J. Pseudo-Differ. Oper. Appl. (2020) 11:2023–2043
https://doi.org/10.1007/s11868-020-00366-8

Nemytskii operator on $(\phi, 2, \alpha)$-bounded variation space in the sense of Riesz

René E. Castillo[1] · Edixon M. Rojas[1] · Eduard Trousselot[2]

Received: 16 April 2020 / Revised: 6 July 2020 / Accepted: 2 September 2020 /
Published online: 15 September 2020
© Springer Nature Switzerland AG 2020

Abstract
In this paper we show that if the Nemytskii operator maps the $(\phi, 2, \alpha)$-bounded variation space into itself and satisfies some Lipschitz condition, then there are two functions g and h belonging to the $(\phi, 2, \alpha)$-bounded variation space such that $f(t, y) = g(t)y + h(t)$ for all $t \in [a, b]$, $y \in \mathbb{R}$.

Keywords Nemytskii operator · $(\phi, 2, \alpha)$-Bounded variation · Riesz p-variation · Embedding

Mathematics Subject Classification 26A45 · 26B30 · 26A16 · 26A24

1 Introduction

According to Lakoto [16], functions of bounded variation were discovered by Camille Jordan around 1880 through a "critical" re-examination of Dirichlet's famous flawed proof that arbitrary functions can be represented by Fourier series (see, [14]). It was Jordan who gave the characterization of such functions as differences of increasing functions, but, as point out by Hawkins [15], the key observation that Dirichlet's proof was valid for differences of increasing functions had already been made by Dubois-Raymond [13]. In the same vein, in 1905 G. Vitali [32] introduce the absolutely continuous functions of one variable. Since then, the concept has been generalized in many ways.

✉ Edixon M. Rojas
emrojass@unal.edu.co

René E. Castillo
recastillo@unal.edu.co

Eduard Trousselot
eddycharles2007@gmail.com

1 Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia

2 Departamento de Matemáticas, Universidad de Oriente, Cumaná, Edo. Sucre, Venezuela

Some of those generalizations were motivated by problems in areas such that geometric measures theory and mathematical physics. (For applications of function variation in mathematical physics, see the monograph [33]). One of those generalizations appeared in 1908, when de la Vallée Poussin [12] defined the second bounded variation of a function f in the interval $[a, b]$ by

$$V^2(f) = V^2(f, [a, b]) = \sup_{\Pi} \sum_{j=1}^{n-1} \left| \frac{f(x_{j+1}) - f(x_j)}{x_{j+1} - x_j} - \frac{f(x_j) - f(x_{j-1})}{x_j - x_{j-1}} \right|,$$

where the supremun is taken over all partition

$$\Pi = \{a = x_0 < x_1 < \cdots < x_n = b\}$$

of the interval $[a, b]$.

Another generalization is due to F. Riesz in his 1910 paper (see, [28]), he defined the p -variation of a function on an interval $[a, b]$ as

$$V_p^R(f) = V_p^R(f, [a, b]) = \sup_{\Pi} \sum_{j=1}^{n} \frac{|f(x_j) - f(x_{j-1})|^p}{|x_j - x_{j-1}|^{p-1}}. \tag{1.1}$$

Again, the supremun is taken over all partition $\Pi = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of the interval $[a, b]$. Riesz proved that, for $1 < p < +\infty$, the class of functions of bound p-variation (i.e., the class of functions for which $V_p^R(f) < +\infty$) coincides with the class of absolutely continuous functions with derivative belonging to $L_p([a, b])$. Moreover, the p-variation of a function f on $[a, b]$ is given by

$$V_p^R(f) = V_p^R(f, [a, b]) = \int_a^b |f'(x)|^p dx. \tag{1.2}$$

One may replace the p-th power in (1.1) by a function ϕ behaves similar to x^p for $p \geq 1$ as follows: A function $\phi : [0, +\infty) \longrightarrow [0, +\infty)$ such that

(a) $\phi(x) = 0$ if and only if $x = 0$.
(b) $\lim_{x \to +\infty} \phi(x) = +\infty$,

is known as Young function. In 1953 Y.T. Medvedev [26] introduced the concept of ϕ-bounded variation in the following way: given a Young function ϕ, a partition $\Pi = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$ and a function $f : [a, b] \longrightarrow \mathbb{R}$, the ϕ-variation of f is defined as

$$V_\phi^R(f) = V_\phi^R(f, [a, b]) = \sup_{\Pi} \sum_{j=1}^{n} \phi\left(\frac{|f(x_j) - f(x_{j-1})|}{|x_j - x_{j-1}|^{p-1}} \right) |x_{j-1} - x_j|, \tag{1.3}$$

where the supremun is taken over all partition Π of $[a, b]$. We might observe that, when $\phi(x) = x^p$, $p \geq 1$, $x \geq 0$ we get back the p-variation concept. In other words, the Medvedev characterization generalizes the one made by Riesz. In such a sense that (1.3) is called the Riesz-Medvedev variation of f on $[a, b]$. Again, in case $V_\phi^R(f) < \infty$, we say that f has bounded Riesz-Medvedev variation (or bounded ϕ-variation in Riesz's sense) on $[a, b]$, and we write $f \in BV_\phi^R([a, b])$.

In the same paper [26], for a convex Young function ϕ which satisfies the ∞_1-condition (that is $\lim_{x\to+\infty} \frac{\phi(x)}{x} = +\infty$), the following remarkable result was proven: $f \in BV_\phi^R([a, b])$ if and only if f is absolutely continuous on $[a, b]$ and $\int_a^b \phi(|f'(x)|)dx < +\infty$.

Moreover, the ϕ-variation of f on $[a, b]$ is given by

$$V_\phi^R(f) = V_\phi^R(f, [a, b]) = \int_a^b \phi(|f'(x)|)dx. \tag{1.4}$$

Also note that (1.4) generalizes (1.2). In [6] the first and third named authors, together with H. Rafeiro, introduced the $(2, \alpha)$-variation in the sense of de la Vallée Poussin, combining the second bounded variation with the (p, α)-variation (see [3] and [8]).

Definition 1.1 Let ϕ be a ϕ-function (Young function), f a real function defined on $[a, b]$ and let α be any strictly increasing continuous function defined on $[a, b]$. Let Π be a block partition of the interval $[a, b]$, that is,

$$\Pi : a = x_{1,1} < x_{1,2} \le x_{1,3} < x_{1,4} = x_{2,1} < x_{2,2} \le x_{2,3} < x_{2,4} = x_{3,1} < \cdots$$
$$< x_{n-1,4} = x_{n,2} \le x_{n,3} < x_{n,4} = b.$$

Let

$$\begin{aligned}
\sigma^R_{(\phi,2,\alpha)}(f, \Pi) &= \sum_{j=1}^{n} \phi\left(\left|\frac{f(x_{j,4}) - f(x_{j,3})}{(\alpha(x_{j,4}) - \alpha(x_{j,3}))(\alpha(x_{j,4}) - \alpha(x_{j,1}))}\right.\right. \\
&\quad \left.\left. - \frac{f(x_{j,2}) - f(x_{j,1})}{(\alpha(x_{j,2}) - \alpha(x_{j,1}))(\alpha(x_{j,4}) - \alpha(x_{j,1}))}\right|\right) |\alpha(x_{j,4}) - \alpha(x_{j,1})| \\
&= \sum_{j=1}^{n} \phi\left(\frac{\left|\frac{f(x_{j,4})-f(x_{j,3})}{\alpha(x_{j,4})-\alpha(x_{j,3})} - \frac{f(x_{j,2})-f(x_{j,1})}{\alpha(x_{j,2})-\alpha(x_{j,1})}\right|}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|}\right) |\alpha(x_{j,4}) - \alpha(x_{j,1})| \\
&= \sum_{j=1}^{n} \phi\left(\frac{|f_\alpha[x_{j,4}, x_{j,3}] - f_\alpha[x_{j,2}, x_{j,1}]|}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|}\right) |\alpha(x_{j,4}) - \alpha(x_{j,1})|
\end{aligned}$$

where

$$f_\alpha[p, q] = \frac{f(q) - f(p)}{\alpha(q) - \alpha(p)}$$

and

$$V^R_{(\phi,2,\alpha)}(f,[a,b]) = V^R_{(\phi,2,\alpha)}(f) = \sup_{\Pi} \sigma^R_{(\phi,2,\alpha)}(f,\Pi),$$

where the supremum is taken over all possible block partition of $[a,b]$.

$V^R_{(\phi,2,\alpha)}(f)$ is called $(\phi,2,\alpha)$-variation in the sense of Riesz on the interval $[a,b]$. If $V^R_{(\phi,2,\alpha)}(f) < \infty$, the function f is said to be of $(\phi,2,\alpha)$-variation in the sense of Riesz. The set of all this functions is denoted by $V^R_{(\phi,2,\alpha)}([a,b])$. $RV_{(\phi,2,\alpha)}([a,b])$ is the space generated by $V^R_{(\phi,2,\alpha)}([a,b])$.

2 Definitions and some needed results

In this section, we gather definitions and notations that will be used throughout the paper. Let α be any strictly increasing continuous function defined on $[a,b]$.

Definition 2.1 Let ϕ be a convex ϕ-function, then

$$RV^0_{(\phi,2,\alpha)}([a,b]) = \{f : [a,b] \longrightarrow \mathbb{R} \ : \ f \in RV_{(\phi,2,\alpha)}([a,b]) \text{ and } f(a) = 0\}$$

is the linear space of $(\phi,2,\alpha)$-bounded variation in the sense of Riesz which vanish at a.

Definition 2.2 Let ϕ be a convex ϕ-function,

$$|\cdot|^R_{(\phi,2,\alpha)} : RV^0_{(\phi,2,\alpha)}([a,b]) \longrightarrow \mathbb{R}^+$$

given by

$$|f|^R_{(\phi,2,\alpha)} = |f'_\alpha(a)| + \inf\{\varepsilon > 0 \ : \ V^R_{(\phi,2,\alpha)}(f/\varepsilon) \leq 1\}.$$

Definition 2.3 Suppose f and α are real-valued functions on the same open interval (bounded or unbounded). Suppose x_0 is a point in this interval. We say f is α-derivable at x_0 if

$$\lim_{\alpha \to x_0} \frac{f(x) - f(x_0)}{\alpha(x) - \alpha(x_0)} \quad \text{exists.}$$

We denote its value by $f'_\alpha(x_0)$, which we call the α-derivative of f at x_0.

Definition 2.4 A function $\upsilon : [a,b] \longrightarrow \mathbb{R}$ is α-convex in $[a,b]$, if for all $a \leq \lambda \leq \xi \leq \mu \leq b$ the following holds

$$\upsilon(\xi) \leq \frac{\alpha(\xi) - \alpha(\lambda)}{\alpha(\mu) - \alpha(\lambda)} \upsilon(\mu) + \frac{\alpha(\mu) - \alpha(\xi)}{\alpha(\mu) - \alpha(\lambda)} \upsilon(\lambda).$$

As a consequence, we have, from the properties of α-convex functions, the existence of the lateral derivatives $f'_{\alpha+}(x_0)$ and $f'_{\alpha-}(x_0)$ in each point $x_0 \in (a, b)$ and the existence of $f'_{\alpha+}(a)$ and $f'_{\alpha-}(b)$.

Theorem 2.1 *Let ϕ be a convex ϕ-function. If $f \in V^R_{(\phi,2,\alpha)}([a, b])$, then $f \in BV^{(2,\alpha)}([a, b])$. Moreover*

$$V^{(2,\alpha)}(f) \leq \frac{1}{\phi(1)} V^R_{(\phi,2,\alpha)}(f) + \alpha(b) - \alpha(a).$$

For the proof see [2].

Theorem 2.2 *If ϕ is a convex ϕ-function such that satisfy the (∞_1)-condition, then we have the following embedding results*

$$RV_{(\phi,2,\alpha)}([a, b]) \subset BV^{(2,\alpha)}([a, b]) \subset \alpha - Lip([a, b]) \subset RV_{(\phi,\alpha)}([a, b]) \subset \cdots \subset \alpha - AC([a, b]) \subset BV([a, b]) \subset B([a, b]).$$

Theorem 2.3 *$f \in BV_{(2,\alpha)}([a, b])$ if and only if $f = f_1 - f_2$ where f_1 and f_2 are α-convex functions.*

Theorem 2.4 *Let ϕ be a convex function, which satisfy the (∞_1)-condition. If $f \in V^R_{(\phi,2,\alpha)}([a, b])$, then there exists $f'_\alpha(x_0)$ on each point $x_0 \in [a, b]$.*

Proof From Theorem 2.2 we know the $V^R_{(\phi,2,\alpha)}([a, b]) \subset BV^{(2,\alpha)}([a, b])$ and by Theorem 2.3 we have that there are lateral α-derivatives at each point of the interval $[a, b]$. Suppose there is $x_0 \in (a, b)$ such that $f_{\alpha^+}(x_0) \neq f_{\alpha^-}(x_0)$. From the definition of $V_{(\phi,2,\alpha)}([a, b])$, let us consider in the partition the points $\cdots \leq x_0 + h \leq x_0 < x_0 + h < \cdots$ in order to obtain

$$V^R_{(\phi,2,\alpha)}([a, b]) \geq \phi\left(\frac{\left|\frac{f(x_0+h)-f(x_0)}{\alpha(x_0+h)-\alpha(x_0)} - \frac{f(x_0)-f(x_0-h)}{\alpha(x_0)-\alpha(x_0-h)}\right|}{|\alpha(x_0+h) - \alpha(x_0-h)|}\right) |\alpha(x_0+h) - \alpha(x_0-h)|$$
$$= \frac{\phi\left(\frac{\left|\frac{f(x_0+h)-f(x_0)}{\alpha(x_0+h)-\alpha(x_0)} - \frac{f(x_0)-f(x_0-h)}{\alpha(x_0)-\alpha(x_0-h)}\right|}{|\alpha(x_0+h)-\alpha(x_0-h)|}\right)}{\frac{|f'_{\alpha+}(x_0)-f_{\alpha-}(x_0)|}{\alpha(x_0+h)-\alpha(x_0+h)}} |f_{\alpha^+}(x_0) - f_{\alpha^-}(x_0)|$$

letting $h \to 0$ and using the fact that ϕ and α are continuous, then we have

$$V^R_{(\phi,2,\alpha)}([a, b]) \geq \frac{\phi\left(\frac{|f'_{\alpha+}(x_0)-f'_{\alpha-}(x_0)|}{\lim\limits_{h\to 0} |\alpha(x_0+h) - \alpha(x_0-h)|}\right)}{\frac{|f'_{\alpha+}(x_0)-f'_{\alpha-}(x_0)|}{\lim\limits_{h\to 0} |\alpha(x_0+h) - \alpha(x_0-h)|}} \cdot |f'_{\alpha+}(x_0) - f_{\alpha^-}(x_0)|$$

$$= \lim_{h\to 0} \frac{\phi\left(\frac{|f'_{\alpha^+}(x_0)-f'_{\alpha^-}(x_0)|}{|\alpha(x_0+h)-\alpha(x_0-h)|}\right)}{\frac{|f'_{\alpha^+}(x_0)-f'_{\alpha^-}(x_0)|}{|\alpha(x_0+h)-\alpha(x_0-h)|}} \cdot |f'_{\alpha^+}(x_0) - f_{\alpha^-}(x_0)| = +\infty.$$

That is,

$$\lim_{h\to 0} \frac{|f'_{\alpha^+}(x_0) - f_{\alpha^-}(x_0)|}{|\alpha(x_0+h) - \alpha(x_0-h)|} = +\infty$$

and

$$|\alpha(x_0+h) - \alpha(x_0-h)| \neq 0.$$

This contradicts the fact that $f \in V^R_{(\phi,2,\alpha)}([a,b])$, so f is α-derivable at each point of (a,b) and there exist $f'_{\alpha^+}(a)$ and $f_{\alpha^-}(b)$. □

Theorem 2.5 *Let ϕ be a convex ϕ-function which satisfies the (∞_1)-condition and $f : [a,b] \longrightarrow \mathbb{R}$. If $f \in V^R_{(\phi,2,\alpha)}([a,b])$, that is $V^R_{(\phi,2,\alpha)}(f) < +\infty$, then $f'_\alpha \in V^R_{(\phi,2,\alpha)}([a,b])$, that is $V^R_{(\phi,2,\alpha)}(f'_\alpha) < +\infty$. Moreover,*

$$\int_a^b \phi(|f''_\alpha(t)|)d\alpha(t) \leq V_{(\phi,2,\alpha)}(f).$$

Proof Let $\Pi : a < x_0 < x_1 \cdots < x_n = b$ be a partition of $[a,b]$ and $0 < h \leq \min\left\{\frac{x_j - x_{j-1}}{2},\ j = 1, 2, \ldots, n\right\}$. We have that

$$\begin{aligned} a = x_0 < x_0 + h < x_1 - h < x_1 < x_1 + h \leq x_2 - h < \\ \cdots < x_{n-1} < x_{n-1} + h \leq x_n - h < x_n = b \end{aligned}$$

is a block partition of $[a,b]$. By definition we have

$$\sum_{j=1}^n \phi\left(\frac{\left|\frac{f(x_j)-f(x_j-h)}{\alpha(x_j)-\alpha(x_j-h)} - \frac{f(x_{j-1}+h)-f(x_{j-1})}{\alpha(x_{j-1}+h)-\alpha(x_{j-1})}\right|}{|\alpha(x_j) - \alpha(x_{j-1})|}\right) |\alpha(x_j) - \alpha(x_{j-1})| \leq V^R_{(\phi,2,\alpha)}(f).$$

Allowing that h goes to 0 in the previous expression, we deduce

$$\lim_{h\to 0}\sum_{j=1}^n \phi\left(\frac{\left|\frac{f(x_j)-f(x_j-h)}{\alpha(x_j)-\alpha(x_j-h)} - \frac{f(x_{j-1}+h)-f(x_{j-1})}{\alpha(x_{j-1}+h)-\alpha(x_{j-1})}\right|}{|\alpha(x_j) - \alpha(x_{j-1})|}\right) |\alpha(x_j) - \alpha(x_{j-1})| \leq V^R_{(\phi,2,\alpha)}(f).$$

By continuity of ϕ and the definition of $f'_{\alpha+}$ and $f'_{\alpha-}$

$$\sum_{j=1}^{n} \phi\left(\frac{|f'_{\alpha-}(x_j) - f'_{\alpha+}(x_{j-1})|}{|\alpha(x_j) - \alpha(x_{j-1})|}\right) |\alpha(x_j) - \alpha(x_{j-1})| \leq V^R_{(\phi,2,\alpha)}(f).$$

This result holds for all partition Π of $[a, b]$. In consequence

$$V^R_{(\phi,\alpha)}(f'_\alpha) \leq V^R_{(\phi,2,\alpha)}(f).$$

Hence

$$f'_\alpha \in V^R_{(\phi,\alpha)}([a, b]).$$

By virtue of the Theorem of Medved'ev (see [26]) we conclude that $f'_\alpha \in \alpha - AC([a, b])$ and therefore there exists f''_α a.e. in $[a, b]$. Moreover

$$\int_a^b \phi\left(|f''_\alpha(t)|\right) d\alpha(t) \leq V^R_{(\phi,\alpha)}(f'_\alpha) \leq V^R_{(\phi,2,\alpha)}(f).$$

□

The next result is the reciprocal.

Theorem 2.6 *Let ϕ be a convex function which satisfy the (∞_1)-condition and $f : [a, b] \longrightarrow \mathbb{R}$. If f'_α is α-absolutely continuous and*

$$\int_a^b \phi(|f''_\alpha(t)|)d\alpha(t) < +\infty,$$

then

$$V^R_{(\phi,2,\alpha)}(f) \leq \int_a^b \phi(|f''_\alpha(t)|)d\alpha(t).$$

Proof Let $\Pi : a = x_{1,1} < x_{1,2} \leq x_{1,3} < x_{1,4} = x_{2,1} < \cdots < x_{n-1,4} = x_{n,1} < x_{n,2} \leq x_{n,3} < x_{n,4} = b$ be a block partition of $[a, b]$.

Since f'_α exists, then f is continuous in (a, b) and using the mean value Theorem, we deduce that there exists $x_j^+ \in (x_{j,3}, x_{j,4})$ and $x_j^- \in (x_{j,1}, x_{j,4})$ such that

$$f'_\alpha(x_j^-) = \frac{f(x_{j,2}) - f(x_{j,3})}{\alpha(x_{j,2}) - \alpha(x_{j,1})}$$

and

$$f_\alpha'(x_j^+) = \frac{f(x_{j,4}) - f(x_{j,3})}{\alpha(x_{j,4}) - \alpha(x_{j,3})}, \quad j = 1, 2, \ldots, n.$$

In this way we obtain the following estimation

$$\begin{aligned}
&\phi\left(\frac{\left|\frac{f(x_{j,4})-f(x_{j-3})}{\alpha(x_{j,4})-\alpha(x_{j,3})} - \frac{f(x_{j,2})-f(x_{j-1})}{\alpha(x_{j,2})-\alpha(x_{j,1})}\right|}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|}\right)|\alpha(x_{j,4}) - \alpha(x_{j,1})| \\
&= \phi\left(\frac{|f'\alpha(x_j^+) - f'\alpha(x_j^-)|}{|\alpha(x_{j,4}) - \alpha(x_{j,1}|}\right)|\alpha(x_{j,4}) - \alpha(x_{j,1})| \\
&= \phi\left(\frac{1}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|}\int_{x_j^-}^{x_j^+} |f_\alpha''(\xi)|d\alpha(\xi)\right)|\alpha(x_{j,4}) - \alpha(x_{j,1})| \\
&\leq \phi\left(\frac{1}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|}\int_{x_{j,1}}^{x_{j,4}} |f_\alpha''(\xi)|d\alpha(\xi)\right)|\alpha(x_{j,4}) - \alpha(x_{j,1})| \\
&= \phi\left(\frac{\int_{x_{j,1}}^{x_{j,4}} |f_\alpha''(\xi)|d\alpha(\xi)}{\int_{x_{j,1}}^{x_{j,4}} d\alpha(\xi)}\right)|\alpha(x_{j,4}) - \alpha(x_{j,1})|.
\end{aligned}$$

By the Jensen inequality

$$\begin{aligned}
&\leq \frac{\int_{x_{j,1}}^{x_{j,4}} \phi(|f_\alpha''(\xi)|)d\alpha(\xi)}{\int_{x_{j,1}}^{x_{j,4}} d\alpha(\xi)}|\alpha(x_{j,4}) - \alpha(x_{j,1})|. \\
&= \frac{1}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|}\left[\int_{x_{j,1}}^{x_{j,4}} \phi(|f_\alpha''(\xi)|)d\alpha(\xi)\right]|\alpha(x_{j,4}) - \alpha(x_{j,1})| \\
&= \int_{x_{j,1}}^{x_{j,4}} \phi(|f_\alpha''(\xi)|)d\alpha(\xi).
\end{aligned}$$

Next, adding for $j = 1, 2, \ldots, n$

$$\sigma^R_{(\phi,2,\alpha)}(f, \Pi) = \sum_{j=1}^{n} \phi\left(\frac{\left|\frac{f(x_{j,4})-f(x_{j-3})}{\alpha(x_{j,4})-\alpha(x_{j,3})} - \frac{f(x_{j,2})-f(x_{j-1})}{\alpha(x_{j,2})-\alpha(x_{j,1})}\right|}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|}\right)|\alpha(x_{j,4}) - \alpha(x_{j,1})|$$

$$\leq \sum_{j=1}^{n} \int_{x_{j,1}}^{x_{j,4}} \phi(|f''_\alpha(\xi)|)d\alpha(\xi)$$

$$= \int_a^b \phi(|f''_\alpha(\xi)|)d\alpha(\xi).$$

From this expression we deduce that

$$V^R_{(\phi,2,\alpha)}(f) \leq \int_a^b \phi(|f''_\alpha(\xi)|)d\alpha(\xi),$$

therefore $f \in V_{(\phi,2,\alpha)}([a, b])$. □

From the previous theorems, we deduce the following.

Corollary 2.7 *Let ϕ be a convex ϕ-function which satisfy the (∞_1)-condition and $f : [a, b] \longrightarrow \mathbb{R}$. Then the following propositions are equivalent*

1. $f \in V^R_{(\phi,2,\alpha)}([a, b])$ *if and only if* $f'_\alpha \in V^R_{(\phi,\alpha)}([a, b])$.
2. $f \in V^R_{(\phi,2,\alpha)}([a, b])$ *if and only if* $f'_\alpha \in \alpha - AC([a, b])$ *and* $\int_a^b \phi(|f'_\alpha(t)|)d\alpha(t) < +\infty$.

Moreover

$$V^R_{(\phi,2,\alpha)}(f) = V^R_{(\phi,\alpha)}(f'_\alpha) = \int_a^b \phi(|f''_\alpha(\xi)|)d\alpha(\xi).$$

Corollary 2.8 *Let ϕ be a convex ϕ-function which satisfy the (∞_1)-condition and $f \in RV^0_{(\phi,2,\alpha)}([a, b])$. Then*

$$|f|^R_{(\phi,2,\alpha)} = |f'_\alpha(a)| + \inf\{\varepsilon > 0 \ : \ \int_a^b \phi\left(\frac{|f''_\alpha(t)|}{\varepsilon}\right)d\alpha(t) \leq 1\}.$$

Proof By Corollary 2.7 and Definition 2.2 we obtain

$$|f|^R_{(\phi,2,\alpha)} = |f'_\alpha(a)| + \inf\{\varepsilon > 0 \ : \ V_{(\phi,2,\alpha)}(f/\varepsilon) \leq 1\}$$

$$=|f'_\alpha(a)| + \inf\{\varepsilon > 0 \; : \; \int_a^b \phi\left(\frac{|f''_\alpha(t)|}{\varepsilon}\right) d\alpha(t) \leq 1\}.$$

□

Remark 2.1 From Corollary 2.8 we might derive the following result:

$$\|f\|^R_{(\phi,2,\alpha)} = |f(a)| + |f'_\alpha(a)| + \inf\{\varepsilon > 0 \; : \; \int_a^b \phi\left(\frac{|f''_\alpha(t)|}{\varepsilon}\right) d\alpha(t) \leq 1\}.$$

Corollary 2.9 *Let ϕ be a convex ϕ-function which satisfy the (∞_1)-condition and $f \in RV^0_{(\phi,2,\alpha)}([a,b])$. Then*

$$|f|^R_{(\phi,2,\alpha)} = \|f'_\alpha\|^R_{(\phi,\alpha)}.$$

Proof Applying Corollary 2.7 we have

$$\begin{aligned}
|f|^R_{(\phi,2,\alpha)} =& |f'_\alpha(a)| + \inf\{\varepsilon > 0 \; : \; V^R_{(\phi,2,\alpha)}(f/\varepsilon) \leq 1\} \\
=& |f'_\alpha(a)| + \inf\{\varepsilon > 0 \; : \; V^R_{(\phi,\alpha)}(f'_\alpha/\varepsilon) \leq 1\} \\
=& \|f'_\alpha\|^R_{(\phi,2,\alpha)}.
\end{aligned}$$

□

Definition 2.5 Let ϕ be a convex ϕ-function, then

$$\begin{aligned}
&\{f : [a,b] \longrightarrow \mathbb{R} \; : \; \exists\lambda > 0 \text{ such that } \lambda f \in V^R_{(\phi,2,\alpha)}([a,b])\} \\
&= \{f : [a,b] \longrightarrow \mathbb{R} \; : \; \exists\lambda > 0 \text{ such that } V^R_{(\phi,2,\alpha)}(\lambda f) < +\infty\}
\end{aligned}$$

it is called the linear space of the $(\phi, 2, \alpha)$-bounded variation functions in the sense of Riesz, and it is denoted by $RV_{(\phi,2,\alpha)}([a,b])$.

Definition 2.6 Let ϕ be a convex ϕ-function, then

$$\begin{aligned}
&\{f : [a,b] \longrightarrow \mathbb{R} \; : \; \exists\lambda > 0 \text{ such that } \lambda f \in V^R_{(\phi,\alpha)}([a,b])\} \\
=&\{f : [a,b] \longrightarrow \mathbb{R} \; : \; \exists\lambda > 0 \text{ such that } V^R_{(\phi,\alpha)}(\lambda f) < +\infty\}
\end{aligned}$$

it is called the linear space of the (ϕ, α)-bounded variation functions in the sense of Riesz, and it is denoted by $RV_{(\phi,\alpha)}([a,b])$.

Corollary 2.10 *Let ϕ be a convex ϕ-function which satisfy the (∞_1)-condition and let $f : [a,b] \longrightarrow \mathbb{R}$, then $f \in RV_{(\phi,2,\alpha)}([a,b])$ if and only if $f'_\alpha \in RV_{(\phi,\alpha)}([a,b])$.*

Proof $f \in RV_{(\phi,2,\alpha)}([a,b])$ iff there exits $\lambda > 0$ such that $V^R_{(\phi,2,\alpha)}(\lambda f) < +\infty$. By Definition 2.5

$$\iff \exists\lambda > 0 \text{ such that } V^R_{(\phi,\alpha)}((\lambda f)'_\alpha) < +\infty$$

by Corollary 2.7

$$\iff \exists\lambda > 0 \text{ such that } V^R_{(\phi,\alpha)}(\lambda f'_\alpha) < +\infty$$
$$\iff f'_\alpha \in RV_{(\phi,\alpha)}([a,b]), \text{ by Definition 2.6.}$$

□

Definition 2.7 If ϕ is a convex ϕ-function $\|\cdot\|^R_{(\phi,2,\alpha)} : RV_{(\phi,2,\alpha)}([a,b]) \longrightarrow \mathbb{R}^+$ given by

$$\|f\|^R_{(\phi,2,\alpha)} = |f(a)| + |f - f(a)|^R_{(\phi,2,\alpha)}.$$

Lemma 2.11 *Let ϕ be a ϕ-function, then $f \in RV_{(\phi,2,\alpha)}([a,b])$ if and only if $f - f(a) \in RV^0_{(\phi,2,\alpha)}([a,b])$.*

Proof By Definition 1.1, we observe that $\sigma_{(\phi,2,\alpha)}(f - f(a), \Pi) = \sigma^R_{(\phi,2,\alpha)}(f, \Pi)$ for any partition Π of $[a,b]$ where

$$V_{(\phi,2,\alpha)}(f - f(a)) = V^R_{(\phi,2,\alpha)}(f).$$

□

Observe that

$$\begin{aligned}
\|f\|^R_{(\phi,2,\alpha)} &= |f(a)| + |f - f(a)|^R_{(\phi,2,\alpha)} \\
&= |f(a)| + |(f - f(a))'_\alpha(a)| + |f - f(a)|^0_{(\phi,2,\alpha)}, \text{ by Definition 2.2} \\
&= |f(a)| + |f'_\alpha(a)| + \inf\{\varepsilon > 0 \;:\; V^R_{(\phi,2,\alpha)}\left(\frac{f - f(a)}{\varepsilon}\right) \le 1\}, \text{ by Definition 2.1} \\
&= |f(a)| + |f'_\alpha(a)| + \inf\{\varepsilon > 0 \;:\; V^R_{(\phi,2,\alpha)}(f/\varepsilon) \le 1\}, \text{ by Lemma 2.11.}
\end{aligned}$$

3 $RV_{(\phi,2,\alpha)}([a,b])$ as a Banach algebra

In this section we will show that $RV_{(\phi,2,\alpha)}([a,b])$ is closed under the product of functions. To attain such a goal, we will use a criterion given in 1987 by L. Maligranda and W. Orlicz [17], which supplies a test to check if some function space is a Banach algebra, namely.

Lemma 3.1 (Maligranda Orlicz criterion) *Let $(X, \|\cdot\|)$ be a Banach space whose elements are bounded functions and the space is closed under multiplication of functions. Let us asssume that*

$$f \cdot g \in X \text{ and } \|fg\| \leq \|f\|_\infty \cdot \|g\| + \|f\| \cdot \|g\|_\infty$$

for any $f, g \in X$. Then the space X equipped with the norm

$$\|f\|_1 = \|f\|_\infty + \|f\|$$

is a normed Banach algebra. Also if $X \hookrightarrow B([a,b])$, then the norms $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent. Moreover, if $\|f\|_\infty \leq M\|f\|$ for $f \in X$, then $(X, \|\cdot\|_2)$ is a normed Banach algebra with $\|f\|_2 = 2M\|f\|$, $f \in X$ and the norms $\|\cdot\|_2$ and $\|\cdot\|$ are equivalent.

In [7] the first and third named authors generalized the Maligranda Orlicz Lemma, in the following way.

Theorem 3.2 (Generalized Maligranda-Orlicz's Lemma) *Let $(X, \|\cdot\|)$ be a Banach space whose elements are bounded functions, which is closed under pointwise multiplication of functions. Let us assume that $f \cdot g \in X$ such that*

$$\|fg\| \leq \|g\|_\infty \cdot \|g\| + \|f\| \cdot \|g\|_\infty + K\|f\| \cdot \|g\|, \quad K > 0.$$

Then $(X, \|\cdot\|)$ equipped with the norm

$$\|f\|_1 = \|f\|_\infty + K\|f\|, \quad f \in X$$

is a Banach algebra, if $X \hookrightarrow B([a,b])$, then $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent.

Theorem 3.3 *Let ϕ be a convex ϕ-convex function which satisfy the (∞_1)-condition. Let $f, g \in RV_{(\phi,2,\alpha)}([a,b])$, then $f \cdot g \in RV_{(\phi,2,\alpha)}([a,b])$.*

Proof Let $f, g \in RV_{(\phi,2,\alpha)}([a,b])$, then by Corollary 2.10, we have $f'_\alpha, g'_\alpha \in RV_{(\phi,\alpha)}([a,b])$ since $RV_{(\phi,2,\alpha)}([a,b]) \subset RV_{\phi,\alpha}([a,b])$ by Theorem 2.2. Also, $f, g \in RV_{(\phi,\alpha)}([a,b])$ since $RV_{(\phi,\alpha)}$ is an algebra, (see [4]), we obtain that

$$f'_\alpha g + g'_\alpha f = (fg)'_\alpha \in RV_{(\phi,\alpha)}([a,b]).$$

One more time from Theorem 2.2 we conclude that $f \cdot g \in RV_{(\phi,2,\alpha)}([a,b])$. □

Lemma 3.4 *Let ϕ be a convex ϕ-function which satisfy the (∞_1)-condition. Let $f, g \in RV^0_{(\phi,2,\alpha)}([a,b])$, then there exists $K > 0$ such that*

$$|f \cdot g|^R_{(\phi,2,\alpha)} \leq \|f\|_\infty \cdot |g|^R_{(\phi,2,\alpha)} + \|g\|_\infty \cdot |f|^R_{(\phi,2,\alpha)} + K|f|^R_{(\phi,2,\alpha)} \cdot |g|^R_{(\phi,2,\alpha)}.$$

Proof Let $f, g \in RV^0_{(\phi,2,\alpha)}([a,b])$, then by Corollary 2.9 we have

$$\begin{aligned}|f\cdot g|^R_{(\phi,2,\alpha)} &= \|(fg)'_\alpha\|^R_{(\phi,\alpha)}\\ &\leq \|fg'_\alpha\|^R_{(\phi,\alpha)} + \|f'_\alpha g\|^R_{(\phi,\alpha)}\\ &\leq |f\|_\infty \cdot \|g'_\alpha\|^R_{(\phi,\alpha)} + \|g'_\alpha\|_\infty \cdot \|f\|^R_{(\phi,\alpha)} + \|g\|_\infty \|f'_\alpha\|^R_{(\phi,\alpha)}\\ &\quad + \|f'\alpha\|_\infty \cdot \|g\|^R_{(\phi,\alpha)}\\ &= \|f\|_\infty \cdot |g|^R_{(\phi,2,\alpha)} + \|g\|_\infty \cdot |f|^R_{(\phi,2,\alpha)} + \|g'_\alpha\|_\infty \|f\|^R_{(\phi,\alpha)}\\ &\quad + \|f'_\alpha\|_\infty \cdot \|g\|^R_{(\phi,\alpha)}.\end{aligned}$$

$f(a) = 0$ implies $\|f\|^R_{(\phi,\alpha)} = |f|^R_{(\phi,\alpha)}$. Then

$$\begin{aligned}|f\cdot g|^R_{(\phi,2,\alpha)} \leq \|f\|_\infty \cdot |g|^R_{(\phi,2,\alpha)} + \|g\|_\infty \cdot |f|^R_{(\phi,2,\alpha)}\\ +\|g'_\alpha\|_\infty |f|^R_{(\phi,\alpha)} + \|f'_\alpha\|_\infty \cdot \|g\|^R_{(\phi,\alpha)}.\end{aligned}$$

Since $RV_{(\phi,\alpha)}([a,b]) \hookrightarrow B([a,b])$, there exists $M_2 > 0$ such that

$$\|f'_\alpha\|_\infty \leq M_2\|f'_\alpha\|^R_{(\phi,\alpha)} = M_2|f|^R_{(\phi,2,\alpha)}, \text{ since } f'_\alpha \in RV_{(\phi,\alpha)}([a,b]).$$

Hence,

$$\begin{aligned}|f\cdot g|^R_{(\phi,2,\alpha)} &\leq \|f\|_\infty |g|^R_{(\phi,2,\alpha)} + \|g\|_\infty |f|^R_{(\phi,2,\alpha)} + M_1M_2|g|^R_{(\phi,2,\alpha)}|f|^R_{(\phi,\alpha)}\\ &\quad + M_1M_2|f|_{(\phi,2,\alpha)}|g|^R_{(\phi,2,\alpha)}\\ &= \|f\|_\infty |g|^R_{(\phi,2,\alpha)} + \|g\|_\infty |f|^R_{(\phi,2,\alpha)} + K|g|^R_{(\phi,2,\alpha)}|f|^R_{(\phi,\alpha)}\end{aligned}$$

with $K = 2M_1M_2$. □

Lemma 3.5 *Let ϕ be a convex ϕ-function which satisfy the (∞_1)-condition, let $f, g \in RV_{(\phi,2,\alpha)}([a,b])$, then there exists $K > 0$ such that*

$$\|fg\|^R_{(\phi,2,\alpha)} \leq \|f\|_\infty \|g\|^R_{(\phi,2,\alpha)} + \|g\|_\infty \|f\|^R_{(\phi,2,\alpha)} + K\|f\|^R_{(\phi,2,\alpha)}\|g\|^R_{(\phi,2,\alpha)}.$$

Proof Let $f, g \in RV^R_{(\phi,2,\alpha)}([a,b])$, then

$$\begin{aligned}\|fg\|^R_{(\phi,2,\alpha)} &= |(fg)(a)| + |fg - (fg)(a)|^R_{(\phi,2,\alpha)} \text{By Definition 2.7}\\ &= |(fg)(a)| + \|[fg - (fg)(a)]'_\alpha\|_{(\phi,2,\alpha)}\\ &\qquad \text{since } fg - (fg)(a) \in RV^0_{(\phi,2,\alpha)}([a,b]) \text{ and by Corollary 2.9, we have}\\ &= |(fg)(a)| + \|(fg)'_\alpha\|^R_{(\phi,\alpha)}\\ &\leq |f(a)| \cdot |g(a)| + \|fg'_\alpha\|^R_{(\phi,\alpha)} + \|gf'_\alpha\|^R_{(\phi,\alpha)}\\ &\leq 2|f(a)| \cdot |g(a)| + \|f\|_\infty \|g'_\alpha\|^R_{(\phi,\alpha)} + \|g'_\alpha\|_\infty \|f\|^R_{(\phi,\alpha)} + \|g\|_\infty \|f'_\alpha\|^R_{(\phi,\alpha)}\\ &\qquad +\|f'_\alpha\|_\infty \|g\|^R_{(\phi,\alpha)}\end{aligned}$$

$$\leq \|f\|_\infty |g(a)| + |f(a)| \cdot \|g\|_\infty + \|f\|_\infty \|(g-g(a))'_\alpha\|^R_{(\phi,\alpha)}$$
$$+\|g\|_\infty \|(f-f(a))'_\alpha\|^R_{(\phi,\alpha)} + \|g'_\alpha\|_\infty \|f\|^R_{(\phi,\alpha)} + \|f'_\alpha\|_\infty \|g\|_{(\phi,\alpha)}.$$

Since $RV_{(\phi,2,\alpha)}([a,b]) \hookrightarrow RV_{(\phi,\alpha)}([a,b])$ (Theorem 2.2) there exists $M_1 > 0$ such that $\|\cdot\|^R_{(\phi,\alpha)} \leq M_1 \|\cdot\|^R_{(\phi,2,\alpha)}$ and by Remark 2.1 we have

$$\|fg\|^R_{(\phi,2,\alpha)} \leq \|f\|_\infty |g(a)| + \|g\|_\infty |f(a)| + \|f\|_\infty |g-g(a)|^R_{(\phi,2,\alpha)}$$
$$+\|g\|_\infty |f-f(a)|^R_{(\phi,2,\alpha)} + M_1 \|g'_\alpha\|_\infty \|f\|^R_{(\phi,2,\alpha)} + M_1 \|f'_\alpha\|_\infty \|g\|^R_{(\phi,2,\alpha)}.$$

Since $RV_{(\phi,\alpha)}([a,b]) \hookrightarrow B([a,b])$ (Theorem 2.2) there exists $M_2 > 0$ such that

$$\|f'_\alpha\|_\infty = \|(f-f(a))'_\alpha\|_\infty \leq M_2 \|(f-f(a))'_\alpha\|_{(\phi,\alpha)} R = M_2 |f-f(a)|^R_{(\phi,2,\alpha)}$$

(by Corollary 2.9). Since $f - f(a) \in RV_{(\phi,\alpha)}([a,b])$ (by Corollary 2.10)

$$\begin{aligned}\|f'_\alpha\|_\infty \leq & \|f\|_\infty \left(|g(a)| + |g-g(a)|^R_{(\phi,2,\alpha)}\right) + \|g\|_\infty \left(|f(a)| + |f-f(a)|^R_{(\phi,2,\alpha)}\right) \\ & + M_1 M_2 |f-f(a)|^R_{(\phi,2,\alpha)} \|g\|^R_{(\phi,2,\alpha)} + M_1 M_2 |g-g(a)|^R_{(\phi,2,\alpha)} \|f\|^R_{(\phi,2,\alpha)} \\ \leq & \|f\|_\infty \|g\|^R_{(\phi,2,\alpha)} + \|g\|_\infty \|f\|^R_{(\phi,2,\alpha)} \\ & + M_1 M_2 \left(|f(a)| + |f-f(a)|^R_{(\phi,2,\alpha)}\right) \|f\|_{(\phi,2,\alpha)} \\ = & \|f\|_\infty \|g\|^R_{(\phi,2,\alpha)} + \|g\|_\infty \|f\|_{(\phi,2,\alpha)} + K \|f\|^R_{(\phi,2,\alpha)} \|g\|^R_{(\phi,2,\alpha)}\end{aligned}$$

with $K = 2M_1 M_2$. □

Theorem 3.6 *Let ϕ be a convex ϕ-function which satisfy the (∞_1)-condition. Then $RV_{(\phi,2,\alpha)}([a,b])$ with the norm*

$$\|f\|^1_{(\phi,2,\alpha)} = \|f\|_\infty + K\|f\|^R_{(\phi,2,\alpha)}, \quad f \in RV_{(\phi,2,\alpha)}([a,b])$$

is a Banach algebra. The norms $\|\cdot\|^R_{(\phi,2,\alpha)}$ and $\|\cdot\|^1_{(\phi,2,\alpha)}$ are equivalents, that is there exists $\gamma, \delta > 0$ such that

$$\gamma \|\cdot\|^1_{(\phi,2,\alpha)} \leq \|\cdot\|^R_{(\phi,2,\alpha)} \leq \delta \|\cdot\|^1_{(\phi,2,\alpha)}.$$

Proof We just need to check the hypotheses of Theorem 3.2. □

4 Nemytskii operator on $RV_{(\phi,2,\alpha)}([a,b])$

The superposition operator, or Nemytskii operator, defined by $F(u(s)) = f(s, u(s))$, is the simplest among the nonlinear operators. It appeared for the first time in 1934 in the paper of V.V. Nemytskii [27], in connection with the study of solutions of some nonlinear integral equations. Due to its simplicity, this operator have largely studied, since it is very useful in diverse modeling applications in differential and integral equations, variational calculus, probability theory and statistics, optimization theory, among others. In the monograph [1] can be found the fundamental properties of this operators like boundedness, compactness etc, in the general setting of ideal spaces of measurable functions. In this section we study under what conditions the Nemytskii operator acts in the (ϕ, s, α)-bounded variation space.

In his 1982 paper, J. Matkowski [18] has show that the operator F generated by $f : [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ maps $Lip([a, b])$ into itself and it is globally Lipschitz, that is, there exists a positive constant K such that

$$\|F(u) - F(v)\|_{Lip([a,b])} \leq K\|u - v\|_{Lip([a,b])}$$

where $u, v \in Lip([a, b])$ if and only if there exist $g, h \in Lip([a, b])$ such that

$$f(t, x) = g(t)x + h(t) \quad \text{for } t \in [a, b],\ x \in \mathbb{R}. \tag{4.1}$$

Remark 4.1 Note that there are function spaces where the Matkowski result does no remain valid. For example, on the spaces $C([a, b])$ and $L_p([a, b])$ with $p \geq 1$ take $g : \mathbb{R} \longrightarrow \mathbb{R}$ given by $g(x) = \sin(x)$ and define $f(t, x) = g(x)$, $t \in [a, b]$, $x \in \mathbb{R}$.

The function g is Lipschitz on $\mathbb{R}$, but does not satisfy the relation (4.1), however, the operator F generated by f maps each the above spaces into itself and

$$\|F(u) - F(v)\| = \|\sin(u(\cdot)) - \sin(v(\cdot))\|_\infty \leq K\|u - v\|_\infty$$

with $u, v \in C([a, b])$, and

$$\|F(u) - F(v)\|_{Lip([a,b])} \leq \left(\int_a^b |\sin(u(t)) - \sin(v(t))|^p dt\right)^{1/p}$$
$$\leq K\|u - v\|_{Lip([a,b])}$$

with $u, v \in L_p([a, b])$, where K is Lipschitz result has been extended in the framework of various function spaces for single-valued as well as multivalued Lipschitzian Nemytskii operators c.f. [9–11,17,19–25,29–31,34]. In this section we extend the Matkowski result in the framework of the function space $RV_{(\phi,2,\alpha)}([a, b])$.

Theorem 4.1 *Let ϕ be a convex ϕ-function which satisfies the (∞_1) condition. Let $f : [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$. Then Nemyskii operator associated to f defined by*

$$F : RV_{(\phi,2,\alpha)}([a, b]) \longrightarrow \mathbb{R}$$
$$u \longmapsto F(u)$$

with $F(u) = f(t, u(t))$, $t \in [a, b]$ act on $RV_{(\phi,2,\alpha)}([a, b])$ and is globally Lipschitz, that is there exists $K > 0$ such that

$$\|F(u_1) - F(u_2)\|^R_{(\phi,2,\alpha)} \leq K\|u_1 - u_2\|^R_{(\phi,2,\alpha)}, \quad t \in [a, b]$$

if and only if there exist $g, h \in RV_{(\phi,2,\alpha)}([a,b])$ *such that*

$$f(t,y) = g(t)y + h(t), \quad t \in [a,b], \ y \in \mathbb{R}.$$

Proof From Theorem 2.2 $RV_{(\phi,2,\alpha)}([a,b]) \hookrightarrow \alpha - Lip([a,b])$, then there exists $N > 0$ such that

$$\|\cdot\|_{\alpha-Lip([a,b])} \leq N\|\cdot\|^R_{(\phi,2,\alpha)}.$$

By hypothesis, there exists $K > 0$ such that

$$\begin{aligned}\|Fu_1 - Fu_2\|_{\alpha-Lip([a,b])} \leq & N\|Fu_1 - Fu_2\|^R_{(\phi,2,\alpha)} \\ \leq & KN\|u_1 - u_2\|^R_{(\phi,2,\alpha)}, \quad u_1, u_2 \in RV_{(\phi,2,\alpha)}([a,b]).\end{aligned}$$

Let us define two particular polynomials u_1, u_2 such a way that $u_1, u_2 \in RV_{(\phi,2,\alpha)}([a,b])$.

To define that fix $t, t' \in [a,b]$, $t < t'$, $y_1, y_2, y'_1, y'_2 \in \mathbb{R}$. Let us define $u_i : [a,b] \longrightarrow \mathbb{R}$, $i = 1, 2$ by

$$\begin{aligned}u_i(s) =& \frac{y'_i - y_i}{2(\alpha(t') - \alpha(t))}(\alpha(s) - \alpha(a))^2 \\ &+ \frac{y'_i - y_i}{\alpha(t') - \alpha(t)}\left[1 - \frac{(\alpha(t') - \alpha(t))^2 - (\alpha(t) - \alpha(a))^2}{2(\alpha(t') - \alpha(t))}\right](\alpha(s) - \alpha(a)) \\ &+ y_i - \frac{y'_i - y_i}{2(\alpha(t') - \alpha(t))}(\alpha(t) - \alpha(a))^2 - \frac{y'_i - y_i}{\alpha(t') - \alpha(t)} \\ &\times \left[1 - \frac{(\alpha(t') - \alpha(t))^2 - (\alpha(t) - \alpha(a))^2}{2(\alpha(t') - \alpha(t))}\right](\alpha(t) - \alpha(a)), \quad s \in [a,b].\end{aligned}$$

The functions u_1 and u_2 satisfies the following conditions:

$$\begin{aligned}u_i(t) =& y_i, \qquad i = 1, 2 \\ u_i(t') =& y'_i, \qquad i = 1, 2.\end{aligned}$$

Moreover,

$$\begin{aligned}(u_i)'(s) =& \frac{y'_i - y_i}{\alpha(t') - \alpha(t)}(\alpha(s) - \alpha(a)) \\ &+ \frac{y'_i - y_i}{\alpha(t') - \alpha(t)}\left[1 - \frac{(\alpha(t') - \alpha(t))^2 - (\alpha(t) - \alpha(a))^2}{2(\alpha(t') - \alpha(t))}\right], \quad s \in [a,b], \ i = 1, 2\end{aligned}$$

and

$$(u_i)''_\alpha(s) = \frac{y'_i - y_i}{\alpha(t') - \alpha(t)}, \quad s \in [a,b], \ i = 1, 2.$$

Let us calculate $\|u_1 - u_2\|^R_{(\phi,2,\alpha)}$. We observe that $(u_i)'_\alpha$, $i = 1, 2$ is absolutely continuous with respect to α in $[a, b]$, and

$$\int_a^b \phi(|(u_i)''_\alpha(s)|)d\alpha(s) = \int_a^b \phi\left(\left|\frac{y'_i - y_i}{\alpha(t') - \alpha(t)}\right|\right) d\alpha(s)$$
$$= \phi\left(\left|\frac{y'_i - y_i}{\alpha(t') - \alpha(t)}\right|\right) (\alpha(b) - \alpha(a)) < +\infty, \quad i = 1, 2.$$

By Corollary 2.7 we conclude that $u_i \in RV_{(\phi,2,\alpha)}([a, b])$, $i = 1, 2$.

To calculate $\|u_1 - u_2\|^R_{(\phi,2,\alpha)}$:

$$u_1(a) - u_2(a) = y_1 - y_2 - \frac{y'_1 - y_1 - y'_2 + y_2}{\alpha(t') - \alpha(t)}(\alpha(t) - \alpha(a))\left(\frac{\alpha(t) - \alpha(t')}{2} + 1\right),$$
$$(u_1)'_\alpha - (u_2)'_\alpha = \frac{y'_1 - y_1 - y'_2 + y_2}{\alpha(t') - \alpha(t)}[2 - \alpha(t') - \alpha(t) + 2\alpha(a)].$$

One more time by Corollary 2.7

$$V^R_{(\phi,2,\alpha)}\left(\frac{u_1 - u_2}{\varepsilon}\right) = \int_a^b \phi\left(\left|\frac{(u_1 - u_2)''_\alpha(s)}{\varepsilon}\right|\right) d\alpha(s)$$
$$= \int_a^b \phi\left(\left|\frac{y'_1 - y_1 - y'_2 + y_2}{\varepsilon(\alpha(t') - \alpha(t))}\right|\right) d\alpha(s)$$
$$= \phi\left(\left|\frac{y'_1 - y_1 - y'_2 + y_2}{\varepsilon(\alpha(t') - \alpha(t))}\right|\right) (\alpha(b) - \alpha(a)).$$

Then

$$V^R_{(\phi,2,\alpha)}\left(\frac{u_1 - u_2}{\varepsilon}\right) \le 1$$
$$\iff \phi\left(\left|\frac{y'_1 - y_1 - y'_2 + y_2}{\varepsilon(\alpha(t') - \alpha(t))}\right|\right) (\alpha(b) - \alpha(a)) \le 1$$
$$\iff \left|\frac{y'_1 - y_1 - y'_2 + y_2}{\varepsilon(\alpha(t') - \alpha(t))}\right| \le \phi^{-1}\left(\frac{1}{\alpha(b) - \alpha(a)}\right)$$
$$\iff \frac{|y'_1 - y_1 - y'_2 + y_2|}{\phi^{-1}\left(\frac{1}{\alpha(b)-\alpha(a)}\right)|\alpha(t') - \alpha(t)|} \le \varepsilon$$

and then

$$\inf\left\{\varepsilon > 0 \ : \ V^R_{(\phi,2,\alpha)}\left(\frac{u_1 - u_2}{\varepsilon}\right) \le 1\right\} = \frac{|y'_1 - y_1 - y'_2 + y_2|}{\phi^{-1}\left(\frac{1}{\alpha(b)-\alpha(a)}\right)|\alpha(t') - \alpha(t)|}.$$

Since Fu_1 and Fu_2 are in $RV_{(\phi,2,\alpha)}([a,b]) \hookrightarrow \alpha - Lip([a,b])$ also $Fu_1 - Fu_2 \in \alpha - Lip([a,b])$ with $Fu_i : [a,b] \longrightarrow \mathbb{R}$ given by

$$(Fu_i)(s) = f(s, u_i(s)), \quad i = 1, 2.$$

In particular,

$$\begin{aligned}(Fu_i)(t) =& f(t, u_i(t)) = f(s, u_i(s)) \quad i = 1, 2\\ (Fu_i)(t') =& f(t', u_i(t')) = f(t', u_i(t')) \quad i = 1, 2.\end{aligned}$$

Then

$$\begin{aligned}\frac{|(Fu_1 - Fu_2)(t') - (Fu_1 - Fu_2)(t)|}{|\alpha(t') - \alpha(t)|} \leq& \|Fu_1 - Fu_2\|_{\alpha - Lip([a,b])}\\ \leq& KN\|u_1 - u_2\|^R_{(\phi,2,\alpha)}.\end{aligned}$$

Replacing

$$\begin{aligned}&\frac{|f(t', y_1') - f(t', y_2') - (f(t, y_1) - f(t, y_2))|}{|\alpha(t') - \alpha(t)|}\\ &\leq KN\left\{\left|y_1 - y_2 - \frac{y_1' - y_1 - y_2' + y_2}{\alpha(t') - \alpha(t)}(\alpha(t) - \alpha(a))\left(\frac{\alpha(t) - \alpha(t')}{2} + 1\right)\right|\right.\\ &\quad + \left|\frac{y_1' - y_2' - (y_1 - y_2)}{2(\alpha(t') - \alpha(t))}[2 - \alpha(t') - \alpha(t) + 2\alpha(a)]\right|\\ &\quad \left.+ \frac{|y_1' - y_1 - y_2' + y_2|}{\phi^{-1}\left(\frac{1}{\alpha(b) - \alpha(a)}\right)|\alpha(t') - \alpha(t)|}\right\}.\end{aligned}$$

Multiplying the inequality by $|\alpha(t') - \alpha(t)|$ and applying the triangular inequality, we obtain

$$\begin{aligned}&|f(t', y_1') - f(t', y_2') - (f(t, y_1) - f(t, y_2))|\\ &\quad \leq KN\left\{|y_1 - y_2||\alpha(t') - \alpha(t)| + |y_1' - y_2' - (y_1 - y_2)||\alpha(t)\right.\\ &\quad\quad -\alpha(a)|\left|\frac{\alpha(t) - \alpha(t')}{2} + 1\right|\\ &\quad\quad \left.+|y_1' - y_2' - (y_1 - y_2)|\left|1 - \frac{\alpha(t') - \alpha(t)}{2} + \alpha(a)\right| + \frac{|y_1' - y_1 - y_2' + y_2|}{\phi^{-1}\left(\frac{1}{|\alpha(b) - \alpha(a)|}\right)}\right\}.\end{aligned}$$

For $y \in \mathbb{R}$ the constant function $u_0(t) = y$, $t \in [a,b]$ belong to $RV_{(\phi,2,\alpha)}([a,b])$ by hypothesis the function $(Fu_0)(t) = f(t, u_0(t)) = f(t, y)$ belong to $RV_{(\phi,2,\alpha)}([a,b])$ and therefore the function $f(\cdot, y)$ is continuous in $[a,b]$. Since α is continuous $\alpha(t') \rightarrow$

$\alpha(t)$ whenever $t' \to t$, then

$$\begin{aligned}
&|f(t', y_1') - f(t', y_2') - (f(t, y_1) - f(t, y_2))| \\
&\quad \le KN|y_1' - y_2' \\
&\qquad - (y_1 - y_2)| \left\{ |\alpha(t) - \alpha(a)| + |\alpha(t) - \alpha(a) - 1| + \frac{1}{\phi^{-1}\left(\frac{1}{|\alpha(b)-\alpha(a)|}\right)} \right\} \\
&\quad \le KN|y_1' - y_2' \\
&\qquad - (y_1 - y_2)| \left\{ |\alpha(b) - \alpha(a)| + |\alpha(b) - \alpha(a) - 1| + \frac{1}{\phi^{-1}\left(\frac{1}{|\alpha(b)-\alpha(a)|}\right)} \right\}.
\end{aligned}$$

Arguing as in Theorem 3.1 in [5] we get the result.

Reciprocally, let $g, h \in RV_{(\phi,2,\alpha)}([a, b])$ such that $f(t, y) = g(t)y + h(t)$. The Nemytskii operator generated by f is given by

$$(Fu)(t) = f(t, u(t)) = g(t)u(t) + h(t), \quad t \in [a, b].$$

Since $RV_{(\phi,2,\alpha)}([a, b])$ is an algebra (Theorem 3.3) we conclude the F acts in the space $RV_{(\phi,2,\alpha)}([a, b])$. We will show that F satisfies a globally Lipschitz condition, let $u_1, u_2 \in RV_{(\phi,2,\alpha)}([a, b])$ and so

$$\begin{aligned}
\|Fu_1 - Fu_2\|^R_{(\phi,2,\alpha)} &= \|f(\cdot, u_1(\cdot)) - f(\cdot, u_2(\cdot))\|^R_{(\phi,2,\alpha)} \\
&= \|g(\cdot)u_1(\cdot) - h(\cdot) - (g(\cdot)u_2(\cdot) - h(\cdot))\|^R_{(\phi,2,\alpha)} \\
&= \|g(\cdot)[u_1(\cdot) - u_2(\cdot)]\|^R_{(\phi,2,\alpha)} = \|g(u_1 - u_2)\|^R_{(\phi,2,\alpha)}.
\end{aligned}$$

By Theorem 3.6 the norms $\|\cdot\|^R_{(\phi,2,\alpha)}$ and $\|\cdot\|^1_{(\phi,2,\alpha)}$ are equivalents, thus

$$\|g(u_1 - u_2)\|^R_{(\phi,2,\alpha)} \le \delta \|g(u_1 - u_2)\|^1_{(\phi,2,\alpha)}.$$

Since $(RV_{(\phi,2,\alpha)}([a, b]), \|\cdot\|^1_{(\phi,2,\alpha)})$ is a Banach algebra, we have

$$\begin{aligned}
\delta\|g(u_1 - u_2)\|^1_{(\phi,2,\alpha)} &\le \delta \|g\|^1_{(\phi,2,\alpha)} \|u_1 - u_2\|^1_{(\phi,2,\alpha)} \\
&\le \delta \left(\frac{1}{\gamma}\right) \|g\|^R_{(\phi,2,\alpha)} \frac{1}{\gamma} \|u_1 - u_2\|^R_{(\phi,2,\alpha)} \\
&= \frac{\delta}{\gamma^2} \|g\|^R_{(\phi,2,\alpha)} \|u_1 - u_2\|^R_{(\phi,2,\alpha)}.
\end{aligned}$$

Considering $\frac{\delta}{\gamma^2}\|g\|^R_{(\phi,2,\alpha)}$ as Lipschitz constant, it is concluded. □

Acknowledgements The authors are grateful to the referee whose comments and suggestions lead to an improvement of this article.

References

1. Appell, J., Zabrejko, P.P.: Nonlineal Superpostion Operators. Cambridge University Press, Cambridge (1990)
2. Castillo, R.E., Chaparro, H.C., Trousselot, E.: On functions of $(\phi, 2, \alpha)$-bounded variation, To appear in Proyecciones (2020)
3. Castillo, R.E., Rafeiro, H., Trousselot, E.: Embeddings on spaces of generalized bounded variation. Rev. Colombiana Mat. **48**(1), 97–109 (2014)
4. Castillo, R.E., Rafeiro, H., Trousselot, E.: A generalization for the Riesz p-variation. Rev. Colombiana Mat. **48**(1), 165–190 (2014)
5. Castillo, R.E., Rafeiro, H., Trousselot, E.: Nemytskii operator on generalized bounded variation space. Rev. Integr. Temas Mat. **32**(1), 71–90 (2014)
6. Castillo, R.E., Rafeiro, H., Trousselot, E.: space of functions with some generalization of bounded variation with some generalization of bounded variation in the sense of de la Vallée Poussin, J. Funct. Spaces, Art. ID 605380, 9pp (2015)
7. Castillo, R.E., Trousselot, E.: A generalization of the Maligranda Orlicz lemma, JIPAM J. Inequal. Pure Appl. Math., 8(**4)**, Article 115, 3 p.p (2007)
8. Castillo, R.E., Trousselot, E.: On functions of (p, α)-bounded variation. Real Anal. Exch. **34**(1), 49–60 (2009)
9. Chistyakov, V.V.: Lipschitzian superposition operators between spaces of functions of bounded generalized variation with weight. J. Appl. Anal. **6**(2), 173–186 (2000)
10. Chistyakov, V.V.: Generalized variation of mappings with applications to composition operators and multifunctions. Positivity **5**(4), 323–358 (2001)
11. Chistyakov, V.V.: Superposition operators in the algebra of functions of two variables with finite total variation. Monatsh. Math. **137**(2), 99–114 (2002)
12. de la Valée Poussin, C.J.: Sur la convergence des formulas d'interpolation entre ordonnées équidistantes. Bull. Cl. Sci. Acad. R. Belg. Série **4**, 319–410 (1908)
13. Dubois-Raymond, P.: Zur Geschite der trigonometrichen Reigen: Eine Entgegnung. H. Laupp, Tübingen (1880)
14. Jordan, C.: Sur la série de Fourier. Comptes Rendus de L'académie des Sciences, Paris **2**, 228–230 (1881)
15. Hawkins, T.: Lebesgue's Theory of Integration: Its Origins and Developments, 2nd edn. Chelsea Publishing, New York (1975)
16. Lakoto, I.: Proofs and Refutations. Cambrigde University Press, New York (1976)
17. Maligranda, L., Orlicz, W.: On some properties of functions of generalized variation. Monatshift für Mathematik **104**, 53–65 (1987)
18. Matkowski, J.: Functional equations and Nemytskii operators. Funkc. Ekvacioj ser Int. **25**, 127–132 (1982)
19. Matkowski, J.: Form of Lipschitz operators of substitution in Banach spaces of differentiable functions. Sci. Bull. Lodz Tech. Univ. **17**, 5–10 (1984)
20. Matkowski, J.: On Nemytskii operator. Math. Japon. **33**(1), 81–86 (1988)
21. Matkowski, J.: Lipschitzian composition operators in some function spaces. Nonlinear Anal. **30**(2), 719–726 (1997)
22. Matkowski, J., Merentes, N.: Characterization of globally Lipschitzian composition operators in the Banach space. Arch. Math. **28**(3–4), 181–186 (1992)
23. Matkowski, J., Miś, J.: On a characterization of Lipschitzian operators of substitution in the space. Math. Nachr. **117**, 155–159 (1984)
24. Merentes, N., Nikodem, K.: On Nemytskii operator and set-valued functions of bounded p-variation. Rad. Mat. **8**(1), 139–145 (1992)
25. Merentes, N., Rivas, S.: On characterization of the Lipschitzian composition operator between spaces of functions of bounded p-variation. Czechoslovak Math. J. **45**(4), 627–637 (1995)
26. Medvede'v, Y.T.: A generalization of certain theorem of Riesz. Uspekhi. Math. Nauk. **6**, 115–118 (1953)
27. Nemytskii, V.V.: On a class of non-linear integral equations. Mat. Sb. **41**, 655–658 (1934)
28. Riesz, F.: Untersuchungen über systeme integrierbarer funktionen. Math. Ann. **69**, 449–497 (2010)
29. Riesz, F., Nagy, B.: Functional Analysis (Translated from the Second), french edn. Ungar, New York (1955)

30. Smajdor, A., Smajdor, W.: Jensen equation and Nemytskii operator for set-valued functions. Rad. Mat. **5**, 311–320 (1989)
31. Smajdor, W.: Note on Jensen and Pexider functional equations. Demonstratio Math. **32**(2), 363–376 (1999)
32. Vitali, G.: Salle Funanzioni Integrali. Atti dela Accademia delle Scienze Fisiche, Matematiche e Naturali **41**, 1021–1034 (1905)
33. Vo'lpert, A.I., Hudjeav, S.I.: Analysis in Class of Discontinuous Functions and Equations of Mathematical Physics, Mechanics: Analysis, 8. Martinus Nijhoff Publisher, Dordrecht (1985)
34. Zawadzka, G.: On Lipschitzian operators of substitution in the space of set-valued functions of bounded variation. Rad. Mat. **6**, 279–293 (1990)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.