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A critical fractional choquard problem involving a singular nonlinearity and a radon measure

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Abstract

This article concerns about the existence of a positive SOLA (Solutions Obtained as Limits of Approximations) for the following singular critical Choquard problem involving fractional power of Laplacian and a critical Hardy potential.

$$
\begin{aligned}
(-\Delta)^s u - \alpha \frac{u}{|x|^{2s}} &= \lambda u + u^{-\gamma} + \beta \left(\int_\Omega \frac{u^{2_b^*}(y)}{|x-y|^b} dy \right) u^{2_b^*-1} + \mu \text{ in } \Omega, \\
u &> 0 \text{ in } \Omega, \\
u &= 0 \text{ in } \mathbb{R}^N \backslash \Omega.
\end{aligned}
\tag{0.1}
$$

Here, Ω is a bounded domain of $\mathbb{R}^N$, $s \in (0, 1)$, α, λ and β are positive real parameters, $N > 2s$, $\gamma \in (0, 1)$, $0 < b < \min\{N, 4s\}$, $2_b^* = \frac{2N-b}{N-2s}$ is the critical exponent in the sense of Hardy–Littlewood–Sobolev inequality and μ is a bounded Radon measure in Ω.

Keywords Choquard equation · Fractional Sobolev spaces · Radon measure · Marcinkiewicz space · Hardy potential

Mathematics Subject Classification 35J60 · 35R11 · 35A15

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1 Introduction

The nonlocal problem involving fractional Laplacian has a remarkable contribution in various fields of science. The fractional Laplacian arise in chemical reactions in liquids, diffusion in plasma, geophysical fluid dynamics, electromagnetism and is the infinitesimal generator of Lévy stable diffusion process, see [3] for instance. Therefore a considerable amount of research is carried out by a numerous scientists, engineers, mathematicians with equal interest. Elliptic PDEs involving singular nonlinearity have been studied by many authors, refer [6,10,22,24,35] and the references therein. In all these referred works the authors have proved the existence of solution to the singular problem with approximation arguments and the solution space depends on the power of the singular term. Recently, Sun & Zhang in [35] explained the role of power 3 for elliptic equations with negative exponents and claimed that for exponent greater than 3, the problem does not possess a solution. In [33], for the multiplicity result is proved with the help of the variational method, where the author proved the existence result by converting the nonlocal problem to a local problem. Similarly, problems involving a Radon measure as a nonhomogeneous term are also treated with approximations, since we can always approximate a Radon measure by sequence of smooth functions. Thus, in this case one can expect the solution space with lesser degree of differentiability or/and integrability. For example, Boccardo et al. ([5,7]) proved the existence of solution in $W_0^{1,m}(\Omega)$ for every $m < \frac{N(p-1)}{N-1}$ for a problem involving p-Laplacian and a Radon measure. Later, Kuusi et al. [23] extended the work of Boccardo to fractional p-Laplacian set up and guarenteed a solution in $W^{\bar{s},m}(\Omega)$ for every $\bar{s} < s < 1$, $m < \min\{\frac{N(p-1)}{N-s}, p\}$. Further search of the literature led us to find similar problems but consisting of both a singularity and a Radon measure. The local case (with Laplace operator) of such problems has been dealt by Panda et al. in [28] and the corresponding problem admits a weak solution in $W_0^{1,m}(\Omega)$ if $\gamma \in (0, 1]$ and in $W_{loc}^{1,m}(\Omega)$ if $\gamma > 1$ for all $m < \frac{N}{N-1}$. The nonlocal case (with fractional Laplacian) with a singularity and a Radon measure has been studied by Ghosh et al. in [18].
In this paper we will consider the following singular fractional elliptic problem with a Choquard type critical nonlinearity and a Radon measure. The motivation to condier this work has been mentioned towards the end of this section.

$$\begin{aligned}(-\Delta)^s u - \alpha\frac{u}{|x|^{2s}} &= \lambda u + u^{-\gamma} + \beta\left(\int_\Omega \frac{u^{2_b^*}(y)}{|x-y|^b}dy\right)u^{2_b^*-1} + \mu \text{ in } \Omega,\\ u &> 0 \text{ in } \Omega,\\ u &= 0 \text{ in } \mathbb{R}^N \setminus \Omega.\end{aligned} \tag{P_β}$$

where Ω be a bounded domain in $\mathbb{R}^N$ with C^2 boundary, $s \in (0, 1)$, $N > 2s$, $0 < \gamma < 1$, $\alpha, \beta, \lambda > 0$, $b < \min\{N, 4s\}$, μ is a bounded Radon measure and $(-\Delta)^s$ is the fractional Laplacian defined by

$$(-\Delta)^s u = \text{P. V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}}dy.$$

Nonlinear problems involving a Choquard term draws its motivation from the Hardy–Littlewood–Sobolev inequality. Buffoni in [9], considered the following Choquard problem and shown the existence of a ground state solution.

$$(-\Delta)u + V(x)u = \left(\frac{1}{|x|^b} * |u|^p\right)|u|^{p-2}u \ \text{ in } \mathbb{R}^N \tag{1.2}$$

for $p > 1$ and $N \geq 3$. S. Pekar in [30] studied the problem (1.2) for $p = 2$ and $b = 1$ as a physical model and described the quantum theory of a polaron at rest. Later, P. Choquard [25] used the Choquard problem of type (1.2) for the modeling of one component plasma. The nonlocal Choquard problem, i.e. the Choquard problem with fractional Laplacian, is known as nonlinear fractional Schrödinger equations with Hartree-type nonlinearity. These problems have a wide application in the quantum mechanical theory, mean field limit of weakly in-teracting molecules, physics of multi particle systems, etc. One can refer [11,26] and the references therein for further study of fractional Choquard problem.

The Brezis–Nirenberg type critical Choquard problem in a bounded domain Ω, that is

$$\begin{aligned} -\Delta u &= \lambda u + \left(\int_\Omega \frac{|u|^{2_b^*}(y)}{|x-y|^b}dy\right)|u|^{2_b^*-2}u \text{ in } \Omega, \\ u &= 0 \text{ in } \mathbb{R}^N \backslash \Omega. \end{aligned} \tag{1.3}$$

has been studied by [16,20,36], etc. Gao & Yang in [16] proved the existence, nonexistence and multiplicity results for a range of λ.

Recently, Giacomoni et al. in [21] dealt with fractional critical Choquard problem with singular nonlinearity, i.e. they considered problem (P_β) with $\alpha, \lambda = 0$ and without Radon measure μ. In [21], the authors have explained a very weak comparison principle, established the existence of two positive weak solution and discussed about the Sobolev regularity of the solutions.

The problem (P_β) involves two critical terms, the Hardy potential in the left hand side and the Choquard nonliner term in the right hand side. The nonlocal problems with a Hardy critical potential has been recently treated in [4,14,15], etc. In 2016, Fiscella & Pucci in [15] studied the following problem with a Hardy term and proved the existence of multiple solutions with the explanation of the asymptotic behavior of solutions.

$$\begin{aligned} (-\Delta)^s u - \alpha\frac{u}{|x|^{2s}} &= \lambda u + \theta f(x,u) + g(x,u) \text{ in } \Omega, \\ u &= 0 \text{ in } \mathbb{R}^N \backslash \Omega, \end{aligned} \tag{1.4}$$

where the function f appears with a subcritical growth while g could be either a critical term or a perturbation.

Motivated by the above works, in this paper, we discuss the problem (P_β) in a bounded domain. To the best of our knowledge, this work is novel, even for the local case (i.e.

for $s = 1$), in the sense that in the literature there is no contribution whatsoever which indicates a study on problem involving a singular nonlinearity, Hardy potential, Choquard nonlinearity and a measure data together. We find a very less number of articles dealing with singular fractional problem with critical exponent and measure data. Amongst them, Panda et al. in [29] have considered the following problem and obatained a positive SOLA via a sequence of approximating problems.

$$\begin{aligned} (-\Delta)^s u &= \frac{1}{u^\gamma} + \lambda u^{2_s^*-1} + \mu \text{ in } \Omega, \\ u &> 0 \text{ in } \Omega, \\ u &= 0 \text{ in } \mathbb{R}^N \backslash \Omega. \end{aligned} \tag{1.5}$$

We have extended the work in [29] by considering a Hardy potential and a Choquard type critical nonlinearity in (P_β). We show the existence of a SOLA to our problem with the method of approximations. We follow the approach closely related to approaches used in [15], [21] and [29].

Turning to the paper organization: In Sect. 2, we provide some functional settings, introduce a suitable notion of solution (SOLA) to (P_β) and further state some auxiliary results and main results. In Sect. 3, we show that the approximating problem to (P_β) admits a positive weak solution for certain range of β. Finally, in Sect. 4, we prove our main result, i.e. Theorem 2.10, and guarantee the existence of a SOLA to (P_β).

2 Functional settings and auxiliary results

The fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$, for $1 \leq p < \infty$ and for $s \in (0, 1)$, is defined as

$$W^{s,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dxdy < \infty \right\}$$

and

$$W_0^{s,p}(\Omega) = \{u \in W^{s,p}(\mathbb{R}^N) : \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dydx < \infty,\ u = 0 \text{ in } \mathbb{R}^N \backslash \Omega\}$$

is a reflexive subspace of $W^{s,p}(\mathbb{R}^N)$ endowed with the following norm

$$\|u\|^p_{W_0^{s,p}(\Omega)} = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dydx.$$

Further, for $p = 2$, we denote the space $W^{s,p}(\mathbb{R}^N)$ as $H^s(\mathbb{R}^N)$ and $W_0^{s,p}(\Omega)$ as $H_0^s(\Omega)$. Actually, $H_0^s(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the following norm

$$\|u\|^2_{H_0^s(\Omega)} = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dydx.$$

Moreover, the space $(H_0^s(\Omega), \|\cdot\|_{H_0^s(\Omega)})$ is a reflexive separable Hilbert space. According to Proposition 3.6 of [27], the norms $\|\cdot\|_{H_0^s(\Omega)}$ and $\|(-\Delta)^{s/2}\cdot\|_{L^2(\mathbb{R}^N)}$ are norm equivalent.
We now state the well known fractional Sobolev embedding theorem, Theorem 6.5 of [27], which will be used frequently throughout this article.

Theorem 2.1 *Let $s \in (0,1)$ and $1 \le p < \infty$ with $N > sp$. Then there exists $C = C(s,N,p) > 0$ such that for every $u \in W_0^{s,p}(\Omega)$,*

$$\|u\|_{L^q(\Omega)} \le C\|u\|_{W_0^{s,p}(\Omega)}$$

for all $1 \le q \le p_s^* = \frac{Np}{N-sp}$. Further, the embedding from $W_0^{s,p}(\Omega)$ to $L^q(\Omega)$ is compact for every $q \in [1, p_s^*)$.

Define $\mathbb{S}_{s,p}$, the best Sobolev constant in the Sobolev embedding theorem, by

$$\mathbb{S}_{s,p} = \inf_{u \in W_0^{s,p}(\Omega)\setminus\{0\}} \frac{\|u\|^p_{W_0^{s,p}(\Omega)}}{\|u\|^p_{L^{p_s^*}(\Omega)}} \tag{2.6}$$

We now define some function spaces which will be further used in this article.
Choose $b < \min\{N, 4s\}$. Let us denote $2_b^* = \frac{2N-b}{N-2s}$ and for any $u \in L^{2_s^*}(\mathbb{R}^N)$, define

$$\|u\|_C = \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^{2_b^*}(x)u^{2_b^*}(y)}{|x-y|^b}dxdy\right)^{1/22_b^*}.$$

According to Lemma 2.2 of [37], $\|\cdot\|_C$ is a norm equivalent to the standard norm $\|\cdot\|_{L^{2_s^*}(\mathbb{R}^N)}$ on $L^{2_s^*}(\mathbb{R}^N)$. Thus, in this sense we can say that the problem (P_β) is a critical Choquard type problem. To understand this sense of criticalness, we need to introduce the Hardy–Littlewood–Sobolev Inequality which is the foundation of the Choquard problem of type (P_β).

Proposition 2.2 (Proposition 2.1 of [20]). *Let $t, r > 1$ and $0 < b < N$ with $1/t + 1/r + b/N = 2$. Further, assume $f \in L^t(\mathbb{R}^N)$ and $g \in L^r(\mathbb{R}^N)$. Then there exists a sharp constant $C(t,r,b,N) > 0$ such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^b}dxdy \le C(t,r,b,N)\|f\|_{L^t(\mathbb{R}^N)}\|g\|_{L^r(\mathbb{R}^N)}.$$

For the choice $f = g = |u|^{2_b^*}$, by using the above inequality we get

$$\|u\|_C^{22_b^*} \le C(N,b)\|u\|^{22_b^*}_{L^{2_s^*}(\mathbb{R}^N)}. \tag{2.7}$$

Define

$$\mathbb{S}_{C,b} = \inf_{u \in H_0^s(\Omega)\setminus\{0\}} \frac{\|u\|^2_{H_0^s(\Omega)}}{\|u\|_C^2}. \tag{2.8}$$

Lemma 2.3 (Lemma 2.5 of [20]).
The constant $\mathbb{S}_{C,b}$ is achieved if and only if

$$u = C\left(\frac{k}{k^2 + |x - x_0|^2}\right)^{\frac{N-2s}{2}}$$

where $C > 0$ is a fixed constant, $x_0 \in \mathbb{R}^N$ and $k \in (0, \infty)$ are parameters. Moreover,

$$\mathbb{S}_{C,b} = \frac{\mathbb{S}_{s,2}}{C(N, b)^{\frac{N-2s}{2N-b}}}$$

Let us define the functional of the elliptic part of our problem (P_β) as

$$\mathbb{H}_{\alpha,\lambda}(u) = \frac{1}{2}\left(\|u\|^2_{H_0^s(\Omega)} - \alpha\|u\|^2_{NH} - \lambda\|u\|^2_{L^2(\Omega)}\right). \tag{2.9}$$

Since the embedding $H_0^s(\Omega) \hookrightarrow L^2(\Omega, |x|^{-2s})$ is continuous but not compact, the Hardy term in the problem is also a critical part. To get rid of this critical term, we look for a range of α and λ such that the functional $\mathbb{H}_{\alpha,\lambda}$ is weakly lower semicontinuous and coercive in $H_0^s(\Omega)$.
Let λ_1 be the first eigenvalue of the fractional Laplacian $(-\Delta)^s$ and hence $\lambda_1 > 0$. Thus, for every $\lambda < \lambda_1$ and every $u \in H_0^s(\Omega)$, we get the following inequalities (refer [15]).

$$m_\lambda\|u\|^2_{H_0^s(\Omega)} \leq \int_\Omega |(-\Delta)^{s/2}u(x)|^2dx - \lambda\int_\Omega |u(x)|^2dx \leq M_\lambda\|u\|^2_{H_0^s(\Omega)}, \tag{2.10}$$

where $m_\lambda = 1 - \frac{\lambda}{\lambda_1}$ and $M_\lambda = 1 + \frac{\lambda}{\lambda_1}$. The best fractional Hardy constant $C_H = C_H(N, s) > 0$, defined below, plays an important role in $H_0^s(\Omega)$.

$$C_H = \inf_{u\in H_0^s(\Omega), u\neq 0} \frac{\|u\|^2_{H_0^s(\Omega)}}{\|u\|^2_{NH}}, \quad \|u\|^2_{NH} = \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}}dx. \tag{2.11}$$

Corollary 2.4 (Corollary 2.3 of [15]). *For any $\lambda \in (-\infty, \lambda_1)$ and $\alpha \in (-\infty, m_\lambda C_H)$, the functional $\mathbb{H}_{\alpha,\lambda} : H_0^s(\Omega) \to \mathbb{R}$, defined in (2.9), is weakly lower semicontinuous and coercive in $H_0^s(\Omega)$. Furthermore,*

$$\mathbb{H}_{\alpha,\lambda}(u) \geq \frac{1}{2}\left(m_\lambda - \frac{\alpha^+}{C_H}\right)\|u\|^2_{H_0^s(\Omega)}.$$

The following theorem is a commonly used variational principle, known as 'Ekeland Variational Principle', to prove the existence of solution to variational problems.

Theorem 2.5 (Ekeland Variational Principle [13]) *Assume H to be a Banach space and the function $J : H \to \mathbb{R} \cup \{+\infty\}$ is Gâteaux-differentiable, lower semi continuous* *and bounded from below. Then for every $\epsilon > 0$ and for every $u \in H$ satisfying $J(u) \leq \inf J + \epsilon$, every $\delta > 0$, there exists $v \in H$ such that $J(v) \leq J(u)$, $\|u - v\| \leq \delta$ and $\|J'(v)\|^* \leq \frac{\epsilon}{\delta}$. The norms $\|.\|$ and $\|.\|^*$ are the norm of V and the dual norm of V^*, respectively.*

Since our problem (P_β) involves a measure data, we do not expect the solution space to be $H_0^s(\Omega)$ but expect the solution to lie in a space with a lower degree of integrability or/and differentiability. Thus, we look for a SOLA (Solutions Obtained as Limits of Approximations). We now define the notion of solution to problem (P_β).

Definition 2.6 Let $\mathcal{M}(\Omega)$ be the set of all finite Radon measures on Ω and $\mu \in \mathcal{M}(\Omega)$. Then a function $u \in W_0^{\bar{s},m}(\Omega)$, for $\bar{s} < s$ and $m < \frac{N}{N-s}$, is said to be SOLA to (P_β) if

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2} u \cdot (-\Delta)^{s/2} \phi - \alpha \int_\Omega \frac{u\phi}{|x|^{2s}} = \lambda \int_\Omega u\phi + \int_\Omega \frac{\phi}{u^\gamma} + \beta \int_\Omega \int_\Omega \frac{u^{2_b^*}(y) u^{2_b^*-1}(x)\phi(x)}{|x-y|^b} dxdy + \int_\Omega \phi d\mu, \tag{2.12}$$

for every $\phi \in C_c^\infty(\Omega)$. Further, for any $\omega \subset\subset \Omega$, there exists a C_ω such that

$$u \geq C_\omega > 0. \tag{2.13}$$

Consider a sequence $(\mu_n) \subset L^\infty(\Omega)$ which is L^1 bounded and converges to μ in the sense of measure as defined in the following definition.

Definition 2.7 Assume $(\mu_n) \subset \mathcal{M}(\Omega)$ to be a sequence of measurable functions. Then (μ_n) converges to $\mu \in \mathcal{M}(\Omega)$ in the sense of measure if

$$\int_\Omega \varphi \mu_n \to \int_\Omega \varphi d\mu, \quad \forall \varphi \in C_0(\bar{\Omega}).$$

We now construct the following sequence of problems which is the approximating problem to (P_β).

$$\begin{aligned} (-\Delta)^s u_n - \alpha \frac{u_n}{|x|^{2s}} &= \lambda u_n + \frac{1}{(u_n + \frac{1}{n})^\gamma} + \beta \left(\int_\Omega \frac{u_n^{2_b^*}(y)}{|x-y|^b} dy \right) u_n^{2_b^*-1} + \mu_n \text{ in } \Omega, \\ u_n &> 0 \text{ in } \Omega, \\ u_n &= 0 \text{ in } \mathbb{R}^N \setminus \Omega, \end{aligned} \tag{$P_{\beta,n}$}$$

Definition 2.8 Let $\mu_n \in L^\infty(\Omega)$ and $\gamma \in (0,1)$. Then $u_n \in H_0^s(\Omega)$ is said to be a weak solution to $(P_{\beta,n})$ if

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2} u_n \cdot (-\Delta)^{s/2} \phi - \alpha \int_\Omega \frac{u_n \phi}{|x|^{2s}}$$

$$
\begin{aligned}
&= \lambda \int_\Omega u_n\phi + \int_\Omega \frac{\phi}{(u_n+1/n)^\gamma} \\
&+ \beta \int_\Omega\int_\Omega \frac{u_n^{2_b^*}(y)u_n^{2_b^*-1}(x)\phi(x)}{|x-y|^b}dxdy + \int_\Omega \mu_n\phi
\end{aligned} \tag{2.14}
$$

for every $\phi \in C_c^\infty(\Omega)$. Further, for any $\omega \subset\subset \Omega$ there exists C_ω such that $u_n \geq C_\omega > 0$.

Denote

$$
H = \{u \in H_0^s(\Omega) : \|u\|_{L^{2_s^*}(\Omega)} = 1\}.
$$

We now state the following existence theorem. We prove this in the next section.

Theorem 2.9 *Let $\gamma \in (0,1)$, $b < \min\{4s, N\}$, $\lambda \in (0,\lambda_1)$ and $\alpha \in (0, m_\lambda C_H)$ such that $\left(\frac{N-b+2s}{2N-b}\right)^{\frac{N-b+2s}{2N-b}} \leq m_\lambda - \frac{\alpha}{C_H}$. Then there exists $\bar\beta \in (0,\infty)$ such that for every $\beta \in (0,\bar\beta)$, $(P_{\beta,n})$ possesses a positive weak solution u_n in H.*

We are now ready to state the main result of the paper.

Theorem 2.10 *Let the assumptions on $\gamma, b, \alpha, \lambda$ are same as in Theorem 2.9. Then there exists $0 < \bar\beta < \infty$ such that for $\beta \in (0,\bar\beta)$ the problem (P_β) has a positive SOLA $u \in W_0^{\bar s,m}(\Omega)$, in the sense of Definition 2.6, for every $\bar s < s$ and $m < \frac{N}{N-s}$.*

3 Existence of weak solution to $(P_{\beta,n})$- Proof of Theorem 2.9

We prove the existence of weak solution to $(P_{\beta,n})$ via two sequence of problems given below, $(P_{\beta,n}^1)$ and $(P_{\beta,n}^2)$. Let us first consider the following problem

$$
\begin{aligned}
(-\Delta)^s w_n - \alpha\frac{w_n}{|x|^{2s}} &= \lambda w_n + \frac{1}{(w_n+\frac{1}{n})^\gamma} + \mu_n \text{ in } \Omega, \\
w_n &> 0 \text{ in } \Omega, \\
w_n &= 0 \text{ in } \mathbb{R}^N\setminus\Omega.
\end{aligned} \tag{$P_{\beta,n}^1$}
$$

We now prove the problem $(P_{\beta,n}^1)$ has a weak solution in $\bar H = \{u \in H_0^s(\Omega) : \|u\|_{L^{2_s^*}(\Omega)} < 1\}$ in the following lemma.

Lemma 3.1 *Let $\gamma \in (0,1)$, $\lambda \in (0,\lambda_1)$ and $\alpha \in (0, m_\lambda C_H)$. Then the problem $(P_{\beta,n}^1)$ admits a positive weak solution w_n in $\bar H$.*

Proof Consider the Euler -Lagrange functional J_n associated to problem $(P_{\beta,n}^1)$, i.e.

$$
J_n(w_n) = \frac{1}{2}\left(\|w_n\|^2_{H_0^s(\Omega)} - \alpha\|w_n\|^2_{NH} - \lambda\|w_n\|^2_{L^2(\Omega)}\right) - \frac{1}{1-\gamma}\int_\Omega \left((w_n+1/n)^{1-\gamma}\right.
$$

$$- \frac{1}{n^{1-\gamma}}\Big) - \int_\Omega \mu_n w_n$$
$$= \mathbb{H}_{\alpha,\lambda}(w_n) - \frac{1}{1-\gamma}\int_\Omega \left((w_n + 1/n)^{1-\gamma} - \frac{1}{n^{1-\gamma}} \right) - \int_\Omega \mu_n w_n$$

for any $w_n \in \bar{H}$. By Corollary 2.4, $\mathbb{H}_{\alpha,\lambda}$ is coercive and weakly lower semicontinuous in $H_0^s(\Omega)$. Since $\gamma \in (0, 1)$, it is easy to prove that J_n is coercive and weakly lower semicontinuous in $\bar{H}$. Thus, by using a standard minimization argument we show the existence of a minimum w_n for J_n in $\bar{H}$ and hence a weak solution to $(P^1_{\beta,n})$ in $\bar{H}$. Fix $\bar{w}_n \in \bar{H}\backslash\{0\}$. Then for sufficiently small $t > 0$,

$$J_n(t\bar{w}_n) = t^2\mathbb{H}_{\alpha,\lambda}(\bar{w}_n) - \frac{1}{1-\gamma}\int_\Omega \left((t\bar{w}_n + 1/n)^{1-\gamma} - \frac{1}{n^{1-\gamma}} \right) - t\int_\Omega \mu_n \bar{w}_n$$
$$< 0.$$

This implies

$$J_n(w_n) = \min_{\bar{w}_n \in \bar{H}} J_n(\bar{w}_n) < 0 = J_n(0)$$

and hence w_n is nontrivial. Let us consider the following problem

$$\begin{aligned} (-\Delta)^s \underline{w}_n &= \frac{1}{(\underline{w}_n + \frac{1}{n})^\gamma} + \mu_n \text{ in } \Omega, \\ \underline{w}_n &> 0 \text{ in } \Omega, \\ \underline{w}_n &= 0 \text{ in } \mathbb{R}^N\backslash\Omega. \end{aligned} \tag{3.15}$$

According to Lemma 2.3 and Lemma 2.4 of Ghosh et.al [18], (3.15) admits a nontrivial positive weak solution in $H_0^s(\Omega)$ and for any $\omega \subset\subset \Omega$, there exists C_ω such that $\underline{w}_n \geq C_\omega > 0$. Using a standrd comparison principle, Lemma 2.4 of [1], we conclude that $\underline{w}_n \leq w_n$. Thus, $w_n \geq \underline{w}_n \geq C_\omega > 0$. This finishes the proof. □

Let us consider the second problem.

$$\begin{aligned} (-\Delta)^s v_n - \alpha\frac{v_n}{|x|^{2s}} + f_n(x, v_n) &= \lambda v_n + \beta \left(\int_\Omega \frac{(w_n + v_n)^{2_b^*}(y)}{|x-y|^b} dy \right) \\ &\quad \times (w_n + v_n)^{2_b^*-1} \text{ in } \Omega, \\ v_n &> 0 \text{ in } \Omega, \\ v_n &= 0 \text{ in } \mathbb{R}^N \setminus \Omega, \end{aligned} \tag{$P^2_{\beta,n}$}$$

where w_n is the positive weak solution of $(P^1_{\beta,n})$ obtained from Lemma 3.1 and the function $f_n : \Omega \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is given by

$$f_n(x,s) = \begin{cases} \frac{1}{(w_n(x)+\frac{1}{n})^\gamma} - \frac{1}{(s+w_n(x)+\frac{1}{n})^\gamma} & \text{if } s + w_n(x) + \frac{1}{n} > 0 \\ -\infty & \text{otherwise.} \end{cases} \tag{3.16}$$

For $(x, s) \in \Omega \times \mathbb{R}$, let us denote $F_n(x,s) = \int_0^s f_n(x,\tau)d\tau$. The corresponding energy functional $J_{\beta,n} : H_0^s(\Omega) \to (-\infty, \infty]$ of $(P^2_{\beta,n})$ is defined by

$$J_{\beta,n}(v_n) = \begin{cases} \frac{1}{2}\left(\int_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^2}{|x-y|^{N+2s}}dxdy - \alpha\int_\Omega \frac{v_n^2}{|x|^{2s}}dx - \lambda\int_\Omega v_n^2 dx\right) & \\ + \int_\Omega F_n(x,v_n)dx - \frac{\beta}{22_s^*}\int_\Omega\int_\Omega \frac{(v_n+w_n)^{2_b^*}(x)(v_n+w_n)^{2_b^*}(y)}{|x-y|^b}dxdy & \text{if } F_n(.,v_n) \in L^1(\Omega) \\ \infty & \text{otherwise.} \end{cases} \tag{3.17}$$

Further,

$$\begin{aligned} \langle J'_{\beta,n}(v_n), v\rangle = &\int_{\mathbb{R}^{2N}} \frac{(v_n(x)-v_n(y))(v(x)-v(y))}{|x-y|^{N+2s}}dxdy \\ &- \alpha\int_\Omega \frac{v_n v}{|x|^{2s}}dx - \lambda\int_\Omega v_n v dx \\ &+ \int_\Omega f_n(x,v_n)v dx \\ &- \beta\int_\Omega\int_\Omega \frac{(v_n+w_n)^{2_b^*}(x)(v_n+w_n)^{2_b^*-1}(y)v(y)}{|x-y|^b}dxdy \end{aligned}$$

for any $v \in H_0^s(\Omega)$. Let us denote

$$H_n = \{u \in \bar{H} : \|u + w_n\|_{L^{2_s^*}(\Omega)} = 1\}.$$

Definition 3.2 We say a function $v_n \in \bar{H}$ is a weak solution of $(P^2_{\beta,n})$ if v_n is a critical point of the corresponding energy functional $J_{\beta,n}$.

Lemma 3.3 *The functional $J_{\beta,n}$ satisfies the Palais-Smale (P-S) condition in H_n for energy level*

$$c < \frac{1}{2}\left(\frac{N-b+2s}{2N-s}\right)\frac{\mathbb{S}_{C,b}^{\frac{2N-b}{N-b+2s}}}{\beta^{\frac{N-2s}{N-b+2s}}} - \frac{C(N,b)\beta}{22_b^*},$$

where $\mathbb{S}_{C,b}$ is the best Sobolev constant defined in (2.8) and $C(N,b)$ is the sharp constant in the Hardy–Littlewood–Sobolev Inequality given in (2.7).

Proof Consider a (P-S) sequence $(v_{n,j})$ of $J_{\beta,n}$ in H_n, i.e. $J_{\beta,n}(v_{n,j}) \to c$ and $J'_{\beta,n}(v_{n,j}) \to 0$ as $j \to \infty$. It is easy to show that $J_{\beta,n}$ is coercive when restricted to H_n. Thus, $(v_{n,j})$ is bounded in $H_0^s(\Omega)$ and there exists v_n in $H_0^s(\Omega)$ such that, up to a subsequential level, $v_{n,j} \to v_n$ weakly in $H_0^s(\Omega)$. We now claim the following.
Claim: $v_{n,j} \to v_n$ strongly in $H_0^s(\Omega)$ and $v_n \in H_n$.

Let $\|v_{n,j} - v_n\|^2_{H_0^s(\Omega)} \to a^2$ and $\int_\Omega \int_\Omega \frac{(v_{n,j}-v_n)^{2_b^*}(x)(v_{n,j}-v_n)^{2_b^*}(y)}{|x-y|^b} dxdy \to d^{22_b^*}$ as $j \to \infty$. Thus, we have

$$\begin{aligned}
&\langle J'_{\beta,n}(v_{n,j}) - J'_{\beta,n}(v_n), v_{n,j} - v_n \rangle \\
&= \|v_{n,j} - v_n\|^2_{H_0^s(\Omega)} - \alpha \|v_{n,j} - v_n\|^2_{NH} - \lambda \|v_{n,j} - v_n\|^2_{L^2(\Omega)} \\
&\quad + \int_\Omega (f_n(x, v_{n,j}) - f_n(x, v_n))(v_{n,j} - v_n) \\
&\quad - \beta \int_\Omega \int_\Omega \frac{(v_{n,j} + w_n)^{2_b^*}(v_{n,j} + w_n)^{2_b^*-1}(v_{n,j} - v_n)}{|x-y|^b} dxdy \\
&\quad + \beta \int_\Omega \int_\Omega \frac{(v_n + w_n)^{2_b^*}(v_n + w_n)^{2_b^*-1}(v_{n,j} - v_n)}{|x-y|^b} dxdy.
\end{aligned}$$

This implies,

$$\begin{aligned}
&\langle J'_{\beta,n}(v_{n,j}) - J'_{\beta,n}(v_n), v_{n,j} - v_n \rangle \\
&\geq \left(m_\lambda - \frac{\alpha}{C_H} \right) \|v_{n,j} - v_n\|^2_{H_0^s(\Omega)} \\
&\quad - \beta \int_\Omega \int_\Omega \frac{(v_{n,j} - v_n)^{2_b^*}(v_{n,j} - v_n)^{2_b^*} + (v_n + w_n)^{2_b^*}(v_n + w_n)^{2_b^*}}{|x-y|^b} dxdy \\
&\quad + \beta \int_\Omega \int_\Omega \frac{(v_{n,j} + w_n)^{2_b^*}(v_{n,j} + w_n)^{2_b^*-1}(v_n + w_n)}{|x-y|^b} dxdy \\
&\quad + \int_\Omega (f_n(x, v_{n,j}) - f_n(x, v_n))(v_{n,j} - v_n) \\
&\quad + \beta \int_\Omega \int_\Omega \frac{(v_n + w_n)^{2_b^*}(v_n + w_n)^{2_b^*-1}(v_{n,j} - v_n)}{|x-y|^b} dxdy.
\end{aligned}$$

On using the Brezis–Lieb Lemma [8], Hardy–Littlewood–Sobolev Inequality (Proposition 2.2), Corollary 2.4, and then passing the limit $j \to \infty$ in the above equation we get

$$\beta d^{22_b^*} \geq \left(m_\lambda - \frac{\alpha}{C_H} \right) a^2. \tag{3.18}$$

From (2.8) we already have $a^2 \geq \mathbb{S}_{C,b} d^2$. Thus, by simplification we obtain

$$d \geq \left(\frac{(m_\lambda - \frac{\alpha}{C_H})\mathbb{S}_{C,b}}{\beta} \right)^{\frac{N-2s}{2(N-b+2s)}}. \tag{3.19}$$

We have the sequence $(v_{n,j})$ is a (P-S) sequence in H_n and by the choice of α, λ, we obtain $\left(\frac{N-b+2s}{2N-b}\right)^{\frac{N-b+2s}{2N-b}} \leq m_\lambda - \frac{\alpha}{C_H}$. Now applying Corollary 2.4, (2.7), (3.18) and (3.19) we have,

$$\begin{aligned}
c &= \lim_{j\to\infty} J_{\beta,n}(v_{n,j}) \\
&= \frac{1}{2} \lim_{j\to\infty} \left(\int_{\mathbb{R}^{2N}} \frac{|v_{n,j}(x) - v_{n,j}(y)|^2}{|x-y|^{N+2s}} dxdy - \alpha \int_\Omega \frac{v_{n,j}^2}{|x|^{2s}} dx - \lambda \int_\Omega v_{n,j}^2 dx \right) \\
&\quad + \lim_{j\to\infty} \left(\int_\Omega F_n(x, v_{n,j}) dx - \frac{\beta}{22_s^*} \int_\Omega \int_\Omega \frac{(v_{n,j} + w_n)^{2_b^*}(x)(v_{n,j} + w_n)^{2_b^*}(y)}{|x-y|^b} dxdy \right) \\
&\geq \lim_{j\to\infty} \left(\frac{1}{2}\left(m_\lambda - \frac{\alpha}{C_H}\right) \|v_{n,j}\|^2 - \frac{C(N,b)\beta}{22_b^*} \|v_{n,j}\|_{L^{2_s^*}(\Omega)}^{22_b^*} \right) \\
&\geq \lim_{j\to\infty} \frac{1}{2}\left(m_\lambda - \frac{\alpha}{C_H}\right) \mathbb{S}_{C,b} \left(\int_\Omega \int_\Omega \frac{v_{n,j}^{2_b^*}(x) v_{n,j}^{2_b^*}(y)}{|x-y|^b} dxdy \right)^{\frac{2}{22_b^*}} - \frac{C(N,b)\beta}{22_b^*} \\
&\geq \frac{1}{2}\left(m_\lambda - \frac{\alpha}{C_H}\right) d^2 \mathbb{S}_{C,b} - \frac{C(N,b)\beta}{22_b^*} \\
&\geq \frac{1}{2}\left(m_\lambda - \frac{\alpha}{C_H}\right) \mathbb{S}_{C,b} \left(\frac{(m_\lambda - \frac{\alpha}{C_H})\mathbb{S}_{C,b}}{\beta} \right)^{\frac{N-2s}{N-b+2s}} - \frac{C(N,b)\beta}{22_b^*} \\
&= \frac{1}{2} \frac{(\mathbb{S}_{C,b}(m_\lambda - \frac{\alpha}{C_H}))^{\frac{2N-b}{N-b+2s}}}{\beta^{\frac{N-2s}{N-b+2s}}} - \frac{C(N,b)\beta}{22_b^*} \\
&\geq \frac{1}{2}\left(\frac{N-b+2s}{2N-b} \right) \frac{\mathbb{S}_{C,b}^{\frac{2N-b}{N-b+2s}}}{\beta^{\frac{N-2s}{N-b+2s}}} - \frac{C(N,b)\beta}{22_b^*}.
\end{aligned}$$

This is a contradiction to our assumption

$$c < \frac{1}{2}\left(\frac{N-b+2s}{2N-b} \right) \frac{\mathbb{S}_{C,b}^{\frac{2N-b}{N-b+2s}}}{\beta^{\frac{N-2s}{N-b+2s}}} - \frac{C(N,b)\beta}{22_b^*}.$$

Thus, $a = 0$ and $\lim_{j\to\infty} \|v_{n,j} - v_n\|_{H_0^s(\Omega)} = 0$. Hence, the claim. □

Let us consider the following sequence (Z_ϵ) given by

$$Z_\epsilon = \epsilon^{-\frac{N-2s}{2}} \mathbb{S}_{s,2}^{\frac{(N-b)(2s-N)}{4(N-b+2s)}} C(N,b)^{\frac{2s-N}{2(N-b+2s)}} z^* \left(\frac{x}{\epsilon}\right), \quad x \in \mathbb{R}^N,$$

where $z^*(x) = \bar{z}\left(\frac{x}{\mathbb{S}_{2,s}^{\frac{1}{2s}}}\right)$, $\bar{z}(x) = \frac{\tilde{z}(x)}{\|\tilde{z}\|_{L^{2_s^*}(\Omega)}}$ and $\tilde{z}(x) = \eta_1(\eta_2^2 + |x|^2)^{-\frac{N-2s}{2}}$, $\eta_1 \in \mathbb{R}^N\backslash\{0\}$, $\eta_2 > 0$. By Lemma 2.3, for each $\epsilon > 0$, corresponding Z_ϵ satisfies the problem

$$(-\Delta)^s z = (|x|^{-b} * |z|^{2_b^*})|z|^{2_b^*-2}v \quad \text{in } \mathbb{R}^N.$$

Let us assume $0 \in \Omega$. Consider $\xi \in C_c^\infty(\mathbb{R}^N)$ such that $0 \leq \xi \leq 1$, for fixed $\delta > 0$, $B_{4\delta} \subset \Omega$, $\xi \equiv 0$ in $\mathbb{R}^N\backslash B_{2\delta}$, $\xi \equiv 1$ in B_δ. Define

$$\Phi_\epsilon(x) = \xi(x)Z_\epsilon(x).$$

Then $\Phi_\epsilon = 0$ in $\mathbb{R}^N\backslash\Omega$. By Proposition 6.2 of Giacomoni et al. [21], there exists $a_1, a_2, a_3, a_4 > 0$ such that for $1 < q < \min\{2, \frac{N}{N-2s}\}$ we have the following four estimates.

$$\int_{\mathbb{R}^{2N}} \frac{|\Phi_\epsilon(x) - \Phi_\epsilon(y)|^2}{|x-y|^{N+2s}} dxdy \leq \mathbb{S}_{C,b}^{\frac{2N-b}{N-b+2s}} + a_1\epsilon^{N-2s},$$

$$\int_\Omega\int_\Omega \frac{|\Phi_\epsilon|^{2_b^*}(x)|\Phi_\epsilon|^{2_b^*}(y)}{|x-y|^b} dxdy \geq \mathbb{S}_{C,b}^{\frac{2N-b}{N-b+2s}} - a_2\epsilon^N,$$

$$\int_\Omega |\Phi_\epsilon|^q dx \leq a_3\epsilon^{(N-2s)q/2},$$

$$\int_\Omega\int_\Omega \frac{|\Phi_\epsilon|^{2_b^*}(x)|\Phi_\epsilon|^{2_b^*}(y)}{|x-y|^b} dxdy \leq \mathbb{S}_{C,b}^{\frac{2N-b}{N-b+2s}} + a_4\epsilon^N.$$

Lemma 3.4 *There exists $\bar{\beta} > 0$ such that for $\beta \in (0, \bar{\beta})$ and for $\epsilon > 0$ sufficienty small,*

$$\sup\{J_{\beta,n}(t\Phi_\epsilon) : t \geq 0\} < \frac{1}{2}\left(\frac{N-b+2s}{2N-b}\right)\frac{\mathbb{S}_{C,b}^{\frac{2N-b}{N-b+2s}}}{\beta^{\frac{N-2s}{N-b+2s}}} - \frac{C(N,b)\beta}{22_b^*}.$$

Proof Clearly for $\beta < \left(\frac{2_b^*}{C(N,b)}\left(\frac{N-b+2s}{2N-b}\right)\right)^{\frac{N-b+2s}{2N-b}}\mathbb{S}_{C,b}$, we have

$$\frac{1}{2}\left(\frac{N-b+2s}{2N-b}\right)\frac{\mathbb{S}_{C,b}^{\frac{2N-b}{N-b+2s}}}{\beta^{\frac{N-2s}{N-b+2s}}} - \frac{C(N,b)\beta}{22_b^*} > 0.$$

For a fixed sufficiently small $\epsilon > 0$ and for any $t \geq 0$,

$$\begin{aligned} &J_{\beta,n}(t\Phi_\epsilon) \\ &= \frac{t^2}{2}\left(\int_{\mathbb{R}^{2N}} \frac{|\Phi_\epsilon(x) - \Phi_\epsilon(y)|^2}{|x-y|^{N+2s}} dxdy - \alpha\int_\Omega \frac{|\Phi_\epsilon|^2}{|x|^{2s}} - \lambda\int_\Omega |\Phi_\epsilon|^2\right) + \int_\Omega F_n(x, t\Phi_\epsilon)dx \end{aligned}$$

$$
\begin{aligned}
&- \frac{\beta}{22_b^*}\int_\Omega\int_\Omega \frac{|t\Phi_\epsilon + w_n|^{2_b^*}|t\Phi_\epsilon + w_n|^{2_b^*}}{|x-y|^b}dxdy \\
&\leq \frac{t^2}{2}\int_{\mathbb{R}^{2N}} \frac{|\Phi_\epsilon(x) - \Phi_\epsilon(y)|^2}{|x-y|^{N+2s}}dxdy + \int_\Omega \frac{|t\Phi_\epsilon|}{(w_n+1/n)^\gamma} \\
&- \frac{1}{1-\gamma}\int_\Omega (t\Phi_\epsilon + w_n + 1/n)^{1-\gamma} - (w_n+1/n)^{1-\gamma} \\
&- \frac{\beta}{22_b^*}\int_\Omega\int_\Omega \frac{|t\Phi_\epsilon + w_n|^{2_b^*}|t\Phi_\epsilon + w_n|^{2_b^*}}{|x-y|^b}dxdy \\
&\leq \frac{t^2}{2}(\mathbb{S}_{C,b}^{\frac{2N-b}{N-b+2s}} + a_1\epsilon^{N-2s}) + tn^\gamma \int_\Omega |\Phi_\epsilon| + \frac{C(N,b)\beta}{22_b^*} - \frac{C(N,b)\beta}{22_b^*} \\
&- \frac{1}{1-\gamma}\int_\Omega (t\Phi_\epsilon + w_n + 1/n)^{1-\gamma} - (w_n+1/n)^{1-\gamma} - \frac{\beta t^{22_b^*}}{22_b^*}\int_\Omega\int_\Omega \frac{|\Phi_\epsilon|^{2_b^*}|\Phi_\epsilon|^{2_b^*}}{|x-y|^b}dxdy \\
&\leq \frac{t^2}{2}(\mathbb{S}_{C,b}^{\frac{2N-b}{N-b+2s}} + a_1\epsilon^{N-2s}) + tn^\gamma a_3^{1/q}\epsilon^{(N-2s)/2} + \frac{C(N,b)\beta}{22_b^*} - \frac{C(N,b)\beta}{22_b^*} \\
&- \frac{1}{1-\gamma}\int_\Omega (t\Phi_\epsilon + w_n + 1/n)^{1-\gamma} - (w_n+1/n)^{1-\gamma} - \frac{\beta t^{22_b^*}}{22_b^*}(\mathbb{S}_{C,b}^{\frac{2N-b}{N-b+2s}} - a_2\epsilon^N).
\end{aligned} \tag{3.20}
$$

Assume $\beta \leq 1$ and define $g : \mathbb{R}^+ \to \mathbb{R}$ as

$$
\begin{aligned}
g(t) &= \frac{C(N,b)\beta}{22_b^*} - \frac{1}{1-\gamma}\int_\Omega (t\Phi_\epsilon + w_n + 1/n)^{1-\gamma} - (w_n+1/n)^{1-\gamma} \\
&\leq \frac{C(N,b)}{22_b^*} - \frac{1}{1-\gamma}\int_\Omega (t\Phi_\epsilon + w_n + 1/n)^{1-\gamma} - (w_n+1/n)^{1-\gamma}.
\end{aligned} \tag{3.21}
$$

Hence, $g(t) \to -\infty$ as $t \to \infty$. Therefore, there exists $\bar{t} > 0$ such that $g(t) \leq 0$ for every $t \geq \bar{t}$. First consider the case $t \geq \bar{t}$ and we have

$$
\begin{aligned}
J_{\beta,n}(t\Phi_\epsilon) &\leq \frac{t^2}{2}(\mathbb{S}_{C,b}^{\frac{2N-b}{N-b+2s}} + a_1\epsilon^{N-2s}) + tn^\gamma a_3^{1/q}\epsilon^{(N-2s)/2} \\
&\quad - \frac{\beta t^{22_b^*}}{22_b^*}(\mathbb{S}_{C,b}^{\frac{2N-b}{N-b+2s}} - a_2\epsilon^N) - \frac{C(N,b)\beta}{22_b^*} \\
&= \bar{g}_\epsilon(t).
\end{aligned}
$$

Clearly, $\bar{g}_\epsilon$ attains the maximum value at

$$
t_\beta = \left(\frac{1}{\beta}\right)^{\frac{2(N-b+2s)}{N-2s}} + o(\epsilon^{(N-2s)/2}).
$$

This implies,

$$
J_{\beta,n}(t\Phi_\epsilon) \leq \frac{1}{2}\left(\frac{N-b+2s}{2N-b}\right)\frac{\mathbb{S}_{C,b}^{\frac{2N-b}{N-b+2s}}}{\beta^{\frac{N-2s}{N-b+2s}}} - \frac{C(N,b)\beta}{22_b^*} + o(\epsilon^{(N-2s)/2})
$$

$$< \frac{1}{2}\left(\frac{N-b+2s}{2N-b}\right)\frac{\mathbb{S}_{C,b}^{\frac{2N-b}{N-b+2s}}}{\beta^{\frac{N-2s}{N-b+2s}}} - \frac{C(N,b)\beta}{22_b^*}. \tag{3.22}$$

For the second case, i.e. $t < \bar{t}$,

$$\begin{aligned} J_{\beta,n}(t\Phi_\epsilon) &\leq \frac{t^2}{2}\int_{\mathbb{R}^{2N}} \frac{|\Phi_\epsilon(x)-\Phi_\epsilon(y)|^2}{|x-y|^{N+2s}}dxdy + \int_\Omega \frac{|t\Phi_\epsilon|}{(w_n+1/n)^\gamma} \\ &\leq \frac{t^2}{2}(\mathbb{S}_{C,b}^{\frac{2N-b}{N-b+2s}} + a_1\epsilon^{N-2s}) + tn^\gamma a_3^{1/q}\epsilon^{(N-2s)/2} \\ &< \frac{\bar{t}^2}{2}(\mathbb{S}_{C,b}^{\frac{2N-b}{N-b+2s}} + a_1\epsilon^{N-2s}) + \bar{t}n^\gamma a_3^{1/q}\epsilon^{(N-2s)/2}. \end{aligned}$$

Choose $\beta^* > 0$ depending on $\bar{t}, N, s, \mathbb{S}_{C,b}$ such that for $\beta \in (0, \beta^*)$ we get

$$J_{\beta,n}(t\Phi_\epsilon) < \frac{1}{2}\left(\frac{N-b+2s}{2N-b}\right)\frac{\mathbb{S}_{C,b}^{\frac{2N-b}{N-b+2s}}}{\beta^{\frac{N-2s}{N-b+2s}}} - \frac{C(N,b)\beta}{22_b^*}.$$

Choose $\bar{\beta} = \min\{1, \left(\frac{2_b^*}{C(N,b)}\left(\frac{N-b+2s}{2N-b}\right)\right)^{\frac{N-b+2s}{2N-b}} \mathbb{S}_{C,b}, \beta^*\}$. Thus, for $\beta \in (0, \bar{\beta})$ we obtain

$$\sup\{J_{\beta,n}(t\Phi_\epsilon) : t \geq 0\} < \frac{1}{2}\left(\frac{N-b+2s}{2N-b}\right)\frac{\mathbb{S}_{C,b}^{\frac{2N-b}{N-b+2s}}}{\beta^{\frac{N-2s}{N-b+2s}}} - \frac{C(N,b)\beta}{22_b^*}.$$

This concludes the proof. □

The following is the existence theorem for $(P^2_{\beta,n})$ in H_n.

Theorem 3.5 *Assume $b < \min\{4s, N\}$, $\lambda \in (0, \lambda_1)$ and $\alpha \in (0, m_\lambda H)$. Then there exists $\bar{\beta} > 0$ such that for every $\beta \in (0, \bar{\beta})$, $(P^2_{\beta,n})$ admits a positive weak solution $v_n \in H_n$.*

Proof The functional $J_{\beta,n}$ is bounded from below and G$\hat{a}$teaux-differentiable on H_n. Hence, it satisfies all the hypotheses of Theorem 2.5, i.e. Ekeland variational principle. Thus, we can produce a Palais-Smale sequence $(v_{n,j})$ in H_n of the functional $J_{\beta,n}$. By Lemma 3.3 and Lemma 3.4, $(v_{n,j})$ satisfies the (P-S) conditions and hence, up to a subsequential level, $(v_{n,j})$ converges strongly to $v_n \in H_n$. This implies v_n is a critical point of $J_{\beta,n}$ and therefore a weak solution of $(P_{\beta,n})$ in H_n for any $\beta \in (0, \bar{\beta})$. □

Proof of Theorem 2.9 According to Theorem 3.5, v_n is a nontrivial weak solution to $(P^2_{\beta,n})$ in $H_n = \{u \in H_0^s(\Omega) : \|u + w_n\|_{L^{2_s^*}(\Omega)} = 1\}$, for every for $\beta \in (0, \bar{\beta})$, where w_n is the weak solution of $(P^1_{\beta,n})$ from Lemma 3.1. Hence, the function $u_n = v_n + w_n$ is a positive weak solution of $(P_{\beta,n})$ in $H = \{u \in H_0^s(\Omega) : \|u\|_{L^{2_s^*}(\Omega)} = 1\}$ in the sense of Definition 2.8. □

4 Existence of SOLA to (P_β)- Proof of Theorem 2.10

From the previous section, Sect. 3, we a weak solution u_n of the approximating problem $(P_{\beta,n})$ in H. In this section, using some apriori estimates, we pass the limit $n \to \infty$ in the weak formulation of $(P_{\beta,n})$, i.e. (2.14), to obtain a SOLA to (P_β).

Lemma 4.1 *Let u_n be a weak solution to $(P_{\beta,n})$ in H. Then the sequence (u_n) is uniformly bounded in $W_0^{\bar{s},m}(\Omega)$ for every $\bar{s} < s$ and $m < \frac{N}{N-s}$.*

Proof Let us fix a $k > 0$ and define a truncation function $T_k : \mathbb{R} \to \mathbb{R}$ by

$$T_k(u_n) = \begin{cases} u_n & \text{if } u_n \leq k \\ k & \text{if } u_n > k. \end{cases}$$

Choose $\phi = T_k(u_n)$ in (2.14) as a test function. Thus, we have

$$\begin{aligned} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} T_k(u_n)|^2 &\leq \int_{\mathbb{R}^N} (-\Delta)^{s/2} u_n \cdot (-\Delta)^{s/2} T_k(u_n) \\ &= \alpha \int_\Omega \frac{u_n T_k(u_n)}{|x|^{2s}} + \lambda \int_\Omega u_n T_k(u_n) + \beta \int_\Omega \int_\Omega \frac{u_n^{2_b^*} u_n^{2_b^*-1} T_k(u_n)}{|x-y|^b} dxdy \\ &\quad + \int_\Omega \frac{1}{(u_n + \frac{1}{n})^\gamma} T_k(u_n) + \int_\Omega \mu_n T_k(u_n). \end{aligned} \tag{4.23}$$

The sequence $(u_n) \subset H$, i.e. $\|u_n\|_{L^{2_s^*}(\Omega)} = 1$ for each n and (μ_n) is bounded in $L^1(\Omega)$. So, equation (4.23) becomes

$$\begin{aligned} &\int_{\mathbb{R}^N} |(-\Delta)^{s/2} T_k(u_n)|^2 \\ &\leq \alpha k \int_\Omega \frac{u_n}{|x|^{2s}} + \lambda \int_\Omega u_n^2 + \int_\Omega u_n^{1-\gamma} + k\|\mu_n\|_{L^1(\Omega)} + \beta C(N,b) \\ &\leq \alpha k \int_\Omega \frac{u_n}{|x|^{2s}} + \lambda C_1 \|u_n\|^2_{L^{2_s^*}(\Omega)} + C_2 \|u_n\|^{1-\gamma}_{L^{2_s^*}(\Omega)} + C_3 k + \beta C(N,b) \\ &\leq \alpha k \int_\Omega \frac{u_n}{|x|^{2s}} + C_4 k. \end{aligned} \tag{4.24}$$

Since $N > 2s$, this implies $(N - 2s(2_s^*)') > 0$ where $(2_s^*)'$ is the Hölder conjugate of 2_s^*. Therefore, using the Hölder's inequality we get

$$\begin{aligned} \int_\Omega \frac{u_n}{|x|^{2s}} &\leq \left\| x|^{-2s} \right\|_{L^{(2_s^*)'}(\Omega)} \|u_n\|_{L^{2_s^*}(\Omega)} \\ &= \left\| |x|^{-2s} \right\|_{L^{(2_s^*)'}(\Omega)} \\ &\leq C_5. \end{aligned} \tag{4.25}$$

Hence, (u_n) is bounded in $L^1(\Omega, |x|^{-2s})$ and from (4.24), we conclude that

$$\int_{\mathbb{R}^N} |(-\Delta)^{s/2} T_k(u_n)|^2 \leq Ck. \tag{4.26}$$

Therefore, $(T_k(u_n))$ is uniformly bounded in $H_0^s(\Omega)$. By following the proof of Lemma 4.1 of Panda et al. [29], we conclude that (u_n) is uniformly bounded in $W_0^{\bar{s},m}(\Omega)$, for all $\bar{s} < s$ and $m < \frac{N}{N-s}$. □

We are now in a position to prove our main result, i.e. the existence of positive SOLA to (P_β).

Proof of Theorem 2.10 Let $\mu \in \mathcal{M}(\Omega)$ and the assumptions on $\gamma, b, \alpha, \lambda$ are same as provided in the statement of Theorem 2.10. From Theorem 2.9, for any $\beta \in (0, \bar{\beta})$, we have a positive weak solution u_n of $(P_{\beta,n})$ in H. Since (u_n) is uniformly bounded in $W_0^{\bar{s},m}(\Omega)$, for every $\bar{s} < s$ and $m < \frac{N}{N-s}$, by Lemma 4.1, there exists $u \in W_0^{\bar{s},m}(\Omega)$ such that $u_n \to u$ weakly in $W_0^{\bar{s},m}(\Omega)$. This implies $u_n \to u$ a.e. in $\mathbb{R}^N$ and $u \equiv 0$ in $\mathbb{R}^N \backslash s\Omega$. From the weak formulation of $(P_{\beta,n})$, i.e. (2.14), we have

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}} dxdy - \alpha \int_\Omega \frac{u_n \phi}{|x|^{2s}} \\
&\quad = \lambda \int_\Omega u_n \phi + \int_\Omega \frac{1}{(u_n + \frac{1}{n})^\gamma} \phi + \beta \int_\Omega \int_\Omega \frac{u_n^{2_b^*} u_n^{2_b^*-1} \phi}{|x-y|^b} dxdy + \int_\Omega \mu_n \phi,
\end{aligned} \tag{4.27}$$

for all $\phi \in C_c^\infty(\Omega)$. Proceeeding on similar lines as in Theorem 1.1. of [29] and using Vitali convergence theorem, we establish

$$\begin{aligned}
&\lim_{n\to\infty} \int_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}} dxdy \\
&= \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}} dxdy.
\end{aligned}$$

On using the definition of convergence in measure, i.e. Definition 2.7, Dominated convergence theorem and the fact that $\|u_n\|_{L^{2_s^*}(\Omega)} = 1$, we can pass the limit $n \to \infty$ in the following integrals.

$$\begin{aligned}
&\lim_{n\to\infty} \int_\Omega \mu_n \phi = \int_\Omega \phi d\mu, \\
&\lim_{n\to\infty} \beta \int_\Omega \int_\Omega \frac{u_n^{2_b^*} u_n^{2_b^*-1} \phi}{|x-y|^b} dxdy = \beta \int_\Omega \int_\Omega \frac{u^{2_b^*} u^{2_b^*-1} \phi}{|x-y|^b} dxdy, \\
&\lim_{n\to\infty} \int_\Omega \frac{u_n \phi}{|x|^{2s}} = \int_\Omega \frac{u\phi}{|x|^{2s}},
\end{aligned}$$

$$\lim_{n\to\infty}\int_\Omega \frac{1}{(u_n+\frac{1}{n})^\gamma}\phi = \int_\Omega \frac{1}{u^\gamma}\phi.$$

Hence, we obtain a SOLA u to (P_β), in the sense of Definition 2.12, as the limit of approximation in (4.27). Thus, for every $\phi \in C_c^\infty(\Omega)$, u satisfies

$$\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}dxdy - \alpha\int_\Omega \frac{u\phi}{|x|^{2s}}$$
$$= \lambda\int_\Omega u\phi + \int_\Omega \frac{1}{u^\gamma}\phi + \beta\int_\Omega\int_\Omega \frac{u^{2_b^*}u^{2_b^*-1}\phi}{|x-y|^b}dxdy + \int_\Omega \mu\phi.$$

This concludes the proof our main result. □

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