\mathcal{G}_h^k(f)\right\|_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})} = t^{2b}\left\|(t||(y,\nu)||^2)^{-2b}(1-\Gamma_t)||(y,\nu)||^{4b}\mathcal{G}_h^k(f)\right\|_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})}.$$

Since $(1-e^{-u})u^{-2b}$ is bounded for $u \geq 0$ if $b \leq \frac{1}{2}$, we obtain

$$\begin{aligned}\left\|\mathcal{G}_h^k(f)\right\|_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})} &\leq C\|h\|_{L^2_k(\mathbb{R}^d)}t^{-2a}\Big[\big\|\,||y||^a f\big\|_{L^2_k(\mathbb{R}^d)}\\ &\quad + \big\|\,||y||^a f\big\|_{L^{2p}_k(\mathbb{R}^d)}\Big] + Ct^{2b}\left\|\,||(y,\nu)||^{4b}\mathcal{G}_h^k(f)\right\|_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})},\end{aligned}$$

from which, optimizing in t, we obtain (4.9) for $0 < a < \frac{2\langle k\rangle+d-1}{2p'}$ and $b \leq \frac{1}{2}$.

Next, we assume that $b > \frac{1}{2}$. For $u \geq 0$ and $b' \leq \frac{1}{2} < b$, we have $u^{4b'} \leq 1 + u^{4b}$, which is for $u = ||(y,\nu)||/\varepsilon$ becomes

$$\left(\frac{||(y,\nu)||}{\varepsilon}\right)^{4b'} < 1 + \left(\frac{||(y,\nu)||}{\varepsilon}\right)^{4b}, \qquad \text{for all } \varepsilon > 0.$$

It follows that

$$\begin{aligned}&\left\|\,||(y,\nu)||^{4b'}\mathcal{G}_h^k(f)\right\|_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})}\\ &\leq \varepsilon^{4b'}\left\|\mathcal{G}_h^k(f)\right\|_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})} + \varepsilon^{4(b'-b)}\left\|\,||(y,\nu)||^{4b}\mathcal{G}_h^k(f)\right\|_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})}.\end{aligned}$$

Optimizing in ε, we obtain that there exist a positive constant C:

$$\begin{aligned}&\left\|\,||(y,\nu)||^{4b'}\mathcal{G}_h^k(f)\right\|_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})}\\ &\leq C\left\|\mathcal{G}_h^k(f)\right\|^{\frac{b-b'}{b}}_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})}\left\|\,||(y,\nu)||^{4b}\mathcal{G}_h^k(f)\right\|^{\frac{b'}{b}}_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})}.\end{aligned}$$

Together with (4.9) for b', we get the result for $b > \frac{1}{2}$. □

Corollary 4.9 *Let* $0 < a < \frac{2\langle k\rangle+d-1}{4}$ *and* $b > 0$. *There exists a positive constant* C *such that, for all* $f \in L^2_k(\mathbb{R}^d)$, *we have*

$$||f||_{L^2_k(\mathbb{R}^d)} \leq C\|h\|^{\frac{-a}{a+b}}_{L^2_k(\mathbb{R}^d)} \left[\left\| ||y||^a f\right\|_{L^2_k(\mathbb{R}^d)} + \left\| ||y||^a f\right\|_{L^4_k(\mathbb{R}^d)}\right]^{\frac{b}{a+b}} \left\| ||(y,\nu)||^{4b} \mathcal{G}^k_h(f)\right\|^{\frac{a}{a+b}}_{L^2_{\mu_k}(\mathbb{R}^{2d})}. \tag{4.10}$$

Proof The statement follows from Theorem 4.8 with $p = 2$ and Plancherel's formula (3.8). □

5 Concentration uncertainty principles for the *k*-Hankel Gabor transforms

In this Section, we derive some concentration uncertainty principles for the k-Hankel Gabor transforms as an analog of the Benedick–Amrein–Berthier and local uncertainty principles in the time–frequency analysis.

5.1 Benedick–Amrein–Berthier's uncertainty principle

Recently Johansen in [35] has proved the Benedicks-Amrein-Berthier uncertainty principle for the k-Hankel transform which states that if E_1 and E_2 are two subsets of $\mathbb{R}^d$ with finite measure, then there exist a positive constant $C_k(E_1, E_2)$ such that for any $f \in L^2_k(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} |f(t)|^2 d\gamma_k(t) \leq C_k(E_1, E_2)\left\{\int_{\mathbb{R}^d\setminus E_1} |f(t)|^2 d\gamma_k(t) + \int_{\mathbb{R}^d\setminus E_2} |\mathcal{F}_k(f)(\xi)|^2 d\gamma_k(\xi)\right\}. \tag{5.1}$$

In this Section, our primary interest is to establish the Benedick–Amrein–Berthier's uncertainty principle for the k-Hankel Gabor transforms by employing the inequality (5.1). In this direction, we have the following main theorem.

Theorem 5.1 *For any arbitrary function $f \in L^2_k(\mathbb{R}^d)$, we have the following uncertainty inequality*

$$\int_{\mathbb{R}^d\setminus E_1}\int_{\mathbb{R}^d} \left|\mathcal{G}^k_h(f)(y,\nu)\right|^2 d\mu_k(y,\nu) + ||h||^2_{L^2_k(\mathbb{R}^d)} \int_{\mathbb{R}^d\setminus E_2} |\mathcal{F}_k(f)(\xi)|^2 d\gamma_k(\xi) \geq \frac{||h||^2_{L^2_k(\mathbb{R}^d)} ||f||^2_{L^2_k(\mathbb{R}^d)}}{C_k(E_1, E_2)} \tag{5.2}$$

where $C_k(E_1, E_2)$ the constant given in relation (5.1).

Proof Since for all $\nu \in \mathbb{R}^d$, $\mathcal{G}^k_h(f)(., \nu) \in L^2_k(\mathbb{R}^d)$, whenever $f \in L^2_k(\mathbb{R}^d)$, so we can replace the function f appearing in (5.1) with $\mathcal{G}^k_h(f)(., \nu)$ to get

$$\int_{\mathbb{R}^d} \left|\mathcal{G}^k_h(f)(y,\nu)\right|^2 d\gamma_k(y) \leq$$

$$C_k(E_1, E_2)\left\{\int_{\mathbb{R}^d\setminus E_1} \left|\mathcal{G}_h^k(f)(y,\nu)\right|^2 d\gamma_k(y) + \int_{\mathbb{R}^d\setminus E_2} \left|\mathcal{F}_k\left[\mathcal{G}_h^k(f)(.,\nu)\right](\xi)\right|^2 d\gamma_k(\xi)\right\}. \tag{5.3}$$

By integrating (5.3) with respect to the measure $d\gamma_k(\nu)$, we obtain

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left|\mathcal{G}_h^k(f)(y,\nu)\right|^2 d\mu_k(y,\nu) \leq C_k(E_1, E_2)\left\{\int_{\mathbb{R}^d\setminus E_1}\int_{\mathbb{R}^d} \left|\mathcal{G}_h^k(f)(y,\nu)\right|^2 d\mu_k(y,\nu) + \int_{\mathbb{R}^d\setminus E_2}\int_{\mathbb{R}^d} \mathcal{F}_k\left[\mathcal{G}_h^k(f)](y,\nu)\right](\xi)\right|^2 d\mu_k(\xi,\nu)\right\}.$$

Using Lemma 3.9 together with Plancherel's formula (3.8), the above inequality becomes

$$\int_{\mathbb{R}^d\setminus E_1}\int_{\mathbb{R}^d} \left|\mathcal{G}_h^k(f)(y,\nu)\right|^2 d\mu_k(y,\nu) + \int_{\mathbb{R}^d\setminus E_2}\int_{\mathbb{R}^d} |\mathcal{F}_k(f)(\xi)\sqrt{\tau_\nu^k|h|^2(\xi)}|^2 d\mu_k(\xi,\nu) \geq \frac{||h||^2_{L_k^2(\mathbb{R}^d)}||f||^2_{L_k^2(\mathbb{R}^d)}}{C_k(E_1, E_2)}$$

which further implies

$$\int_{\mathbb{R}^d\setminus E_1}\int_{\mathbb{R}^d} \left|\mathcal{G}_h^k(f)(y,\nu)\right|^2 d\mu_k(y,\nu) + \int_{\mathbb{R}^d\setminus E_2} |\mathcal{F}_k(f)(\xi)|^2 \left\{\int_{\mathbb{R}^d} \tau_\nu^k|h|^2(\xi)d\gamma_k(\nu)\right\} d\gamma_k(\xi) \geq \frac{||h||^2_{L_k^2(\mathbb{R}^d)}||f||^2_{L_k^2(\mathbb{R}^d)}}{C_k(E_1, E_2)}.$$

Thus using the fact that $h \in L^2_{k,rad}(\mathbb{R}^d)\bigcap L_k^\infty(\mathbb{R}^d)$, Lemma 3.9 and relation (2.24) we obtain

$$\int_{\mathbb{R}^d\setminus E_1}\int_{\mathbb{R}^d} \left|\mathcal{G}_h^k(f)(y,\nu)\right|^2 d\mu_k(y,\nu) + ||h||^2_{L_k^2(\mathbb{R}^d)} \int_{\mathbb{R}^d\setminus E_2} |\mathcal{F}_k(f)(\xi)|^2 d\gamma_k(\xi) \geq \frac{||h||^2_{L_k^2(\mathbb{R}^d)}||f||^2_{L_k^2(\mathbb{R}^d)}}{C_k(E_1, E_2)}$$

which is the desired Benedick–Amrein–Berthier's uncertainty principle for the k-Hankel Gabor transforms. □

Theorem 5.1 allows as to obtain a general form of Heisenberg-type uncertainty inequality for the k-Hankel Gabor transforms.

Corollary 5.2 *Let $p, q > 0$. Then there exist a positive constant $\mathcal{C}_k(p,q)$ such that for any arbitrary function $f \in L^2_k(\mathbb{R}^d)$, we have the following uncertainty inequality*

$$\left(\int_{\mathbb{R}^{2d}} ||y||^{2p} \left|\mathcal{G}_h^k(f)(y,\nu)\right|^2 d\mu_k(y,\nu)\right)^{\frac{q}{2}}$$
$$\left(\int_{\mathbb{R}^d} ||\xi||^{2q} |\mathcal{F}_k(f)(\xi)|^2 d\gamma_k(\xi)\right)^{\frac{p}{2}} \geq \mathcal{C}_k(p,q)||h||^q_{L^2_k(\mathbb{R}^d)}||f||^{p+q}_{L^2_k(\mathbb{R}^d)}.$$

Proof Let $p, q > 0$ and let $f \in L^2_k(\mathbb{R}^d)$. Take $E_1 = E_2 = B_d(0,1)$ the unit ball in $\mathbb{R}^d$. Then by (5.2)

$$\int_{B^c_d(0,1)}\int_{\mathbb{R}^d} \left|\mathcal{G}_h^k(f)(y,\nu)\right|^2 d\mu_k(y,\nu) + ||h||^2_{L^2_k(\mathbb{R}^d)} \int_{B^c_d(0,1)} |\mathcal{F}_k(f)(\xi)|^2 d\gamma_k(\xi)$$
$$\geq \frac{||h||^2_{L^2_k(\mathbb{R}^d)}||f||^2_{L^2_k(\mathbb{R}^d)}}{C(k)}.$$

Here $C(k) := C_k(E_1, E_2)$.

It follows that

$$\int_{\mathbb{R}^{2d}} ||y||^{2p} \left|\mathcal{G}_h^k(f)(y,\nu)\right|^2 d\mu_k(y,\nu) + ||h||^2_{L^2_k(\mathbb{R}^d)} \int_{\mathbb{R}^d} ||\xi||^{2q} |\mathcal{F}_k(f)(\xi)|^2 d\gamma_k(\xi)$$
$$\geq \frac{||h||^2_{L^2_k(\mathbb{R}^d)}||f||^2_{L^2_k(\mathbb{R}^d)}}{C(k)}.$$

Now replacing f by f_λ and h by $h_{\frac{1}{\lambda}}$, we get by (3.6)

$$\int_{\mathbb{R}^{2d}} ||y||^{2p} \left|\mathcal{G}_h^k(f)(\frac{y}{\lambda},\lambda\nu)\right|^2 d\mu_k(y,\nu) + \lambda^{2\langle k\rangle+d-1}||h||^2_{L^2_k(\mathbb{R}^d)} \int_{\mathbb{R}^d} ||\xi||^{2q} |\mathcal{F}_k(f)(\lambda\xi)|^2 d\gamma_k(\xi)$$
$$\geq \frac{||h||^2_{L^2_k(\mathbb{R}^d)}||f||^2_{L^2_k(\mathbb{R}^d)}}{C(k)}.$$

Thus

$$\lambda^{2p}\int_{\mathbb{R}^{2d}} ||y||^{2p} \left|\mathcal{G}_h^k(f)(y,\nu)\right|^2 d\mu_k(y,\nu) + \lambda^{-2q}||h||^2_{L^2_k(\mathbb{R}^d)} \int_{\mathbb{R}^d} ||\xi||^{2q} |\mathcal{F}_k(f)(\xi)|^2 d\gamma_k(\xi)$$
$$\geq \frac{||h||^2_{L^2_k(\mathbb{R}^d)}||f||^2_{L^2_k(\mathbb{R}^d)}}{C(k)}.$$

The desired result follows by minimizing the right hand side over $\lambda > 0$. □

5.2 Local-type uncertainty principles

We begin this subsection by recalling the local uncertainty principle for the k-Hankel transforms.

Proposition 5.3 ([27]) *Let E be a subset of $\mathbb{R}^d$ such that*
$0 < \gamma_k(E) := \int_E d\gamma_k(x) < \infty.$

For $0 < s < \frac{2\langle k\rangle+d-1}{2}$, there exist a positive constant $\mathfrak{C}(k,s)$ such that for any $f \in L^2_k(\mathbb{R}^d)$

$$\int_E |\mathcal{F}_k(f)(\xi)|^2\, d\gamma_k(\xi) \le \mathfrak{C}(k,s)\,(\gamma_k(E))^{\frac{2s}{2\langle k\rangle+d-1}}\, ||\,||x||^s f||^2_{L^2_k(\mathbb{R}^d)}. \tag{5.4}$$

The main objective of this Subsection is to establish the local uncertainty principles for the k-Hankel Gabor transforms in arbitrary space dimensions by employing the previous inequality.

Theorem 5.4 *Let E be a subset of $\mathbb{R}^d$ with finite measure $0 < \gamma_k(E) < \infty$ and let $0 < s < \frac{2\langle k\rangle+d-1}{2}$. For any $f \in L^2_k(\mathbb{R}^d)$, we have*

$$\int_E |\mathcal{F}_k(f)(\xi)|^2 d\gamma_k(\xi) \le \frac{\mathfrak{C}(k,s)(\gamma_k(E))^{\frac{2s}{2\langle k\rangle+d-1}}}{||h||^2_{L^2_k(\mathbb{R}^d)}} \int_{\mathbb{R}^{2d}} ||y||^{2s} \left|\mathcal{G}^k_h(f)(y,\nu)\right|^2 d\mu_k(y,\nu), \tag{5.5}$$

where $\mathfrak{C}(k,s)$ the constant given in Proposition 5.3.

Proof Let $\nu \in \mathbb{R}^d$. Since $\mathcal{G}^k_h(f)(.,\nu) \in L^2_k(\mathbb{R}^d)$, whenever $f \in L^2_k(\mathbb{R}^d)$, so we can replace the function f appearing in (5.4) with $\mathcal{G}^k_h(f)(.,\nu)$ to get

$$\int_E \left|\mathcal{F}_k\left[\mathcal{G}^k_h(f)(.,\nu)\right](\xi)\right|^2 d\gamma_k(\xi) \le \mathfrak{C}(k,s)\,(\gamma_k(E))^{\frac{2s}{2\langle k\rangle+d-1}}\, ||\,||y||^s \mathcal{G}^k_h(f)(.,\nu)||^2_{L^2_k(\mathbb{R}^d)},\ \text{for all } \nu \in \mathbb{R}^d. \tag{5.6}$$

For explicit expression of (5.6), we shall integrate this inequality with respect to the measure $d\gamma_k(\nu)$ to get

$$\int_E \int_{\mathbb{R}^d} \left|\mathcal{F}_k\left[\mathcal{G}^k_h(f)(.,\nu)\right](\xi)\right|^2 d\mu_k(\xi,\nu) \le \mathfrak{C}(k,s)\,(\gamma_k(E))^{\frac{2s}{2\langle k\rangle+d-1}} \int_{\mathbb{R}^{2d}} ||y||^{2s} \left|\mathcal{G}^k_h(f)(y,\nu)\right|^2 d\mu_k(y,\nu)$$

which together with Lemma 3.9 gives

$$\int_E \int_{\mathbb{R}^d} |\mathcal{F}_k(f)(\xi)|^2 \tau^k_\nu |h|^2(\xi) d\gamma_k(\xi) d\gamma_k(\nu) \le \mathfrak{C}(k,s) \quad (\gamma_k(E))^{\frac{2s}{2\langle k\rangle+d-1}} \int_{\mathbb{R}^{2d}} ||y||^{2s} \left|\mathcal{G}^k_h(f)(y,\nu)\right|^2 d\mu_k(y,\nu). \tag{5.7}$$

Using the hypothesis on h, inequality (5.7) reduces to

$$||h||^2_{L^2_k(\mathbb{R}^d)} \int_E |\mathcal{F}_k(f)(\xi)|^2 \, d\gamma_k(\xi) \leq$$
$$\mathfrak{C}(k,s)\,(\gamma_k(E))^{\frac{2s}{2\langle k\rangle+d-1}} \int_{\mathbb{R}^{2d}} ||y||^{2s} \left|\mathcal{G}^k_h(f)(y,\nu)\right|^2 d\mu_k(y,\nu).$$

Or equivalently,

$$\int_{\mathbb{R}^{2d}} ||y||^{2s} \left|\mathcal{G}^k_h(f)(y,\nu)\right|^2 d\mu_k(y,\nu) \geq \frac{||h||^2_{L^2_k(\mathbb{R}^d)}}{\mathfrak{C}(k,s)\,(\gamma_k(E))^{\frac{2s}{2\langle k\rangle+d-1}}}$$
$$\int_E |\mathcal{F}_k(f)(\xi)|^2 d\gamma_k(\xi), \quad 0 < s < \frac{2\langle k\rangle + d - 1}{2}. \tag{5.8}$$

This completes the proof of (5.5). □

Let E be a subset of $\mathbb{R}^d$. We define the Paley–Wiener space $PW_k(E)$ as follow:

$$PW_k(E) := \left\{ f \in L^2_k(\mathbb{R}^d) : \; supp\, \mathcal{F}_k(f) \subset E \right\}.$$

Involving Plancherel's formula (2.13), definition of the Paley–Wiener space $PW_k(E)$ and the previous theorem we obtain the following:

Corollary 5.5 *Let E be a subset of $\mathbb{R}^d$ with finite measure $0 < \gamma_k(E) < \infty$. Let $0 < s < \frac{2\langle k\rangle+d-1}{2}$. For any $f \in PW_k(E)$, we have*

$$||f||^2_{L^2_k(\mathbb{R}^d)} \leq \frac{\mathfrak{C}(k,s)(\gamma_k(E))^{\frac{2s}{2\langle k\rangle+d-1}}}{||h||^2_{L^2_k(\mathbb{R}^d)}}$$
$$\int_{\mathbb{R}^{2d}} ||y||^{2s} \left|\mathcal{G}^k_h(f)(y,\nu)\right|^2 d\mu_k(y,\nu), \tag{5.9}$$

where $\mathfrak{C}(k,s)$ the constant given in Proposition 5.3.

By interchanging the roles of f and $\mathcal{F}_k(f)$ in Proposition 5.3, we get the following:

Corollary 5.6 *Let F be a subset of $\mathbb{R}^d$ with finite measure $0 < \gamma_k(F) < \infty$. For $0 < t < \frac{2\langle k\rangle+d-1}{2}$ and for any $f \in L^2_k(\mathbb{R}^d)$, we have*

$$\int_F |f(y)|^2 \, d\gamma_k(y) \leq \mathfrak{C}(k,t)\,(\gamma_k(F))^{\frac{2t}{2\langle k\rangle+d-1}} \, || \, ||\xi||^t \mathcal{F}_k(f)||^2_{L^2_k(\mathbb{R}^d)}, \tag{5.10}$$

where $\mathfrak{C}(k,t)$ the constant given in Proposition 5.3.

Involving Corollary 5.6 and using similar ideas given in the proof of Theorem 5.4, we prove the following.

Corollary 5.7 *Let F be a subset of $\mathbb{R}^d$ with finite measure $0 < \gamma_k(F) < \infty$. Let $0 < t < \frac{2\langle k\rangle+d-1}{2}$. For any $f \in L^2_k(\mathbb{R}^d)$, we have*

$$\int_{\mathbb{R}^d}\int_F \left|\mathcal{G}^k_h(f)(y,\nu)\right|^2 d\mu_k(y,\nu) \le \mathfrak{C}(k,t)(\gamma_k(F))^{\frac{2t}{2\langle k\rangle+d-1}}\,||h||^2_{L^2_k(\mathbb{R}^d)} \int_{\mathbb{R}^d} ||\xi||^{2t}\,|\mathcal{F}_k(f)(\xi)|^2\,d\gamma_k(\xi), \tag{5.11}$$

where $\mathfrak{C}(k,t)$ the constant given in Proposition 5.3.

Let F be a subset of $\mathbb{R}^d$. We define the generalized Paley–Wiener space $GPW_k(F)$ as follow:

$$GPW_k(F) := \left\{ f \in L^2_k(\mathbb{R}^d) : \forall\, \nu \in \mathbb{R}^d,\ supp\, \mathcal{G}^k_h(f)(.,\nu) \subset F \right\}.$$

Applying Plancherel's formula (3.8), definition of generalized Paley–Wiener space $GPW_k(F)$ and the previous corollary we obtain the following:

Corollary 5.8 *Let E and F be two subsets of $\mathbb{R}^d$ with finite measures $0 < \gamma_k(E), \gamma_k(F) < \infty$. Let $0 < s, t < \frac{2\langle k\rangle+d-1}{2}$.*

i) *For any $f \in GPW_k(F)$, we have*

$$||f||^2_{L^2_k(\mathbb{R}^d)} \le \mathfrak{C}(k,t)(\gamma_k(F))^{\frac{2t}{2\langle k\rangle+d-1}} \int_{\mathbb{R}^d} ||\xi||^{2t}\,|\mathcal{F}_k(f)(\xi)|^2\,d\gamma_k(\xi). \tag{5.12}$$

ii) *For any $f \in PW_k(E) \bigcap GPW_k(F)$, we have*

$$||f||^{s+t}_{L^2_k(\mathbb{R}^d)} \le (\mathfrak{C}(k,t))^{\frac{s}{2}}\,(\mathfrak{C}(k,s))^{\frac{t}{2}}\,(\gamma_k(E)\gamma_k(F))^{\frac{2ts}{2\langle k\rangle+d-1}} \left(\int_{\mathbb{R}^d} ||\xi||^{2t}\,|\mathcal{F}_k(f)(\xi)|^2\,d\gamma_k(\xi)\right)^{\frac{s}{2}} \left(\int_{\mathbb{R}^{2d}} ||y||^{2s}\left|\mathcal{G}^k_h(f)(y,\nu)\right|^2 d\mu_k(y,\nu)\right)^{\frac{t}{2}}, \tag{5.13}$$

where $\mathfrak{C}(k,t)$ the constant given in Proposition 5.3.

Our next endeavour is to obtain another version of Heisenberg-type uncertainty inequality for the k-Hankel Gabor transforms in arbitrary space dimensions.

Theorem 5.9 *Let $0 < p < \frac{2\langle k\rangle+d-1}{2}$ and $q > 0$. Then for any $f \in L^2_k(\mathbb{R}^d)$, we have*

$$||f||^2_{L^2_k(\mathbb{R}^d)} \le \mathcal{C}(k,p,q)\,\Big|\Big|\, ||y||^p\,\mathcal{G}^k_h(f)\Big|\Big|^{\frac{2q}{p+q}}_{L^2_{\mu_k}(\mathbb{R}^{2d})}\,\Big|\Big|\, ||\xi||^q\,\mathcal{F}_k(f)\Big|\Big|^{\frac{2p}{p+q}}_{L^2_k(\mathbb{R}^d)}, \tag{5.14}$$

where

$$\mathcal{C}(k,p,q) = \left(\frac{\mathfrak{C}(k,p)}{(d_k(2\langle k\rangle+d-1))^{\frac{2p}{2\langle k\rangle+d-1}}\,||h||^2_{L^2_k(\mathbb{R}^d)}}\right)^{\frac{q}{p+q}} \left[(\tfrac{p}{q})^{\frac{q}{p+q}} + (\tfrac{q}{p})^{\frac{p}{p+q}}\right].$$

Proof Let $0 < p < \frac{2\langle k\rangle+d-1}{2}$, $q > 0$ and $r > 0$. Then

$$||f||^2_{L^2_k(\mathbb{R}^d)} = ||\mathcal{F}_k(f)||^2_{L^2_k(\mathbb{R}^d)} = \int_{B_d(0,r)} |\mathcal{F}_k(f)(\xi)|^2 d\gamma_k(\xi) + \int_{B^c_d(0,r)} |\mathcal{F}_k(f)(\xi)|^2 d\gamma_k(\xi), \quad (5.15)$$

where $B_d(0, r)$ denotes the ball of $\mathbb{R}^d$ of center 0 and radius r.

From Theorem 5.4 and by simple calculation, we have

$$\int_{B_d(0,r)} |\mathcal{F}_k(f)(\xi)|^2 d\gamma_k(\xi) \leq \frac{\mathfrak{C}(k,p)}{(d_k(2\langle k\rangle + d - 1))^{\frac{2p}{2\langle k\rangle+d-1}} ||h||^2_{L^2_k(\mathbb{R}^d)}} r^{2p} \int_{\mathbb{R}^{2d}} ||y||^{2p} \left|\mathcal{G}^k_h(f)(y,\nu)\right|^2 d\mu_k(y,\nu). \quad (5.16)$$

Moreover it is easy to see that

$$\int_{B^c_d(0,r)} |\mathcal{F}_k(f)(\xi)|^2 d\gamma_k(\xi) \leq r^{-2q} \int_{\mathbb{R}^d} ||\xi||^{2q} |\mathcal{F}_k(f)(\xi)|^2 d\gamma_k(\xi). \quad (5.17)$$

Combining the relations (5.15), (5.16) and (5.17), we get

$$||f||^2_{L^2_k(\mathbb{R}^d)} \leq \frac{\mathfrak{C}(k,p)}{(d_k(2\langle k\rangle + d - 1))^{\frac{2p}{2\langle k\rangle+d-1}} ||h||^2_{L^2_k(\mathbb{R}^d)}} r^{2p} \Big|\Big| \, ||y||^p \big|\mathcal{G}^k_h(f)||^2_{L^2_{\mu_k}(\mathbb{R}^{2d})} + r^{-2q} || \, ||\xi||^q \mathcal{F}_k(f)||^2_{L^2_k(\mathbb{R}^d)}.$$

We choose

$$r = \left[\frac{q(d_k(2\langle k\rangle + d - 1))^{\frac{2p}{2\langle k\rangle+d-1}} ||h||^2_{L^2_k(\mathbb{R}^d)}}{p\mathfrak{C}(k,p)}\right]^{\frac{1}{2p+2q}} \left(\frac{\Big|\Big| \, ||y||^p \mathcal{G}^k_h(f)||_{L^2_{\mu_k}(\mathbb{R}^{2d})}}{|| \, ||\xi||^q \mathcal{F}_k(f)||_{L^2_k(\mathbb{R}^d)}}\right)^{\frac{-1}{p+q}},$$

we obtain the desired inequality. □

We apply the same arguments that used in [47] we derive the following local uncertainty principles for the k-Hankel Gabor transform on $\mathbb{R}^d$.

Theorem 5.10 *We assume that* $h \in L^2_{k,rad}(\mathbb{R}^d)$. *Let* $1 < p \le 2$, $a > 0$ *and a measurable subset* $T \subset \mathbb{R}^{2d}$ *satisfying* $0 < \mu_k(T) := \int_T d\mu_k(x, y) < \infty$. *Then for all* $f \in L^2_k(\mathbb{R}^d)$, *we have*

$$||1_T \mathcal{G}^k_h(f)||_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})} \le \begin{cases} C_1(a,h,k)\,(\mu_k(T))^{\frac{2a}{2\langle k\rangle+d-1}} \\ \quad \Big[||\,||y||^a f||_{L^2_k(\mathbb{R}^d)} + ||\,||y||^a f||_{L^{2p}_k(\mathbb{R}^d)}\Big], \quad 0 < a < \frac{2\langle k\rangle+d-1}{2p'}, \\ C_2(a,h,k)\,(\mu_k(T))^{\frac{1}{p'}}\,||f||^{1-\frac{2\langle k\rangle+d-1}{2ap'}}_{L^{2p}_k(\mathbb{R}^d)}\,||\,||y||^a f||^{\frac{2\langle k\rangle+d-1}{2ap'}}_{L^{2p}_k(\mathbb{R}^d)}, \quad a > \frac{2\langle k\rangle+d-1}{2p'}, \\ C_3(a,h,k)\,(\mu_k(T))^{\frac{1}{2p'}} \\ \quad \Big[||f||^{\frac12}_{L^2_k(\mathbb{R}^d)}\,||\,||y||^a f||^{\frac12}_{L^2_k(\mathbb{R}^d)} + ||f||^{\frac12}_{L^{2p}_k(\mathbb{R}^d)}\,||\,||y||^a f||^{\frac12}_{L^{2p}_k(\mathbb{R}^d)}\Big], \; a = \frac{2\langle k\rangle+d-1}{2p'}, \end{cases}$$

where

$$C_1(a,h,k) = c_k^{1-\frac{2}{p}-\frac{4a}{2\langle k\rangle+d-1}}\,(d_k\,(2\langle k\rangle + d - 1 - 2ap'))^{\frac{-a}{2\langle k\rangle+d-1}}\,||h||_{L^2_k(\mathbb{R}^d)},$$
$$C_2(a,h,k) = \Big(\frac{2ap'}{2p'a-2\langle k\rangle-d+1}\Big)^{\frac{1}{2p}}\Big(\frac{2ap'}{2\langle k\rangle+d-1} - 1\Big)^{\frac{2\langle k\rangle+d-1}{4app'}}\,(C(a,p,k))^{\frac{1}{2p'}}\,\frac{||h||_{L^2_k(\mathbb{R}^d)}}{c_k},$$
$$C_3(a,h,k) = \frac{2}{c_k^{\frac{1}{p}}}\Big(\frac{2}{(2\langle k\rangle+d-1)d_k}\Big)^{\frac{1}{4p'}}\,||h||_{L^2_k(\mathbb{R}^d)}$$

and

$$C(a,p,k) := \frac{\Gamma(\frac{2\langle k\rangle+d-1}{2ap})\Gamma(\frac{2p'a-2\langle k\rangle-d+1}{2pa})}{2ap\,d_k\Gamma(\frac{p'}{p})}. \tag{5.18}$$

Applying Theorem 5.10 and using the same arguments that used in [47], we obtain another version of the Heisenberg's uncertainty principle for the k-Hankel Gabor transform on $\mathbb{R}^d$.

Theorem 5.11 *We assume that* $h \in L^2_{k,rad}(\mathbb{R}^d)$. *Let* $a, b > 0$ *and* $1 < p \le 2$. *Then for all* $f \in L^2_k(\mathbb{R}^d)$, *we have*

$$||\mathcal{G}^k_h(f)||_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})} \le \begin{cases} C_1(a,b,h,k)\Big[||\,||y||^a f||_{L^2_k(\mathbb{R}^d)} + ||\,||y||^a f||_{L^{2p}_k(\mathbb{R}^d)}\Big]^{\frac{b}{4a+b}} \\ \Big|\Big|\,||(x,\nu)||^b \mathcal{G}^k_h(f)\Big|\Big|^{\frac{4a}{4a+b}}_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})}, \quad 0 < a < \frac{2\langle k\rangle+d-1}{2p'}, \\ C_2(a,b,h,k)\left(||f||^{1-\frac{2\langle k\rangle+d-1}{2ap'}}_{L^{2p}_k(\mathbb{R}^d)}\,\Big|\Big|\,||y||^a f\Big|\Big|^{\frac{2\langle k\rangle+d-1}{2ap'}}_{L^{2p}_k(\mathbb{R}^d)}\right)^{\frac{bp'}{4\langle k\rangle+2d-2+bp'}} \\ \quad \Big|\Big|\,||(x,\nu)||^b \mathcal{G}^k_h(f)\Big|\Big|^{\frac{4\langle k\rangle+2d-2}{4\langle k\rangle+2d-2+bp'}}_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})}, \quad a > \frac{2\langle k\rangle+d-1}{2p'}, \\ C_3(a,b,h,k)\Big[||f||^{\frac12}_{L^2_k(\mathbb{R}^d)}\Big|\Big|\,||y||^a f\Big|\Big|^{\frac12}_{L^2_k(\mathbb{R}^d)} + ||f||^{\frac12}_{L^{2p}_k(\mathbb{R}^d)}\Big|\Big|\,||y||^a f\Big|\Big|^{\frac12}_{L^{2p}_k(\mathbb{R}^d)}\Big]^{\frac{b}{2a+b}} \\ \quad \Big|\Big|\,||(x,\nu)||^b \mathcal{G}^k_h(f)\Big|\Big|^{\frac{2a}{2a+b}}_{L^{p'}_{\mu_k}(\mathbb{R}^{2d})}, \quad a = \frac{2\langle k\rangle+d-1}{2p'}, \end{cases}$$

where

$$
\begin{aligned}
C_1(a,b,h,k) &= \Big[(\tfrac{b}{4a})^{\frac{4a}{4a+b}} + (\tfrac{4a}{b})^{\frac{b}{4a+b}}\Big]^{\frac{1}{p'}} \Big(C_1(a,h,k)\Big(\frac{\big(\Gamma(\frac{2\langle k\rangle+d-1}{2})\big)^2}{4d_k^2\Gamma(2\langle k\rangle+d)}\Big)^{\frac{2a}{2\langle k\rangle+d-1}}\Big)^{\frac{b}{4a+b}},\\
C_2(a,b,h,k) &= [(\frac{bp'}{4\langle k\rangle+2d-2})^{\frac{4\langle k\rangle+2d-2}{4\langle k\rangle+2d-2+bp'}} + (\frac{4\langle k\rangle+2d-2}{bp'})^{\frac{bp'}{4\langle k\rangle+2d-2+bp'}}]^{\frac{1}{p'}}\\
&\quad \Big(\Big(\frac{\big(\Gamma(\frac{2\langle k\rangle+d-1}{2})\big)^2}{4d_k^2\Gamma(2\langle k\rangle+d)}\Big)\big(C_2(a,h,k)\big)^{p'}\Big)^{\frac{b}{4\langle k\rangle+2d-2+bp'}},\\
C_3(a,b,h,k) &= [(\tfrac{b}{2a})^{\frac{2a}{2a+b}} + (\tfrac{2a}{b})^{\frac{b}{2a+b}}]^{\frac{1}{p'}}\Big(\Big(\frac{\big(\Gamma(\frac{2\langle k\rangle+d-1}{2})\big)^2}{4d_k^2\Gamma(2\langle k\rangle+d)}\Big)^{\frac{1}{2p'}} C_3(a,h,k)\Big)^{\frac{b}{2a+b}},
\end{aligned}
$$

and $C_j(a,h,k)$, $j = 1-3$, the constants given in Theorem 5.10.

Corollary 5.12 *We assume that $0 \neq h \in L^2_{k,rad}(\mathbb{R}^d)$. Let $a, b > 0$. Then for all $f \in L^2_k(\mathbb{R}^d)$, we have*

$$
||f||_{L^2_k(\mathbb{R}^d)} \leq \begin{cases}
\frac{C_1(a,b,h,k)}{||h||_{L^2_k(\mathbb{R}^d)}}\Big[\big|\big|\,||y||^a f\big|\big|_{L^2_k(\mathbb{R}^d)} + \big|\big|\,||y||^a f\big|\big|_{L^4_k(\mathbb{R}^d)}\Big]^{\frac{b}{4a+b}} & \\
\quad \big|\big|\,||(x,\nu)||^b \mathcal{G}_h^k(f)\big|\big|_{L^2_{\mu_k}(\mathbb{R}^{2d})}^{\frac{4a}{4a+b}}, & 0 < a < \frac{2\langle k\rangle+d-1}{4},\\
\frac{C_2(a,b,h,k)}{||h||_{L^2_k(\mathbb{R}^d)}}\Big(||f||_{L^4_k(\mathbb{R}^d)}^{1-\frac{2\langle k\rangle+d-1}{4a}}\big|\big|\,||y||^a f\big|\big|_{L^4_k(\mathbb{R}^d)}^{\frac{2\langle k\rangle+d-1}{4a}}\Big)^{\frac{b}{2\langle k\rangle+d-1+b}} & \\
\quad \big|\big|\,||(x,\nu)||^b \mathcal{G}_h^k(f)\big|\big|_{L^2_{\mu_k}(\mathbb{R}^{2d})}^{\frac{2\langle k\rangle+d-1}{2\langle k\rangle+d-1+b}}, & a > \frac{2\langle k\rangle+d-1}{4},\\
\frac{C_3(a,b,h,k)}{||h||_{L^2_k(\mathbb{R}^d)}}\Big[||f||_{L^2_k(\mathbb{R}^d)}^{\frac12}\big|\big|\,||y||^a f\big|\big|_{L^2_k(\mathbb{R}^d)}^{\frac12} + ||f||_{L^4_k(\mathbb{R}^d)}^{\frac12}\big|\big|\,||y||^a f\big|\big|_{L^4_k(\mathbb{R}^d)}^{\frac12}\Big]^{\frac{b}{2a+b}} & \\
\quad \big|\big|\,||(x,\nu)||^b \mathcal{G}_h^k(f)\big|\big|_{L^2_{\mu_k}(\mathbb{R}^{2d})}^{\frac{2a}{2a+b}}, & a = \frac{2\langle k\rangle+d-1}{4}.
\end{cases}
$$

We close this subsection by the following local uncertainty principle version:

Theorem 5.13 *(Faris-Price's uncertainty principle for $\mathcal{G}_h^k$) Let η, p be two real numbers such that $0 < \eta < 2\langle k\rangle + d - 1$ and $p \geq 1$. Then, there is a positive constant $C_k(\eta, p)$ such that for every function f in $L^2_k(\mathbb{R}^d)$ and for every measurable subset $T \subset \mathbb{R}^{2d}$ such that $0 < \mu_k(T) := \int_T d\mu_k(y,\nu) < \infty$, we have*

$$\left(\int_T |\mathcal{G}_h^k(f)(y,\nu)|^p d\mu_k(y,\nu)\right)^{\frac{1}{p}} \leq C_k(\eta,p)\,(\mu_k(T))^{\frac{1}{p(p+1)}} \left\| \,||(y,\nu)||^\eta \mathcal{G}_h^k(f)\right\|_{L^2_{\mu_k}(\mathbb{R}^{2d})}^{\frac{4\langle k\rangle+2d-2}{(2\langle k\rangle+d-1+\eta)(p+1)}} \left(\|f\|_{L^2_k(\mathbb{R}^d)}\|h\|_{L^2_k(\mathbb{R}^d)}\right)^{\frac{(2\langle k\rangle+d-1+\eta)(p+1)-(4\langle k\rangle+2d-2)}{(2\langle k\rangle+d-1+\eta)(p+1)}}.$$

Proof One can assume that $\|f\|_{L^2_k(\mathbb{R}^d)} = \|h\|_{L^2_k(\mathbb{R}^d)} = \sqrt{c_k}$, then for every positive real number $s > 1$, we have

$$\|\mathcal{G}_h^k(f)\|_{L^p_{\mu_k}(T)} \leq \|\mathcal{G}_h^k(f)\mathbf{1}_{B_{2d}(0,s)}\|_{L^p_{\mu_k}(T)} + \|\mathcal{G}_h^k(f)\mathbf{1}_{B^c_{2d}(0,s)}\|_{L^p_{\mu_k}(T)},$$

where $B_{2d}(0,s)$ denotes the ball of $\mathbb{R}^{2d}$ of radius s given by

$$B_{2d}(0,s) := \left\{(y,\nu) \in \mathbb{R}^{2d} : \quad ||(y,\nu)|| \leq s\right\}.$$

However, by Hölder's inequality and (3.7) we get for every $\eta \in (0, 2\langle k\rangle + d - 1)$

$$\begin{aligned}\|\mathcal{G}_h^k(f)\mathbf{1}_{B_{2d}(0,s)}\|_{L^p_{\mu_k}(T)} &= \left(\int_{\mathbb{R}^{2d}} |\mathcal{G}_h^k(f)(y,\nu)|^p \mathbf{1}_{B_{2d}(0,s)}(y,\nu)\mathbf{1}_T(y,\nu)d\mu_k(y,\nu)\right)^{\frac{1}{p}}\\
&\leq \|\mathcal{G}_h^k(f)\|_{L^\infty_{\mu_k}(\mathbb{R}^{2d})}^{\frac{p}{p+1}} \left(\int_{\mathbb{R}^{2d}} |\mathcal{G}_h^k(f)(y,\nu)|^{\frac{p}{p+1}} \mathbf{1}_{B_{2d}(0,s)}(y,\nu)\mathbf{1}_T(y,\nu)d\mu_k(y,\nu)\right)^{\frac{1}{p}}\\
&\leq (\mu_k(T))^{\frac{1}{p(p+1)}} \|\mathcal{G}_h^k(f)\mathbf{1}_{B_{2d}(0,s)}\|_{L^1_{\mu_k}(\mathbb{R}^{2d})}^{\frac{1}{p+1}}\\
&\leq (\mu_k(T))^{\frac{1}{p(p+1)}} \left\| \,||(y,\nu)||^\eta \mathcal{G}_h^k(f)\right\|_{L^2_{\mu_k}(\mathbb{R}^{2d})}^{\frac{1}{p+1}} \left\| \,||(y,\nu)||^{-\eta}\mathbf{1}_{B_{2d}(0,s)}\right\|_{L^2_{\mu_k}(\mathbb{R}^{2d})}^{\frac{1}{p+1}}.\end{aligned}$$

On the other hand by simple calculation we see that

$$\left\| \,||(y,\nu)||^{-\eta}\mathbf{1}_{B_{2d}(0,s)}\right\|_{L^2_{\mu_k}(\mathbb{R}^{2d})} \leq \left(\frac{\Gamma(\frac{2\langle k\rangle+d-1}{2})}{2d_k\sqrt{(2\langle k\rangle+d-1-\eta)\Gamma(2\langle k\rangle+d-1)}}\right)s^{2\langle k\rangle+d-1-\eta}.$$

Thus we get

$$\|\mathcal{G}_h^k(f)\mathbf{1}_{B_{2d}(0,s)}\|_{L^p_{\mu_k}(T)} \leq (\mu_k(T))^{\frac{1}{p(p+1)}} \left(\frac{\Gamma(\frac{2\langle k\rangle+d-1}{2})}{2d_k\sqrt{(2\langle k\rangle+d-1-\eta)\Gamma(2\langle k\rangle+d-1)}}\right)^{\frac{1}{p+1}} s^{\frac{2\langle k\rangle+d-1-\eta}{p+1}} \left\| \,||(y,\nu)||^\eta \mathcal{G}_h^k(f)\right\|_{L^2_{\mu_k}(\mathbb{R}^{2d})}^{\frac{1}{p+1}}.$$

On the other hand, and again by Hölder's inequality and Relation (3.7), we deduce that

$$\begin{aligned}\|\mathcal{G}_h^k(f)\mathbf{1}_{B^c_{2d}(0,s)}\|_{L^p_{\mu_k}(T)} &\leq \|\mathcal{G}_h^k(f)\|^{\frac{p-1}{p+1}}_{L^\infty_{\mu_k}(\mathbb{R}^{2d})}\left(\int_{\mathbb{R}^{2d}}|\mathcal{G}_h^k(f)(y,\nu)|^{\frac{2p}{p+1}}\mathbf{1}_{B^c_{2d}(0,s)}(y,\nu)\mathbf{1}_T(y,\nu)d\mu_k(y,\nu)\right)^{\frac{1}{p}}\\ &\leq (\mu_k(T))^{\frac{1}{p(p+1)}}\left(\int_{\mathbb{R}^{2d}}|\mathcal{G}_h^k(f)(y,\nu)|^2\mathbf{1}_{B^c_{2d}(0,s)}(y,\nu)d\mu_k(y,\nu)\right)^{\frac{1}{p+1}}\\ &\leq (\mu_k(T))^{\frac{1}{p(p+1)}}\,\|\,|(y,\nu)|^\eta\mathcal{G}_h^k(f)\|^{\frac{2}{p+1}}_{L^2_{\mu_k}(\mathbb{R}^{2d})}s^{-\frac{2\eta}{p+1}}.\end{aligned}$$

Hence, for every $\eta\in(0,2\langle k\rangle+d-1)$,

$$\begin{aligned}&\left(\int_T|\mathcal{G}_h^k(f)(y,\nu)|^pd\mu_k(y,\nu)\right)^{\frac{1}{p}}\leq(\mu_k(T))^{\frac{1}{p(p+1)}}\,\|\,||(y,\nu)||^\eta\mathcal{G}_h^k(f)\|^{\frac{1}{p+1}}_{L^2_{\mu_k}(\mathbb{R}^{2d})}\\ &\left(\left(\frac{\Gamma(\frac{2\langle k\rangle+d-1}{2})}{2d_k\sqrt{(2\langle k\rangle+d-1-\eta)\Gamma(2\langle k\rangle+d-1)}}\right)^{\frac{1}{p+1}}s^{\frac{2\langle k\rangle+d-1-\eta}{p+1}}+\|\,||(y,\nu)||^\eta\mathcal{G}_h^k(f)\|^{\frac{1}{p+1}}_{L^2_{\mu_k}(\mathbb{R}^{2d})}s^{-\frac{2\eta}{p+1}}\right).\end{aligned}$$

In particular the inequality holds for

$$\begin{aligned}s_0=&\left(\frac{\Gamma(\frac{2\langle k\rangle+d-1}{2})}{2d_k\sqrt{(2\langle k\rangle+d-1-\eta)\Gamma(2\langle k\rangle+d-1)}}\right)^{\frac{-1}{2\langle k\rangle+d-1+\eta}}\left(\frac{2\eta}{2\langle k\rangle+d-1-\eta}\right)^{\frac{p+1}{2\langle k\rangle+d-1+\eta}}\\ &\|\,||(y,\nu)||^\eta\mathcal{G}_h^k(f)\|^{\frac{1}{2\langle k\rangle+d-1+\eta}}_{L^2_{\mu_k}(\mathbb{R}^{2d})}\end{aligned}$$

and therefore

$$\begin{aligned}\left(\int_T|\mathcal{G}_h^k(f)(y,\nu)|^pd\mu_k(y,\nu)\right)^{\frac{1}{p}}&\leq(\mu_k(T))^{\frac{1}{p(p+1)}}\left(\frac{\Gamma(\frac{2\langle k\rangle+d-1}{2})}{2d_k\sqrt{(2\langle k\rangle+d-1-\eta)\Gamma(2\langle k\rangle+d-1)}}\right)^{\frac{2\eta}{(2\langle k\rangle+d-1+\eta)(p+1)}}\\ &\|\,||(y,\nu)||^\eta\mathcal{G}_h^k(f)\|^{\frac{4\langle k\rangle+2d-2}{(2\langle k\rangle+d-1+\eta)(p+1)}}_{L^2_{\mu_k}(\mathbb{R}^{2d})}\left(\frac{2\langle k\rangle+d-1-\eta}{2\eta}\right)^{\frac{2\eta}{2\langle k\rangle+d-1+\eta}}\\ &\left(\frac{2\langle k\rangle+d-1+\eta}{2\langle k\rangle+d-1-\eta}\right).\end{aligned}$$

□

6 Shapiro's dispersion theorem

In this section we will assume that h is a fixed function in $L^2_{k,rad}(\mathbb{R}^d)$ such that $||h||_{L^2_k(\mathbb{R}^d)}=1$.

The proof of the statement bellow requires the following notation:

- Let P_h be the orthogonal projection from $L^2_{\mu_k}(\mathbb{R}^{2d})$ onto the space $\mathcal{G}_h^k(L^2_k(\mathbb{R}^d))\subset L^2_{\mu_k}(\mathbb{R}^{2d})$.
- Let P_U be the orthogonal projection from $L^2_{\mu_k}(\mathbb{R}^{2d})$ onto the subspace of function in $L^2_{\mu_k}(\mathbb{R}^{2d})$ supported in the subset $U\subset\mathbb{R}^{2d}$ where $0<\mu_k(U)<\infty$.

Definition 6.1 Let $0 < \varepsilon < 1$ and $U \subset \mathbb{R}^{2d}$ be a measurable subset. For $f \in L^2_k(\mathbb{R}^d)$, we say that $\mathcal{G}^k_h(f)$ is ε-concentrated on U if

$$\left\| \mathcal{G}^k_h(f) \right\|_{L^2_{\mu_k}(U^c)} \leq \varepsilon \left\| \mathcal{G}^k_h(f) \right\|_{L^2_{\mu_k}(\mathbb{R}^{2d})},$$

where U^c is the complement of U in $\mathbb{R}^{2d}$.

Proposition 6.2 *Let $(\varphi_n)_{n\in\mathbb{N}}$ be an orthonormal sequence in $L^2_k(\mathbb{R}^d)$ and U be a measurable subset of $\mathbb{R}^{2d}$ such that $\mu_k(U) < \infty$. For every nonempty finite subset $\mathcal{E} \subset \mathbb{N}$, we have*

$$\sum_{n\in\mathcal{E}} \left(1 - \left\| \mathbb{1}_{U^c} \mathcal{G}^k_h(\varphi_n) \right\|_{L^2_{\mu_k}(\mathbb{R}^{2d})} \right) \leq \frac{\mu_k(U)}{c_k^2}.$$

Proof Since $(\varphi_n)_{n\in\mathbb{N}}$ is an orthonormal sequence in $L^2_k(\mathbb{R}^d)$, by (3.8) we deduce that $\left(\mathcal{G}^k_h(\varphi_n)\right)_{n\in\mathbb{N}}$ is an orthonormal sequence in $L^2_{\mu_k}(\mathbb{R}^{2d})$. Moreover, since the operator $P_U P_h$ is of Hilbert–Schmidt type, then, by (2.33) and (2.31), it is easy to see that

$$\begin{aligned}
&\sum_{n\in\mathcal{E}} \langle P_U \mathcal{G}^k_h(\varphi_n), \mathcal{G}^k_h(\varphi_n)\rangle_{L^2_{\mu_k}(\mathbb{R}^{2d})} \\
&\quad = \sum_{n\in\mathcal{E}} \langle P_h P_U P_h \mathcal{G}^k_h(\varphi_n), \mathcal{G}^k_h(\varphi_n)\rangle_{L^2_{\mu_k}(\mathbb{R}^{2d})} \\
&\quad \leq \operatorname{tr}(P_h P_U P_h) \\
&\quad = \| P_U P_h \|^2_{HS}.
\end{aligned}$$

Further, proceeding as in [40] and involving [[49], Inequality (5.1)], we get

$$\| P_U P_h \|_{HS} \leq \frac{\sqrt{\mu_k(U)}}{c_k}.$$

Thus,

$$\sum_{n\in\mathcal{E}} \langle P_U \mathcal{G}^k_h(\varphi_n), \mathcal{G}^k_h(\varphi_n)\rangle_{L^2_{\mu_k}(\mathbb{R}^{2d})} \leq \frac{\mu_k(U)}{c_k^2}. \tag{6.1}$$

On the other hand, by Cauchy–Schwarz's inequality we have for every $n \in \mathcal{E}$,

$$\begin{aligned}
&\langle P_U \mathcal{G}^k_h(\varphi_n), \mathcal{G}^k_h(\varphi_n)\rangle_{L^2_{\mu_k}(\mathbb{R}^{2d})} \\
&\quad = 1 - \langle P_{U^c} \mathcal{G}^k_h(\varphi_n), \mathcal{G}^k_h(\varphi_n)\rangle_{L^2_{\mu_k}(\mathbb{R}^{2d})} \\
&\quad \geq 1 - \| \mathbb{1}_{U^c} \mathcal{G}^k_h(\varphi_n) \|_{L^2_{\mu_k}(\mathbb{R}^{2d})}.
\end{aligned}$$

In particular, by relation (6.1), we obtain

$$\sum_{n\in\mathcal{E}}\Big(1-\|\mathbb{1}_{U^c}\mathcal{G}_h^k(\varphi_n)\|_{L^2_{\mu_k}(\mathbb{R}^{2d})}\Big)$$
$$\leq \sum_{n\in\mathcal{E}}\langle P_U\mathcal{G}_h^k(\varphi_n),\mathcal{G}_h^k(\varphi_n)\rangle_{L^2_{\mu_k}(\mathbb{R}^{2d})}\leq \frac{\mu_k(U)}{c_k^2}.$$

□

Next, we shall use Proposition 6.2 to prove that if the k-Hankel Gabor transform of an orthonormal sequence is ε-concentrated on a given centered ball in $\mathbb{R}^{2d}$, then a such sequence is necessary finite

Proposition 6.3 *Let ε and δ be two positive real numbers such that $0<\varepsilon<1$. Let $\mathcal{E}\subset\mathbb{N}$ be a nonempty subset and $(\varphi_n)_{n\in\mathcal{E}}$ be an orthonormal sequence in $L_k^2(\mathbb{R}^d)$. If, for every $n\in\mathcal{E}$, $\mathcal{G}_h^k(\varphi_n)$ is ε-concentrated on the ball*

$$B_{2d}(0,\delta):=\{(y,\nu)\in\mathbb{R}^{2d}:||(y,\nu)||\leq\delta\},$$

then the set $\mathcal{E}$ is finite and

$$\mathrm{Card}(\mathcal{E})\leq\frac{\Big(\Gamma(\frac{2\langle k\rangle+d-1}{2})\Big)^2}{4(\Gamma(2\langle k\rangle+d-1))^2\Gamma(2\langle k\rangle+d)(1-\varepsilon)}\delta^{4\langle k\rangle+2d-2}. \tag{6.2}$$

Proof Let $\mathcal{M}\subset\mathcal{E}$ be a nonempty finite subset, then by Proposition 6.2, we deduce that

$$\sum_{n\in\mathcal{M}}\Big(1-\|\mathbb{1}_{B_{2d}(0,\delta)^c}\mathcal{G}_h^k(\varphi_n)\|_{L^2_{\mu_k}(\mathbb{R}^{2d})}\Big)\leq\frac{\mu_k(B_{2d}(0,\delta))}{c_k^2}. \tag{6.3}$$

However, for every $n\in\mathcal{M}$, we have

$$\|\mathbb{1}_{B_{2d}(0,\delta)^c}\mathcal{G}_h^k(\varphi_n)\|_{L^2_{\mu_k}(\mathbb{R}^{2d})}\leq\varepsilon\quad\text{and} \tag{6.4}$$

$$\mu_k(B_{2d}(0,\delta))=\frac{\Big(\Gamma(\frac{2\langle k\rangle+d-1}{2})\Big)^2}{4d_k^2\Gamma(2\langle k\rangle+d)}\delta^{4\langle k\rangle+2d-2}. \tag{6.5}$$

Hence, by combining relations (6.3), (6.4) and (2.5), we deduce that

$$\mathrm{Card}(\mathcal{M})\leq\frac{\Big(\Gamma(\frac{2\langle k\rangle+d-1}{2})\Big)^2}{4(\Gamma(2\langle k\rangle+d-1))^2\Gamma(2\langle k\rangle+d)(1-\varepsilon)}\delta^{4\langle k\rangle+2d-2},$$

which means that $\mathcal{E}$ is finite and satisfies relation (6.2). □

For a positive real number p, the generalized p^{th} time–frequency dispersion of $\mathcal{G}_h^k(f)$ is defined by

$$\rho_p(\mathcal{G}_h^k(f)) = \left(\int_{\mathbb{R}^{2d}} ||(y,\nu)||^p \left| \mathcal{G}_h^k(f)(y,\nu) \right|^2 d\mu_k(y,\nu) \right)^{\frac{1}{p}}.$$

Corollary 6.4 *Let A and p be two positive real numbers. Let $\mathcal{E} \subset \mathbb{N}$ be a nonempty subset and $(\varphi_n)_{n\in\mathcal{E}}$ be an orthonormal sequence in $L_k^2(\mathbb{R}^d)$. Assume that for every $n \in \mathcal{E}$,*

$$\rho_p(\mathcal{G}_h^k(\varphi_n)) \leq A.$$

Then $\mathcal{E}$ is finite and

$$\mathrm{Card}(\mathcal{E}) \leq M'(k,p) A^{4\langle k\rangle + 2d - 2},$$

where $M'(k,p) = 2^{\frac{8\langle k\rangle+4d-4}{p}-1} \frac{\left(\Gamma(\frac{2\langle k\rangle+d-1}{2})\right)^2}{(\Gamma(2\langle k\rangle+d-1))^2 \Gamma(2\langle k\rangle+d)}$.

Proof Since $\rho_p(\mathcal{G}_h^k(\varphi_n)) \leq A$ for every $n \in \mathcal{E}$, it follows

$$\int_{B_{2d}^c(0, A2^{\frac{2}{p}})} |\mathcal{G}_h^k(\varphi_n)(y,\nu)|^2 d\mu_k(y,\nu) \leq \frac{1}{\left(A2^{\frac{2}{p}}\right)^p} \rho_p^p(\mathcal{G}_h^k(\varphi_n)) \leq \frac{1}{4}. \quad (6.6)$$

The inequality (6.6) means that for every $n \in \mathcal{E}$, $\mathcal{G}_h^k(\varphi_n)$ is $\frac{1}{2}$-concentrated in the ball $B_{2d}(0, A2^{\frac{2}{p}})$. According to Proposition 6.3, we deduce that $\mathcal{E}$ is finite and

$$\mathrm{Card}(\mathcal{E}) \leq M'(k,p) A^{4\langle k\rangle + 2d - 2}.$$

□

Lemma 6.5 *Let p be a positive real number. If $(\varphi_n)_{n\in\mathbb{N}}$ is an orthonormal sequence in $L_k^2(\mathbb{R}^d)$, then there exists $j_0 \in \mathbb{Z}$ such that*

$$\rho_p^p(\mathcal{G}_h^k(\varphi_n)) \geq 2^{p(j_0-1)}, \qquad \forall n \in \mathbb{N}.$$

Proof Involving uncertainty inequality (4.3), the assumptions $||h||_{L_k^2(\mathbb{R}^d)} = 1$ and the fact that $(\varphi_n)_{n\in\mathbb{N}}$ is an orthonormal sequence in $L_k^2(\mathbb{R}^d)$, we infer that there exist a positive constant $C_1(k,p)$ such that

$$\rho_p^p(\mathcal{G}_h^k(\varphi_n)) \geq \frac{1}{C_1^2(k,p)}.$$

Moreover it is easy to see that there exists $j_0 \in \mathbb{Z}$ such that

$$\frac{1}{C_1^2(k, p)} \geq 2^{p(j_0-1)}.$$

Thus the desired result is proved. □

Theorem 6.6 (Shapiro's dispersion theorem for $\mathcal{G}_h^k$) *Let $(\varphi_n)_{n\in\mathbb{N}}$ be an orthonormal sequence in $L_k^2(\mathbb{R}^d)$. For every positive real numbers p and for every nonempty finite subset $\mathcal{E} \subset \mathbb{N}$, we have*

$$\sum_{n\in\mathcal{E}} \left(\rho_p(\mathcal{G}_h^k(\varphi_n))\right)^p \geq \frac{1}{2}\left(\frac{3}{2^{8\langle k\rangle+4d-3}M'(k, p)}\right)^{\frac{p}{4\langle k\rangle+2d-2}} (\mathrm{Card}(\mathcal{E}))^{1+\frac{p}{4\langle k\rangle+2d-2}}. \tag{6.7}$$

Proof For every $j \in \mathbb{Z}$, let

$$P_j = \left\{n \in \mathbb{N} \ : \ \rho_p(\mathcal{G}_h^k(\varphi_n)) \in [2^{j-1}, 2^j)\right\}.$$

Then, for every $n \in P_j$,

$$\int_{\mathbb{R}^{2d}} ||(y, \nu)||^p \left|\mathcal{G}_h^k(\varphi_n)(y, \nu)\right|^2 d\mu_k(y, \nu) \leq 2^{jp}.$$

That is the sequence $(\varphi_n)_{n\in P_j}$ satisfies the conditions of Corollary 6.4, and therefore P_j is finite with

$$\mathrm{Card}(P_j) \leq M'(k, p)2^{(4\langle k\rangle+2d-2)j}. \tag{6.8}$$

For $m \in \mathbb{Z}, \ m \geq j_0$, we denote by $Q_m := \bigcup_{j=j_0}^m P_j$. According to (6.8), we have

$$\mathrm{Card}\,(Q_m) = \sum_{j=j_0}^m \mathrm{Card}(P_j) \leq \frac{M'(k, p)2^{4\langle k\rangle+2d-2}}{3} 2^{(4\langle k\rangle+2d-2)m}.$$

Now, if $\mathrm{Card}(\mathcal{E}) > \frac{M'(k, p)2^{4\langle k\rangle+2d-1}}{3} 2^{(4\langle k\rangle+2d-2)j_0}$, then we can choose an integer $m > j_0$ such that

$$\frac{M'(k, p)2^{4\langle k\rangle+2d-1}}{3} 2^{4(m-1)k} < \mathrm{Card}(\mathcal{E}) \leq \frac{M'(k, p)2^{4\langle k\rangle+2d-1}}{3} 2^{(4\langle k\rangle+2d-2)m}. \tag{6.9}$$

Thus, by (6.9), we get

$$\sum_{n\in\mathcal{E}} \left(\rho_p(\mathcal{G}_h^k(\varphi_n))\right)^p \geq \frac{\mathrm{Card}(\mathcal{E})}{2} 2^{(m-1)p}$$

$$\geq \frac{1}{2}\left(\mathrm{Card}(\mathcal{E})\right)^{1+\frac{p}{4\langle k\rangle+2d-2}}\left(\frac{3}{2^{8\langle k\rangle+4d-3}M'(k,p)}\right)^{\frac{p}{4\langle k\rangle+2d-2}}.$$

Finally, if $\mathrm{Card}(\mathcal{E}) \leq \dfrac{M'(k,p)2^{4\langle k\rangle+2d-1}}{3}2^{(4\langle k\rangle+2d-2)j_0}$, then

$$\sum_{n\in\mathcal{E}}\left(\rho_p(\mathcal{G}_h^k(\varphi_n))\right)^p \geq \mathrm{Card}(\mathcal{E})2^{(j_0-1)p} \geq \left(\mathrm{Card}(\mathcal{E})\right)^{1+\frac{p}{4\langle k\rangle+2d-2}}\left(\frac{3}{2^{8\langle k\rangle+4d-3}M'(k,p)}\right)^{\frac{p}{4\langle k\rangle+2d-2}}.$$

□

Remark 6.7 By taking $\mathrm{Card}(\mathcal{E}) = 1$, relation (6.7) appears as a general version of Heisenberg–Pauli–Weyl inequality for the k-Hankel Gabor transform including the p^{th} dispersion.

Corollary 6.8 *Let $p > 0$ and let $(\varphi_n)_{n\in\mathbb{N}}$ be an orthonormal sequence in $L_k^2(\mathbb{R}^d)$. Then for every $\mathcal{E} \subset \mathbb{N}$*

$$\sum_{n\in\mathcal{E}}\left(\left\| \,||\nu||^{\frac{p}{2}}\mathcal{G}_h^k(\varphi_n)\right\|^2_{L^2_{\mu_k}(\mathbb{R}^{2d})} + \left\| \,||y||^{\frac{p}{2}}\mathcal{G}_h^k(\varphi_n)\right\|^2_{L^2_{\mu_k}(\mathbb{R}^{2d})}\right)$$
$$\geq \frac{1}{2}\left(\frac{3}{M'(k,p)2^{12\langle k\rangle+6d-5}}\right)^{\frac{p}{4\langle k\rangle+2d-2}}\left(\mathrm{Card}(\mathcal{E})\right)^{1+\frac{p}{4\langle k\rangle+2d-2}}. \tag{6.10}$$

Proof The result is an immediate consequence of Theorem 6.6 together with the fact that

$$||(y,\nu)||^p \leq 2^p(||\nu||^p + ||y||^p).$$

□

The dispersion inequality (6.10) implies that there is no infinite sequence $(\varphi_n)_{n\in\mathcal{E}}$ in $L_k^2(\mathbb{R}^d)$ such that both sequences

$$\left\|\,||\nu||^{\frac{p}{2}}\mathcal{G}_h^k(\varphi_n)\right\|_{L^2_{\mu_k}(\mathbb{R}^{2d})} \quad \text{and} \quad \left\|\,||y||^{\frac{p}{2}}\mathcal{G}_h^k(\varphi_n)\right\|_{L^2_{\mu_k}(\mathbb{R}^{2d})}$$

are bounded. More precisely:

Corollary 6.9 *Let $p > 0$ and let $(\varphi_n)_{n\in\mathbb{N}}$ be an orthonormal sequence in $L_k^2(\mathbb{R}^d)$. For every $\mathcal{E} \subset \mathbb{N}$, we have*

$$\sup_{n\in\mathcal{E}}\left(\left\|\,||\nu||^{\frac{p}{2}}\mathcal{G}_h^k(\varphi_n)\right\|^2_{L^2_{\mu_k}(\mathbb{R}^{2d})}, \left\|\,||y||^{\frac{p}{2}}\mathcal{G}_h^k(\varphi_n)\right\|^2_{L^2_{\mu_k}(\mathbb{R}^{2d})}\right)$$
$$\geq \frac{1}{4}\left(\frac{3}{M'(k,p)2^{12\langle k\rangle+6d-5}}\right)^{\frac{p}{4\langle k\rangle+2d-2}}\left(\mathrm{Card}(\mathcal{E})\right)^{\frac{p}{4\langle k\rangle+2d-2}}. \tag{6.11}$$

In particular,

$$\sup_{n\in\mathcal{E}}\left(\left|\left|\, ||\nu||^{\frac{p}{2}}\mathcal{G}_h^k(\varphi_n)\right|\right|^2_{L^2_{\mu_k}(\mathbb{R}^{2d})}+\left|\left|\, ||x||^{\frac{p}{2}}\mathcal{G}_h^k(\varphi_n)\right|\right|^2_{L^2_{\mu_k}(\mathbb{R}^{2d})}\right)=\infty.$$

Theorem 6.10 (Shapiro's Umbrella theorem for $\mathcal{G}_h^k$) *Let $\mathcal{E}\subset\mathbb{N}$ be a nonempty subset and $(\varphi_n)_{n\in\mathcal{E}}$ be an orthonormal sequence in $L^2_k(\mathbb{R}^d)$. If there is a positive function $g\in L^2_{\mu_k}(\mathbb{R}^{2d})$ such that*

$$|\mathcal{G}_h^k(\varphi_n)(y,\nu)|\leq g(y,\nu)$$

for every $n\in\mathcal{E}$ and for almost every $(y,\nu)\in\mathbb{R}^{2d}$, then $\mathcal{E}$ is finite.

Proof Following the idea of Malinnikova [39], for every $0<\varepsilon<1$, there is a subset $\Delta_{g,\varepsilon}\subset\mathbb{R}^{2d}$ such that

$$\mu_k(\Delta_{g,\varepsilon})=\inf\left\{\mu_k(U):\int_{U^c}|g(y,\nu)|^2\,d\mu_k(y,\nu)\leq\varepsilon^2\right\},$$

and

$$\int_{\Delta^c_{g,\varepsilon}}|g(y,\nu)|^2\,d\mu_k(y,\nu)=\varepsilon^2.$$

Hence, according to the hypothesis, for every $n\in\mathcal{E}$ we have

$$\int_{\Delta^c_{g,\varepsilon}}\left|\mathcal{G}_h^k\left(\varphi_n\right)(y,\nu)\right|^2 d\mu_k(y,\nu)\leq\varepsilon^2,$$

and by Proposition 6.2, we get $\mathrm{Card}(\mathcal{E})(1-\varepsilon)\leq\mu_k(\Delta_{g,\varepsilon})$. □

7 Weighted inequalities for the *k*-Hankel Gabor transform

The Pitt inequality in the k-Hankel setting expresses a fundamental relationship between a sufficiently smooth function and the corresponding k-Hankel transform. This subject was studied by Gorbachev et all in [29], where the authors have given the Sharp Pitt's inequality and logarithmic uncertainty principle for k-Hankel transform on $\mathbb{R}^d$. More precisely they proved that, for every $f\in\mathcal{S}(\mathbb{R}^d)\subseteq L^2_k(\mathbb{R}^d)$

$$\begin{aligned}&\int_{\mathbb{R}^d}||\xi||^{-2\lambda}|\mathcal{F}_k(f)(\xi)|^2d\gamma_k(\xi)\\&\leq C_k(\lambda)\int_{\mathbb{R}^d}||x||^{2\lambda}|f(x)|^2d\gamma_k(x),\;\;0\leq\lambda<\frac{2\langle k\rangle+d-1}{2},\end{aligned}\tag{7.1}$$

where

$$C_k(\lambda) := \left[\frac{\Gamma(\frac{2\langle k\rangle+d-1-2\lambda}{2})}{\Gamma(\frac{2\langle k\rangle+d-1+2\lambda}{2})} \right]^2 \tag{7.2}$$

and Γ denotes the well known Euler's Gamma function.

The first main objective of this section is to formulate an analogue of Pitt's inequality (7.1) for the k-Hankel Gabor transform.

Theorem 7.1 *For any arbitrary $f \in \mathcal{S}(\mathbb{R}^d) \subseteq L^2_k(\mathbb{R}^d)$, the Pitt inequality for the k-Hankel Gabor transform is given by:*

$$\begin{aligned} &||h||^2_{L^2_k(\mathbb{R}^d)} \int_{\mathbb{R}^d} ||\xi||^{-\lambda} \left|\mathcal{F}_k(f)(\xi)\right|^2 d\gamma_k(\xi) \\ &\quad \leq C_k(\lambda) \int_{\mathbb{R}^{2d}} ||y||^{\lambda} \left|\mathcal{G}^k_h(f)(y,\nu)\right|^2 d\mu_k(y,\nu), \ 0 \leq \lambda < \frac{2\langle k\rangle + d - 1}{2}, \end{aligned} \tag{7.3}$$

where $C_k(\lambda)$ is given by (7.2).

Proof As a consequence of the inequality (7.1), we can write

$$\begin{aligned} &\int_{\mathbb{R}^d} ||\xi||^{-2\lambda} \left|\mathcal{F}_k[\mathcal{G}^k_h(f)(.,\nu)](\xi)\right|^2 d\gamma_k(\xi) \\ &\quad \leq C_k(\lambda) \int_{\mathbb{R}^d} ||y||^{2\lambda} \left|\mathcal{G}^k_h(f)(y,\nu)\right|^2 d\gamma_k(y), \ \text{for all } \nu \in \mathbb{R}^d \end{aligned} \tag{7.4}$$

which upon integration with respect to the Haar measure $d\gamma_k(\nu)$ yields

$$\begin{aligned} &\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} ||\xi||^{-2\lambda} \left|\mathcal{F}_k[\mathcal{G}^k_h(f)(.,\nu)](\xi)\right|^2 d\mu_k(\xi,\nu) \\ &\quad \leq C_k(\lambda) \int_{\mathbb{R}^{2d}} ||y||^{2\lambda} \left|\mathcal{G}^k_h(f)(y,\nu)\right|^2 d\mu_k(y,\nu). \end{aligned} \tag{7.5}$$

Invoking Lemma 3.9, we can express the inequality (7.5) in the following manner:

$$\begin{aligned} &\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} ||\xi||^{-2\lambda} |\mathcal{F}_k(f)(\xi)|^2 \tau^k_\nu |h|^2(\xi) d\mu_k(\xi,\nu) \\ &\quad \leq C_k(\lambda) \int_{\mathbb{R}^{2d}} ||y||^{2\lambda} \left|\mathcal{G}^k_h(f)(y,\nu)\right|^2 d\mu_k(y,\nu). \end{aligned}$$

Equivalently, we have

$$\begin{aligned} &\int_{\mathbb{R}^d} ||\xi||^{-2\lambda} \left|\mathcal{F}_k(f)(\xi)\right|^2 \left\{ \int_{\mathbb{R}^d} \tau^k_\nu |h|^2(\xi) d\gamma_k(\nu) \right\} d\gamma_k(\xi) \\ &\quad \leq C_k(\lambda) \int_{\mathbb{R}^{2d}} ||y||^{2\lambda} \left|\mathcal{G}^k_h(f)(y,\nu)\right|^2 d\mu_k(y,\nu) \end{aligned}$$

Using the hypothesis on h, the relation (2.24) becomes

$$||h||^2_{L^2_k(\mathbb{R}^d)} \int_{\mathbb{R}^d} ||\xi||^{-2\lambda} \big|\mathcal{F}_k(f)(\xi)\big|^2 d\gamma_k(\xi) \\ \leq C_k(\lambda) \int_{\mathbb{R}^{2d}} ||y||^{2\lambda} \big|\mathcal{G}^k_h(f)(y,\nu)\big|^2 d\mu_k(y,\nu) \tag{7.6}$$

which establishes the Pitt inequality for the k-Hankel Gabor transform. □

Remark 7.2 For $\lambda = 0$, equality holds in (7.3), which is in consonance with the Plancherel formula (3.8).

The k-Hankel Beckner's inequality [29] is given by

$$\int_{\mathbb{R}^d} \log ||y|| \, |f(y)|^2 d\gamma_k(y) + \int_{\mathbb{R}^d} \log ||\xi|| \, |\mathcal{F}_k(f)(\xi)|^2 d\gamma_k(\xi) \geq \\ 2\frac{\Gamma'(\frac{2\langle k\rangle+d-1}{2})}{\Gamma(\frac{2\langle k\rangle+d-1}{2})} \int_{\mathbb{R}^d} |f(t)|^2 d\gamma_k(t), \tag{7.7}$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$. This inequality is related to the Heisenberg's uncertainty principle and for that reason it is often referred as the logarithmic uncertainty principle. Considerable attention has been paid to this inequality for its various generalizations, improvements, analogues, and their applications [35]. The second main objective of this section is to formulate an analogue of Beckner's inequality (7.7) for the k-Hankel Gabor transform.

Theorem 7.3 *For any function $f \in \mathcal{S}(\mathbb{R}^d)$, the following inequality holds:*

$$\int_{\mathbb{R}^{2d}} \log ||y|| \, |\mathcal{G}^k_h(f)(y,\nu)|^2 d\mu_k(y,\nu) + ||h||^2_{L^2_k(\mathbb{R}^d)} \int_{\mathbb{R}^d} \log ||\xi|| \, |\mathcal{F}_k(f)(\xi)|^2 d\gamma_k(\xi) \\ \geq 2\frac{\Gamma'(\frac{2\langle k\rangle+d-1}{2})}{\Gamma(\frac{2\langle k\rangle+d-1}{2})} ||h||^2_{L^2_k(\mathbb{R}^d)} ||f||^2_{L^2_k(\mathbb{R}^d)}. \tag{7.8}$$

Proof We replace f in (7.7) with $\mathcal{G}^k_h(f)(.,\nu)$, so that

$$\int_{\mathbb{R}^d} \log ||y|| \, |\mathcal{G}^k_h(f)(y,\nu)|^2 d\gamma_k(y) + \int_{\mathbb{R}^d} \log ||\xi|| \, \big|\mathcal{F}_k[\mathcal{G}^k_h(f)(.,\nu)](\xi)\big|^2 d\gamma_k(\xi) \geq \\ 2\frac{\Gamma'(\frac{2\langle k\rangle+d-1}{2})}{\Gamma(\frac{2\langle k\rangle+d-1}{2})} \int_{\mathbb{R}^d} \big|\mathcal{G}^k_h(f)(y,\nu)\big|^2 d\gamma_k(y), \text{ for all } \nu \in \mathbb{R}^d. \tag{7.9}$$

Integrating (7.9) with respect to the measure $d\gamma_k(\nu)$, we obtain

$$\int_{\mathbb{R}^{2d}} \log ||y|| \, |\mathcal{G}^k_h(f)(y,\nu)|^2 d\mu_k(y,\nu) + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \log ||\xi|| \, \big|\mathcal{F}_k[\mathcal{G}^k_h(f)(.,\nu)](\xi)\big|^2 d\mu_k(\xi,\nu) \\ \geq 2\frac{\Gamma'(\frac{2\langle k\rangle+d-1}{2})}{\Gamma(\frac{2\langle k\rangle+d-1}{2})} \int_{\mathbb{R}^{2d}} \big|\mathcal{G}^k_h(f)(y,\nu)\big|^2 d\mu_k(y,\nu). \tag{7.10}$$

Using Plancherel's formula (3.8), we get

$$\int_{\mathbb{R}^{2d}} \log ||y|| \, |\mathcal{G}_h^k(f)(y,\nu)|^2 d\mu_k(y,\nu) + \int_{\mathbb{R}^{2d}} \log ||\xi|| \, |\mathcal{F}_k[\mathcal{G}_h^k(f)(.,\nu)](\xi)|^2 d\mu_k(\xi,\nu) \\ \geq 2\frac{\Gamma'(\frac{2\langle k\rangle+d-1}{2})}{\Gamma(\frac{2\langle k\rangle+d-1}{2})} ||h||^2_{L_k^2(\mathbb{R}^d)} ||f||^2_{L_k^2(\mathbb{R}^d)}. \tag{7.11}$$

We shall now simplify the second integral of (7.11). By using Lemma 3.9 and relation (2.24) we infer that

$$\begin{aligned} &\int_{\mathbb{R}^{2d}} \log ||\xi|| \, |\mathcal{F}_k[\mathcal{G}_h^k(f)(.,\nu)](\xi)|^2 d\mu_k(\xi,\nu) \\ &= \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} \log ||\xi|| \, \big|\mathcal{F}_k[\mathcal{G}_h^k(f)(.,\nu)](\xi)\big|^2 d\gamma_k(\xi) \Big) d\gamma_k(\nu) \\ &= \Big(\int_{\mathbb{R}^d} \log ||\xi|| \, |\mathcal{F}_k(f)(\xi)|^2 d\gamma_k(\xi) \Big) ||h||^2_{L_k^2(\mathbb{R}^d)}. \end{aligned} \tag{7.12}$$

Plugging the estimate (7.12) in (7.11) gives the desired inequality for the k-Hankel Gabor transforms as

$$\int_{\mathbb{R}^{2d}} \log ||y|| \, |\mathcal{G}_h^k(f)(y,\nu)|^2 d\mu_k(y,\nu) + ||h||^2_{L_k^2(\mathbb{R}^d)} \int_{\mathbb{R}^d} \log ||\xi|| \, |\mathcal{F}_k(f)(\xi)|^2 d\gamma_k(\xi) \geq \\ 2\frac{\Gamma'(\frac{2\langle k\rangle+d-1}{2})}{\Gamma(\frac{2\langle k\rangle+d-1}{2})} ||h||^2_{L_k^2(\mathbb{R}^d)} ||f||^2_{L_k^2(\mathbb{R}^d)}.$$

The previous inequality is the desired Beckner's uncertainty principle for the k-Hankel Gabor transform. □

We now present an alternate proof of Theorem 7.3. The strategy of the proof is obtained from the k-Hankel Pitt's inequality (7.3).

Proof of Theorem 7.3 For every $0 \leq \lambda < \frac{2\langle k\rangle+d-1}{2}$, we define

$$\begin{aligned} S(\lambda) = &||h||^2_{L_k^2(\mathbb{R}^d)} \int_{\mathbb{R}^d} ||\xi||^{-2\lambda} \big|\mathcal{F}_k(f)(\xi)\big|^2 d\gamma_k(\xi) \\ &- C_k(\lambda) \int_{\mathbb{R}^{2d}} ||y||^{2\lambda} \big|\mathcal{G}_h^k(f)(y,\nu)\big|^2 d\mu_k(y,\nu). \end{aligned} \tag{7.13}$$

On differentiating (7.13) with respect to λ, we obtain

$$\begin{aligned} S'(\lambda) = &-2||h||^2_{L_k^2(\mathbb{R}^d)} \int_{\mathbb{R}^d} ||\xi||^{-2\lambda} \log ||\xi|| \, \big|\mathcal{F}_k(f)(\xi)\big|^2 d\gamma_k(\xi) \\ &-2C_k(\lambda) \int_{\mathbb{R}^{2d}} ||y||^{2\lambda} \log ||y|| \, |\mathcal{G}_h^k(f)(y,\nu)|^2 d\mu_k(y,\nu) \\ &-C_k'(\lambda) \int_{\mathbb{R}^{2d}} ||y||^{2\lambda} \big|\mathcal{G}_h^k(f)(y,\nu)\big|^2 d\mu_k(y,\nu), \end{aligned} \tag{7.14}$$

where

$$C_k'(\lambda) = -2C_k(\lambda)\Big(\frac{\Gamma'(\frac{2\langle k\rangle+d-1-2\lambda}{2})}{\Gamma(\frac{2\langle k\rangle+d-1-2\lambda}{2})} + \frac{\Gamma'(\frac{2\langle k\rangle+d-1+2\lambda}{2})}{\Gamma(\frac{2\langle k\rangle+d-1+2\lambda}{2})}\Big). \tag{7.15}$$

For $\lambda = 0$, equation (7.15) yields

$$C_k'(0) = -4\frac{\Gamma'(\frac{2\langle k\rangle+d-1}{2})}{\Gamma(\frac{2\langle k\rangle+d-1}{2})}. \tag{7.16}$$

By virtue of k-Hankel Pitt's inequality (7.3), it follows that $S(\lambda) \leq 0$, for all λ belongs to $[0, \frac{2\langle k\rangle+d-1+2\lambda}{2})$ and

$$\begin{aligned} S(0) &= ||h||^2_{L^2_k(\mathbb{R}^d)} \int_{\mathbb{R}^d} \big|\mathcal{F}_k(f)(\xi)\big|^2 d\gamma_k(\xi) - C_k(0)\int_{\mathbb{R}^{2d}} \big|\mathcal{G}_h^k(f)(y,\nu)\big|^2 d\mu_k(y,\nu) \\ &= ||h||^2_{L^2_k(\mathbb{R}^d)}||f||^2_{L^2_k(\mathbb{R}^d)} - ||h||^2_{L^2_k(\mathbb{R}^d)}||f||^2_{L^2_k(\mathbb{R}^d)} = 0. \end{aligned}$$

Therefore we deduce that

$$S'(0^+) := \lim_{\lambda\to 0^+} \frac{S(\lambda)}{\lambda} \leq 0.$$

Equivalently we have

$$\begin{aligned} &-2||h||^2_{L^2_k(\mathbb{R}^d)} \int_{\mathbb{R}^d} \log||\xi||\,\big|\mathcal{F}_k(f)(\xi)\big|^2 d\gamma_k(\xi) \\ &-2C_k(0)\int_{\mathbb{R}^{2d}} \log||y||\,|\mathcal{G}_h^k(f)(y,\nu)|^2 d\mu_k(y,\nu) \\ &-C_k'(0)\int_{\mathbb{R}^{2d}} \big|\mathcal{G}_h^k(f)(y,\nu)\big|^2 d\mu_k(y,\nu) \leq 0. \end{aligned} \tag{7.17}$$

Applying Plancherel's formula (3.8) and the obtained estimate (7.16) of $C_k'(0)$, we get

$$\begin{aligned} -2||h||^2_{L^2_k(\mathbb{R}^d)} \int_{\mathbb{R}^d} \log||\xi||\,\big|\mathcal{F}_k(f)(\xi)\big|^2 d\gamma_k(\xi) - 2\int_{\mathbb{R}^{2d}} \log||y||\,\big|\mathcal{G}_h^k(f)(y,\nu)\big|^2 d\mu_k(y,\nu) \\ +4\frac{\Gamma'(\frac{2\langle k\rangle+d-1}{2})}{\Gamma(\frac{2\langle k\rangle+d-1}{2})}||h||^2_{L^2_k(\mathbb{R}^d)}||f||^2_{L^2_k(\mathbb{R}^d)} \leq 0 \end{aligned}$$

or equivalently,

$$\begin{aligned} \int_{\mathbb{R}^{2d}} \log||y||\,\big|\mathcal{G}_h^k(f)(y,\nu)\big|^2 d\mu_k(y,\nu) + ||h||^2_{L^2_k(\mathbb{R}^d)} \int_{\mathbb{R}^d} \log||\xi||\,|\mathcal{F}_k(f)(\xi)|^2 d\gamma_k(\xi) \\ \geq 2\frac{\Gamma'(\frac{2\langle k\rangle+d-1}{2})}{\Gamma(\frac{2\langle k\rangle+d-1}{2})}||h||^2_{L^2_k(\mathbb{R}^d)}||f||^2_{L^2_k(\mathbb{R}^d)}. \end{aligned}$$

This completes the second proof of Theorem 7.3. □

Corollary 7.4 *Let* $h \in L^2_{k,rad}(\mathbb{R}^d) \bigcap L^\infty_k(\mathbb{R}^d)$ *such that* $||h||_{L^2_k(\mathbb{R}^d)} = 1$. *For any function* $f \in \mathcal{S}(\mathbb{R}^d)$, *the following inequality holds:*

$$\Big\{ \int_{\mathbb{R}^{2d}} ||y||^2 \left|\mathcal{G}^k_h(f)(y,\nu)\right|^2 d\mu_k(y,\nu)\Big\}^{1/2} \Big\{ \int_{\mathbb{R}^d} ||\xi||^2 |\mathcal{F}_k(f)(\xi)|^2 d\gamma_k(\xi)\Big\}^{1/2} \geq \exp\Big(2\frac{\Gamma'(\frac{2\langle k\rangle+d-1}{2})}{\Gamma(\frac{2\langle k\rangle+d-1}{2})}\Big)||f||^2_{L^2_k(\mathbb{R}^d)}.$$

Proof Using Jensen's inequality in (7.8) and the fact that $||h||_{L^2_k(\mathbb{R}^d)} = 1$, we obtain an analogue of the classical Heisenberg's uncertainty inequality for the k-Hankel Gabor transforms as

$$\begin{aligned} &\log\Big\{ \int_{\mathbb{R}^{2d}} ||y||^2 \frac{\left|\mathcal{G}^k_h(f)(y,\nu)\right|^2}{||f||^2_{L^2_k(\mathbb{R}^d)}} d\mu_k(y,\nu) \int_{\mathbb{R}^d} ||\xi||^2 \frac{|\mathcal{F}_k(f)(\xi)|^2}{||f||^2_{L^2_k(\mathbb{R}^d)}} d\gamma_k(\xi)\Big\}^{1/2} \\ &= \log\Big\{ \int_{\mathbb{R}^{2d}} ||y||^2 \frac{\left|\mathcal{G}^k_h(f)(y,\nu)\right|^2}{||f||^2_{L^2_k(\mathbb{R}^d)}} d\mu_k(y,\nu)\Big\}^{1/2} + \log\Big\{ \int_{\mathbb{R}^d} ||\xi||^2 \frac{|\mathcal{F}_k(f)(\xi)|^2}{||f||^2_{L^2_k(\mathbb{R}^d)}} d\gamma_k(\xi)\Big\}^{1/2} \\ &\geq \int_{\mathbb{R}^{2d}} \log||y|| \frac{\left|\mathcal{G}^k_h(f)(y,\nu)\right|^2}{||f||^2_{L^2_k(\mathbb{R}^d)}} d\mu_k(y,\nu) + \int_{\mathbb{R}^d} \log||\xi|| \frac{|\mathcal{F}_k(f)(\xi)|^2}{||f||^2_{L^2_k(\mathbb{R}^d)}} d\gamma_k(\xi) \\ &\geq 2\frac{\Gamma'(\frac{2\langle k\rangle+d-1}{2})}{\Gamma(\frac{2\langle k\rangle+d-1}{2})}, \end{aligned}$$

which upon simplification with yields the result. □

Remark 7.5 i) Using the approximation identity

$$\frac{\Gamma'(z)}{\Gamma(z)} = \log z - \frac{1}{2z} - 2\int_0^\infty \frac{t}{(t^2+z^2)(e^{2\pi t}-1)}dt \tag{7.18}$$

we infer

$$\exp\left[2\frac{\Gamma'(\frac{2\langle k\rangle+d-1}{2})}{\Gamma(\frac{2\langle k\rangle+d-1}{2})}\right] \approx \Big(\frac{2\langle k\rangle+d-1}{2}\Big)^2 \quad \text{for } 2\langle k\rangle+d-1 \gg 1, \tag{7.19}$$

which is the constant of the Heisenberg uncertainty principle for the k-Hankel Gabor transform given in Theorem 4.2.

ii) Proceeding as above in logarithmic uncertainty inequality (7.7) we deduce the following Heisenberg uncertainty inequality

$$\Big\{ \int_{\mathbb{R}^d} \|t\|^2 |f(t)|^2 d\gamma_k(t)\Big\}^{\frac{1}{2}} \Big\{ \int_{\mathbb{R}^d} \|\xi\|^2 |\mathcal{F}_k(f)(\xi)|^2 d\gamma_k(\xi)\Big\}^{\frac{1}{2}} \geq \exp\left[2\frac{\Gamma'(\frac{2\langle k\rangle+d-1}{2})}{\Gamma(\frac{2\langle k\rangle+d-1}{2})}\right] \int_{\mathbb{R}^d} |f(t)|^2 d\gamma_k(t). \tag{7.20}$$

iii) Using the approximation relation (7.18) we deduce that the constant in the right-hand side of (7.20),

$$\exp\left[2\frac{\Gamma'(\frac{2\langle k\rangle+d-1}{2})}{\Gamma(\frac{2\langle k\rangle+d-1}{2})}\right] \approx \left(\frac{2\langle k\rangle+d-1}{2}\right)^2 \quad \text{for } 2\langle k\rangle+d-1 \gg 1$$

which is the constant of the Heisenberg uncertainty principle for the k-Hankel transform given in Proposition 4.1.

8 Localization operators associated to the continuous *k*-Hankel Gabor transform

Definition 8.1 Let u, v be measurable even functions on $\mathbb{R}^d$, σ be measurable function on $\mathbb{R}^{2d}$, we define the two-Gabor localization operator, associated to the continuous k-Hankel Gabor transform on $\mathbb{R}^d$, noted by $\mathcal{L}_{u,v}(\sigma)$, on $L^2_k(\mathbb{R}^d)$, by

$$\mathcal{L}_{u,v}(\sigma)(f)(x) = \frac{1}{c_k}\int_{\mathbb{R}^{2d}} \sigma(y,\nu)\mathcal{G}^k_u(f)(y,\nu)\ v_{y,\nu}(y)d\mu_k(y,\nu),\ \ x \in \mathbb{R}^d. \tag{8.1}$$

It is often more convenient to interpret the definition of $\mathcal{L}_{u,v}(\sigma)$ in a weak sense, that is, for f in $L^2_k(\mathbb{R}^d)$ and g in $L^2_k(\mathbb{R}^d)$,

$$\langle \mathcal{L}_{u,v}(\sigma)(f), g\rangle_{L^2_k(\mathbb{R}^d)} = \int_{\mathbb{R}^{2d}} \sigma(y,\nu)\mathcal{G}^k_u(f)(y,\nu)\overline{\mathcal{G}^k_v(g)(y,\nu)}d\mu_k(y,\nu). \tag{8.2}$$

Proposition 8.2 *The adjoint of linear operator*

$$\mathcal{L}_{u,v}(\sigma) : L^2_k(\mathbb{R}^d) \to L^2_k(\mathbb{R}^d)$$

is $\mathcal{L}_{v,u}(\overline{\sigma}) : L^2_k(\mathbb{R}^d) \to L^2_k(\mathbb{R}^d)$.

Proof For all f in $L^2_k(\mathbb{R}^d)$ and g in $L^2_k(\mathbb{R}^d)$ it immediately follows from (8.2)

$$\begin{aligned}\langle \mathcal{L}_{u,v}(\sigma)(f), g\rangle_{L^2_k(\mathbb{R}^d)} &= \int_{\mathbb{R}^{2d}} \sigma(y,\nu)\mathcal{G}^k_u(f)(y,\nu)\overline{\mathcal{G}^k_v(g)(y,\nu)}d\mu_k(y,\nu)\\ &= \overline{\int_{\mathbb{R}^{2d}} \overline{\sigma(y,\nu)}\mathcal{G}^k_v(g)(y,\nu)\overline{\mathcal{G}^k_u(f)(y,\nu)}d\mu_k(y,\nu)}\\ &= \overline{\langle \mathcal{L}_{v,u}(\overline{\sigma})(g), f\rangle_{L^2_k(\mathbb{R}^d)}} = \langle f, \mathcal{L}_{v,u}(\overline{\sigma})(g)\rangle_{L^2_k(\mathbb{R}^d)}.\end{aligned}$$

Thus we get

$$\mathcal{L}^*_{u,v}(\sigma) = \mathcal{L}_{v,u}(\overline{\sigma}). \tag{8.3}$$

□

In what follows, such operator $\mathcal{L}_{u,v}(\sigma)$ will be named localization operator for the sake of simplicity.

In the rest of this section we assume that u and v belong to $L^2_{k,rad}(\mathbb{R}^d)$ such that

$$\|u\|_{L^2_k(\mathbb{R}^d)} = \|v\|_{L^2_k(\mathbb{R}^d)} = 1.$$

8.1 Boundedness for $\mathcal{L}_{u,v}(\sigma)$ on S_∞

The main result of this subsection is to prove that the linear operators

$$\mathcal{L}_{u,v}(\sigma) : L^2_k(\mathbb{R}^d) \to L^2_k(\mathbb{R}^d)$$

are bounded for all symbols $\sigma \in L^p_{\mu_k}(\mathbb{R}^{2d})$, $1 \leq p \leq \infty$. We first consider this problem for σ in $L^1_{\mu_k}(\mathbb{R}^{2d})$ and next in $L^\infty_{\mu_k}(\mathbb{R}^{2d})$ and we then conclude by using interpolation theory.

Proposition 8.3 *Let σ be in $L^1_{\mu_k}(\mathbb{R}^{2d})$, then the localization operator $\mathcal{L}_{u,v}(\sigma)$ is in S_∞ and we have*

$$||\mathcal{L}_{u,v}(\sigma)||_{S_\infty} \leq \frac{1}{c_k^2}\|\sigma\|_{L^1_{\mu_k}(\mathbb{R}^{2d})}. \tag{8.4}$$

Proof For every functions f and g in $L^2_k(\mathbb{R}^d)$, we have from the relations (8.2) and (3.7),

$$\begin{aligned}|\langle \mathcal{L}_{u,v}(\sigma)(f), g\rangle_{L^2_k(\mathbb{R}^d)}| &\leq \int_{\mathbb{R}^{2d}} |\sigma(y,\nu)||\mathcal{G}^k_u(f)(y,\nu)||\overline{\mathcal{G}^k_v(g)(y,\nu)}|d\mu_k(y,\nu)\\ &\leq \|\mathcal{G}^k_u(f)\|_{L^\infty_{\mu_k}(\mathbb{R}^{2d})}\|\mathcal{G}^k_v(g)\|_{L^\infty_{\mu_k}(\mathbb{R}^{2d})}\|\sigma\|_{L^1_{\mu_k}(\mathbb{R}^{2d})}\\ &\leq \frac{1}{c_k^2}\|f\|_{L^2_k(\mathbb{R}^d)}\|g\|_{L^2_k(\mathbb{R}^d)}\|\sigma\|_{L^1_{\mu_k}(\mathbb{R}^{2d})}.\end{aligned}$$

Thus,

$$||\mathcal{L}_{u,v}(\sigma)||_{S_\infty} \leq \frac{1}{c_k^2}\|\sigma\|_{L^1_{\mu_k}(\mathbb{R}^{2d})}.$$

□

Proposition 8.4 *Let σ be in $L^\infty_{\mu_k}(\mathbb{R}^{2d})$, then the localization operator $\mathcal{L}_{u,v}(\sigma)$ is in S_∞ and we have*

$$||\mathcal{L}_{u,v}(\sigma)||_{S_\infty} \leq \|\sigma\|_{L^\infty_{\mu_k}(\mathbb{R}^{2d})}.$$

Proof For all functions f and g in $L^2_k(\mathbb{R}^d)$, we have from Hölder's inequality

$$|\langle \mathcal{L}_{u,v}(\sigma)(f), g\rangle_{L^2_k(\mathbb{R}^d)}| \leq \int_{\mathbb{R}^{2d}} |\sigma(y,\nu)||\mathcal{G}^k_u(f)(y,\nu)||\overline{\mathcal{G}^k_v(g)(y,\nu)}|d\mu_k(y,\nu)$$

$$\leq \|\sigma\|_{L^\infty_{\mu_k}(\mathbb{R}^{2d})} \|\mathcal{G}^k_u(f)\|_{L^2_{\mu_k}(\mathbb{R}^{2d})} \|\mathcal{G}^k_v(g)\|_{L^2_{\mu_k}(\mathbb{R}^{2d})}.$$

Using Plancherel's formula for $\mathcal{G}^k_u$ and $\mathcal{G}^k_v$, given by the relation (3.8), we get

$$|\langle \mathcal{L}_{u,v}(\sigma)(f), g\rangle_{L^2_k(\mathbb{R}^d)}| \leq \|f\|_{L^2_k(\mathbb{R}^d)} \|g\|_{L^2_k(\mathbb{R}^d)} \|\sigma\|_{L^\infty_{\mu_k}(\mathbb{R}^{2d})}.$$

Thus,

$$||\mathcal{L}_{u,v}(\sigma)||_{S_\infty} \leq \|\sigma\|_{L^\infty_{\mu_k}(\mathbb{R}^{2d})}.$$

□

We can now associate a localization operator

$$\mathcal{L}_{u,v}(\sigma) : L^2_k(\mathbb{R}^d) \to L^2_k(\mathbb{R}^d)$$

to every symbol σ in $L^p_{\mu_k}(\mathbb{R}^{2d})$, $1 \leq p \leq \infty$ and prove that $\mathcal{L}_{u,v}(\sigma)$ is in S_∞. The precise result is the following theorem.

Theorem 8.5 *Let σ be in $L^p_{\mu_k}(\mathbb{R}^{2d})$, $1 \leq p \leq \infty$. Then there exists a unique bounded linear operator*

$$\mathcal{L}_{u,v}(\sigma) : L^2_k(\mathbb{R}^d) \to L^2_k(\mathbb{R}^d),$$

such that

$$||\mathcal{L}_{u,v}(\sigma)||_{S_\infty} \leq \frac{1}{c_k^{\frac{2}{p}}} \|\sigma\|_{L^p_{\mu_k}(\mathbb{R}^{2d})}.$$

Proof Let f be in $L^2_k(\mathbb{R}^d)$. We consider the following operator

$$\mathcal{T}\colon L^1_{\mu_k}(\mathbb{R}^{2d} \bigcap L^\infty_{\mu_k}(\mathbb{R}^{2d}) \to L^2_k(\mathbb{R}^d),$$

given by

$$\mathcal{T}(\sigma) := \mathcal{L}_{u,v}(\sigma)(f).$$

Then by Proposition 8.3 and Proposition 8.4

$$||\mathcal{T}(\sigma)||_{L^2_k(\mathbb{R}^d)} \leq \frac{1}{c_k^2} ||f||_{L^2_k(\mathbb{R}^d)} \|\sigma\|_{L^1_{\mu_k}(\mathbb{R}^{2d})} \tag{8.5}$$

and

$$||\mathcal{T}(\sigma)||_{L^2_k(\mathbb{R}^d)} \leq ||f||_{L^2_k(\mathbb{R}^d)} \|\sigma\|_{L^\infty_{\mu_k}(\mathbb{R}^{2d})}. \tag{8.6}$$

Therefore, by (8.5), (8.6) and the Riesz–Thorin interpolation theorem (see [[61], Theorem 2] and [[65], Theorem 2.11]), $\mathcal{T}$ may be uniquely extended to a linear operator on $L^p_{\mu_k}(\mathbb{R}^{2d})$, $1 \leq p \leq \infty$ and we have

$$||\mathcal{L}_{u,v}(\sigma)(f)||_{L^2_k(\mathbb{R}^d)} = ||\mathcal{T}(\sigma)||_{L^2_k(\mathbb{R}^d)} \leq \frac{1}{c_k^{\frac{2}{p}}} ||f||_{L^2_k(\mathbb{R}^d)} \|\sigma\|_{L^p_{\mu_k}(\mathbb{R}^{2d})}. \tag{8.7}$$

Since (8.7) is true for arbitrary functions f in $L^2_k(\mathbb{R}^d)$, then we obtain the desired result. □

8.2 Schatten-von Neumann properties for $\mathcal{L}_{u,v}(\sigma)$

The main result of this subsection is to prove that, the localization operator

$$\mathcal{L}_{u,v}(\sigma) : L^2_k(\mathbb{R}^d) \to L^2_k(\mathbb{R}^d)$$

is in the Schatten class S_p.

Proposition 8.6 *Let σ be in $L^1_{\mu_k}(\mathbb{R}^{2d})$, then the localization operator $\mathcal{L}_{u,v}(\sigma)$ is in S_2 and we have*

$$\|\mathcal{L}_{u,v}(\sigma)\|_{S_2} \leq \frac{1}{c_k^2} \|\sigma\|_{L^1_{\mu_k}(\mathbb{R}^{2d})}.$$

Proof Let $\{\phi_j, j = 1, 2 \ldots\}$ be an orthonormal basis for $L^2_k(\mathbb{R}^d)$. Then by (8.2), Fubini's theorem, Parseval's identity and the relations (3.4) and (8.3), we have

$$\begin{aligned}
\sum_{j=1}^{\infty} ||\mathcal{L}_{u,v}(\sigma)(\phi_j)||^2_{L^2_k(\mathbb{R}^d)} &= \sum_{j=1}^{\infty} \langle \mathcal{L}_{u,v}(\sigma)(\phi_j), \mathcal{L}_{u,v}(\sigma)(\phi_j)\rangle_{L^2_k(\mathbb{R}^d)} \\
&= \frac{1}{c_k^2} \sum_{j=1}^{\infty} \int_{\mathbb{R}^{2d}} \sigma(y,\nu) \langle \phi_j, u_{y,\nu}\rangle_{L^2_k(\mathbb{R}^d)} \overline{\langle \mathcal{L}_{u,v}(\sigma)(\phi_j), v_{y,\nu}\rangle}_{L^2_k(\mathbb{R}^d)} d\mu_k(y,\nu) \\
&= \frac{1}{c_k^2} \int_{\mathbb{R}^{2d}} \sigma(y,\nu) \sum_{j=1}^{\infty} \langle \phi_j, u_{y,\nu}\rangle_{L^2_k(\mathbb{R}^d)} \langle \mathcal{L}^*_{u,v}(\sigma)(v_{y,\nu}), \phi_j\rangle_{L^2_k(\mathbb{R}^d)} d\mu_k(y,\nu) \\
&= \frac{1}{c_k^2} \int_{\mathbb{R}^{2d}} \sigma(y,\nu) \langle \mathcal{L}^*_{u,v}(\sigma)\, v_{y,\nu}, u_{y,\nu}\rangle_{L^2_k(\mathbb{R}^d)} d\mu_k(y,\nu).
\end{aligned}$$

Thus, from (8.3) and (8.4), we get

$$\begin{aligned}
&\sum_{j=1}^{\infty} ||\mathcal{L}_{u,v}(\sigma)(\phi_j)||^2_{L^2_k(\mathbb{R}^d)} \\
&\leq \frac{1}{c_k^2} \int_{\mathbb{R}^{2d}} |\sigma(y,\nu)|\, ||\mathcal{L}^*_{u,v}(\sigma)||_{S_\infty} d\mu_k(y,\nu) \leq \frac{1}{c_k^4} \|\sigma\|^2_{L^1_{\mu_k}(\mathbb{R}^{2d})} < \infty.
\end{aligned} \tag{8.8}$$

So, by (8.8) and Proposition 2.8 in the book [65], by Wong

$$\mathcal{L}_{u,v}(\sigma) : L_k^2(\mathbb{R}^d) \to L_k^2(\mathbb{R}^d)$$

is in the Hilbert-Schmidt class S_2 and hence compact. □

Proposition 8.7 *Let σ be a symbol in $L_{\mu_k}^p(\mathbb{R}^{2d})$, $1 \leq p < \infty$. Then the localization operator $\mathcal{L}_{u,v}(\sigma)$ is compact.*

Proof Let σ be in $L_{\mu_k}^p(\mathbb{R}^{2d})$ and let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of functions in $L_{\mu_k}^1(\mathbb{R}^{2d} \bigcap L_{\mu_k}^\infty(\mathbb{R}^{2d})$ such that $\sigma_n \to \sigma$ in $L_{\mu_k}^p(\mathbb{R}^{2d})$ as $n \to \infty$. Then by Theorem 8.5

$$||\mathcal{L}_{u,v}(\sigma_n) - \mathcal{L}_{u,v}(\sigma)||_{S_\infty} \leq \frac{1}{c_k^{\frac{2}{p}}}||\sigma_n - \sigma||_{L_{\mu_k}^p(\mathbb{R}^{2d})}.$$

Hence $\mathcal{L}_{u,v}(\sigma_n) \to \mathcal{L}_{u,v}(\sigma)$ in S_∞ as $n \to \infty$. On the other hand, as by Proposition 8.6, $\mathcal{L}_{u,v}(\sigma_n)$ is in S_2 hence compact, it follows that $\mathcal{L}_{u,v}(\sigma)$ is compact. □

Theorem 8.8 *Let σ be in $L_{\mu_k}^1(\mathbb{R}^{2d})$. Then $\mathcal{L}_{u,v}(\sigma) : L_k^2(\mathbb{R}^d) \to L_k^2(\mathbb{R}^d)$ is in S_1 and we have*

$$\frac{1}{c_k^2}\|\widetilde{\sigma}\|_{L_{\mu_k}^1(\mathbb{R}^{2d})} \leq \|\mathcal{L}_{u,v}(\sigma)\|_{S_1} \leq \frac{1}{c_k^2}\|\sigma\|_{L_{\mu_k}^1(\mathbb{R}^{2d})}, \tag{8.9}$$

where $\widetilde{\sigma}$ is given by

$$\forall\,(y,\nu) \in \mathbb{R}^{2d}, \quad \widetilde{\sigma}(y,\nu) = \langle \mathcal{L}_{u,v}(\sigma)\, u_{y,\nu},\; v_{y,\nu}\rangle_{L_k^2(\mathbb{R}^d)}.$$

Proof Since σ is in $L_{\mu_k}^1(\mathbb{R}^{2d})$, by Proposition 8.6, $\mathcal{L}_{u,v}(\sigma)$ is in S_2, then from the canonical form for compact operators given in [65, Theorem 2.2], there exists an orthonormal basis $\{\phi_j,\; j = 1, 2\ldots\}$ for the orthogonal complement of the kernel of the operator $\mathcal{L}_{u,v}(\sigma)$, consisting of eigenvectors of $|\mathcal{L}_{u,v}(\sigma)|$ and $\{\varphi_j,\; j = 1, 2\ldots\}$ an orthonormal set in $L_k^2(\mathbb{R}^d)$, such that

$$\mathcal{L}_{u,v}(\sigma)(f) = \sum_{j=1}^{\infty} s_j \langle f, \phi_j\rangle_{L_k^2(\mathbb{R}^d)}\varphi_j, \tag{8.10}$$

where $s_j,\; j = 1, 2\ldots$ are the positive singular values of $\mathcal{L}_{u,v}(\sigma)$ corresponding to ϕ_j. Then, we get

$$\|\mathcal{L}_{u,v}(\sigma)\|_{S_1} = \sum_{j=1}^{\infty} s_j = \sum_{j=1}^{\infty}\langle \mathcal{L}_{u,v}(\sigma)(\phi_j), \varphi_j\rangle_{L_k^2(\mathbb{R}^d)}.$$

Thus, by Fubini's theorem, Cauchy–Schwarz's inequality, Bessel inequality, relations (3.4) and (3.3), we get

$$\begin{aligned}
\|\mathcal{L}_{u,v}(\sigma)\|_{S_1} &= \sum_{j=1}^{\infty}\langle \mathcal{L}_{u,v}(\sigma)(\phi_j), \varphi_j\rangle_{L^2_k(\mathbb{R}^d)} \\
&= \sum_{j=1}^{\infty}\int_{\mathbb{R}^{2d}} \sigma(y,\nu)\mathcal{G}^k_u(\phi_j)(y,\nu)\overline{\mathcal{G}^k_v(\varphi_j)(y,\nu)}d\mu_k(y,\nu) \\
&\leq \int_{\mathbb{R}^{2d}} |\sigma(y,\nu)| \left(\sum_{j=1}^{\infty}|\mathcal{G}^k_u(\phi_j)(y,\nu)|^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{\infty}|\mathcal{G}^k_v(\varphi_j)(y,\nu)|^2\right)^{\frac{1}{2}} d\mu_k(y,\nu) \\
&\leq \frac{1}{c_k^2}\int_{\mathbb{R}^{2d}} |\sigma(y,\nu)| \| u_{y,\nu}\|_{L^2_k(\mathbb{R}^d)} \| v_{y,\nu}\|_{L^2_k(\mathbb{R}^d)} d\mu_k(y,\nu) \\
&\leq \frac{1}{c_k^2}\|\sigma\|_{L^1_{\mu_k}(\mathbb{R}^{2d})}.
\end{aligned}$$

Thus

$$\|\mathcal{L}_{u,v}(\sigma)\|_{S_1} \leq \frac{1}{c_k^2}\|\sigma\|_{L^1_{\mu_k}(\mathbb{R}^{2d})}.$$

We now prove that $\mathcal{L}_{u,v}(\sigma)$ satisfies the first member of (8.9). It is easy to see that $\widetilde{\sigma}$ belongs to $L^1_k(\mathbb{R}^d)$, and using formula (8.10), we get

$$\begin{aligned}
|\widetilde{\sigma}(y,\nu)| &= \left|\langle \mathcal{L}_{u,v}(\sigma)(u_{y,\nu}), \ v_{y,\nu}\rangle_{L^2_k(\mathbb{R}^d)}\right| \\
&= \left|\sum_{j=1}^{\infty} s_j \langle u_{y,\nu}, \phi_j\rangle_{L^2_k(\mathbb{R}^d)}\langle \varphi_j, \ v_{y,\nu}\rangle_{L^2_k(\mathbb{R}^d)}\right| \\
&\leq \frac{1}{2}\sum_{j=1}^{\infty} s_j \left(\left|\langle u_{y,\nu}, \phi_j\rangle_{L^2_k(\mathbb{R}^d)}\right|^2 + \left|\langle v_{y,\nu}, \varphi_j\rangle_{L^2_k(\mathbb{R}^d)}\right|^2\right).
\end{aligned}$$

Then from Fubini's theorem, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^{2d}} |\widetilde{\sigma}(y,\nu)| d\mu_k(y,\nu) \leq \frac{1}{2}\sum_{j=1}^{\infty} s_j \Big(&\int_{\mathbb{R}^{2d}} |\langle u_{y,\nu}, \phi_j\rangle_{L^2_k(\mathbb{R}^d)}|^2 d\mu_k(y,\nu) \\
&+ \int_{\mathbb{R}^{2d}} |\langle v_{y,\nu}, \varphi_j\rangle_{L^2_k(\mathbb{R}^d)}|^2 d\mu_k(y,\nu)\Big).
\end{aligned}$$

Thus using Plancherel's formula for $\mathcal{G}^k_u$, $\mathcal{G}^k_v$, we get

$$\int_{\mathbb{R}^{2d}} |\widetilde{\sigma}(y,\nu)| d\mu_k(y,\nu) \leq c_k^2 \sum_{j=1}^{\infty} s_j = c_k^2 \|\mathcal{L}_{u,v}(\sigma)\|_{S_1}.$$

The proof is completed. □

Corollary 8.9 *For σ in $L^1_{\mu_k}(\mathbb{R}^{2d})$, we have the following trace formula*

$$tr(\mathcal{L}_{u,v}(\sigma)) = \frac{1}{c_k^2}\int_{\mathbb{R}^{2d}} \sigma(y,\nu)\langle\, v_{y,\nu},\ u_{y,\nu}\rangle_{L^2_k(\mathbb{R}^d)} d\mu_k(y,\nu). \tag{8.11}$$

Proof Let $\{\phi_j,\ j = 1, 2\ldots\}$ be an orthonormal basis for $L^2_k(\mathbb{R}^d)$. From Theorem 8.8, the localization operator $\mathcal{L}_{u,v}(\sigma)$ belongs to S_1, then by the definition of the trace given by the relation (2.31), Fubini's theorem and Parseval's identity, we have

$$\begin{aligned} tr(\mathcal{L}_{u,v}(\sigma)) &= \sum_{j=1}^{\infty}\langle\mathcal{L}_{u,v}(\sigma)(\phi_j), \phi_j\rangle_{L^2_k(\mathbb{R}^d)} \\ &= \frac{1}{c_k^2}\sum_{j=1}^{\infty}\int_{\mathbb{R}^{2d}} \sigma(y,\nu)\langle\phi_j,\ u_{y,\nu}\rangle_{L^2_k(\mathbb{R}^d)}\overline{\langle\phi_j,\ v_{y,\nu}\rangle}_{L^2_k(\mathbb{R}^d)} d\mu_k(y,\nu) \\ &= \frac{1}{c_k^2}\int_{\mathbb{R}^{2d}} \sigma(y,\nu)\sum_{j=1}^{\infty}\langle\phi_j,\ u_{y,\nu}\rangle_{L^2_k(\mathbb{R}^d)}\overline{\langle\phi_j,\ v_{y,\nu}\rangle}_{L^2_k(\mathbb{R}^d)} d\mu_k(y,\nu) \\ &= \frac{1}{c_k^2}\int_{\mathbb{R}^{2d}} \sigma(y,\nu)\langle v_{y,\nu}, u_{y,\nu}\rangle_{L^2_k(\mathbb{R}^d)} d\mu_k(y,\nu), \end{aligned}$$

and the proof is completed. □

In the following we give the main result of this subsection.

Corollary 8.10 *Let σ be in $L^p_{\mu_k}(\mathbb{R}^{2d})$, $1 \le p \le \infty$. Then, the localization operator*

$$\mathcal{L}_{u,v}(\sigma) : L^2_k(\mathbb{R}^d) \longrightarrow L^2_k(\mathbb{R}^d)$$

is in S_p and we have

$$\|\mathcal{L}_{u,v}(\sigma)\|_{S_p} \le \frac{1}{c_k^{\frac{2}{p}}}\|\sigma\|_{L^p_{\mu_k}(\mathbb{R}^{2d})}.$$

Proof The result follows from Proposition 8.4, Theorem 8.8 and by interpolation (see [65, Theorem 2.10 and Theorem 2.11]). □

Remark 8.11 If $u = v$ and if σ is a real valued and nonnegative function in $L^1_{\mu_k}(\mathbb{R}^{2d})$ then

$$\mathcal{L}_{u,v}(\sigma) : L^2_k(\mathbb{R}^d) \to L^2_k(\mathbb{R}^d)$$

is a positive operator. So, by (2.32) and Corollary 8.9

$$||\mathcal{L}_{u,v}(\sigma)||_{S_1} = \int_{\mathbb{R}^{2d}} \sigma(y,\nu)||u_{y,\nu}||^2_{L^2_k(\mathbb{R}^d)} d\mu_k(y,\nu). \tag{8.12}$$

Now we state a result concerning the trace of products of localization operators.

Corollary 8.12 *Let* σ_1 *and* σ_2 *be any real-valued and non-negative functions in* $L^1_{\mu_k}(\mathbb{R}^{2d})$. *We assume that* $u = v$ *and* u *is a function in* $L^2_k(\mathbb{R}^d)$ *such that* $||u||_{L^2_k(\mathbb{R}^d)} = 1$. *Then, the localization operators* $\mathcal{L}_{u,v}(\sigma_1)$, $\mathcal{L}_{u,v}(\sigma_2)$ *are positive trace class operators and*

$$\begin{aligned}\left|\left|\Big(\mathcal{L}_{u,v}(\sigma_1)\,\mathcal{L}_{u,v}(\sigma_2)\Big)^n\right|\right|_{S_1} &= tr\Big(\mathcal{L}_{u,v}(\sigma_1)\,\mathcal{L}_{u,v}(\sigma_2)\Big)^n \\ &\leq \Big(tr\big(\mathcal{L}_{u,v}(\sigma_1)\big)\Big)^n\Big(tr\big(\mathcal{L}_{u,v}(\sigma_2)\big)\Big)^n \\ &= \left|\left|\mathcal{L}_{u,v}(\sigma_1)\right|\right|^n_{S_1}\left|\left|\mathcal{L}_{u,v}(\sigma_2)\right|\right|^n_{S_1},\end{aligned}$$

for any natural number n.

Proof By Theorem 1 in the paper [38] by Liu we know that if A and B are in the trace class S_1 and are positive operators, then

$$\forall n \in \mathbb{N}, \quad tr(AB)^n \leq \Big(tr(A)\Big)^n\Big(tr(B)\Big)^n.$$

So, if we take $A = \mathcal{L}_{u,v}(\sigma_1)$, $B = \mathcal{L}_{u,v}(\sigma_2)$ and we invoke the previous remark, the desired result is obtained and the proof is completed. □

Remark 8.13 i) When $W = \mathbb{Z}_2^d$, all results of this paper for the k-Hankel Gabor transform transform $\mathcal{G}_h^k$ are true if we replace the hypothesis h radial by

$$h := h_1 \otimes \ldots \otimes h_d,$$

where the functions h_i, $i = 1, \ldots, d$ are even functions on $\mathbb{R}$.

ii) We note that we have studied these types of time-frequency analysis problems and others for some integral transforms as the Dunkl Gabor transform on $\mathbb{R}^d$, the (k, a)-generalized wavelet transform on $\mathbb{R}^d$, the deformed Hankel Gabor transform on $\mathbb{R}$, the generalized Stockwell transforms and others integral transforms. (See as examples [51–53]).

9 Perspectives

In [49], we have studied the concentration operator $\mathcal{L}_h(U)$ associated with the k-Hankel Gabor transform defined as

$$\mathcal{L}_h(U)(f)(y) = \frac{1}{c_k}\int_U \mathcal{G}_h^k(f)(x,\nu)\,\tau_y^k h_\nu(x)d\mu_k(x,\nu),\ \ y \in \mathbb{R}^d,$$

where U is a subset of $\mathbb{R}^{2d}$ with finite measure. We have proved that this operator is bounded, compact, even trace class and self-adjoint operator with spectral representation:

$$\mathcal{L}_h(U)(f) = \sum_{n=1}^{\infty} s_n(U) \left\langle f, v_n^U \right\rangle_{L_k^2(\mathbb{R}^d)} v_n^U, \quad f \in L_k^2(\mathbb{R}^d),$$

where $\{s_n(U)\}_{n=1}^{\infty}$ are the positive eigenvalues arranged in a nonincreasing manner and $\{v_n^U\}_{n=1}^{\infty}$ is the corresponding orthonormal set of eigenfunctions. Thus, using eigenfunctions and eigenvalues of the concentration operator $\mathcal{L}_h(U)$, we have proved a characterization of functions that are time-frequency concentrated in U, and we obtain approximation inequalities for such functions using a finite linear combination of eigenfunctions, since they are maximally time-frequency-concentrated in the region of interest. As perspective, involving the concentration operator $\mathcal{L}_h(U)$ and the ε-concentration of the k-Hankel Gabor transform, we will latter prove an uncertainty principle of Donoho-Stark type. Moreover, we will study functions whose time-frequency content are concentrated in a compact region in phase space using time-frequency localization operators as a main tool. We claim to obtain approximation inequalities for such functions using a finite linear combination of eigenfunctions of these operators, as well as a local Gabor system covering the region of interest. These would allow the construction of modified time-frequency dictionaries concentrated in the region. The results presented in the perspective section are pre-published [50].

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References

1. Baccar, C., Hamadi, N.B., Herch, H.: Time-frequency analysis of localization operators associated to the windowed Hankel transform. Integral Transforms Spec. Funct. **27**, 245–258 (2016)
2. Bahri, M., Shah, F.A., Tantary, A.Y.: Uncertainty principles for the continuous Shearlet transforms in arbitrary space dimensions. Integral Transforms Spec. Funct. **31**(7), 538–555 (2020)
3. Bansal, A., Kumar, A.: Heisenberg uncertainty inequality for Gabor transform. J. Math. Inequal. **10**(3), 737–749 (2016)
4. Benedicks, M.: On Fourier transforms of functions supported on sets of finite Lebesgue measure. J. Math. Anal. Appl. **106**, 180–183 (1985)
5. Ben Saïd, S., Kobayashi, T., Ørsted, B.: Laguerre semigroup and k-Hankel operators. Compos. Math. **148**(04), 1265–1336 (2012)
6. Ben Saïd, S.: A product formula and a convolution structure for a k-Hankel transform on $\mathbb{R}$. J. Math. Anal. Appl. **463**(2), 1132–1146 (2018)
7. Ben Saïd, S., Deleaval, L.: Translation operator and maximal function for the $(k, 1)$-generalized Fourier transform. J. Funct. Anal. (2020). https://doi.org/10.1016/j.jfa.2020.108706
8. Ben Salem, N., Nasr, A.R.: Shapiro type inequalities for the Weinstein and the Weinstein Gabor transforms. Konuralp J. Math. **5**(1), 68–76 (2017)
9. Boggiatto, P., Cordero, E.: Anti-Wick quantization with symbols in L^p spaces. Proc. Am. Math. Soc. **130**(9), 2679–2685 (2002)
10. Cohen, L.: Time-frequency distributions—a review. Proc IEEE **77**, 941–981 (1989)
11. Constales, D., De Bie, H., Lian, P.: Explicit formulas for the Dunkl dihedral kernel and the (κ, a)-generalized Fourier kernel. J. Math. Anal. Appl. **460**(2), 900–926 (2018)

12. Cordero, E., Gröchenig, K.: Time-frequency analysis of localization operators. J. Funct. Anal. **205**(1), 107–131 (2003)
13. Czaja, W., Gigante, G.: Continuous Gabor transform for strong hypergroups. J. Fourier Anal. Appl. **9**, 321–339 (2003)
14. Daubechies, I.: Time-frequency localization operators: a geometric phase space approach. IEEE Trans. Inf. Theory **34**(4), 605–612 (1988)
15. Daubechies, I.: The wavelet transform, time-frequency localization and signal analysis. IEEE Trans. Inf. Theory **36**(5), 961–1005 (1990)
16. De Bie, H., Ørsted, B., Somberg, P., Souček, V.: Dunkl operators and a family of realizations of osp(1|2). Trans. Am. Math. Soc. **364**, 3875–3902 (2012)
17. Debnath, L.: Wavelet Transforms and Their Applications. Birkhäuser, Boston (2002)
18. De Mari, F., Feichtinger, H., Nowak, K.: Uniform eigenvalue estimates for time-frequency localization operators. J. Lond. Math. Soc. **65**(3), 720–732 (2002)
19. De Mari, F., Nowak, K.: Localization type Berezin–Toeplitz operators on bounded symmetric domains. J. Geom. Anal. **12**(1), 9–27 (2002)
20. Donoho, D.L., Stark, P.B.: Uncertainty principles and signal recovery. SIAM J. Appl. Math. **49**, 906–931 (1989)
21. Dunkl, C.F.: Differential-difference operators associated to reflection groups. Trans. Am. Math. Soc. **311**, 167–183 (1989)
22. Dunkl,C. F.: Hankel transforms associated to finite reflection groups. In: Proceedings of the special session on hypergeometric functions on domains of positivity, Jack polynomials and applications. (Tampa, FL, 1991), Contemporary Mathematic, vol 138. pp. 123–138 (1992)
23. Fernandez, C., Galbis, A.: Annihilating sets for the short time Fourier transform. Adv. Math. **224**(5), 1904–1926 (2010)
24. Fernandez, C., Galbis, A., Martinez, J.: Localization operators and an uncertainty principle for the discrete short time Fourier transform. Abstr. Appl. Anal. (2014); Art. ID 131459
25. Folland, G.B., Sitaram, A.: The uncertainty principle: a mathematical survey. J Fourier Anal Appl. **3**(3), 207–238 (1997)
26. Gabor, D.: Theory of communication part 1: the analysis of information. J. Inst. Electr. Eng. Part III Radio. Commun. Eng. **93**(26), 429–441 (1946)
27. Ghobber, S., Jaming, P.: Uncertainty principles for integral orperators. Studia Math. **220**, 197–220 (2014)
28. Ghobber, S.: Phase space localized functions in the Dunkl setting. J. Pseudo Differ. Oper. Appl. **7**, 571–593 (2016)
29. Gorbachev, D., Ivanov, V., Tikhonov, S.: Pitt's inequalities and uncertainty principle for generalized Fourier transform. Int. Math. Res. Not. **23**, 7179–7200 (2016)
30. Gröchenig, K.: Aspects of Gabor analysis on locally compact abelian groups. Gabor analysis and algorithms, pp. 211–231, Applied and Numerical Harmonic Analysis, Birkhäuser Boston, Boston(1998)
31. Gröchenig, K.: Foundations of Time-Frequency Analysis. Applied and Numerical Harmonic Analysis. Birkhäuser, Boston (2001)
32. Havin, V., Jöricke, B.: The Uncertainty Principle in Harmonic Analysis, vol. 24. Springer, Berlin (1994)
33. He, Z., Wong, M.W.: Localization operators associated to square integrable group representations. Panamer. Math. J. **6**(1), 93–104 (1996)
34. Howe, R.: The oscillator semigroup. The mathematical heritage of Hermann Weyl (Durham, NC, 1987), Proceedings of the Symposium on Pure Mathematical, 48. American Mathematical Society, pp. 61–132. Providence (1988)
35. Johansen, T.R.: Weighted inequalities and uncertainty principles for the (k, a)-generalized Fourier transform. Int. J. Math. **27**(3), 1650019 (2016)
36. Kawazoe, T., Mejjaoli, H.: Uncertainty principles for the Dunkl transform. Hiroshima Math. J. **40**(2), 241–268 (2010)
37. Kobayashi, T., Mano, G.: The Schrödinger model for the minimal representation of the indefinite orthogonal group $O(p, q)$. vi + 132pp. The Memoirs of the American Mathematical Society, vol. 212, No. 1000 (2011)
38. Liu, L.: A trace class operator inequality. J. Math. Anal. Appl. **328**, 1484–1486 (2007)
39. Malinnikova, E.: Orthonormal sequences in $L^2(\mathbb{R}^d)$ and time frequency localization. J. Fourier Anal. Appl. **16**(6), 983–1006 (2010)

40. Mejjaoli, H., Sraieb, N.: Uncertainty principles for the continuous Dunkl Gabor transform and the Dunkl continuous Gabor transform. Mediterr. J. Math. **5**, 443–466 (2008)
41. Mejjaoli, H., Sraieb, N.: Gabor transform in quantum calculus and applications. Fracti. Calc. Appl. Anal. **12**(3), 319–336 (2009)
42. Mejjaoli, H.: Uncertainty principles for the generalized Fourier transform associated to a Dunkl-type operator. Appl Anal. **95**(9), 1930–1956 (2017)
43. Mejjaoli, H.: Harmonic analysis associated with the generalized differential-difference operator on the real line and quantitative uncertainty principles for its Hartley transform. Appl Anal. **96**(7), 1146–1169 (2017)
44. Mejjaoli, H.: Wavelet-multipliers analysis in the framework of the k-Laguerre theory. Linear Multilinear Algebra **67**(1), 1–24 (2017)
45. Mejjaoli, H.: Spectral theorems associated with the (k, a)-generalized wavelet multipliers. J. Pseudo Differ. Oper. Appl. **9**, 735–762 (2018)
46. Mejjaoli, H.: (k, a)-generalized wavelet transform and applications. J. Pseudo Differ. Oper. Appl. **11**, 55–92 (2020)
47. Mejjaoli, H., Sraieb, N., Trimèche, K.: L^p local uncertainty principles for the Dunkl Gabor transform on $\mathbb{R}^d$. Boletín de la Sociedad Matemática Mexicana **26**, 1163–1182 (2020)
48. Mejjaoli, H., Trimèche, K.: k-Hankel two-wavelet theory and localization operators. Integral Transform Spec. Funct. **31**(8), 620–644 (2020)
49. Mejjaoli, H.: k-Hankel Gabor transform on $\mathbb{R}^d$ and its applications to the reproducing kernel theory. Complex Anal. Oper. Theory **15**(14), 1–54 (2021)
50. Mejjaoli, H.: Time-frequency concentration associated with the k-Hankel Gabor transform on $\mathbb{R}^d$ and its applications to the approximation theory Preprint (2020)
51. Mejjaoli, H.: Dunkl-Stockwell transform and its applications to the time-frequency analysis. J. Pseudo-Di er. Oper. Appl. https://doi.org/10.1007/s11868-021-00378-y
52. Mejjaoli, H., Trimeche, K.: Quantitative uncertainty principles associated with the k-generalized Stockwell transform. To appear in Mediterr. J. Math
53. Mejjaoli, H.: Deformed Stockwell transform and applications on the reproducing kernel theory. To appear in International Journal of Reproducing Kernels
54. Ramanathan, J., Topiwala, P.: Time-frequency localization via the Weyl correspondence. SIAM J. Math. Anal. **24**(5), 1378–1393 (1993)
55. Rösler, M.: Positivity of Dunkl's intertwining operator. Duke Math. J. **98**, 445–463 (1999)
56. Shah, F.A., Tantary, A.Y.: Polar wavelet transform and the associated uncertainty principles. Int. J. Theor. Phys. **57**(6), 1774–1786 (2018)
57. Shah, F.A., Tantary, A.Y.: Non-isotropic angular Stockwell transform and the associated uncertainty principles. Appl. Anal. https://doi.org/10.1080/00036811.2019.1622681 (to appear)
58. Shannon, C.E.: A mathematical theory of communication. ACM SIGMOBILE Mobile Comput. Commun. Rev. **5**(1), 3–55 (2001)
59. Slepian, D., Pollak, H.O.: Prolate spheroidal wave functions, Fourier analysis and uncertainty I. Bell. Syst. Technol. J. **40**, 43–63 (1961)
60. Slepian, D.: Prolate spheroidal wave functions, Fourier analysis and uncertainty IV: extensions to many dimensions, generalized prolate spheroidal functions. Bell. Syst. Technol. J. **43**, 3009–3057 (1964)
61. Stein, E.M.: Interpolation of linear operators. Trans. Am. Math. Soc. **83**, 482–492 (1956)
62. Trimèche, K.: Generalized Wavelets and Hypergroups. Gordon and Breach Science Publishers, Amsterdam (1997)
63. Vasilevski,N. L.: Commutative algebras of Toeplitz operators on the Bergman space. Operator Theory: Advances and Applications 185. Basel: Birkhäuser. xxix (2008)
64. Wilczok, E.: New uncertainty principles for the continuous Gabor transform and the continuous Gabor transform. Doc. Math. J. DMV (Electronic) **5**, 201–226 (2000)
65. Wong, M.W.: Wavelet Transforms and Localization Operators. Springer, Berlin (2002)
66. Yakubovich, S.B.: Uncertainty principles for the Kontorovich–Lebedev transform. Math. Model. Anal. **13**(2), 289–302 (2008)