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Microlocal regularity of nonlinear PDE in quasi-homogeneous Fourier–Lebesgue spaces

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Abstract
We study the continuity in weighted Fourier–Lebesgue spaces for a class of pseudodifferential operators, whose symbol has finite Fourier–Lebesgue regularity with respect to x and satisfies a quasi-homogeneous decay of derivatives with respect to the ξ variable. Applications to Fourier–Lebesgue microlocal regularity of linear and nonlinear partial differential equations are given.

Keywords Microlocal analysis · Pseudodifferential operators · Fourier–Lebesgue spaces

Mathematics Subject Classification 35J60 · 35J62 · 35S05

1 Introduction

In [13] we studied inhomogeneuos local and microlocal propagation of singularities of generalized Fourier–Lebesgue type for a class of semilinear partial differential equations (shortly written PDE); other results on the topic may be found in [6,18, 19]. The present paper is a natural continuation of the same subject, where Fourier–Lebesgue microlocal regularity for nonlinear PDE is considered. To introduce the problem, let us first consider the following general equation

$$F(x, \partial^\alpha u)_{\alpha\in\mathcal{I}} = 0, \tag{1}$$

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where $\mathcal{I}$ is a finite set of multi-indices $\alpha \in \mathbb{Z}_+^n$, $F(x, \zeta) \in C^\infty(\mathbb{R}^n \times \mathbb{C}^N)$ is a nonlinear function of $x \in \mathbb{R}^n$ and $\zeta = (\zeta^\alpha)_{\alpha \in \mathcal{I}} \in \mathbb{C}^N$. In order to study the regularity of solutions of (1), we can move the investigation to the *linearized equations* obtained from differentiation with respect to x_j

$$\sum_{\alpha \in \mathcal{I}} \frac{\partial F}{\partial \zeta^\alpha}(x, \partial^\beta u)_{\beta \in \mathcal{I}} \partial^\alpha \partial_{x_j} u = -\frac{\partial F}{\partial x_j}(x, \partial^\beta u)_{\beta \in \mathcal{I}}, \quad j = 1, \ldots, n.$$

Notice that the regularity of the coefficients $a_\alpha(x) := \frac{\partial F}{\partial \zeta^\alpha}(x, \partial^\beta u)_{\beta \in \mathcal{I}}$ depends on some a priori smoothness of the solution $u = u(x)$ and the nonlinear function $F(x, \zeta)$. This naturally leads to the study of linear PDE whose coefficients have only limited regularity, in our case they will belong to some generalized Fourier–Lebesgue space.

Results about local and microlocal regularity for semilinear and nonlinear PDE in Sobolev and Besov framework may be found in [7,12].

Failing of any symbolic calculus for pseudodifferential operators with symbols $a(x, \xi)$ with limited smoothness in x, one needs to refer to paradifferential calculus of Bony–Meyer [2,17] or decompose the non smooth symbols according to the general technique introduced by M.Taylor in [23, Proposition 1.3 B]; here we will follow this second approach. By the way both methods rely on the dyadic decomposition of distributions, based on a partition of the frequency space $\mathbb{R}^n_\xi$ by means of suitable family of crowns, see again Bony [2].

In this paper we consider a natural framework where such a decomposition method can be adapted, namely we deal with symbols which exhibit a behavior at infinity of quasi-homogeneous type, called in the following *quasi-homogeneous symbols*. When the behavior of symbols at infinity does not satisfy any kind of homogeneity, the dyadic decomposition method seems to fail.

In general the technique of Taylor quoted above splits the symbols $a(x, \xi)$ with limited smoothness in x into

$$a(x, \xi) = a^\#(x, \xi) + a^\natural(x, \xi). \tag{2}$$

While $a^\natural(x, \xi)$ keeps the same regularity of $a(x, \xi)$, with a slightly improved decay at infinitive, $a^\#(x, \xi)$ is a smooth symbols of type $(1, \delta)$, with $\delta > 0$.

From Sugimoto–Tomita [21], it is known that, in general, pseudodifferential operators with symbol in $S^0_{1,\delta}$, are not bounded on modulation spaces $M^{p,q}$ as long as $0 < \delta \leq 1$ and $q \neq 2$. Since the Fourier–Lebesgue and modulation spaces are *locally* the same, see [14] for details, it follows from [21] that the operators $a^\#(x, D)$ are generally unbounded on Fourier–Lebesgue spaces, when the exponent is different of 2. We are able to avoid this difficulty by carefully analyzing the behavior of the term $a^\#(x, \xi)$ as described in the next Sects. 5, 6.

In the first section all the main results of the paper are presented. The proofs are postponed in the subsequent sections. Precisely in Sect. 3 a generalization to the quasi-homogeneous framework of the characterization of Fourier–Lebesgue spaces, by means of dyadic decomposition is detailed. Section 4 is completely devoted to the proof of Thoerem 1. The symbolic calculus of pseudodifferential operators with smooth symbols is developed in Sect. 5, while Sect. 6 is devoted to the generalization of the Taylor splitting technique. In the last section we study the microlocal behavior of pseudodifferential operators with smooth symbols, jointly with their applications to nonlinear PDE.

2 Main results

2.1 Notation

In this preliminary section we give the main definitions and notation most frequently used in the paper. $\mathbb{R}_+$ and $\mathbb{N}$ are respectively the sets of strictly positive real and integer numbers. For $M=(\mu_1,\dots,\mu_n)\in\mathbb{R}^n_+$, $\xi\in\mathbb{R}^n$ we define:

$$\langle\xi\rangle_M := \left(1+|\xi|^2_M\right)^{1/2} \quad (M-\ \mathit{weight}), \tag{3}$$

where

$$|\xi|^2_M := \sum_{j=1}^n |\xi_j|^{2\mu_j} \quad (M-\mathit{norm}). \tag{4}$$

For $t>0$ and $\alpha\in\mathbb{Z}^n_+$, we set

$$\begin{aligned} t^{1/M}\xi &:= (t^{1/\mu_1}\xi_1,\dots,t^{1/\mu_n}\xi_n)\,;\\ \langle\alpha,1/M\rangle &:= \sum_{j=1}^n \alpha_j/\mu_j\,;\\ \mu_* := \min_{1\le j\le n}\mu_j, &\qquad \mu^* := \max_{1\le j\le n}\mu_j. \end{aligned} \tag{5}$$

We call μ_* and μ^* respectively the *minimum* and the *maximum order* of $\langle\xi\rangle_M$; furthermore, we will refer to $\langle\alpha,1/M\rangle$ as the *M-order* of α. In the case of $M=(1,\dots,1)$, (4) reduces to the *Euclidean* norm $|\xi|$, and the M-weight (3) reduces to the standard *homogeneous weight* $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$.

The following properties can be easily proved, see [8] and the references therein.

Lemma 1 *For any $M\in\mathbb{R}^n_+$, there exists a suitable positive constant C such that the following hold for any $\xi\in\mathbb{R}^n$:*

$$\frac{1}{C}\langle\xi\rangle^{\mu_*} \le \langle\xi\rangle_M \le C\langle\xi\rangle^{\mu^*},\ \mathit{Polynomial\ growth}; \tag{6}$$

$$|\xi+\eta|_M \le C\left\{|\xi|_M+|\eta|_M\right\},\ M-\mathit{sub\text{-}additivity}; \tag{7}$$

$$|t^{1/M}\xi|_M = t|\xi|_M, t>0,\ M-\mathit{homogeneity}. \tag{8}$$

For ϕ in the space of *rapidly decreasing functions* $\mathcal{S}(\mathbb{R}^n)$, the Fourier transform is defined by $\hat\phi(\xi)=\mathcal{F}\phi(\xi)=\int e^{-ix\cdot\xi}\phi(x)\,dx$, $x\cdot\xi=\sum_{j=1}^n x_j\xi_j$; $\hat u=\mathcal{F}u$, defined by $\langle\hat u,\phi\rangle=\langle u,\hat\phi\rangle$, is its analogous in the dual space of *tempered distributions* $\mathcal{S}'(\mathbb{R}^n)$

2.2 Pseudodifferential operators with symbols in Fourier–Lebesgue spaces

Definition 1 For $s \in \mathbb{R}$ and $p \in [1, +\infty]$ we denote by $\mathcal{F}L^p_{s,M}$ the class of all $u \in \mathcal{S}'(\mathbb{R}^n)$ such that $\hat{u}$ is a measurable function in $\mathbb{R}^n$ and $\langle\cdot\rangle^s_M \hat{u} \in L^p(\mathbb{R}^n)$. $\mathcal{F}L^p_{s,M}$, endowed with the natural norm

$$\|u\|_{\mathcal{F}L^p_{s,M}} := \|\langle\cdot\rangle^s_M \hat{u}\|_{L^p}, \tag{9}$$

is a Banach space, said *M-homogeneous Fourier–Lebesgue space* of order s and exponent p.

Notice that for $p = 2$, Plancherel's Theorem yields that $\mathcal{F}L^2_{s,M}$ reduces to the M-homogeneous *Sobolev space* of order s, see [8] for details; in this case $\mathcal{F}L^2_{s,M}$ inherits from $L^2(\mathbb{R}^n)$ the structure of Hilbert space, with inner product $(u, v)_{\mathcal{F}L^2_{s,M}} := (\langle\cdot\rangle^s_M \hat{u}, \langle\cdot\rangle^s_M \hat{v})_{L^2}$.

In the case $M = (1, \dots, 1)$, $\mathcal{F}L^p_{s,M}$ reduces to the *homogeneous* Fourier–Lebesgue space $\mathcal{F}L^p_s$ and, in particular, we set $\mathcal{F}L^p := \mathcal{F}L^p_0$.

The *pseudodifferential operator* $a(x, D)$ with symbol $a(x, \xi) \in \mathcal{S}'(\mathbb{R}^{2n})$ and standard Kohn–Nirenberg quantization is the bounded linear map

$$\begin{aligned} a(x, D) : \mathcal{S}(\mathbb{R}^n) &\to \mathcal{S}'(\mathbb{R}^n) \\ u &\to \ a(x, D)u(x) := (2\pi)^{-n} \int e^{ix\cdot\xi} a(x, \xi)\widehat{u}(\xi)d\xi, \end{aligned} \tag{10}$$

where the integral above must be understood in the distributional sense.

We introduce here some classes of symbols $a(x, \xi)$, of M-homogeneous type, with limited Fourier–Lebesgue smoothness with respect to the space variable x.

Definition 2 For $m, r \in \mathbb{R}$, $\delta \in [0, 1]$, $p \in [1, +\infty]$ and $N \in \mathbb{N}$, we denote by $\mathcal{F}L^p_{r,M} S^m_{M,\delta}(N)$ the set of $a(x, \xi) \in \mathcal{S}'(\mathbb{R}^{2n})$ such that for all $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| \le N$, the map $\xi \mapsto \partial^\alpha_\xi a(\cdot, \xi)$ is measurable in $\mathbb{R}^n$ with values in $\mathcal{F}L^p_{r,M} \cap \mathcal{F}L^1$ and satisfies for any $\xi \in \mathbb{R}^n$ the following estimates

$$\|\partial^\alpha_\xi a(\cdot, \xi)\|_{\mathcal{F}L^1} \le C\langle\xi\rangle^{m-\langle\alpha,1/M\rangle}_M, \tag{11}$$

$$\|\partial^\alpha_\xi a(\cdot, \xi)\|_{\mathcal{F}L^p_{r,M}} \le C\langle\xi\rangle_M^{m-\langle\alpha,1/M\rangle+\delta\left(r-\frac{n}{\mu_* q}\right)}, \tag{12}$$

where C is a suitable positive constant and q is the conjugate exponent of p.

When $\delta = 0$, we will write for shortness $\mathcal{F}L^p_{r,M} S^m_M(N)$.

The first result concerns with the Fourier–Lebesgue boundedness of pseudodifferential operators with symbol in $\mathcal{F}L^p_{s,M} S^m_{M,\delta}(N)$.

Theorem 1 *Consider $p \in [1, +\infty]$, q its conjugate exponent, $r > \frac{n}{\mu_* q}$, $\delta \in [0, 1]$, $m \in \mathbb{R}$, $N > n + 1$ and $a(x, \xi) \in \mathcal{F}L^p_{r,M} S^m_{M,\delta}(N)$. Then for all s satisfying*

$$(\delta - 1)\left(r - \frac{n}{\mu_* q}\right) < s < r$$

the pseudodifferential operator $a(x, D)$ extends to a bounded operator

$$a(x, D) : \mathcal{F}L^p_{s+m,M} \to \mathcal{F}L^p_{s,M}. \tag{13}$$

If $\delta < 1$ then the above continuity property holds true also for $s = r$.

The proof is given in the next Sect.4.

Remark 1 Observe that in the case of $\delta = 0$, the above result was already proved in [13, Proposition 6], where a much more general setting than the framework of M-homogeneous symbols was considered and very weak growth conditions on symbols with respect to ξ were assumed.

2.3 *M*-homogeneous smooth symbols

Smooth symbols satisfying M-quasi-homogenous decay of derivatives at infinity are useful for the study of microlocal propagation of singularities for pseudodifferential operators with non smooth symbols and nonlinear PDE.

Definition 3 For $m \in \mathbb{R}$ and $\delta \in [0, 1]$, $S^m_{M,\delta}$ is the class of the functions $a(x, \xi) \in C^\infty(\mathbb{R}^{2n})$ such that for all $\alpha, \beta \in \mathbb{Z}^n_+$, and $x, \xi \in \mathbb{R}^n$

$$|\partial^\alpha_\xi \partial^\beta_x a(x, \xi)| \le C_{\alpha,\beta} \langle \xi \rangle_M^{m-\langle \alpha, 1/M \rangle + \delta \langle \beta, 1/M \rangle}, \tag{14}$$

for a suitable constant $C_{\alpha,\beta}$.

In the following, we set for shortness $S_M := S_{M,0}$. Notice that for any $\delta \in [0, 1]$ we have $\bigcap_{m\in\mathbb{R}} S^m_{M,\delta} \equiv S^{-\infty}$, where $S^{-\infty}$ denotes the set of the functions $a(x, \xi) \in C^\infty(\mathbb{R}^{2n})$ such that for all $\mu > 0$ and $\alpha, \beta \in \mathbb{Z}^n_+$

$$|\partial^\alpha_\xi \partial^\beta_x a(x, \xi)| \le C_{\mu,\alpha,\beta} \langle \xi \rangle^{-\mu}, \quad x, \xi \in \mathbb{R}^n, \tag{15}$$

for a suitable positive constant $C_{\mu,\alpha,\beta}$.

We recall that a pseudodifferential operator $a(x, D)$ with symbol $a(x, \xi) \in S^{-\infty}$ is *smoothing*, namely it extends as a linear bounded operator from $\mathcal{S}'(\mathbb{R}^n)$ ($\mathcal{E}'(\mathbb{R}^n)$) to $\mathcal{P}(\mathbb{R}^n)$ ($\mathcal{S}(\mathbb{R}^n)$), where $\mathcal{P}(\mathbb{R}^n)$ and $\mathcal{E}'(\mathbb{R}^n)$ are respectively the space of smooth functions polynomially bounded together with their derivatives and the space of compactly supported distributions.

As long as $0 \le \delta < \mu_*/\mu^*$, for the M-homogeneous classes $S^m_{M,\delta}$ a complete symbolic calculus is available, see e.g. Garello–Morando [9,10] for details.

Pseudodifferential operators with symbol in S^0_M are known to be locally bounded on Fourier–Lebesgue spaces $\mathcal{F}L^p_{s,M}$ for all $s \in \mathbb{R}$ and $1 \le p \le +\infty$, see e.g. Tachizawa [22] and Rochberg–Tachizawa [20]. For continuity of Fourier Integral Operators on Fourier–Lebesgue spaces see [4]. On the other hand, by easily adapting the arguments used in the homogeneous case $M = (1, \dots, 1)$ by Sugimoto–Tomita [21], it is known that pseudodifferential operators with symbol in $S^0_{M,\delta}$ are not locally bounded on $\mathcal{F}L^p_{s,M}$, as long as $0 < \delta \le 1$ and $p \ne 2$.

For this reason we introduce suitable subclasses of M-homogeneous symbols in $S^m_{M,\delta}$, $\delta \in [0, 1]$, whose related pseudodifferential operators are (locally) well-behaved on weighted Fourier–Lebesgue spaces. These symbols will naturally come into play in the splitting method presented in Sect. 6 and used in Sect. 7 to derive local and microlocal Fourier–Lebesgue regularity of linear PDE with non smooth coefficients.

In view of such applications, it is useful that the vector $M = (\mu_1, \dots, \mu_n)$ has strictly positive integer components. Let us assume it for the rest of Sect. 2, unless otherwise explicitly stated.

In the following $t_+ := \max\{t, 0\}$, $[t] := \max\{n \in \mathbb{Z}; n \le t\}$ are respectively the *positive part* and the *integer part* of $t \in \mathbb{R}$.

Definition 4 For $m \in \mathbb{R}$, $\delta \in [0, 1]$ and $\kappa > 0$ we denote by $S^m_{M,\delta,\kappa}$ the class of all functions $a(x, \xi) \in C^\infty(\mathbb{R}^{2n})$ such that for $\alpha, \beta \in \mathbb{Z}^n_+$ and $x, \xi \in \mathbb{R}^n$

$$|\partial^\alpha_\xi \partial^\beta_x a(x, \xi)| \le C_{\alpha,\beta} \langle \xi \rangle_M^{m - \langle \alpha, 1/M \rangle + \delta(\langle \beta, 1/M \rangle - \kappa)_+}, \quad \text{if } \langle \beta, 1/M \rangle \ne \kappa, \tag{16}$$

$$|\partial^\alpha_\xi \partial^\beta_x a(x, \xi)| \le C_{\alpha,\beta} \langle \xi \rangle_M^{m - \langle \alpha, 1/M \rangle} \log\left(1 + \langle \xi \rangle^\delta_M\right), \quad \text{if } \langle \beta, 1/M \rangle = \kappa, \tag{17}$$

holds with some positive constant $C_{\alpha,\beta}$.

Remark 2 It is easy to see that for any $\kappa > 0$, the symbol class $S^m_{M,\delta,\kappa}$ defined above is included in $S^m_{M,\delta}$ for all $m \in \mathbb{R}$ and $\delta \in [0, 1]$ (notice in particular that $S^m_{M,0,\kappa} \equiv S^m_{M,0} \equiv S^m_M$ whatever is $\kappa > 0$). Compared to Definition 3, symbols in $S_{M,\delta,\kappa}$ display a better behavior face to the growth at infinity of derivatives; the loss of decay $\delta\langle \beta, 1/M \rangle$, connected to the x derivatives when $\delta > 0$, does not occur when the M- order of β is less than κ; for the subsequent derivatives the loss is decreased of the fixed amount κ.

Since for $M = (\mu_1, \dots, \mu_n)$, with positive integer components, the M-order of any multi-index $\alpha \in \mathbb{Z}^n_+$ is a rational number, we notice that symbol derivatives never exhibit the "logarithmic growth" (17) for an irrational $\kappa > 0$.

Theorem 2 *Assume that*

$$\kappa > [n/\mu_*] + 1 \tag{18}$$

Then for all $p \in [1, +\infty]$ a pseudodifferential operator with symbol $a(x, \xi) \in S^m_{M,\delta,\kappa}$, satisfying the localization condition

$$\operatorname{supp} a(\cdot, \xi) \subseteq \mathcal{K}, \quad \forall \xi \in \mathbb{R}^n, \tag{19}$$

for a suitable compact set $\mathcal{K} \subset \mathbb{R}^n$, extends as a linear bounded operator

$$a(x, D) : \mathcal{F}L^p_{s+m,M} \to \mathcal{F}L^p_{s,M}, \quad \forall s \in \mathbb{R}, \quad \textit{if } 0 \le \delta < 1, \tag{20}$$

$$a(x, D) : \mathcal{F}L^p_{s+m,M} \to \mathcal{F}L^p_{s,M}, \quad \forall s > 0, \quad \textit{if } \delta = 1. \tag{21}$$

The proof of Theorem 2 is postponed to Sect. 5.3.

Taking $\delta = 0$, we directly obtain the boundedness property (20), for any pseudodifferential operator with symbol in S^m_M.

The following result concerning the Fourier multipliers readily follows from Hölder's inequaltity.

Proposition 1 *Let a tempered distribution $a(\xi) \in \mathcal{S}'(\mathbb{R}^n)$ satisfy*

$$\langle \xi \rangle_M^{-m} a(\xi) \in L^\infty(\mathbb{R}^n)$$

for $m \in \mathbb{R}$. Then the Fourier multiplier $a(D)$ extends as a linear bounded operator from $\mathcal{F}L^p_{s+m,M}$ to $\mathcal{F}L^p_{s,M}$, for all $p \in [1, +\infty]$ and $s \in \mathbb{R}$.

2.4 Microlocal propagation of Fourier–Lebesgue singularities

Consider a vector $M = (\mu_1, \ldots, \mu_n) \in \mathbb{N}^n$ and set $T^\circ\mathbb{R}^n := \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\})$.

We say that a set $\Gamma_M \subset \mathbb{R}^n \backslash \{0\}$ is *M-conic*, if $t^{1/M}\xi \in \Gamma_M$ for any $\xi \in \Gamma_M$ and $t > 0$.

Definition 5 For $s \in \mathbb{R}$, $p \in [1, +\infty]$, $u \in \mathcal{S}'(\mathbb{R}^n)$, we say that $(x_0, \xi^0) \in T^\circ\mathbb{R}^n$ does not belong to the M-conic *wave front set* $WF_{\mathcal{F}L^p_{s,M}} u$, if there exist $\phi \in C_0^\infty(\mathbb{R}^n)$, $\phi(x_0) \neq 0$, and a symbol $\psi(\xi) \in S^0_M$, satisfying $\psi(\xi) \equiv 1$ on $\Gamma_M \cap \{|\xi|_M > \varepsilon_0\}$, for suitable M-conic neighborhood $\Gamma_M \subset \mathbb{R}^n \backslash \{0\}$ of ξ^0 and $0 < \varepsilon_0 < |\xi^0|_M$, such that

$$\psi(D)(\phi u) \in \mathcal{F}L^p_{s,M}. \tag{22}$$

We say in this case that u is $FL^p_{s,M}-$ microlocally regular at the point (x_0, ξ^0) and we write $u \in \mathcal{F}L^p_{s,M,\mathrm{mcl}}(x_0, \xi^0)$.

We say that $u \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $\mathcal{F}L^p_{s,M,\mathrm{loc}}(x_0)$ if there exists a smooth function $\phi \in C_0^\infty(\mathbb{R}^n)$ satisfying $\phi(x_0) \neq 0$ such that

$$\phi u \in \mathcal{F}L^p_{s,M}.$$

Remark 3 In view of Definition 1, it is easy to verify that $u \in \mathcal{F}L^p_{s,M,\mathrm{mcl}}(x_0, \xi^0)$ if and only if

$$\chi_{\varepsilon_0, \Gamma_M} \langle \cdot \rangle^r_M \widehat{\phi u} \in L^p(\mathbb{R}^n), \tag{23}$$

where ϕ and Γ_M are considered as in Definition 5 and $\chi_{\varepsilon_0, \Gamma_M}$ is the characteristic function of $\Gamma_M \cap \{|\xi|_M > \varepsilon_0\}$.

Definition 6 We say that a symbol $a(x, \xi) \in S^m_{M,\delta}$ is *microlocally M-elliptic* at $(x_0, \xi^0) \in T^\circ\mathbb{R}^n$ if there exist an open neighborhood U of x_0 and an M-conic open neighborhood Γ_M of ξ^0 such that for $c_0 > 0$, $\rho_0 > 0$:

$$|a(x, \xi)| \geq c_0 \langle \xi \rangle^m_M, \quad (x, \xi) \in U \times \Gamma_M, \quad |\xi|_M > \rho_0. \tag{24}$$

Moreover the *characteristic set* of $a(x,\xi)$ is $\mathrm{Char}(a) \subset T^\circ\mathbb{R}^n$ defined by

$$(x_0,\xi^0) \in T^\circ\mathbb{R}^n \backslash \mathrm{Char}(a) \;\; \Leftrightarrow \;\; a \text{ is microlocally M-elliptic at } (x_0,\xi^0). \tag{25}$$

Theorem 3 *For* $0 \le \delta < \mu_*/\mu^*$, $\kappa > [n/\mu_*]+1$, $m \in \mathbb{R}$, $a(x,\xi) \in S^m_{M,\delta,\kappa}$ *and* $u \in \mathcal{S}'(\mathbb{R}^n)$*, the following inclusions*

$$WF_{\mathcal{F}L^p_{s,M}}(a(x,D)u) \subset WF_{\mathcal{F}L^p_{s+m,M}}(u) \subset WF_{\mathcal{F}L^p_{s,M}}(a(x,D)u) \cup \mathrm{Char}(a)$$

hold true for every $s \in \mathbb{R}$ *and* $p \in [1,+\infty]$.

The proof of Theorem 3 will be given in Sect. 7.3.

2.5 Linear PDE with non smooth coefficients

In this section we discuss the M-homogeneous Fourier–Lebesgue microlocal regularity for linear PDE of the type

$$a(x,D)u := \sum_{\langle\alpha,1/M\rangle\le 1} c_\alpha(x)D^\alpha u = f(x), \tag{26}$$

where $D^\alpha := (-i)^{|\alpha|}\partial^\alpha$, while the coefficients c_α, as well as the source f in the right-hand side, are assumed to have suitable *local* M-homogeneous Fourier–Lebesgue regularity.[1]

Let $(x_0,\xi^0) \in T^\circ\mathbb{R}^n$, $p \in [1,+\infty]$ and $r > \frac{n}{\mu_* q} + \left[\frac{n}{\mu_*}\right] + 1$ (where q is the conjugate exponent of p) be given. We make on $a(x,D)$ in (26) the following assumptions:

(i) $c_\alpha \in \mathcal{F}L^p_{r,M,\mathrm{loc}}(x_0)$ for $\langle\alpha,1/M\rangle \le 1$;

(ii) $a_M(x_0,\xi^0) \ne 0$, where $a_M(x,\xi) := \sum_{\langle\alpha,1/M\rangle=1} c_\alpha(x)\xi^\alpha$ is the M*-principal symbol* of $a(x,D)$.

Arguing on continuity and M-homogeneity in ξ of $a_M(x,\xi)$, it is easy to prove that, for suitable open neighborhood $U \subset \mathbb{R}^n$ of x_0 and open M-conic neighborhood $\Gamma_M \subset \mathbb{R}^n\backslash\{0\}$ of ξ^0

$$a_M(x,\xi) \ne 0, \quad \text{for } (x,\xi) \in U \times \Gamma_M. \tag{27}$$

Theorem 4 *Consider* $(x_0,\xi^0) \in T^\circ\mathbb{R}^n$, $p \in [1,+\infty]$ *and* q *its conjugate exponent,* $r > \frac{n}{\mu_* q} + \left[\frac{n}{\mu_*}\right] + 1$ *and* $0 < \delta < \mu_*/\mu^*$. *Assume moreover that*

$$1 + (\delta - 1)\left(r - \frac{n}{\mu_* q}\right) < s \le r + 1. \tag{28}$$

[1] Without loss of generality, we assume that derivatives involved in the expression of the linear partial differential operator $a(x,D)$ in the left-hand side of (26) have M-order not larger than one, since for any finite set $\mathcal{A}$ of multi-indices $\alpha \in \mathbb{Z}^n_+$ it is always possible selecting a vector $M = (\mu_1,\dots,\mu_n) \in \mathbb{R}^n_+$ so that $\langle\alpha,1/M\rangle \le 1$ for all $\alpha \in \mathcal{A}$.

Let $u \in \mathcal{F}L^p_{s-\delta\left(r-\frac{n}{\mu * q}\right),M,\mathrm{loc}}(x_0)$ *be a solution of the equation (26), with given source* $f \in \mathcal{F}L^p_{s-1,M,\mathrm{mcl}}(x_0,\xi^0)$. *Then* $u \in \mathcal{F}L^p_{s,M,\mathrm{mcl}}(x_0,\xi^0)$, *that is*

$$WF_{\mathcal{F}L^p_{s,M}}(u) \subset WF_{\mathcal{F}L^p_{s-1,M}}(f) \cup \mathrm{Char}(a). \tag{29}$$

The proof of Theorem 4 is postponed to Sect. 7.4. We end up by illustrating a simple application of Theorem 4.

Example. Consider the linear partial differential operator in $\mathbb{R}^2$

$$P(x,D) = c(x)\partial_{x_1} + i\partial_{x_1} - \partial^2_{x_2}, \tag{30}$$

where

$$c(x) := \frac{x_1^{k_1}}{k_1!}\frac{x_2^{k2}}{k_2!}e^{-a_1x_1}e^{-a_2x_2}H(x_1)H(x_2), \quad x = (x_1,x_2) \in \mathbb{R}^2,$$

being $H(t) = \chi_{(0,\infty)}(t)$ the Heaviside function, k_1, k_2 some positive integers and a_1, a_2 positive real numbers.

It tends out that $c \in L^1(\mathbb{R}^2)$ and a direct computation gives:

$$\widehat{c}(\xi) = \frac{1}{(a_1+i\xi_1)^{k_1+1}(a_2+i\xi_2)^{k_2+1}}, \quad \xi = (\xi_1,\xi_2) \in \mathbb{R}^2.$$

Let us consider the vector $M = (1,2)$ and the related M-weight function $\langle\xi\rangle_M := (1+\xi_1^2+\xi_2^4)^{1/2}$.

For any $p \in [1,+\infty]$ and $r > 2/q+3$, $\frac{1}{p}+\frac{1}{q} = 1$, one easily proves, for a suitable constant $C = C(a_1,a_2,k_1,k_2,r)$

$$\langle\xi\rangle_M^r|\widehat{c}(\xi)| \le \frac{C}{(1+|\xi_1|)^{k_1+1-r}(1+|\xi_2|)^{k_2+1-2r}},$$

thus $c \in \mathcal{F}L^p_{r,M}(\mathbb{R}^2)$, provided that k_1, k_2 satisfy

$$k_1 > r - 1/q \quad \text{and} \quad k_2 > 2r - 1/q. \tag{31}$$

Then, under condition (31), the symbol $P(x,\xi) = ic(x)\xi_1 - \xi_1 + \xi_2^2$ of the operator $P(x,D)$ defined in (30) belongs to $\mathcal{F}L^p_{r,M}S^1_M$, cf. Definition 2.

Let us set $\Omega := \mathbb{R}^2\backslash\mathbb{R}^2_+$. Since $|P(x,\xi)|^2 = c^2(x)\xi_1^2 + (-\xi_1+\xi_2^2)^2$, the characteristic set of P is just $\mathrm{Char}(P) = \Omega \times \{(\xi_1,\xi_2) \in \mathbb{R}^2\backslash\{(0,0)\} : \xi_1 = \xi_2^2\}$ (cf. Definition 6) or, equivalently, P is microlocally M-elliptic at a point $(x_0,\xi^0) = (x_{0,1},x_{0,2},\xi_1^0,\xi_2^0) \in T^\circ\mathbb{R}^2$ if and only if

$$x_{0,1} > 0, \;\; x_{0,2} > 0 \quad \text{or} \quad \xi_1^0 \neq (\xi_2^0)^2.$$

Applying Theorem 4, for any such a point (x_0, ξ^0) we have

$$\begin{matrix} u \in \mathcal{F}L^p_{s-\delta\left(r-\frac{2}{q}\right),M,\mathrm{loc}}(x_0) \\ P(x,D)u \in \mathcal{F}L^p_{s-1,M,\mathrm{mcl}}(x_0,\xi^0) \end{matrix} \quad \Rightarrow \quad u \in \mathcal{F}L^p_{s,M,\mathrm{mcl}}(x_0,\xi^0),$$

as long as $0 < \delta < 1/2$ and $1 + (\delta - 1)\left(r - \frac{2}{q}\right) < s \leq r + 1$.

2.6 Quasi-linear PDE

In the last two sections, we consider few applications to the study of M-homogeneous Fourier–Lebesgue singularities of solutions to certain classes of nonlinear PDEs.

Let us start with the *M-quasi-linear* equations. Namely consider

$$\sum_{\langle\alpha,1/M\rangle\leq 1} a_\alpha(x, D^\beta u)_{\langle\beta,1/M\rangle\leq 1-\epsilon} D^\alpha u = f(x), \tag{32}$$

where $a_\alpha = a_\alpha(x, D^\beta u)$ are given suitably regular functions of x and partial derivatives of the unknown u with M-order $\langle\beta, 1/M\rangle$ less than or equal to $1-\epsilon$, for a given $0 < \epsilon \leq 1$, and where the source $f = f(x)$ is sufficiently smooth.

We define the *M-principal part* of the differential operator in the left-hand side of (32) by

$$A_M(x,\xi,\zeta) := \sum_{\langle\alpha,1/M\rangle=1} a_\alpha(x,\zeta)\xi^\alpha, \tag{33}$$

where $x, \xi \in \mathbb{R}^n$, $\zeta = (\zeta_\beta)_{\langle\beta,1/M\rangle\leq 1-\epsilon} \in \mathbb{C}^N$, $N = N(\epsilon) := \#\{\beta \in \mathbb{Z}^n_+ : \langle\beta, 1/M\rangle \leq 1-\epsilon\}$. It is moreover assumed that a_α is not identically zero for at least one multi-index α with $\langle\alpha, 1/M\rangle = 1$.

Let us take a point $(x_0, \xi^0) \in T^\circ\mathbb{R}^n$; we make on the equation (32) the following assumptions:

(a) for all $\alpha \in \mathbb{Z}^n_+$ satisfying $\langle\alpha, 1/M\rangle \leq 1$, the coefficients $a_\alpha(x,\zeta)$ are *locally smooth with respect to x and entire analytic with respect to ζ uniformly in x*; that is, for some open neighborhood U_0 of x_0

$$a_\alpha(x,\zeta) = \sum_{\gamma\in\mathbb{Z}^N_+} a_{\alpha,\gamma}(x)\zeta^\gamma, \qquad a_{\alpha,\gamma} \in C^\infty(U_0),\ \zeta \in \mathbb{C}^N, \tag{34}$$

where for any $\beta \in \mathbb{Z}^n_+$, $\gamma \in \mathbb{Z}^N_+$ and suitable $c_{\alpha,\beta} > 0$, $\sup_{x\in U_0} |\partial_x^\beta a_{\alpha,\gamma}(x)| \leq c_{\alpha,\beta}\lambda_\gamma$ and the expansion $F_1(\zeta) := \sum_{\gamma\in\mathbb{Z}^N_+} \lambda_\gamma\zeta^\gamma$ defines an entire analytic function;

(b) (32) is *microlocally M-elliptic* at (x_0, ξ^0), that is the M-principal part (33) satisfies, for some Γ_M M-conic neighborhood of ξ^0,

$$A_M(x,\xi,\zeta) \neq 0, \quad \text{for } (x,\xi) \in U_0 \times \Gamma_M,\ \zeta \in \mathbb{C}^N. \tag{35}$$

Under the previous assumptions, we may prove the following

Theorem 5 *Let* $p \in [1, +\infty]$, $r > \frac{n}{\mu_* q} + \left[\frac{n}{\mu_*}\right] + 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \epsilon \leq 1$ *and* $(x_0, \xi^0) \in T^\circ\mathbb{R}^n$ *be given, consider the quasi-linear* M*-homogeneous PDE (32), satisfying assumptions (a) and (b). For any* s *such that*

$$r + 1 + \delta\left(r - \frac{n}{\mu_* q}\right) - \epsilon \leq s \leq r + 1, \tag{36}$$

with

$$0 < \delta \leq \frac{\epsilon}{r - \frac{n}{\mu_* q}} \quad \text{and} \quad 0 < \delta < \frac{\mu_*}{\mu^*}, \tag{37}$$

consider $u \in \mathcal{F}L^p_{s-\delta\left(r-\frac{n}{\mu_* q}\right), M, \mathrm{loc}}(x_0)$ *a solution to (32) with source term*

$$f \in \mathcal{F}L^p_{s-1, M, \mathrm{mcl}}(x_0, \xi^0);$$

then $u \in \mathcal{F}L^p_{s, M, \mathrm{mcl}}(x_0, \xi^0)$.

Proof From (36) and the other assumptions on r, in view of Proposition 1 (see also [13, Proposition 8]) and [13, Corollary 2], from $u \in \mathcal{F}L^p_{s-\delta\left(r-\frac{n}{\mu_* q}\right), M, \mathrm{loc}}(x_0)$ it follows that

$$D^\beta u \in \mathcal{F}L^p_{s-\delta\left(r-\frac{n}{\mu_* q}\right)-1+\epsilon, M, \mathrm{loc}}(x_0) \hookrightarrow \mathcal{F}L^p_{r, M, \mathrm{loc}}(x_0),$$

as long as $\langle \beta, 1/M \rangle \leq 1 - \epsilon$, hence $a_\alpha(\cdot, D^\beta u)_{\langle \beta, 1/M \rangle \leq 1-\epsilon} \in \mathcal{F}L^p_{r, M, \mathrm{loc}}(x_0)$ for $\langle \alpha, 1/M \rangle \leq 1$.

Notice that conditions (37) ensure that δ belongs to the interval $\left]0, \frac{\mu_*}{\mu^*}\right[$ as required by Theorem 4, see Remark 4 below. Notice also that, for r satisfying the condition required by Theorem 5, $(\delta - 1)\left(r - \frac{n}{\mu_* q}\right) + 1 < r + 1 + \delta\left(r - \frac{n}{\mu_* q}\right) - \epsilon$. Hence the range of s in (36) is included in the range of s in the statement of Theorem 4. Therefore, we are in the position to apply Theorem 4 to the symbol

$$A_u(x, \xi) := \sum_{\langle \alpha, 1/M \rangle \leq 1} a_\alpha(x, D^\beta u)_{\langle \beta, 1/M \rangle \leq 1-\epsilon} \xi^\alpha, \tag{38}$$

which is of the type involved in (26) and, in particular, is microlocally M-elliptic at (x_0, ξ^0) in the sense of (27). This shows the result. □

Remark 4 According to the proof, we underline that in the statement of Theorem 5 the assumption (b) could be relaxed to the weaker assumption that the symbol (38) of the linear operator, which is obtained by making explicit the expression of the operator in the left-hand side of (32) at the given solution $u = u(x)$, is microlocally M-elliptic at (x_0, ξ^0) in the sense of (27).

Concerning the assumptions (37) on δ, we note that $\frac{\mu_*}{\mu^*} \leq \frac{\epsilon}{r-\frac{n}{\mu_* q}}$ if and only if $r \leq \frac{n}{\mu_* q} + \frac{\mu^*}{\mu_*}\epsilon$, otherwise $\frac{\epsilon}{r-\frac{n}{\mu_* q}}$ is strictly smaller than $\frac{\mu_*}{\mu^*}$. Since $0 < \epsilon \leq 1$ and $\frac{\mu^*}{\mu_*} \geq 1$, in principle $r > \frac{n}{\mu_* q} + \left[\frac{n}{\mu_*}\right] + 1$ could be either smaller or greater than $\frac{n}{\mu_* q} + \frac{\mu^*}{\mu_*}\epsilon$, therefore the two assumptions on δ in (37) cannot be unified.

Assuming in particular $r > \frac{n}{\mu_* q} + \frac{\mu^*}{\mu_*}\epsilon$ and taking, in the statement of Theorem 5, $s = r+1$ and the best (that is biggest) amount of microlocal regularity of u, quantified by $\delta = \frac{\epsilon}{r-\frac{n}{\mu_* q}}$, we obtain

$$f \in \mathcal{F}L^p_{r,M,\text{mcl}}(x_0, \xi^0) \quad \Rightarrow \quad u \in \mathcal{F}L^p_{r+1,M,\text{mcl}}(x_0, \xi^0) \tag{39}$$

for any solution u to the equation (32) belonging a priori to $\mathcal{F}L^p_{r+1-\epsilon,M,\text{loc}}(x_0)$.

Assume now $r \leq \frac{n}{\mu_* q} + \frac{\mu^*}{\mu_*}\epsilon$ and set again $s = r + 1$ in the statement of Theorem 5; since $\frac{\epsilon}{r-\frac{n}{\mu_* q}} \geq \frac{\mu_*}{\mu^*}$, in this case the value $\frac{\epsilon}{r-\frac{n}{\mu_* q}}$ cannot be attained by $\delta \in \left]0, \frac{\mu_*}{\mu^*}\right[$, and we get that (39) remains true for any solution belonging a priori to $\mathcal{F}L^p_{r+1-\delta\left(r-\frac{n}{\mu_* q}\right),M,\text{loc}}(x_0)$ for any positive $\delta < \frac{\mu_*}{\mu^*}$.

Remark 5 As in the case of linear PDEs (see e.g. Theorem 7), also in the framework of quasi-linear PDEs the result of Theorem 5 can be stated for a M-homogeneous quasi-linear equation of *arbitrary* positive order m, namely

$$\sum_{\langle \alpha, 1/M \rangle \leq m} a_\alpha(x, D^\beta u)_{\langle \beta, 1/M \rangle \leq m-\epsilon} D^\alpha u = f(x), \tag{40}$$

with $m > 0$ and $0 < \epsilon \leq m$. In this case, the range (36) of s will be replaced by

$$r + m + \delta\left(r - \frac{n}{\mu_* q}\right) - \epsilon \leq s \leq r + m \tag{41}$$

with δ satisfying (37), and the result becomes

$$f \in \mathcal{F}L^p_{s-m,M,\text{mcl}}(x_0, \xi^0) \quad \Rightarrow \quad u \in \mathcal{F}L^p_{s,M,\text{mcl}}(x_0, \xi^0)$$

for any solution $u \in \mathcal{F}L^p_{s-\delta\left(r-\frac{n}{\mu_* q}\right),M,\text{loc}}(x_0)$ of (40).

2.7 Nonlinear PDE

Let us consider now the fully nonlinear equation

$$F(x, D^\alpha u)_{\langle \alpha, 1/M \rangle \leq 1} = f(x), \tag{42}$$

where $F(x, \zeta)$ is locally smooth with respect to $x \in \mathbb{R}^n$ and entire analytic in $\zeta \in \mathbb{C}^N$, uniformly in x. Namely, for $N = \#\{\alpha \in \mathbb{Z}^n_+ : \langle \alpha, \frac{1}{M} \rangle \leq 1\}$ and some open neighborhood U_0 of x_0,

$$F(x, \zeta) = \sum_{\gamma \in \mathbb{Z}^M_+} c_\gamma(x) \zeta^\gamma, \qquad c_\gamma \in C^\infty(U_0),\ \zeta \in \mathbb{C}^N, \tag{43}$$

where for any $\beta \in \mathbb{Z}^n_+, \gamma \in \mathbb{Z}^N_+$ and some positive a_β, λ_γ, $\sum\limits_{\gamma \in \mathbb{Z}^N_+} \lambda_\gamma \zeta^\gamma$ is entire analytic in $\mathbb{C}^N$ and $\sup\limits_{x \in U_0} |\partial^\beta_x c_\gamma(x)| \leq a_\beta \lambda_\gamma$.

Let the equation (42) be microlocally M-elliptic at $(x_0, \xi^0) \in T^\circ \mathbb{R}^n$, that is the *linearized* M-principal symbol $A_M(x, \xi, \zeta) := \sum\limits_{\langle \alpha, 1/M \rangle = 1} \frac{\partial F}{\partial \zeta_\alpha}(x, \zeta) \xi^\alpha$ satisfies

$$\sum_{\langle \alpha, 1/M \rangle = 1} \frac{\partial F}{\partial \zeta_\alpha}(x, \zeta) \xi^\alpha \neq 0 \ \text{ for } (x, \xi) \in U_0 \times \Gamma_M, \tag{44}$$

for Γ_M a suitable M-conic neighborhood of ξ_0.

Theorem 6 *Assume that equation (42) is microlocally M-elliptic at $(x_0, \xi^0) \in T^\circ \mathbb{R}^n$. For $1 \leq p \leq +\infty$, $r > \frac{n}{\mu_* q} + \left[\frac{n}{\mu_*}\right] + 1$, $0 < \delta < \frac{\mu_*}{\mu^*}$, let $u \in \mathcal{F}L^p_{M, r+1, \mathrm{loc}}(x_0)$ be a solution to (42), satisfying in addition*

$$\partial_{x_j} u \in \mathcal{F}L^p_{M, r+1-\delta(r - \frac{n}{\mu_* q}), \mathrm{loc}}(x_0), \quad j = 1, \dots, n. \tag{45}$$

If moreover the forcing term satisfies

$$\partial_{x_j} f \in \mathcal{F}L^p_{r, M, \mathrm{mcl}}(x_0, \xi^0), \quad j = 1, \dots, n, \tag{46}$$

we obtain

$$\partial_{x_j} u \in \mathcal{F}L^p_{r+1, M, \mathrm{mcl}}(x_0, \xi^0), \quad j = 1, \dots, n. \tag{47}$$

Proof For each $j = 1, \dots, n$, we differentiate (42) with respect to x_j finding that $\partial_{x_j} u$ must solve the linearized equation

$$\sum_{\langle \alpha, 1/M \rangle \leq 1} \frac{\partial F}{\partial \zeta_\alpha}(x, D^\beta u)_{\langle \beta, 1/M \rangle \leq 1} D^\alpha \partial_{x_j} u = \partial_{x_j} f - \frac{\partial F}{\partial x_j}(x, D^\beta u)_{\langle \beta, 1/M \rangle \leq 1}. \tag{48}$$

From Theorems 2 and [13, Corollary 2], $u \in \mathcal{F}L^p_{M, r+1, \mathrm{loc}}(x_0)$ yields that

$$\frac{\partial F}{\partial \zeta_\alpha}(\cdot, D^\beta u)_{\beta \cdot 1/M \leq m} \in \mathcal{F}L^p_{M, r, \mathrm{loc}}(x_0).$$

Because of hypotheses (45), (46), for each $j = 1, \dots, n$, Theorem 4 applies to $\partial_{x_j} u$, as a solution of the equation (48) (which is microlocally M-elliptic at (x_0, ξ^0) in view of (44)), taking $s = r + 1$. This proves the result. □

Lemma 2 *For every $M \in \mathbb{R}^n_+$, $s \in \mathbb{R}$, $1 \le p \le +\infty$, assume that $u, \partial_{x_j} u \in \mathcal{F}L^p_{s,M}(\mathbb{R}^n)$ for all $j = 1, \dots, n$. Then $u \in \mathcal{F}L^p_{s+\frac{\mu_*}{\mu^*},M}(\mathbb{R}^n)$. The same is still true if the Fourier–Lebesgue spaces $\mathcal{F}L^p_{s,M}(\mathbb{R}^n)$, $\mathcal{F}L^p_{s+\frac{\mu_*}{\mu^*},M}(\mathbb{R}^n)$ are replaced by $\mathcal{F}L^p_{s,M,\mathrm{mcl}}(x_0, \xi^0)$, $\mathcal{F}L^p_{s+\frac{\mu_*}{\mu^*},M,\mathrm{mcl}}(x_0, \xi^0)$ at a given point $(x_0, \xi^0) \in T^\circ\mathbb{R}^n$.*

Proof Let us argue for simplicity in the case of the spaces $\mathcal{F}L^p_{s,M}(\mathbb{R}^n)$, the microlocal case being completely analogous.

Notice that $u \in \mathcal{F}L^p_{s+\frac{\mu_*}{\mu^*},M}(\mathbb{R}^n)$ is equivalent to $\langle D\rangle_M^{\mu_*/\mu^*} u \in \mathcal{F}L^p_{s,M}(\mathbb{R}^n)$. By using the known properties of the Fourier transform, we may rewrite $\langle D\rangle_M^{\mu_*/\mu^*} u$ in the form

$$\langle D\rangle_M^{\mu_*/\mu^*} u = \langle D\rangle_M^{\mu_*/\mu^*-2} u + \sum_{j=1}^n \Lambda_{j,M}(D)(D_{x_j} u),$$

where $\Lambda_{j,M}(D)$ is the Fourier multiplier with symbol $\langle\xi\rangle_M^{\mu_*/\mu^*-2}\xi_j^{2\mu_j-1}$, that is

$$\Lambda_{j,M}(D)v := \mathcal{F}^{-1}\left(\langle\xi\rangle_M^{\mu_*/\mu^*-2}\xi_j^{2\mu_j-1}\widehat{v}\right), \quad j = 1, \dots, n.$$

Since $\langle\xi\rangle_M^{\mu_*/\mu^*-2}\xi_j^{2\mu_j-1} \in S_M^{\mu_*/\mu^*-\mu_*/\mu_j}$, the result follows at once from Proposition 1. □

As a straightforward application of the previous lemma, the following consequence of Theorem 6 can be proved.

Corollary 1 *Under the same assumptions of Theorem 6 we have that $u \in \mathcal{F}L^p_{r+1+\frac{\mu_*}{\mu^*},M,\mathrm{mcl}}(x_0, \xi^0)$.*

Remark 6 Notice that if $\left(r - \frac{n}{\mu_* q}\right)\delta \ge 1$ then any $u \in \mathcal{F}L^p_{r+1,M,\mathrm{loc}}(x_0)$ rightly satisfies (45).

Thus $\partial_{x_j} u \in \mathcal{F}L^p_{r+1-\frac{\mu_*}{\mu_j},M,\mathrm{loc}}(x_0) \hookrightarrow \mathcal{F}L^p_{M,r+1-\delta(r-\frac{n}{\mu_* q}),\mathrm{loc}}(x_0)$ being $\mu_*/\mu_j \le 1 \le \left(r - \frac{n}{\mu_* q}\right)\delta$ for each $j = 1, \dots, n$. Notice that for $r > \frac{n}{\mu_* q} + \frac{\mu^*}{\mu_*}$ we can find $\delta^* \in]0, \mu_*/\mu^*[$ such that $\left(r - \frac{n}{\mu_* q}\right)\delta \ge 1$: it suffices to choose an arbitrary $\delta^* \in [\,\frac{1}{r-\frac{n}{\mu_* q}}, \frac{\mu_*}{\mu^*}\,[$. Hence, applying Theorem 6 with such a δ^* we conclude that if $r > \frac{n}{\mu_* q} + \frac{\mu^*}{\mu_*}$ and the right-hand side f of equation (42) obeys to condition (46) at a point $(x_0, \xi^0) \in T^\circ\mathbb{R}^n$, then every solution $u \in \mathcal{F}L^p_{r+1,M,\mathrm{loc}}(x_0)$ to such an equation satisfies condition (47); in particular $u \in \mathcal{F}L^p_{r+1+\frac{\mu_*}{\mu^*},M,\mathrm{mcl}}(x_0, \xi^0)$.

3 Dyadic decomposition

In the following we will provide a useful characterization of M-homogeneous Fourier–Lebesgue spaces, based on a quasi-homogenous dyadic partition of unity.

Namely for fixed $K \geq 1$ we set

$$\begin{aligned} \mathcal{C}_{-1}^{M,K} &:= \left\{\xi \in \mathbb{R}^n \,:\, |\xi|_M \leq K\right\}, \\ \mathcal{C}_{h}^{M,K} &:= \{\xi \in \mathbb{R}^n \,:\, \frac{1}{K}2^{h-1} \leq |\xi|_M \leq K2^{h+1}\}, \quad h = 0, 1, \dots . \end{aligned} \tag{49}$$

It is clear that the crowns (shells) $\mathcal{C}_h^{M,K}$, for $h \geq -1$, provide a covering of $\mathbb{R}^n$. For the sequel of our analysis, a fundamental property of this covering is that the number of overlapping crowns does not increase with the index h; precisely there exists a positive number $N_0 = N_0(K)$ such that

$$\mathcal{C}_p^{M,K} \cap \mathcal{C}_q^{M,K} = \emptyset, \quad \text{for } |p - q| > N_0. \tag{50}$$

Consider now a real-valued function $\Phi = \Phi(t) \in C^\infty([0, +\infty[)$ satisfying

$$\begin{aligned} &0 \leq \Phi(t) \leq 1, \quad \forall t \geq 0, \\ &\Phi(t) = 1 \quad \text{for } 0 \leq t \leq \frac{1}{2K}, \quad \Phi(t) = 0 \quad \text{for } t > K, \end{aligned} \tag{51}$$

and define the sequence $\{\varphi_h\}_{h=-1}^{+\infty}$ in $C^\infty(\mathbb{R}^n)$ by setting for $\xi \in \mathbb{R}^n$

$$\varphi_{-1}(\xi) := \Phi(|\xi|_M), \quad \varphi_h(\xi) := \Phi\left(\frac{|\xi|_M}{2^{h+1}}\right) - \Phi\left(\frac{|\xi|_M}{2^h}\right), \quad h = 0, 1, \dots . \tag{52}$$

It is easy to check that the sequence $\{\varphi_h\}_{h=-1}^{\infty}$ defined above enjoys the following properties:

$$\operatorname{supp} \varphi_h \subseteq \mathcal{C}_h^{M,K}, \quad \text{for } h \geq -1; \tag{53}$$

$$\sum_{h=-1}^{\infty} \varphi_h(\xi) = 1, \quad \text{for all } \xi \in \mathbb{R}^n; \tag{54}$$

$$\sum_{h=-1}^{\infty} u_h = u, \quad \text{with convergence in } \mathcal{S}'(\mathbb{R}^n), \tag{55}$$

where it is set $u_h := \varphi_h(D)u$, for $h \geq -1$.

As a consequence of (50), for any fixed $\xi \in \mathbb{R}^n$ the sum in (54) reduces to a finite number of terms independently of the choice of ξ itself. Namely, for some positive integers N_0 independent of ξ and $h_0 = h_0(\xi) \geq -1$, we have

$$\sum_{h=-1}^{\infty} \varphi_h(\xi) \equiv \sum_{h=\tilde{h}_0}^{h_0+N_0} \varphi_h(\xi), \quad \text{where } \tilde{h}_0 = \tilde{h}_0(\xi) := \max\{-1, h_0 - N_0\}. \tag{56}$$

The sequence $\{\varphi_h\}_{h=-1}^{+\infty}$ above introduced is referred to as a *M-homogeneous dyadic partition of unity*, and the expansion in the left-hand side of (55) will be called *M-homogeneous dyadic decomposition* of $u \in \mathcal{S}'(\mathbb{R}^n)$; in the homogeneous case $M = (1, \dots, 1)$, such a decomposition reduces to the classical *Littlewood–Paley decomposition* of u, cf. for example [1].

Proposition 2 *For $M = (\mu_1, \dots, \mu_n) \in \mathbb{R}^n_+$, $s \in \mathbb{R}$ and $p \in [1. + \infty]$, a distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ belongs to the space $\mathcal{F}L^p_{s,M}$ if and only if*

$$\widehat{u}_h \in L^p(\mathbb{R}^n), \qquad \textit{for all } h \geq -1, \tag{57}$$

and

$$\sum_{h=-1}^{+\infty} 2^{shp} \|\widehat{u}_h\|^p_{L^p} < +\infty. \tag{58}$$

Under the above assumptions,

$$\left(\sum_{h=-1}^{+\infty} 2^{shp} \|\widehat{u}_h\|^p_{L^p} \right)^{1/p} \tag{59}$$

provides a norm in $\mathcal{F}L^p_{s,M}$ equivalent to (9).

For $p = +\infty$, condition (58) (as well as the norm (59)) must be suitably modified.

Proof Let us first observe that the M-weight $\langle\cdot\rangle_M$ is equivalent to 2^h on the support of φ_h; indeed

$$\begin{aligned} &1 \leq \langle\xi\rangle_M \leq (1 + K^2)^{1/2}, \qquad \text{for } \xi \in \operatorname{supp} \varphi_{-1}\,; \\ &\frac{1}{2K} 2^h \leq \langle\xi\rangle_M \leq (1 + 4K^2)^{1/2} 2^h, \qquad \text{for } \xi \in \operatorname{supp} \varphi_h \ \text{ and } \ h \geq 0, \end{aligned} \tag{60}$$

being K the positive constant involved in (51).

For $p \in [1, +\infty[$, it is enough arguing on smooth functions $u \in \mathcal{S}(\mathbb{R}^n)$ in view of density of $\mathcal{S}(\mathbb{R}^n)$ in $\mathcal{F}L^p_{s,M}$. For $\xi \in \mathbb{R}^n$, from (54), (56) we derive

$$\begin{aligned} \sum_{h=-1}^{\infty} |\widehat{u}_h(\xi)|^p &\equiv \sum_{h=\tilde{h}_0}^{h_0+N_0} \varphi_h(\xi)^p |\widehat{u}(\xi)|^p \leq |\widehat{u}(\xi)|^p \equiv \left(\sum_{h=\tilde{h}_0}^{h_0+N_0} \varphi_h(\xi) |\widehat{u}(\xi)| \right)^p \\ &\leq C_{N_0,p} \sum_{h=\tilde{h}_0}^{h_0+N_0} \varphi_h(\xi)^p |\widehat{u}(\xi)|^p \equiv C_{N_0,p} \sum_{h=-1}^{\infty} |\widehat{u}_h(\xi)|^p, \end{aligned} \tag{61}$$

where $h_0 = h_0(\xi)$, $\tilde{h}_0 = \tilde{h}_0(\xi)$ are the integers in (56) and $C_{N_0,p} > 1$ depends only on N_0 and p. Hence, multiplying each side of (61) by $\langle\xi\rangle_M^{sp}$, making use of (60) and integrating on $\mathbb{R}^n$, it yields

$$\frac{1}{C_{s,p,K}}\sum_{h=-1}^{\infty}2^{shp}\|\widehat{u}_h\|^p \le \|u\|^p_{\mathcal{F}L^p_{s,M}} \le C_{s,p,K}\sum_{h=-1}^{\infty}2^{shp}\|\widehat{u}_h\|^p,$$

for a suitable constant $C_{s,p,K} > 1$ depending only on s, p and K. This proves the statement of Proposition 2, for $1 \le p < +\infty$.

In the absence of the density of $\mathcal{S}(\mathbb{R}^n)$ in $\mathcal{F}L^\infty_{s,M}$, let us now argue directly. Thus for arbitrary $u \in \mathcal{F}L^\infty_{s,M}$ and every $h \ge -1$, writing

$$\widehat{u}_h = \frac{\varphi_h}{\langle\cdot\rangle_M^s}\langle\cdot\rangle_M^s\widehat{u}, \tag{62}$$

we get $\widehat{u}_h \in L^\infty(\mathbb{R}^n)$, since $\langle\cdot\rangle_M^s\widehat{u} \in L^\infty(\mathbb{R}^n)$ and, in view of (60) and (53),

$$2^{sh}\left|\frac{\varphi_h(\xi)}{\langle\xi\rangle_M^s}\right| \le C_{s,K}, \qquad \forall\xi\in\mathbb{R}^n, \tag{63}$$

where the constant $C_{s,K}$ depends only on s and K. From (62) and (63)

$$2^{sh}|\widehat{u}_h(\xi)| \le C_{s,K}\|u\|_{\mathcal{F}L^\infty_{s,M}}, \quad \forall\xi\in\mathbb{R}^n, \quad h\ge -1,$$

follows at once and implies (58) with $p = +\infty$.

Conversely, let us suppose that $u \in \mathcal{S}'(\mathbb{R}^n)$ satisfies (57), (58). From (50), (55) and (60) we get for an arbitrary $\ell \ge -1$ and every $\xi \in \mathcal{C}_\ell^{M,K}$:

$$\begin{aligned}|\langle\xi\rangle_M^s\widehat{u}(\xi)| &\le \sum_{h=-1}^{+\infty}|\langle\xi\rangle_M^s\widehat{u}_h(\xi)| = \sum_{h=\ell-N_0}^{\ell+N_0}|\langle\xi\rangle_M^s\widehat{u}_h(\xi)| \le C_{s,K}\sum_{h=\ell-N_0}^{\ell+N_0}2^{sh}|\widehat{u}_h(\xi)|\\ &\le C_{s,K}(2N_0+1)\sup_{h\ge-1}2^{sh}\|\widehat{u}_h\|_{L^\infty},\end{aligned}$$

noticing that u belongs to $\mathcal{F}L^\infty_{s,M}$ and satisfies

$$\|u\|_{\mathcal{F}L^\infty_{s,M}} \le C_{s,K}(2N_0+1)\sup_{h\ge-1}2^{sh}\|\widehat{u}_h\|_{L^\infty}.$$

The proof is complete. □

Remark 7 Arguing along the same lines followed in the proof of estimates (63), one can prove the following estimates for the derivatives of functions φ_h: for all $\nu \in \mathbb{Z}^n_+$ a positive constant C_ν exists such that

$$|D^\nu_\xi\varphi_h(\xi)| \le C_\nu 2^{-h\langle\nu,1/M\rangle}, \quad \forall\xi\in\mathbb{R}^n, \quad h = -1, 0, 1, \ldots. \tag{64}$$

Notice also that, in view of (60), estimates (64) can be stated in the equivalent form

$$|D_\xi^\nu \varphi_h(\xi)| \le C_\nu \langle \xi \rangle_M^{-\langle \nu, 1/M\rangle}, \quad \forall \xi \in \mathbb{R}^n, \quad h = -1, 0, 1, \dots .$$

Along the same arguments of Bony [2], one can show the following

Proposition 3 *Let $M = (\mu_1, \dots, \mu_n) \in \mathbb{R}^n_+$ and $p \in [1. +\infty]$.*

(i) *For $s \in \mathbb{R}$, let $\{u_h\}_{h=-1}^{+\infty}$ be a sequence of distributions $u_h \in \mathcal{S}'(\mathbb{R}^n)$ satisfying the following conditions:*

(a) *there exists a constant $K \ge 1$ such that*

$$\operatorname{supp} \widehat{u}_h \subseteq \mathcal{C}_h^{M,K}, \qquad \textit{for all } h \ge -1\,;$$

(b)

$$\sum_{h=-1}^{+\infty} 2^{shp} \|\widehat{u}_h\|_{L^p}^p < +\infty \tag{65}$$

(with obvious modification for $p = +\infty$).

Then $u = \sum_{h=-1}^{+\infty} u_h \in \mathcal{F}L^p_{s,M}$, where the series is convergent in $\mathcal{S}'(\mathbb{R}^n)$. Moreover, for some positive constant $C_{s,p,K}$ depending only on s, p, K,

$$\|u\|_{\mathcal{F}L^p_{s,M}} \le C_{s,p,K} \left(\sum_{h=-1}^{+\infty} 2^{shp} \|\widehat{u}_h\|_{L^p}^p \right)^{1/p}, \tag{66}$$

(ii) *If $s > 0$, the same result stated in (i) is still valid when a distribution sequence $\{u_h\}_{h=-1}^{+\infty}$ satisfies the condition (b) and*

(a') *there exists a constant $K \ge 1$ such that*

$$\operatorname{supp} \widehat{u}_h \subseteq \mathcal{B}_h^{M,K} := \{\xi \in \mathbb{R}^n \,:\, |\xi|_M \le K 2^{h+1}\}, \qquad \textit{for all } h \ge -1,$$

instead of (a) (notice that $\mathcal{B}_{-1}^{M,K} \equiv \mathcal{C}_{-1}^{M,K}$).

4 Proof of Theorem 1

Following closely the arguments in Coifmann–Meyer [3], see also Garello–Morando [7], one proves that every zero order symbol in $\mathcal{F}L^p_{r,M} S^0_{M,\delta}(N)$ can be expanded into a series of "elementary terms".

Lemma 3 *For $p \in [1, +\infty]$, $r > \frac{n}{\mu_* q}$ (being q the conjugate exponent of p), $N > n+1$ positive integer and $\delta \in [0, 1]$, let $a(x, \xi) \in \mathcal{F}L^p_{r,M} S^0_{M,\delta}(N)$. Then there exist a* *sequence* $\{c_k\}_{k\in\mathbb{Z}^n} \subset \mathbb{R}_+$ *satisfying* $\sum_{k\in\mathbb{Z}^n} c_k < +\infty$ *such that*

$$a(x,\xi) = \sum_{k\in\mathbb{Z}^n} c_k a_k(x,\xi), \tag{67}$$

with absolute convergence in $L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$.

More precisely for each $k \in \mathbb{Z}^n$

$$a_k(x,\xi) = \sum_{h=-1}^{+\infty} d_h^k(x)\psi_h^k(\xi), \tag{68}$$

with suitable sequences $\{d_h^k\}_{h=-1}^{+\infty}$ *in* $\mathcal{F}L^1 \cap \mathcal{F}L^p_{r,M}$ *and* $\{\psi_h^k\}_{h=-1}^{+\infty}$ *in* $C_0^\infty(\mathbb{R}^n)$, *obeying for some positive constants* C, H *and* $K > 1$ *the following conditions:*

(a) $\|d_h^k\|_{\mathcal{F}L^1} \le H, \quad \|d_h^k\|_{\mathcal{F}L^p_{r,M}} \le H2^{h\delta\left(r-\frac{n}{\mu_* q}\right)}$ *for all* $h = -1, 0, \dots;$

(b) $\operatorname{supp}\psi_h^k \subseteq \mathcal{C}_h^{M,K}$, $h = -1, 0, \dots;$

(c) $|\partial^\alpha\psi_h^k(\xi)| \le C2^{-\langle\alpha,1/M\rangle h}$, $\forall\,\xi \in \mathbb{R}^n$, $|\alpha| \le N$.

In view of (50) and condition (b) above, the expansions in the right-hand side of (68) has only finitely many nonzero terms at each point (x,ξ). Conditions (a)-(c) above also imply that $a_k(x,\xi)$ defined by (68) belongs to $\mathcal{F}L^p_{r,M}S^0_{M,\delta}(N)$ for each $k \in \mathbb{Z}^n$. A symbol of the form (68) will be referred to as an *elementary symbol*.

The proof of Theorem 1 follows the same arguments as in [9]. Without loss of generality, we may reduce to prove the statement of the theorem in the case of a symbol $a(x,\xi) \in \mathcal{F}L^p_{r,M}S^0_{M,\delta}(N)$. Also, because of Lemma 3, it will be enough to show the result in the case when $a(x,\xi)$ is an elementary symbol, namely

$$a(x,\xi) = \sum_{h=-1}^{+\infty} d_h(x)\psi_h(\xi),$$

where the sequences $\{d_h\}_{h=-1}^{+\infty}$ and $\{\psi_h\}_{h=-1}^{+\infty}$ obey the assumptions (a)–(c).

In view of Lemma 3 there holds

$$a(x,D)u(x) = \sum_{h=-1}^{+\infty} d_h(x)u_h(x), \quad \forall\, u \in \mathcal{S}(\mathbb{R}^n), \tag{69}$$

where

$$u_h := \psi_h(D)u, \qquad h = -1, 0, \dots. \tag{70}$$

Let $\{\varphi_\ell\}_{\ell\ge -1}$ be an M-homogeneous dyadic partition of unity; then we may decompose (69) as follows

$$a(x,D)u(x) = \sum_{h=-1}^{+\infty}\sum_{\ell=-1}^{+\infty} d_{h,\ell}(x)u_h(x) = T_1u(x) + T_2u(x) + T_3u(x), \tag{71}$$

where it is set

$$T_1u(x) := \sum_{h=N_0-1}^{+\infty} \sum_{\ell=-1}^{h-N_0} d_{h,\ell}(x)u_h(x), \tag{72}$$

$$T_2u(x) := \sum_{h=-1}^{+\infty} \sum_{\ell=\ell_h}^{h+N_0-1} d_{h,\ell}(x)u_h(x) \qquad (\ell_h := \max\{-1, h-N_0+1\}), \tag{73}$$

$$T_3u(x) := \sum_{\ell=N_0-1}^{+\infty} \sum_{h=-1}^{\ell-N_0} d_{h,\ell}(x)u_h(x), \tag{74}$$

with sufficiently large integer $N_0 > 0$, and

$$d_{h,\ell} := \varphi_\ell(D)d_h, \qquad h, \ell = -1, 0, \dots . \tag{75}$$

The proof of Theorem 1 follows from combining the following continuity results concerning the different operators T_1, T_2, T_3.

Henceforth, the following general notation will be adopted: for every pair of Banach spaces X, Y, we will write $\|T\|_{X\to Y}$ to mean the operator norm of every linear bounded operator T from X into Y.

Lemma 4 *For all $s \in \mathbb{R}$, T_1 extends to a linear bounded operator*

$$T_1 : \mathcal{F}L^p_{s,M} \to \mathcal{F}L^p_{s,M} \tag{76}$$

and there exists a positive constant $C = C_{s,p}$ such that

$$\|T_1\|_{\mathcal{F}L^p_{s,M}\to\mathcal{F}L^p_{s,M}} \le C \sup_{h\ge -1} \|d_h\|_{\mathcal{F}L^1} \tag{77}$$

Proof Taking $N_0 > 0$ sufficiently large, we find a suitable $T > 1$ such that

$$\operatorname{supp} \widehat{d_{h,\ell}u_h} \subseteq \mathcal{C}^{M,T}_h, \quad \text{for } -1 \le \ell \le h-N_0 \text{ and } h \ge N_0-1.$$

Then in view of Proposition 3 (i), for every $s \in \mathbb{R}$ a positive constant $C = C_{s,p}$ exists such that

$$\|T_1u\|^p_{\mathcal{F}L^p_{s,M}} \le C \sum_{h\ge N_0-1} 2^{shp} \left\| \sum_{\ell=-1}^{h-N_0} \widehat{d_{h,\ell}u_h} \right\|^p_{L^p};$$

on the other hand

$$\sum_{\ell=-1}^{h-N_0} \widehat{d_{h,\ell}u_h} = (2\pi)^{-n} \sum_{\ell=-1}^{h-N_0} \widehat{d_{h,\ell}} * \widehat{u_h} = (2\pi)^{-n} \sum_{\ell=-1}^{h-N_0} \varphi_\ell \widehat{d_h} * \widehat{u_h}$$

hence Young's inequality yields

$$\left\| \sum_{\ell=-1}^{h-N_0} \widehat{d_{h,\ell} u_h} \right\|_{L^p} \leq (2\pi)^{-n} \left\| \sum_{\ell=-1}^{h-N_0} \varphi_\ell \widehat{d_h} \right\|_{L^1} \|\widehat{u_h}\|_{L^p}$$

and, in view of (54),

$$\left\| \sum_{\ell=-1}^{h-N_0} \varphi_\ell \widehat{d_h} \right\|_{L^1} = \int \sum_{\ell=-1}^{h-N_0} \varphi_\ell(\xi) |\widehat{d_h}(\xi)| d\xi \leq \int |\widehat{d_h}(\xi)| d\xi = \|\widehat{d_h}\|_{L^1}.$$

Combining the preceding estimates and thanks to Lemma 3 and Proposition 2 we get

$$\begin{aligned} \|T_1 u\|^p_{\mathcal{F}L^p_{s,M}} &\leq C \left(\sup_{h \geq -1} \|d_h\|_{\mathcal{F}L^1} \right)^p \sum_{h \geq N_0 - 1} 2^{shp} |\widehat{u_h}|^p_{L^p} \\ &\leq C \left(\sup_{h \geq -1} \|d_h\|_{\mathcal{F}L^1} \right)^p \|u\|^p_{\mathcal{F}L^p_{s,M}}. \end{aligned}$$

This ends the proof of lemma. □

Lemma 5 *For all $s > (\delta - 1)\left(r - \frac{n}{\mu_* q}\right)$, T_2 extends to a linear bounded operator*

$$T_2 : \mathcal{F}L^p_{s,M} \to \mathcal{F}L^p_{s+(1-\delta)\left(r-\frac{n}{\mu_* q}\right),M} \tag{78}$$

and there exists a positive constant $C = C_{N_0,p,r,s}$ such that

$$\|T_2\|_{\mathcal{F}L^p_{s,M} \to \mathcal{F}L^p_{s+(1-\delta)\left(r-\frac{n}{\mu_* q}\right)}} \leq C \sup_{h \geq -1} 2^{-\delta\left(r-\frac{n}{\mu_* q}\right)h} \|d_h\|_{\mathcal{F}L^p_{r,M}}. \tag{79}$$

Proof Taking $N_0 > 0$ sufficiently large, we find a suitable $T > 1$ such that

$$\operatorname{supp} \widehat{d_{h,\ell} u_h} \subseteq \{\xi \ : \ |\xi|_M \leq T 2^{h+1}\}, \quad \text{for } \ell_h \leq \ell \leq h + N_0 - 1, \ \ h \geq -1 \tag{80}$$

and where $\ell_h := \max\{-1, h - N_0 + 1\}$. From Proposition 3 (ii), for $s > (\delta - 1)\left(r - \frac{n}{\mu_* q}\right)$ we get

$$\|T_2 u\|^p_{\mathcal{F}L^p_{s+(1-\delta)\left(r-\frac{n}{\mu_* q}\right),M}} \leq C \sum_{h \geq -1} 2^{\left(s+(1-\delta)\left(r-\frac{n}{\mu_* q}\right)\right)hp} \left\| \sum_{\ell=\ell_h}^{h+N_0-1} \widehat{d_{h,\ell} u_h} \right\|^p_{L^p};$$

and again from Young's inequality

$$\left\| \sum_{\ell=\ell_h}^{h+N_0-1} \widehat{d_{h,\ell} u_h} \right\|_{L^p} \le (2\pi)^{-n} \sum_{\ell=\ell_h}^{h+N_0-1} \|\widehat{d_{h,\ell}} * \widehat{u_h}\|_{L^p} \le (2\pi)^{-n} \sum_{\ell=\ell_h}^{h+N_0-1} \|\widehat{d_{h,\ell}}\|_{L^p} \|\widehat{u_h}\|_{L^1} ;$$

thus, since the number of indices ℓ such that $\ell_h \le \ell \le h + N_0 - 1$ is bounded independently of h one has

$$\begin{aligned}
&\|T_2 u\|^p_{\mathcal{F}L^p_{s+(1-\delta)\left(r-\frac{n}{\mu_* q}\right),M}} \\
&\quad \le C \sum_{h\ge -1} 2^{\left(s+(1-\delta)\left(r-\frac{n}{\mu_* q}\right)\right)hp} \left(\sum_{\ell=\ell_h}^{h+N_0-1} \|\widehat{d_{h,\ell}}\|_{L^p} \|\widehat{u_h}\|_{L^1} \right)^p \\
&\quad \le C_{N_0,p} \sum_{h\ge -1} 2^{\left(s+(1-\delta)\left(r-\frac{n}{\mu_* q}\right)\right)hp} \|\widehat{u_h}\|^p_{L^1} \sum_{\ell=\ell_h}^{h+N_0-1} \|\widehat{d_{h,\ell}}\|^p_{L^p} \\
&\quad = C_{N_0,p} \sum_{h\ge -1} 2^{shp}\, 2^{-\frac{n}{\mu_* q}hp} \|\widehat{u_h}\|^p_{L^1}\, 2^{rhp} 2^{-\delta\left(r-\frac{n}{\mu_* q}\right)hp} \sum_{\ell=\ell_h}^{h+N_0-1} \|\widehat{d_{h,\ell}}\|^p_{L^p}.
\end{aligned}$$

Notice also that Hölder's inequality yields

$$\|\widehat{u_h}\|_{L^1} = \int_{\mathcal{C}_h^{M,K}} |\widehat{u_h}(\xi)| d\xi \le \|\widehat{u_h}\|_{L^p} \left(\int_{\mathcal{C}_h^{M,K}} d\xi \right)^{1/q} \le C \|\widehat{u_h}\|_{L^p} 2^{\frac{nh}{\mu_* q}},$$

hence

$$2^{-\frac{nhp}{\mu_* q}} \|\widehat{u_h}\|^p_{L^1} \le C \|\widehat{u_h}\|^p_{L^p}.$$

Moreover, for a suitable constant $C_{N_0} > 0$ depending only on N_0,

$$2^h \le C_{N_0} 2^\ell, \quad \text{for } \ell_h \le \ell \le h + N_0 - 1.$$

Hence we get

$$\begin{aligned}
&2^{rhp} 2^{-\delta\left(r-\frac{n}{\mu_* q}\right)hp} \sum_{\ell=\ell_h}^{h+N_0-1} \|\widehat{d}_{h,\ell}\|^p_{L^p} \\
&\quad \le C_{N_0,r,p} 2^{-\delta\left(r-\frac{n}{\mu_* q}\right)hp} \sum_{\ell=\ell_h}^{h+N_0-1} 2^{r\ell p} \|\widehat{d}_{h,\ell}\|^p_{L^p} \\
&\quad \le \widetilde{C}_{N_0,r,p} 2^{-\delta\left(r-\frac{n}{\mu_* q}\right)hp} \|d_h\|^p_{\mathcal{F}L^p_{r,M}} \le \widetilde{C}_{N_0,r,p} H^p,
\end{aligned}$$

where

$$H := \sup_{h\geq -1} 2^{-\delta\left(r-\frac{n}{\mu_* q}\right)h} \|d_h\|_{\mathcal{F}L^p_{r,M}}, \tag{81}$$

and, in view of Proposition 2,

$$\|T_2 u\|^p_{\mathcal{F}L^p_{s+(1-\delta)\left(r-\frac{n}{\mu_* q}\right),M}} \leq \widetilde{C}_{N_0,r,p} H^p \sum_{h=-1}^{+\infty} 2^{shp} \|\widehat{u}_h\|^p_{L^p} \leq \widehat{C}_{N_0,p,r,s} H^p \|u\|^p_{\mathcal{F}L^p_{s,M}}.$$

This ends the proof of Lemma 5. □

Remark 8 Since for $0 \leq \delta \leq 1$ and $r > \frac{n}{\mu_* q}$ we have $s + (1-\delta)\left(r - \frac{n}{\mu_* q}\right) \geq s$, as an immediate consequence of Lemma 5, we get the boundedness of T_2 as a linear operator $T_2 : \mathcal{F}L^p_{s,M} \to \mathcal{F}L^p_{s,M}$.

Lemma 6 *For all $s < r$, T_3 extends to a linear bounded operator*

$$T_3 : \mathcal{F}L^p_{s+(\delta-1)\left(r-\frac{n}{\mu_* q}\right),M} \to \mathcal{F}L^p_{s,M} \tag{82}$$

and there exists a positive constant $C = C_{s,p,r}$ such that

$$\|T_3\|_{\mathcal{F}L^p_{s+(\delta-1)\left(r-\frac{n}{\mu_* q}\right),M} \to \mathcal{F}L^p_{s,M}} \leq C \sup_{h\geq -1} 2^{-\delta\left(r-\frac{n}{\mu_* q}\right)h} \|d_h\|_{\mathcal{F}L^p_{r,M}}. \tag{83}$$

Moreover for $0 \leq \delta < 1$ and arbitrary $\varepsilon > 0$, T_3 extends to a linear bounded operator

$$T_3 : \mathcal{F}L^p_{\varepsilon+\delta r-(\delta-1)\frac{n}{\mu_* q},M} \to \mathcal{F}L^p_{r,M} \tag{84}$$

and there exists a positive constant $C = C_{r,p,\varepsilon}$ such that:

$$\|T_3\|_{\mathcal{F}L^p_{\varepsilon+\delta r-(\delta-1)\frac{n}{\mu_* q},M} \to \mathcal{F}L^p_{r,M}} \leq C \sup_{h\geq -1} 2^{-\delta\left(r-\frac{n}{\mu_* q}\right)h} \|d_h\|_{\mathcal{F}L^p_{r,M}}. \tag{85}$$

Proof Let us prove the first statement. For $N_0 > 0$ sufficiently large we have

$$\text{supp}\, \widehat{d_{h,\ell} u_h} \subseteq \mathcal{C}^T_\ell, \quad \text{for } \ell \geq N_0 - 1,\ -1 \leq h \leq \ell - N_0.$$

Hence Proposition 3 and Young's inequality imply, for finite $p \geq 1$,

$$\|T_3 u\|^p_{\mathcal{F}L^p_M} \leq C \sum_{\ell=N_0-1}^{+\infty} 2^{s\ell p} \left\| \sum_{h=-1}^{\ell-N_0} \widehat{d_{h,\ell} u_h} \right\|^p_{L^p} = C \sum_{\ell=N_0-1}^{+\infty} 2^{s\ell p} \left\| \sum_{h=-1}^{\ell-N_0} \widehat{d_{h,\ell}} * \widehat{u_h} \right\|^p_{L^p}$$
$$\leq C \sum_{\ell=N_0-1}^{+\infty} 2^{s\ell p} \left(\sum_{h=-1}^{\ell-N_0} \|\widehat{d_{h,\ell}}\|_{L^p} \|\widehat{u_h}\|_{L^1} \right)^p$$
$$= C \sum_{\ell=N_0-1}^{+\infty} \left(\sum_{h=-1}^{\ell-N_0} 2^{(s-r)\ell} 2^{r\ell} \|\widehat{d_{h,\ell}}\|_{L^p} \|\widehat{u_h}\|_{L^1} \right)^p \tag{86}$$

(with obvious modifications in the case of $p = +\infty$); on the other hand, condition (a) and Proposition 2 yield

$$\sum_{\ell=-1}^{+\infty} 2^{r\ell p} \|\widehat{d}_{h,\ell}\|^p_{L^p} \leq H 2^{\delta\left(r-\frac{n}{\mu * q}\right)ph}, \quad \text{for } h \geq -1,$$

hence

$$2^{r\ell} \|\widehat{d}_{h,\ell}\|_{L^p} \leq H 2^{\delta\left(r-\frac{n}{\mu * q}\right)h}, \quad \text{for } \ell \geq -1, \tag{87}$$

where H is the constant in (81).

Combining (86), (87) and using Bernstein's inequality

$$2^{-\frac{n}{\mu * q}h} \|\widehat{u}_h\|_{L^1} \leq C \|\widehat{u}_h\|_{L^p} \tag{88}$$

we get

$$\|T_3 u\|_{\mathcal{F}L^p_M} \leq C H^p \sum_{\ell=N_0-1}^{+\infty} \left(\sum_{h=-1}^{\ell-N_0} 2^{(s-r)\ell} 2^{\delta\left(r-\frac{n}{\mu * q}\right)h} \|\widehat{u}_h\|_{L^1} \right)^p$$
$$= C H^p \sum_{\ell=N_0-1}^{+\infty} \left(\sum_{h=-1}^{\ell-N_0} 2^{(s-r)(\ell-h)} 2^{(s-r)h} 2^{\delta\left(r-\frac{n}{\mu * q}\right)h} 2^{\frac{n}{\mu * q}h} 2^{-\frac{n}{\mu * q}h} \|\widehat{u}_h\|_{L^1} \right)^p$$
$$= C H^p \sum_{\ell=N_0-1}^{+\infty} \left(\sum_{h=-1}^{\ell-N_0} 2^{(s-r)(\ell-h)} 2^{\left(s+(\delta-1)\left(r-\frac{n}{\mu * q}\right)\right)h} 2^{-\frac{n}{\mu * q}h} \|\widehat{u}_h\|_{L^1} \right)^p$$
$$\leq C H^p \sum_{\ell=N_0-1}^{+\infty} \left(\sum_{h=-1}^{\ell-N_0} 2^{(s-r)(\ell-h)} 2^{\left(s+(\delta-1)\left(r-\frac{n}{\mu * q}\right)\right)h} \|\widehat{u}_h\|_{L^p} \right)^p.$$

The last quantity above is the general term of the discrete convolution of the sequences

$$b := \{2^{(s-r)k}\}_{k \geq N_0-1}, \qquad c := \{2^{\left(s+(\delta-1)\left(r-\frac{n}{\mu * q}\right)\right)k} \|\widehat{u_k}\|_{L^p}\}_{k \geq N_0-1}.$$

Since $b \in \ell^1$, for $s < r$, discrete Young's inequality and Proposition 2 yield

$$\|T_3 u\|^p_{\mathcal{F}L^p_M} \leq C H^p \|b\|_{\ell^1} \|c\|_{\ell^p} \leq \tilde{C} H^p \sum_{\ell \geq -1} 2^{\left(s+(\delta-1)\left(r-\frac{n}{\mu_* q}\right)\right)\ell p} \|\widehat{u_\ell}\|^p_{L^p}$$
$$\leq \hat{C} H^p \|u\|_{\mathcal{F}L^p_{s+(\delta-1)\left(r-\frac{n}{\mu_* q}\right),M}}.$$

This proves the first continuity property (82) together with estimate (83).

Let us now prove the second statement of Lemma 6, so we assume that $\delta \in [0, 1[$. For an arbitrary $\varepsilon > 0$ similar arguments to those used above give the following estimate

$$\begin{aligned}
\|T_3 u\|^p_{\mathcal{F}L^p_{r,M}} &\leq C \sum_{\ell=N_0-1}^{+\infty} 2^{r\ell p} \left\| \sum_{h=-1}^{\ell-N_0} \widehat{d_{h,\ell}} * \widehat{u_h} \right\|^p_{L^p} \\
&\leq C \sum_{\ell=N_0-1}^{+\infty} 2^{r\ell p} \left(\sum_{h=-1}^{\ell-N_0} \|\widehat{d_{h,\ell}}\|_{L^p} \|\widehat{u}_h\|_{L^1} \right)^p \\
&= C \sum_{\ell=N_0-1}^{+\infty} \left(\sum_{h=-1}^{\ell-N_0} 2^{-\varepsilon h} 2^{\varepsilon h} 2^{r\ell} \|\widehat{d_{h,\ell}}\|_{L^p} \|\widehat{u}_h\|_{L^1} \right)^p \\
&\leq C \sum_{\ell=N_0-1}^{+\infty} \left(\sum_{h=-1}^{\ell-N_0} 2^{-\varepsilon h q} \right)^{p/q} \left(\sum_{h=-1}^{\ell-N_0} 2^{\varepsilon h p} 2^{r\ell p} \|\widehat{d_{h,\ell}}\|^p_{L^p} \|\widehat{u}_h\|^p_{L^1} \right) \\
&\leq C_{\varepsilon,p} \sum_{\ell=N_0-1}^{+\infty} \sum_{h=-1}^{\ell-N_0} 2^{\varepsilon h p} 2^{r\ell p} \|\widehat{d_{h,\ell}}\|^p_{L^p} \|\widehat{u}_h\|^p_{L^1} \\
&= C_{\varepsilon,p} \sum_{h=-1}^{+\infty} 2^{\varepsilon h p} \sum_{\ell \geq h+N_0} 2^{r\ell p} \|\widehat{d_{h,\ell}}\|^p_{L^p} \|\widehat{u}_h\|^p_{L^1},
\end{aligned} \tag{89}$$

where in the last quantity above the summation index order was interchanged.

Again from condition (a) and Proposition 2

$$\sum_{\ell \geq h+N_0} 2^{r\ell p} \|\widehat{d_{h,\ell}}\|^p_{L^p} \leq C_{r,p} \|d_h\|_{\mathcal{F}L^p_{r,M}} \leq C_{r,p} H^p 2^{\delta\left(r-\frac{n}{\mu_* q}\right)hp},$$

with H defined in (81). Using the above to estimate the right-hand side of (89), Bernstein's inequality (88) and Proposition 2 we obtain

$$\begin{aligned}\|T_3u\|^p_{\mathcal{F}L^p_{r,M}} &\le C_{r,\varepsilon,p}H^p\sum_{h=-1}^{+\infty}2^{\varepsilon hp}2^{\delta\left(r-\frac{n}{\mu_*q}\right)hp}\|\widehat{u}_h\|^p_{L^1}\\ &\le C_{r,\varepsilon,p}H^p\sum_{h=-1}^{+\infty}2^{\left(\varepsilon+\delta r-(\delta-1)\frac{n}{\mu_*q}\right)hp}\|\widehat{u}_h\|^p_{L^p}\\ &\le C_{r,p,\varepsilon}H^p\|u\|^p_{\mathcal{F}L^p_{\varepsilon+\delta r-(\delta-1)\frac{n}{\mu_*q},M}}.\end{aligned}$$

This completes the proof of the continuity (84) together with estimate (85). □

Remark 9 Let us collect some observations concerning Lemma 6.

We first notice that for $s < r$ the boundedness of T_3 as a linear operator $T_3 : \mathcal{F}L^p_{s,M} \to \mathcal{F}L^p_{s,M}$ follows as an immediate consequence of (82), since $\mathcal{F}L^p_{s,M} \hookrightarrow \mathcal{F}L^p_{s+(\delta-1)\left(r-\frac{n}{\mu_*q}\right),M}$ for δ and r under the assumptions of Lemma 6.

Regarding the second part of Lemma 6 (see (84)), we notice that the Fourier-Lebesgue esponent $\varepsilon+\delta r-(\delta-1)\frac{n}{\mu_*q}$, with any positive ε, is a little more restrictive than the one that should be recovered from the exponent $s+(\delta-1)\left(r-\frac{n}{\mu_*q}\right)$, in the first part of the Lemma, in the limiting case as $s \to r$.

Notice eventually that when $0<\varepsilon<(1-\delta)\left(r-\frac{n}{\mu_*q}\right)$ is considered in the second part of the statement of Lemma 6, then $\varepsilon+\delta r-(\delta-1)\frac{n}{\mu_*q} < r$. Hence we get the boundedness of T_3, as a linear operator $T_3 : \mathcal{F}L^p_{r,M} \to \mathcal{F}L^p_{r,M}$, as long as $0 \le \delta < 1$, as an immediate consequence of the boundedness (84).

5 Calculus for pseudodifferential operators with smooth symbols

In this section we investigate the properties of pseudodifferential operators with M-homogeneous smooth symbols introduced in Sect. 2.3.

At first notice that, despite M-weight (3) is not smooth in $\mathbb{R}^n$, for an arbitrary vector $M=(\mu_1,\dots,\mu_n)\in\mathbb{R}^n_+$, one can always find an *equivalent* weight which is also a smooth symbol in the class S^1_M.

More precisely, in view of [11, Proposition 2.9], the following proposition holds true.

Proposition 4 *For any vector $M=(\mu_1,\dots,\mu_n)\in\mathbb{R}^n_+$ there exists a symbol $\pi=\pi_M(\xi)\in S^1_M$, independent of x,which is equivalent to the M-weight (3), in the sense that a positive constant C exists such that*

$$\frac{1}{C}\pi_M(\xi)\le\langle\xi\rangle_M\le C\pi_M(\xi),\quad \forall\,\xi\in\mathbb{R}^n. \tag{90}$$

In view of the subsequent analysis, it is worth noticing that in the case when the vector M has positive integer components, in Proposition 4 we can take $\pi_M(\xi)=\langle\xi\rangle_M$.

5.1 Symbolic calculus in $S^m_{M,\delta,\kappa}$

The symbolic calculus can be developed for classes $S^m_{M,\delta,\kappa}$, thus pseudodifferential operators with symbol in $S^m_{M,\delta,\kappa}$ form a self-contained sub-algebra of the algebra of operators with symbols in $S^m_{M,\delta}$, for $m \in \mathbb{R}$, $\kappa > 0$ and $0 \le \delta < \mu_*/\mu^*$. The main properties of symbolc calculus are summarized in the following result.

Proposition 5 (i) *For $m, m' \in \mathbb{R}$, $\kappa > 0$ and $\delta, \delta' \in [0, 1]$, consider $a(x,\xi) \in S^m_{M,\delta,\kappa}$, $b(x,\xi) \in S^{m'}_{M,\delta',\kappa}$, $\theta, \nu \in \mathbb{Z}^n_+$. Then*

$$\partial^\theta_\xi \partial^\nu_x a(x,\xi) \in S^{m-\langle\theta,1/M\rangle+\delta\langle\nu,1/M\rangle}_{M,\delta,\kappa}, \quad (ab)(x,\xi) \in S^{m+m'}_{M,\max\{\delta,\delta'\},\kappa}. \tag{91}$$

(ii) *Let $\{m_j\}_{j=0}^{+\infty}$ be a sequence of real numbers satisfying:*

$$m_j > m_{j+1}, \quad j = 0, 1, \dots \quad \textit{and} \quad \lim_{j\to+\infty} m_j = -\infty \tag{92}$$

and $\{a_j\}_{j=0}^{+\infty}$ be a sequence of symbols $a_j(x,\xi) \in S^{m_j}_{M,\delta,\kappa}$ for each integer $j \ge 0$. Then there exists a unique (up to a remainder in $S^{-\infty}$) symbol $a(x,\xi) \in S^{m_0}_{M,\delta,\kappa}$ such that

$$a - \sum_{j<N} a_j \in S^{m_N}_{M,\delta,\kappa}, \quad \textit{for all integers } N > 0. \tag{93}$$

(iii) *Let $a(x,\xi)$ and $b(x,\xi)$ be two symbols as in (i), and assume that $0 \le \delta' < \mu_*/\mu^*$. Then the product $c(x, D) := a(x, D)b(x, D)$ is a pseudodifferential operator with symbol $c(x,\xi) = (a\sharp b)(x,\xi) \in S^{m+m'}_{M,\delta'',\kappa}$, where $\delta'' := \max\{\delta, \delta'\}$; moreover this symbol satisfies*

$$a\sharp b - \sum_{|\alpha|<N} \frac{(-i)^{|\alpha|}}{\alpha!} \partial^\alpha_\xi a\, \partial^\alpha_x b \in S^{m+m'-(1/\mu^*-\delta'/\mu_*)N}_{M,\delta'',\kappa}, \quad \textit{for all integers } N > 0. \tag{94}$$

Proof *(i)*: From estimates (16), (17), it is very easy to check that for any multi-index $\theta \in \mathbb{Z}^n_+$

$$a(x,\xi) \in S^m_{M,\delta,\kappa} \quad \text{implies} \quad \partial^\theta_\xi a(x,\xi) \in S^{m-\langle\theta,1/M\rangle}_{M,\delta,\kappa};$$

hence we can limit the proof of (i) to $\theta = 0$ and an arbitrary $\nu \in \mathbb{Z}^n_+$, $\nu \ne 0$.

Let $\alpha, \beta \in \mathbb{Z}^n_+$ be arbitrary multi-indices and assume, for the first, that $\langle\beta, 1/M\rangle \ne \kappa$; if $\langle\nu + \beta, 1/M\rangle \ne \kappa$, we then get

$$\begin{aligned}|\partial^\alpha_\xi \partial^\beta_x \left(\partial^\nu_x a\right)(x,\xi)| &\le C_{\nu,\alpha,\beta} \langle\xi\rangle_M^{m-\langle\alpha,1/M\rangle+\delta(\langle\nu+\beta,1/M\rangle-\kappa)_+} \\ &\le C_{\nu,\alpha,\beta} \langle\xi\rangle_M^{m-\langle\alpha,1/M\rangle+\delta\langle\nu,1/M\rangle+\delta(\langle\beta,1/M\rangle-\kappa)_+},\end{aligned} \tag{95}$$

in view of (16) and the sub-additivity inequality $(x + y)_+ \le x_+ + y_+$.

Assume now that $\langle \nu+\beta, 1/M\rangle = \kappa$; then

$$|\partial_\xi^\alpha \partial_x^\beta \left(\partial_x^\nu a\right)(x,\xi)| \le C_{\nu,\alpha,\beta} \langle \xi\rangle_M^{m-\langle \alpha, 1/M\rangle} \log(1+\langle \xi\rangle_M^\delta), \tag{96}$$

in view of (17). Since $\langle \nu+\beta, 1/M\rangle = \kappa$ and $\langle \beta, 1/M\rangle \ne \kappa$ imply $\langle \beta, 1/M\rangle < \kappa$ and $\langle \nu, 1/M\rangle > 0$, then

$$\log(1+\langle \xi\rangle_M^\delta) \le C_{\nu,\beta} \langle \xi\rangle_M^{\delta\langle \nu, 1/M\rangle} \equiv C_{\nu,\beta} \langle \xi\rangle_M^{\delta\langle \nu, 1/M\rangle + \delta(\langle \beta, 1/M\rangle - \kappa)_+},$$

which, combined with (96), leads again to (95).

Assume now that $\langle \beta, 1/M\rangle = \kappa$. Since also $\langle \nu, 1/M\rangle > 0$, from (16) we get

$$\begin{aligned} |\partial_\xi^\alpha \partial_x^\beta \left(\partial_x^\nu a\right)(x,\xi)| &\le C_{\nu,\alpha,\beta} \langle \xi\rangle_M^{m-\langle \alpha, 1/M\rangle + \delta(\langle \nu+\beta, 1/M\rangle - \kappa)_+} \\ &\le C'_{\nu,\alpha,\beta} \langle \xi\rangle_M^{m-\langle \alpha, 1/M\rangle + \delta\langle \nu, 1/M\rangle} \log(1+\langle \xi\rangle_M^\delta), \end{aligned}$$

because $(\langle \nu+\beta, 1/M\rangle - \kappa)_+ = \langle \nu+\beta, 1/M\rangle - \kappa = \langle \nu, 1/M\rangle$ and we also use the trivial inequality

$$\log 2 \le \log(1+\langle \xi\rangle_M^\delta), \quad \forall \xi \in \mathbb{R}^n. \tag{97}$$

The preceding calculations show that $\partial_x^\nu a(x,\xi) \in S_{M,\delta,\kappa}^{m+\delta\langle \nu, 1/M\rangle}$.

Similar trivial, while overloading, arguments can be used to prove the second statement of (i) concerning the product of symbols.

(ii) It is known from the symbolic calculus in classes $S^m_{M,\delta}$, cf. [10, Proposition 2.3], that for a sequence of symbols $\{a_j\}_{j=0}^{+\infty}$, obeying the assumptions made in (ii), there exists $a(x,\xi) \in S^{m_0}_{M,\delta}$, which is unique up to a remainder in $S^{-\infty}$, such that

$$a - \sum_{j<N} a_j \in S^{m_N}_{M,\delta}, \quad \text{for all integers } \ N > 0. \tag{98}$$

It remains to check that $a(x,\xi)$ actually belongs to $S^{m_0}_{M,\delta,\kappa}$, namely its derivatives satisfy inequalities (16), (17). In view of (98), for any positive integer N, the symbol $a(x,\xi)$ can be represented in the form

$$a(x,\xi) = a_N(x,\xi) + R_N(x,\xi), \tag{99}$$

where $a_N := \sum\limits_{j<N} a_j$ e $R_N \in S^{m_N}_{M,\delta}$.

Since $\lim\limits_{j\to+\infty} m_j = -\infty$, for all $\alpha, \beta \in \mathbb{Z}^n_+$ an integer $N_{\alpha,\beta} > 0$ can be found such that

$$\begin{aligned} &m_{N_{\alpha,\beta}} + \delta\langle \beta, 1/M\rangle \le m_0 + \delta(\langle \beta, 1/M\rangle - \kappa)_+, \quad \text{if } \langle \beta, 1/M\rangle \ne \kappa, \\ &m_{N_{\alpha,\beta}} + \delta\kappa \le m_0, \quad \text{if } \langle \beta, 1/M\rangle = \kappa, \end{aligned} \tag{100}$$

hence let a be represented in form (99) with $N = N_{\alpha,\beta}$ (from the above inequalities $N_{\alpha,\beta}$ can be chosen independent of α, as a matter of fact). Since $\{m_j\}$ is decreasing, from $a_j \in S^{m_j}_{M,\delta,\kappa}$ for every $j \geq 0$, we deduce at once that $a_{N_{\alpha,\beta}} \in S^{m_0}_{M,\delta,\kappa}$. As for the remainder $R_{N_{\alpha,\beta}}$, from $R_{N_{\alpha,\beta}} \in S^{m_{N_{\alpha,\beta}}}_{M,\delta}$, inequalities (100) and (97), we deduce

$$|\partial_\xi^\alpha \partial_x^\beta R_{N_{\alpha,\beta}}(x,\xi)| \leq C_{\alpha,\beta} \langle \xi \rangle_M^{m_{N_{\alpha,\beta}} - \langle \alpha, 1/M \rangle + \delta \langle \beta, 1/M \rangle}$$
$$\leq \begin{cases} C'_{\alpha,\beta} \langle \xi \rangle_M^{m_0 - \langle \alpha, 1/M \rangle + \delta(\langle \beta, 1/M \rangle - \kappa)_+}, & \text{if } \langle \beta, 1/M \rangle \neq \kappa, \\ C'_{\alpha,\beta} \langle \xi \rangle_M^{m_0 - \langle \alpha, 1/M \rangle} \log(1 + \langle \xi \rangle_M^\delta), & \text{if } \langle \beta, 1/M \rangle = \kappa. \end{cases}$$

From (99) with $N = N_{\alpha,\beta}$ and estimates above, we deduce

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \leq |\partial_\xi^\alpha \partial_x^\beta a_{N_{\alpha,\beta}}(x,\xi)| + |\partial_\xi^\alpha \partial_x^\beta R_{N_{\alpha,\beta}}(x,\xi)|$$
$$\leq \begin{cases} C''_{\alpha,\beta} \langle \xi \rangle_M^{m_0 - \langle \alpha, 1/M \rangle + \delta(\langle \beta \rangle - \kappa)_+}, & \text{if } \langle \beta, 1/M \rangle \neq \kappa, \\ C''_{\alpha,\beta} \langle \xi \rangle_M^{m_0 - \langle \alpha, 1/M \rangle} \log(1 + \langle \xi \rangle_M^\delta), & \text{if } \langle \beta, 1/M \rangle = \kappa \end{cases}$$

and, because of the arbitatriness of α and β, this shows that $a \in S^{m_0}_{M,\delta,\kappa}$.

(iii) By still referring to the symbolic calculus in classes $S^m_{M,\delta}$, cf [10, Proposition 2.5], it is known that the product of two pseudodifferential operators $a(x, D)$ and $b(x, D)$ with symbols like in the statement (iii) is again a pseudodifferential operator $c(x, D) = a(x, D)b(x, D)$ with symbol $c(x,\xi) = (a\sharp b)(x,\xi) \in S^{m+m'}_{M,\delta''}$, if $0 \leq \delta' < \mu_*/\mu^*$; moreover, such a symbol satisfies

$$c(x,\xi) - \sum_{|\alpha|<N} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha a(x,\xi) \partial_x^\alpha b(x,\xi) \in S^{m+m'-(1/\mu^* - \delta'/\mu_*)N}_{M,\delta''}, \quad N \geq 1. \tag{101}$$

To end up, it sufficient applying statements (i) and (ii) above to the sequence $\{c_k\}_{k=0}^{+\infty}$ of symbols

$$c_k(x,\xi) := \sum_{|\alpha|=k} \frac{(-i)^k}{\alpha!} \partial_\xi^\alpha a(x,\xi) \partial_x^\alpha b(x,\xi), \quad k = 0, 1, \ldots.$$

From statement (i) it is immediately seen that $c_k(x,\xi) \in S^{m+m'-(1/\mu^*-\delta'/\mu_*)k}_{M,\delta'',\kappa}$ for all integers $k \geq 0$. Since the sequence $\{m_k\}_{k=0}^{+\infty}$ of orders $m_k := m+m' - (1/\mu^* - \delta'/\mu_*)k$ is decreasing, in view of $0 \leq \delta' < \mu_*/\mu^*$, it follows from (ii) that a symbol $\tilde{c}(x,\xi) \in S^{m+m'}_{M,\delta'',\kappa}$ exists such that the same as (101) holds true with $\tilde{c}(x,\xi)$ instead of $c(x,\xi)$; moreover, from uniqueness of $c(x,\xi)$ (up to a symbol in $S^{-\infty}$), it also follows that $\tilde{c}(x,\xi) - c(x,\xi) \in S^{-\infty}$, hence the symbol $c(x,\xi)$ actually belongs to $S^{m+m'}_{M,\delta'',\kappa}$. □

5.2 Parametrix of an elliptic operator with symbol in $S^m_{M,\delta,\kappa}$

In order to perform the analysis of local and microlocal propagation of singularities of PDE on M-Fourier–Lebesgue spaces, cf. Sect. 7, this section is devoted to the construction of the parametrix of a M-elliptic operator with symbol in $S^m_{M,\delta,\kappa}$.

We first recall the notion of M-elliptic symbol, we are going to deal with, see [9,10].

Definition 7 We say that $a(x,\xi) \in S^m_{M,\delta}$, or the related operator $a(x,D)$, is M-elliptic if there are constants $c_0 > 0$ and $R > 1$ satisfying

$$|a(x,\xi)| \geq c_0 \langle \xi \rangle^m_M, \quad \forall (x,\xi) \in \mathbb{R}^{2n}, \quad |\xi|_M \geq R. \tag{102}$$

Proposition 6 *For $m \in \mathbb{R}$, $\kappa > 0$ and $0 \leq \delta < \mu_*/\mu^*$, let the symbol $a(x,\xi) \in S^m_{M,\delta,\kappa}$ be M-elliptic. Then there exists $b(x,\xi) \in S^{-m}_{M,\delta,\kappa}$ such that $b(x,D)$ is a parametrix of the operator $a(x,D)$, i.e.*

$$b(x,D)a(x,D) = I + l(x,D), \qquad a(x,D)b(x,D) = I + r(x,D), \tag{103}$$

where I is the identity operator and $l(x,D)$, $r(x,D)$ are pseudodifferential operators with symbols $l(x,\xi), r(x,\xi) \in S^{-\infty}$.

Proof The proof follows the standard arguments employed in construcing the parametrix of an elliptic operator, see e.g. [5].

The first step consists to define a symbol $b_0(x,\xi)$ to be the inverse of $a(x,\xi)$, for sufficiently large ξ, that is

$$b_0(x,\xi) := \langle \xi \rangle^{-m}_M F\left(\langle \xi \rangle^{-m}_M a(x,\xi)\right), \tag{104}$$

with some function $F = F(z) \in C^\infty(\mathbb{C})$ satisfying $F(z) = 1/z$ for $|z| \geq c_0$ and where c_0 is the positive constant from (102). From the symbolic calculus in the framework of $S^\infty_{M,\delta}$ (cf. [10]), it is easily shown that $b_0(x,\xi) \in S^{-m}_{M,\delta}$ and $\rho_1(x,\xi) := (a\sharp b_0)(x,\xi) - 1 \in S^{m-(1/\mu^*-\delta/\mu_*)}_{M,\delta}$, where, according to the notation introduced in Proposition 5-(iii), $a\sharp b_0$ stands for the symbols of the product $a(x,D)b_0(x,D)$.

Then an operator $b(x,D)$ satisfying the second identity in (103) (that is a right-parametrix of $a(x,D)$) is defined as $b(x,D) := b_0(x,D)\rho(x,D)$ and where $\rho(x,D)$ is given by the Neumann-type series $\rho(x,D) = \sum_{j=0}^{+\infty} \rho_1^j(x,D)$; more precisely, $\rho(x,D)$ is the pseudodifferential operator with symbol associated to the sequence of symbols $\rho_j(x,\xi) \in S^{-(1/\mu^*-\delta/\mu_*)j}_{M,\delta}$ recursively defined by

$$\rho_0 := 1 \quad \text{and} \quad \rho_j := \rho_1 \sharp \rho_{j-1}, \qquad \text{for } j = 1, 2, \ldots. \tag{105}$$

Since the sequence of orders $-(1/\mu^* - \delta/\mu_*)j$ tends to $-\infty$, once again in view of the symbolic calculus in $S^\infty_{M,\delta}$ (cf. [10]), a symbol $\rho(x,\xi) \in S^0_{M,\delta}$ such that

$$\rho - \sum_{j<N} \rho_j \in S^{-(1/\mu^*-\delta/\mu_*)N}_{M,\delta}, \quad \text{for all integers } N \geq 1, \tag{106}$$

is defined uniquely, up to symbols in $S^{-\infty}$.

One can finally show that $b(x, D)$, constructed as above, is a (two sided) parametrix of $a(x, D)$, see e.g. [5, Ch. 4] for more details.

In view of Proposition 5, to end up it is sufficient to show that the symbol $b_0(x,\xi) \in S^{-m}_{M,\delta}$, defined in (104), actually belongs to $S^{-m}_{M,\delta,\kappa}$, that is its derivatives satisfy estimates (16), (17). Since these estimates only require a more specific behavior of $x-$derivatives, compared to a generic symbol in $S^{\infty}_{M,\delta}$, we may reduce to check their validity for $x-$derivatives alone. Because $\langle\xi\rangle^{-m}_{M} a(x,\xi) \in S^0_{M,\delta,\kappa}$, we are going to only treat the case of a symbol $a(x,\xi) \in S^0_{M,\delta,\kappa}$.

For an arbitrary nonzero multi-index $\beta \neq 0$, from Faà di Bruno's formula, we first recover

$$|\partial^{\beta}_x b_0(x,\xi)| \leq \sum_{k=1}^{|\beta|} C_k \sum_{\beta^1+\dots+\beta^k=\beta} |\partial^{\beta^1}_x a(x,\xi)| \dots |\partial^{\beta^k}_x a(x,\xi)|, \tag{107}$$

where C_k is a suitable positive constant depending only on $k \geq 0$ (notice that the function F is bounded in $\mathbb{C}$ together with all its derivatives), and where, for each integer k satisfying $1 \leq k \leq |\beta|$, the second sum in the right-hand side above is extended over all systems $\{\beta^1, \dots, \beta^k\}$ of nonzero multi-indices β^j ($j = 1, \dots, k$) such that $\beta^1 + \dots + \beta^k = \beta$.

To apply estimates (16), (17), different cases must be considered separately.

Let us first assume that $\langle\beta, 1/M\rangle \neq \kappa$. Since $a \in S^0_{M,\delta,\kappa}$, we have

$$|\partial^{\beta^j}_x a(x,\xi)| \leq C_j \langle\xi\rangle^{\delta(\langle\beta^j,1/M\rangle-\kappa)_+}_M \quad \text{or} \quad |\partial^{\beta^j}_x a(x,\xi)| \leq C_j \log(1+\langle\xi\rangle^{\delta}_M), \tag{108}$$

for all integers $1 \leq k \leq |\beta|$ and $1 \leq j \leq k$, according to whether $\langle\beta^j, 1/M\rangle \neq \kappa$ or $\langle\beta^j, 1/M\rangle = \kappa$, and suitable constants $C_j > 0$.

If $\langle\beta, 1/M\rangle < \kappa$ then $\langle\beta^j, 1/M\rangle < \kappa$ for all $j = 1, \dots, k$ and every $1 \leq k \leq |\beta|$, and

$$|\partial^{\beta}_x b_0(x,\xi)| \leq C_\beta \equiv C_\beta \langle\xi\rangle^{\delta((\langle\beta,1/M\rangle-\kappa)_+}_M$$

follows at once from (107) and (108), with suitable $C_\beta > 0$.

Assume now $\langle\beta, 1/M\rangle > \kappa$, so that, for a given integer $1 \leq k \leq |\beta|$ and an arbitrary system $\{\beta^1, \dots, \beta^k\}$ of multi-indices satisfying $\beta^1 + \dots + \beta^k = \beta$, it could be either $\langle\beta^j, 1/M\rangle \neq \kappa$ or $\langle\beta^j, 1/M\rangle = \kappa$ for different indices $j = 1, \dots, k$; up to a reordering of its elements, let $\{\beta^1, \dots, \beta^k\}$ be split into the sub-systems $\{\beta^1, \dots, \beta^{k'}\}$ and $\{\beta^{k'+1}, \dots, \beta^k\}$ (for an integer k' with $1 \leq k' < k$) such that $\langle\beta^j, 1/M\rangle \neq \kappa$ for all $1 \leq j \leq k'$ and $\langle\beta^\ell, 1/M\rangle = \kappa$ for all $k'+1 \leq \ell \leq k$.[2] In such a case, from (107) and (108) we get

$$\begin{aligned}|\partial^{\beta}_x b_0(x,\xi)| \leq \sum_{k=1}^{|\beta|} C_k \sum_{\beta^1+\dots+\beta^k=\beta} &\langle\xi\rangle^{\delta\{(\langle\beta^1,1/M\rangle-\kappa)_+ + \dots + (\langle\beta^{k'},1/M\rangle-\kappa)_+\}}_M \\ &\times \left(\log(1+\langle\xi\rangle^{\delta}_M)\right)^{k-k'}.\end{aligned} \tag{109}$$

[2] Of course when $k = 1$ then only $\langle\beta^1, 1/M\rangle \equiv \langle\beta, 1/M\rangle > \kappa$ can occur.

Under the previous assumptions, it can be shown that

$$(\langle\beta^1, 1/M\rangle - \kappa)_+ + \cdots + (\langle\beta^{k'}, 1/M\rangle - \kappa)_+ \le (\langle\beta', 1/M\rangle - \kappa)_+,$$

where we have set $\beta' := \beta^1 + \cdots + \beta^{k'}$. Suppose $\langle\beta', 1/M\rangle \le \kappa$ (thus $(\langle\beta', 1/M\rangle - \kappa)_+ = 0$); since $\langle\beta, 1/M\rangle > \kappa$, we have

$$\begin{aligned} &\langle\xi\rangle_M^{\delta\{(\langle\beta^1,1/M\rangle-\kappa)_+ + \cdots + (\langle\beta^{k'},1/M\rangle-\kappa)_+\}} \left(\log(1+\langle\xi\rangle_M^{\delta})\right)^{k-k'} \\ &\quad \le \langle\xi\rangle_M^{\delta(\langle\beta',1/M\rangle-\kappa)_+} \left(\log(1+\langle\xi\rangle_M^{\delta})\right)^{k-k'} \equiv \left(\log(1+\langle\xi\rangle_M^{\delta})\right)^{k-k'} \\ &\quad \le c_{\beta,k,k'}\langle\xi\rangle_M^{\delta(\langle\beta,1/M\rangle-\kappa)} \equiv c_{\beta,k,k'}\langle\xi\rangle_M^{\delta(\langle\beta,1/M\rangle-\kappa)_+}. \end{aligned} \tag{110}$$

Suppose now $\langle\beta', 1/M\rangle > \kappa$ (hence $(\langle\beta', 1/M\rangle - \kappa)_+ = \langle\beta', 1/M\rangle - \kappa$). Since $\langle\beta, 1/M\rangle > \langle\beta', 1/M\rangle$, we get

$$\begin{aligned} &\langle\xi\rangle_M^{\delta\{(\langle\beta^1,1/M\rangle-\kappa)_+ + \cdots + (\langle\beta^{k'},1/M\rangle-\kappa)_+\}} \left(\log(1+\langle\xi\rangle_M^{\delta})\right)^{k-k'} \\ &\quad \le \langle\xi\rangle_M^{\delta(\langle\beta',1/M\rangle-\kappa)_+} \left(\log(1+\langle\xi\rangle_M^{\delta})\right)^{k-k'} \\ &\quad \equiv \langle\xi\rangle_M^{\delta(\langle\beta',1/M\rangle-\kappa)} \left(\log(1+\langle\xi\rangle_M^{\delta})\right)^{k-k'} \\ &\quad \le c_{\beta,\beta',k,k'}\langle\xi\rangle_M^{\delta\{(\langle\beta',1/M\rangle-\kappa)+(\langle\beta,1/M\rangle-\langle\beta',1/M\rangle)\}} \\ &\quad = c_{\beta,\beta',k,k'}\langle\xi\rangle_M^{\delta(\langle\beta,1/M\rangle-\kappa)} \equiv c_{\beta,\beta',k,k'}\langle\xi\rangle_M^{\delta(\langle\beta,1/M\rangle-\kappa)_+}. \end{aligned} \tag{111}$$

In the boarder cases of a system $\{\beta^1, \ldots, \beta^k\}$ where either $\langle\beta^j, 1/M\rangle \neq \kappa$ for all j or $\langle\beta^j, 1/M\rangle = \kappa$ for all j,[3] all preceding arguments can be repeated, by formally taking $k' = k$ in (110) or $\beta' = 0$ and $k' = 0$ in (111) respectively; thus we end up with the same estimates as above. Using (110), (111) in the right-hand side of (109) leads to

$$|\partial_x^\beta b_0(x, \xi)| \le C_\beta \langle\xi\rangle_M^{\delta((\langle\beta,1/M\rangle-\kappa)_+}.$$

□

5.3 Continuity of pseudodifferential operators with symbols in $S^m_{M,\delta,\kappa}$

Throughout the rest of this section, we assume that $M \in \mathbb{R}^n_+$ has all integer components. The Fourier-Lebesgue continuity of pseudodifferential operators with symbols in $S^m_{M,\delta,\kappa}$ is recovered as a consequence of Theorem 1.

Taking advantage from growing estimates (16), (17), we first analyze the relations between smooth *local* symbols of type $S^m_{M,\delta,\kappa}$ and symbols of limited Fourier–Lebesgue smoothness introduced in Sect. 2.2.

Proposition 7 *For* $M = (\mu_1, \ldots, \mu_n) \in \mathbb{N}^n$, $m \in \mathbb{R}$, $\delta \in [0, 1]$ *and* $\kappa > 0$, *let the symbol* $a(x, \xi) \in S^m_{M,\delta,\kappa}$ *satisfy the localization condition (19) for some compact set*

[3] Notice that, under $\langle\beta, 1/M\rangle > \kappa$, this second case can only occur when $k \ge 2$.

$\mathcal{K} \subset \mathbb{R}^n$. The for all integers $N \geq 0$ and multi-indices $\alpha \in \mathbb{Z}^n_+$ there exists a postive constant $C_{\alpha,N,\mathcal{K}}$ such that:

$$\langle \eta \rangle_M^N |\partial_\xi^\alpha \hat{a}(\eta, \xi)| \leq C_{\alpha,N,\mathcal{K}} \langle \xi \rangle_M^{m-\langle \alpha, 1/M \rangle + \delta(N-\kappa)_+}, \quad \text{if } N \neq \kappa, \tag{112}$$

$$\langle \eta \rangle_M^N |\partial_\xi^\alpha \hat{a}(\eta, \xi)| \leq C_{\alpha,N,\mathcal{K}} \langle \xi \rangle_M^{m-\langle \alpha, 1/M \rangle} \log(1 + \langle \xi \rangle_M^\delta), \quad \text{if } N = \kappa, \tag{113}$$

where $\hat{a}(\eta, \xi)$ is the partial Fourier transform of $a(x, \xi)$ with respect to x:

$$\hat{a}(\eta, \xi) := \widehat{a(\cdot, \xi)}(\eta), \quad \forall (\eta, \xi) \in \mathbb{R}^{2n}.$$

Proof For an arbitrary integer $N \geq 0$ we estimate

$$\langle \eta \rangle_M^N \leq C_N \sum_{\langle \beta, 1/M \rangle \leq N} |\eta^\beta|, \quad \forall \eta \in \mathbb{R}^n, \tag{114}$$

with some positive constant $C_N > 0$ (independent of M), hence for any $\alpha \in \mathbb{Z}^n_+$

$$\langle \eta \rangle_M^N |\partial_\xi^\alpha \hat{a}(\eta, \xi)| \leq C_N \sum_{\langle \beta, 1/M \rangle \leq N} |\eta^\beta \partial_\xi^\alpha \hat{a}(\eta, \xi)| = C_N \sum_{\langle \beta, 1/M \rangle \leq N} |\widehat{\partial_x^\beta \partial_\xi^\alpha a}(\eta, \xi)|$$

$$= C_N \sum_{\langle \beta, 1/M \rangle \leq N} \left| \int_{\mathcal{K}} e^{-i\eta \cdot x} \partial_x^\beta \partial_\xi^\alpha a(x, \xi) dx \right| \leq C_N \sum_{\langle \beta, 1/M \rangle \leq N} \int_{\mathcal{K}} |\partial_x^\beta \partial_\xi^\alpha a(x, \xi)| dx.$$

Thus, we end up by using estimates (16), (17) under the integral sign above. □

Remark 10 Notice that estimates (113) are satisfied only when $\kappa > 0$ is an integer number.

As a consequence of Proposition 7 we get the proof of Theorem 2

Proof of Theorem 2 For κ satisfying (18), consider the estimates (112), (113) of $\hat{a}(\eta, \xi)$ with $N = N_* := [n/\mu_*]+1$. For sure, estimates (113) cannot occur, since N_* is smaller than κ, whereas estimates (112) reduce to

$$|\partial_\xi^\alpha \hat{a}(\eta, \xi)| \leq C_{\alpha,N_*,\mathcal{K}} \langle \xi \rangle_M^{m-\langle \alpha, 1/M \rangle} \langle \eta \rangle_M^{-N_*}, \quad \forall (\eta, \xi) \in \mathbb{R}^n. \tag{115}$$

On the other hand, the left inequality in (6) yields

$$\langle \eta \rangle_M^{-N_*} \leq C \langle \eta \rangle^{-\mu_* N_*}, \quad \forall \eta \in \mathbb{R}^n,$$

from which, $\langle \cdot \rangle_M^{-N_*} \in L^1(\mathbb{R}^n)$ follows, since $\mu_* N_* > n$. Then integrating in $\mathbb{R}^n_\eta$ both sides of (115) leads to

$$\|\partial_\xi^\alpha a(\cdot, \xi)\|_{\mathcal{F}L^1} \leq \tilde{C}_{\alpha,N,\mathcal{K}} \langle \xi \rangle_M^{m-\langle \alpha, 1/M \rangle}, \quad \forall \xi \in \mathbb{R}^n, \tag{116}$$

which are just estimates (11).

For an arbitrary integer $r > 0$, we consider again estimates (112), (113) of $\hat{a}(\eta, \xi)$ with $N = N_r := r + [n/\mu_*] + 1$. Notice that from (18)

$$N_r - \kappa < N_r - [n/\mu_*] - 1 = r, \quad \text{hence} \quad (N_r - \kappa)_+ \leq r_+ = r.$$

Then (112), (113) lead to

$$\langle \eta \rangle_M^r |\partial_\xi^\alpha \hat{a}(\eta, \xi)| \leq C_{\alpha, N_r, \mathcal{K}} \langle \eta \rangle_M^{-N_*} \langle \xi \rangle_M^{m - \langle \alpha, 1/M \rangle + \delta r}, \qquad \text{if } N_r \neq \kappa,$$

$$\langle \eta \rangle_M^r |\partial_\xi^\alpha \hat{a}(\eta, \xi)| \leq C_{\alpha, N_r, \mathcal{K}} \langle \eta \rangle_M^{-N_*} \langle \xi \rangle_M^{m - \langle \alpha, 1/M \rangle} \log(1 + \langle \xi \rangle_M^\delta), \text{ otherwise},$$

where $N_* = [n/\mu_*] + 1$ as before. Then using the trivial estimate

$$\log(1 + \langle \xi \rangle_M^\delta) \leq C_r \langle \xi \rangle_M^{\delta r}, \quad \forall \xi \in \mathbb{R}^n \tag{117}$$

and integrating in $\mathbb{R}^n_\eta$ both sides of inequalities above gives

$$\|\partial_\xi^\alpha a(\eta, \xi)\|_{\mathcal{F}L^1_{r,M}} \leq C_{\alpha, N_r, \mathcal{K}} \langle \xi \rangle_M^{m - \langle \alpha, 1/M \rangle + \delta r}, \quad \forall \xi \in \mathbb{R}^n, \tag{118}$$

which are nothing else estimates (12) with $p = 1$ (so $q = +\infty$). Together with (116), estimates above prove that $a(x, \xi) \in \mathcal{F}L^1_{r,M} S^m_{M,\delta}(N)$, for all integer numbers $r > 0$ and $N > 0$ arbitrarily large.

Then applying to $a(x, \xi)$ the result of Theorem 1 with $p = 1$ and an arbitrary integer $r > 0$ shows that $a(x, D)$ fulfils the boundedness in (20) with $p = 1$.

Now we are going to prove that the same symbol $a(x, \xi)$ also belongs to the class $\mathcal{F}L^\infty_{r,M}(N)$ with an arbitrary integer number $r > n/\mu_*$ and $N > 0$ arbitrarily large, so as to apply again Theorem 1 to $a(x, D)$ with $p = +\infty$. To do so, it is enough considering once again estimates (112) for $\hat{a}(\eta, \xi)$ with an arbitrary integer $N \equiv r > \kappa$; noticing that, under the assumption (18),

$$r - \kappa < r - n/\mu_*, \quad \text{hence} \quad (r - \kappa)_+ \leq (r - n/\mu_*)_+ = r - n/\mu_*,$$

estimates (112) just reduce to

$$\|\partial_\xi^\alpha a(\cdot, \xi)\|_{\mathcal{F}L^\infty_{r,M}} \leq C_{\alpha, r, \mathcal{K}} \langle \xi \rangle_M^{m - \langle \alpha, 1/M \rangle + \delta(r - n/\mu_*)}, \quad \forall \xi \in \mathbb{R}^n, \tag{119}$$

which are exactly estimates (12) with $p = +\infty$ (the number of $\xi-$derivatives which these estimates apply to can be chosen here arbitrarily large). So, as announced before, Theorem 1 can be applied to make the conclusion that the boundedness property (20) holds true for $a(x, D)$ with $p = +\infty$ and an arbitrary integer $r > \kappa$, and this shows that $a(x, D)$ also exhibits the boundedness in (20) with $p = +\infty$.

To recover (20) with an arbitrary summability exponent $1 < p < +\infty$ it is then enough to argue by complex interpolation through Riesz-Thorin's Theorem. □

Remark 11 Let us remark that assumption (19) on the x support of the symbol $a(x,\xi)$ amounts to say that the continuous prolongement of $a(x,D)$ on $\mathcal{F}L^p_{s+m,M}$ takes values in $\mathcal{F}L^p_{s,M}$ only *locally*, see the next Definition 8.

6 Decomposition of *M*-Fourier–Lebesgue symbols

As in the preceding Sect. 5, we will assume later on that vector $M=(\mu_1,\dots,\mu_n)$ has strictly positive integer components.

For $m,r\in\mathbb{R}$, $p\in[1,+\infty]$, $\delta\in[0,1]$, we set

$$\mathcal{F}L^p_{r,M}S^m_{M,\delta}:=\bigcap_{N=1}^{\infty}\mathcal{F}L^p_{r,M}S^m_{M,\delta}(N)$$

and $\mathcal{F}L^p_{r,M}S^m_M:=\mathcal{F}L^p_{r,M}S^m_{M,0}$. In order to develop a regularity theory of M-elliptic linear PDEs with M-homogeneous Fourier–Lebesgue coefficients, in the absence of a symbolic calculus for pseudodifferential operators with Fourier–Lebesgue symbols (in particular the lack of a parametrix of an M-elliptic operator with non smooth coefficients), following the approach of Taylor [23, §1.3], we introduce here a decomposition of a M-Fourier–Lebesgue symbol $a(x,\xi)\in\mathcal{F}L^p_{r,M}S^m_M$ as the sum of two terms: one is a M-homogeneous smooth symbol in $S^m_{M,\delta}$ and the other is still a Fourier–Lebesgue symbol of lower order, decreased from m by a positive quantity proportional to δ, where $0<\delta<1$ is given, while arbitrary.

Such a decomposition is made by applying to the symbol $a(x,\xi)$ a suitable "cut-off" Fourier multiplier, "splitting in the frequency space the (nonsmooth) coefficients of $a(x,\xi)$ as a sum of two contributions".

Let us first consider a C^∞−function ϕ such that $\phi(\xi)=1$ for $\langle\xi\rangle_M\le 1$ and $\phi(\xi)=0$ for $\langle\xi\rangle_M>2$. With a given $\varepsilon>0$, we set $\phi(\varepsilon^{\frac{1}{M}}\xi):=\phi(\varepsilon^{\frac{1}{m_1}}\xi_1,\dots,\varepsilon^{\frac{1}{m_n}}\xi_n)$ and let $\phi(\varepsilon^{\frac{1}{M}}D)$ denote the associated Fourier multiplier.

The following M-homogeneous version of [23, Lemma 1.3.A], shows the behavior of $\phi(\varepsilon^{\frac{1}{M}}D)$ on M-homogeneous Fourier–Lebesgue spaces.

Lemma 7 *Let $p\in[1,+\infty]$ and $\varepsilon>0$ be arbitrarily fixed.*

(i) *For every $\beta\in\mathbb{Z}^n_+$ and $r\in\mathbb{R}$, the Fourier multiplier $D^\beta\phi(\varepsilon^{\frac{1}{M}}D)$ extends as a bounded linear operator $D^\beta\phi(\varepsilon^{\frac{1}{M}}D):\mathcal{F}L^p_{r,M}\to\mathcal{F}L^p_{r,M}$ and there is a positive constant C_β, independent of ε, such that:*

$$\|D^\beta\phi(\varepsilon^{\frac{1}{M}}D)u\|_{\mathcal{F}L^p_{r,M}}\le C_\beta\varepsilon^{-\langle\beta,\frac{1}{M}\rangle}\|u\|_{\mathcal{F}L^p_{r,M}},\quad\forall u\in\mathcal{F}L^p_{r,M}\,;\tag{120}$$

(ii) *For all $r\in\mathbb{R}$ and $t\ge 0$, the Fourier multiplier $I-\phi(\varepsilon^{\frac{1}{M}}D)$ (where I denotes the identity operator) extends as a bounded linear operator $I-\phi(\varepsilon^{\frac{1}{M}}D):\mathcal{F}L^p_{r,M}\to$*

$\mathcal{F}L^p_{r-t,M}$ *and there exists a constant* $C_t > 0$*, independent of* ε*, such that:*

$$\|u - \phi(\varepsilon^{\frac{1}{M}} D)u\|_{\mathcal{F}L^p_{r-t,M}} \le C_t \varepsilon^t \|u\|_{\mathcal{F}L^p_{r,M}}, \quad \forall u \in \mathcal{F}L^p_{r,M}\,; \tag{121}$$

(iii) *If* $r > \frac{n}{\mu_* q}$*, where* $\frac{1}{p} + \frac{1}{q} = 1$*, and* $\beta \in \mathbb{Z}^n_+$*, then* $D^\beta \phi(\varepsilon^{1/M} D)$ *and* $I - \phi(\varepsilon^{\frac{1}{M}} D)$ *extend as bounded linear operators* $D^\beta \phi(\varepsilon^{1/M} D)$*,* $I - \phi(\varepsilon^{\frac{1}{M}} D) : \mathcal{F}L^p_{r,M} \to \mathcal{F}L^1$ *and there are constants* $C_{r,\beta}$ *and* C_r*, independent of* ε*, such that:*

$$\begin{aligned}
\|D^\beta \phi(\varepsilon^{1/M} D)u\|_{\mathcal{F}L^1} &\le C_{r,\beta}\varepsilon^{-\left(\langle \beta, 1/M\rangle - (r - \frac{n}{\mu_* q})\right)_+} \|u\|_{\mathcal{F}L^p_{M,r}}, \\
&\qquad \text{if } \langle \beta, 1/M\rangle \ne r - \frac{n}{\mu_* q}, \\
\|D^\beta \phi(\varepsilon^{1/M} D)u\|_{\mathcal{F}L^1} &\le C_r \log^{1/q}(1 + \varepsilon^{-1}) \|u\|_{\mathcal{F}L^p_{M,r}}, \\
&\qquad \text{if } \langle \beta, 1/M\rangle = r - \frac{n}{\mu_* q}, \\
\|u - \phi(\varepsilon^{\frac{1}{M}} D)u\|_{\mathcal{F}L^1} &\le C_r \varepsilon^{r - \frac{n}{\mu_* q}} \|u\|_{\mathcal{F}L^p_{r,M}}, \quad \forall u \in \mathcal{F}L^p_{r,M}.
\end{aligned} \tag{122}$$

Proof (i): From the properties of function ϕ, one can readily show that for any $\beta \in \mathbb{Z}^n_+$ there exists a constant $C_\beta > 0$ such that:

$$|\xi^\beta \phi(\varepsilon^{\frac{1}{M}} \xi)| \le C_\beta \varepsilon^{-\langle \beta, 1/M\rangle}, \quad \forall \xi \in \mathbb{R}^n, \quad \forall \varepsilon \in]0, 1].$$

Then estimate (120) follows at once from Hölder's inequality.

(ii): Similarly as for (i), for $t \ge 0$, one can find a positive constant C_t such that:

$$\left|\langle \xi\rangle_M^{-t} (1 - \phi(\varepsilon^{1/M} \xi))\right| \le C_t \varepsilon^t, \quad \forall \xi \in \mathbb{R}^n, \quad \forall \varepsilon \in]0, 1],$$

then estimate (121) follows once again from Hölder's inequality.

(iii): The extension of $D^\beta \phi(\varepsilon^{\frac{1}{M}} D)$ and $I - \phi(\varepsilon^{\frac{1}{M}} D)$ as linear bounded operators from $\mathcal{F}L^p_{r,M}$ to $\mathcal{F}L^1$ follows at once from a combination of the continuity properties stated in (i), (ii) and the fact that the space $\mathcal{F}L^p_{r,M}$ is imbedded into $\mathcal{F}L^1$ when $r > \frac{n}{\mu_* q}$.

For $\langle \beta, 1/M\rangle < r - \frac{n}{\mu_* q}$, we directly have

$$\|D^\beta \phi(\varepsilon^{1/M} D)u\|_{\mathcal{F}L^1} = \int |\xi^\beta| \phi(\varepsilon^{1/M} \xi) |\widehat{u}(\xi)| d\xi,$$

and $0 \le \phi \le 1$ implies $|\xi^\beta| \phi(\varepsilon^{1/M} \xi) \le \langle \xi\rangle_M^{\langle \beta, 1/M\rangle}$. Combining the above and since $\langle \cdot\rangle_M^{\langle \beta, 1/M\rangle - r} \in L^q$ as $r - \langle \beta, 1/M\rangle > \frac{n}{\mu_* q}$, Hölder's inequality yields

$$\|D^\beta \phi(\varepsilon^{1/M} D)u\|_{\mathcal{F}L^1} \le C_{r,\beta,p} \|u\|_{\mathcal{F}L^p_{r,M}},$$

where $C_{r,\beta,p} := \left(\int \frac{1}{\langle\xi\rangle_M^{(r-\langle\beta,1/M\rangle)q}} d\xi\right)^{1/q}$. The above formula is $(122)_1$ for $\langle\beta, 1/M\rangle < r - \frac{n}{\mu_* q}$.

For $\langle\beta, 1/M\rangle > r - \frac{n}{\mu_* q}$, we first write

$$\|D^\beta \phi(\varepsilon^{1/M} D)u\|_{\mathcal{F}L^1} = \left\|\xi^\beta \sum_{h=-1}^{+\infty} \phi(\varepsilon^{1/M}\xi)\widehat{u}_h\right\|_{L^1}, \tag{123}$$

where, for every integer $h \geq -1$, we set $\widehat{u}_h = \varphi_h \widehat{u}$, being $\{\varphi_h\}_{h=-1}^{\infty}$ the dyadic partition of unity introduced in Sect. 3.

Since $\phi(\varepsilon^{1/M}\xi)\widehat{u}_h \equiv 0$, as long as the integer $h \geq 0$ satisfies $2\varepsilon^{-1} < \frac{1}{K}2^{h-1}$ (that is $h > \log_2(4K/\varepsilon)$), cf. (49), (51), from (123), $0 \leq \phi \leq 1$,

$$|\xi^\beta| \leq |\xi|_M^{\langle\beta,1/M\rangle} \leq C_{K,\beta} 2^{h\langle\beta,1/M\rangle}, \quad \text{for } \xi \in \mathcal{C}_h^{M,K}, \tag{124}$$

with a constant $C_{K,\beta} > 0$ independent of h, and Hölder's inequality, it follows

$$\begin{aligned}
\|D^\beta \phi(\varepsilon^{1/M} D)u\|_{\mathcal{F}L^1} &\leq \sum_{h=-1}^{[\log_2(4K/\varepsilon)]} \int_{\mathcal{C}_h^{M,k}} |\xi^\beta|\phi(\varepsilon^{1/M}\xi)|\widehat{u}_h(\xi)| d\xi \\
&\leq C_{K,\beta} \sum_{h=-1}^{[\log_2(4K/\varepsilon)]} 2^{h\langle\beta,1/M\rangle} \int_{\mathcal{C}_h^{M,k}} |\widehat{u}_h(\xi)| d\xi \\
&= C_{K,\beta} \sum_{h=-1}^{[\log_2(4K/\varepsilon)]} 2^{h\sigma} \int_{\mathcal{C}_h^{M,k}} 2^{-h\frac{n}{\mu_* q}} 2^{hr} |\widehat{u}_h(\xi)| d\xi \\
&\leq C_{K,\beta} \sum_{h=-1}^{[\log_2(4K/\varepsilon)]} 2^{h\sigma} \left(\int_{\mathcal{C}_h^{M,k}} 2^{-h\frac{n}{\mu_*}}\right)^{1/q} \|2^{hr}\widehat{u}_h\|_{L^p} \\
&\leq C_{K,\beta,n,p} \sum_{h=-1}^{[\log_2(4K/\varepsilon)]} 2^{h\sigma} \|2^{hr}\widehat{u}_h\|_{L^p},
\end{aligned} \tag{125}$$

where we used $\int_{\mathcal{C}_h^{M,k}} d\xi \leq C_{*,K,n} 2^{h\frac{n}{\mu_*}}$, for a constant $C_{*,K,n}$ independent of h, and it is set $C_{K,\beta,n,p} := C_{K,\beta} C_{*,K,n}^{1/q}$ and $\sigma := \langle\beta, 1/M\rangle - (r - \frac{n}{\mu_* q})$. Hence, we use discrete Hölder's inequality with conjugate exponents (p, q) and the characterization of M-homogeneous Fourier–Lebesgue spaces provided by Proposition 2 to end up with

$$\|D^\beta \phi(\varepsilon^{1/M} D)u\|_{\mathcal{F}L^1} \le C_{K,\beta,n,p} \left(\sum_{h=-1}^{[\log_2(4K/\varepsilon)]} 2^{h\sigma q} \right)^{1/q} \|u\|_{\mathcal{F}L^p_{r,M}}, \tag{126}$$

and

$$\begin{aligned} \sum_{h=-1}^{[\log_2(4K/\varepsilon)]} 2^{h\sigma q} &= 2^{\sigma q([\log_2(4K/\varepsilon)])} \sum_{h=-1}^{[\log_2(4K/\varepsilon)]} 2^{-\sigma q([\log_2(4K/\varepsilon)]-h)} \\ &\le (4K/\varepsilon)^{\sigma q} C_{\sigma,q} = C_{K,\sigma,q}\varepsilon^{-\sigma q}, \end{aligned} \tag{127}$$

where $C_{\sigma,q} := \sum\limits_{j\ge 0} 2^{-\sigma q j}$ is convergent, as $\sigma > 0$, and $C_{K,\sigma,q} := 4K C_{\sigma,q}$ is independent of ε. Inequality $(122)_1$ for $\langle \beta, 1/M\rangle > r - \frac{n}{\mu_* q}$ follows from combining (126), (127).

To prove $(122)_2$, we repeat the arguments leading to (123)–(126) where $\langle \beta, 1/M\rangle = r - \frac{n}{\mu_* q}$ (that is $\sigma = 0$), use discrete Hölder's inequality and Proposition 2, to get:

$$\begin{aligned} \|D^\beta \phi(\varepsilon^{1/M} D)u\|_{\mathcal{F}L^1} &\le C_{K,r,n,p} \sum_{h=-1}^{[\log_2(4K/\varepsilon)]} \|2^{hr}\widehat{u}_h\|_{L^p} \\ &\le \tilde{C}_{K,r,n,p} \left(\sum_{h=-1}^{[\log_2(4K/\varepsilon)]} 1 \right)^{1/q} \|u\|_{\mathcal{F}L^p_{r,M}} \\ &= \tilde{C}_{K,r,n,p} \left(2 + [\log_2(4K/\varepsilon)]\right)^{1/q} \|u\|_{\mathcal{F}L^p_{r,M}} \\ &\le C'_{K,r,n,p} \log^{1/q}(1+\varepsilon^{-1})\|u\|_{\mathcal{F}L^p_{r,M}}. \end{aligned} \tag{128}$$

The proof of inequality $(122)_3$ follows along the same arguments used above. We resort once again to Proposition 2 and Hölder's inequality to get

$$\begin{aligned} \|(I-\phi(\varepsilon^{1/M}D))u\|_{\mathcal{F}L^1} &= \|(1-\phi(\varepsilon^{1/M}\cdot))\widehat{u}\|_{L^1} = \left\| (1-\phi(\varepsilon^{1/M}\cdot)) \sum_{h=-1}^{\infty} \widehat{u}_h \right\|_{L^1} \\ &\le \sum_{h>\log_2\left(\frac{1}{2K\varepsilon}\right)} \|(1-\phi(\varepsilon^{1/M}\cdot))\widehat{u}_h\|_{L^1} \\ &\le \sum_{h>\log_2\left(\frac{1}{2K\varepsilon}\right)} \left\| \frac{(1-\phi(\varepsilon^{1/M}\cdot))\chi_h}{\langle\cdot\rangle_M^r} \right\|_{L^q} \|\langle\cdot\rangle_M^r \widehat{u}_h\|_{L^p} \end{aligned}$$

where for an integer $h \ge -1$, χ_h is the characteristic function of $\mathcal{C}_h^{M,K}$ and we use $(1-\phi(\varepsilon^{1/M}\cdot))\varphi_h \equiv 0$ for $K2^{h+1} \le 1/\varepsilon$, cf. (49), (51). Arguing as in the proof of

Proposition 2 yields

$$\|\langle\cdot\rangle_M^r \widehat{u}_h\|_{L^p} \le C_{r,p} 2^{rh} \|\widehat{u}_h\|_{L^p}, \quad \forall\, h \ge -1,$$

with positive constant $C_{r,p}$ depending only on r and p. Using again the properties of functions ϕ and φ_h's, we also get, for any $h \ge -1$,

$$\left\| \frac{(1-\phi(\varepsilon^{1/M}\cdot))\chi_h}{\langle\cdot\rangle_M^r} \right\|_{L^q}^q = \int_{\mathcal{C}_h^{M,K}} \left| \frac{(1-\phi(\varepsilon^{1/M}\xi))}{\langle\xi\rangle_M^r} \right|^q d\xi \le \int_{\mathcal{C}_h^{M,K}} \frac{1}{\langle\xi\rangle_M^{rq}} d\xi$$
$$\le C_{r,q} 2^{-rhq} \int_{\mathcal{C}_h^{M,K}} d\xi \le C_{r,p,\mu_*,K,n} 2^{h(-rq+n/\mu_*)}$$

(with obvious modifications in the case of $q=\infty$, that is $p=1$); here and later on, $C_{r,p,\mu_*,K,n}$ will denote some positive constant, depending only on r, p, μ_*, K and the dimension n, that may be different from an occurrence to another.

Using the above inequalities in the previous estimate of the L^1-norm of $(1-\phi(\varepsilon^{1/M}\cdot))\widehat{u}$, together with Hölder's inequality and Proposition 2, we end up with

$$\|(I-\phi(\varepsilon^{1/M}D))u\|_{\mathcal{F}L^1} \le C_{r,p,\mu_*,K,n} \sum_{h>\log_2\left(\frac{1}{2K\varepsilon}\right)} 2^{h(-r+\frac{n}{\mu_* q})} 2^{rh} \|\widehat{u}_h\|_{L^p}$$
$$\le C_{r,p,\mu_*,K,n} \left(\sum_{h>\log_2\left(\frac{1}{2K\varepsilon}\right)} 2^{h(-rq+n/\mu_*)} \right)^{1/q} \left(\sum_{h\ge -1} 2^{rhp} \|\widehat{u}_h\|_{L^p}^p \right)^{1/p}$$
$$\le C_{r,p,\mu_*,K,n} \left(\frac{1}{2K\varepsilon}\right)^{-r+\frac{n}{\mu_* q}} \left(\sum_{\ell>0} 2^{\ell(-rq+n/\mu_*)} \right)^{1/q} \|u\|_{\mathcal{F}L^p_{M,r}}$$
$$\le C_{r,p,\mu_*,K,n} \varepsilon^{r-\frac{n}{\mu_* q}} \|u\|_{\mathcal{F}L^p_{M,r}},$$

since the geometric series $\sum_{\ell>0} 2^{\ell(-rq+n/\mu_*)}$ is convergent for $r > \frac{n}{\mu_* q}$. □

Remark 12 As already noticed in the proof of the above Lemma 7, for $r > \frac{n}{\mu_* q}$ the continuity of the operator $D^\beta \phi(\varepsilon^{\frac{1}{M}} D)$ from $\mathcal{F}L^p_{r,M}$ to $\mathcal{F}L^1$ readily follows from the continuity of the same operator in $\mathcal{F}L^p_{r,M}$ and the validity of the continuous imbedding of $\mathcal{F}L^p_{r,M}$ into $\mathcal{F}L^1$; combining the above with the inequality (120) also gives the following continuity estimate

$$\|D^\beta \phi(\varepsilon^{\frac{1}{M}} D)u\|_{\mathcal{F}L^1} \le C_\beta \varepsilon^{-\langle\beta,\frac{1}{M}\rangle} \|u\|_{\mathcal{F}L^p_{r,M}}, \quad \forall\, u \in \mathcal{F}L^p_{r,M}.$$

Notice however that inequalities $(122)_{1,2}$ provide an improvement of the continuity estimate above, as they give a sharper control of the norm of $D^\beta \phi(\varepsilon^{\frac{1}{M}} D)$, with respect to ε, as a linear bounded operator in $\mathcal{L}(\mathcal{F}L^p_{r,M}; \mathcal{F}L^1)$.

Remark 13 In the case of $r > \frac{n}{\mu_* q}$, applying statement (ii) of Lemma 7 with $0 \le t < r - \frac{n}{\mu_* q}$ and taking account of $\mathcal{F}L^p_{r,M} \subset \mathcal{F}L^1$, with continuous imbedding, yields that

$$\|u - \phi(\varepsilon^{\frac{1}{M}} D)u\|_{\mathcal{F}L^1} \le C_t \varepsilon^t \|u\|_{\mathcal{F}L^p_{r,M}}, \quad \forall u \in \mathcal{F}L^p_{r,M}, \tag{129}$$

holds true with some positive constant C_t, independent of ε. Notice, however, that the endpoint case $t = r - \frac{n}{\mu_* q}$ (corresponding to statement (iii) of Lemma 7) cannot be reached by treating it along the same arguments used to prove statement (ii) above; indeed, in general, $\mathcal{F}L^p_{\frac{n}{\mu_* q},M}$ is not imbedded in $\mathcal{F}L^1$ (that is $\langle\cdot\rangle^{-\frac{n}{\mu_* q}} \notin L^q$).

Let $a(x,\xi)$ belong to $\mathcal{F}L^p_{r,M} S^m_M$ and take $\delta \in]0,1]$; we define

$$a^\#(x,\xi) := \sum_{h=-1}^{\infty} \phi(2^{-\frac{h\delta}{M}} D_x) a(x,\xi) \varphi_h(\xi), \tag{130}$$

and set

$$a^\natural(x,\xi) := a(x,\xi) - a^\#(x,\xi). \tag{131}$$

As a consequence of Lemma 7, one can prove the following result, which will play a fundamental role in the analysis made in Sect. 7.4.

Proposition 8 *For $r > \frac{n}{\mu_* q}$ and $m \in \mathbb{R}$, let $a(x,\xi) \in \mathcal{F}L^p_{r,M} S^m_M$ and take an arbitrary $\delta \in]0,1]$. Then*

$$a^\#(x,\xi) \in S^m_{M,\delta,\kappa},$$

where $\kappa = r - \frac{n}{\mu_* q}$; moreover $a^\natural(x,\xi) \in \mathcal{F}L^p_{r,M} S^{m-\delta\left(r-\frac{n}{\mu_* q}\right)}_{M,\delta}$.

Proof For arbitrary $\alpha, \beta \in \mathbb{Z}^n_+$, from Leibniz's rule we get

$$\begin{aligned} &\|D^\beta_x D^\alpha_\xi a^\#(\cdot,\xi)\|_{\mathcal{F}L^1} \\ &\quad \le \sum_{\nu\le\alpha} \binom{\alpha}{\nu} \sum_{h=-1}^{+\infty} \|D^\beta_x \phi(2^{-\frac{\delta h}{M}} D) D^{\alpha-\nu}_\xi a(\cdot,\xi)\|_{\mathcal{F}L^1} |D^\nu_\xi \varphi_h(\xi)| \\ &\quad = \sum_{\nu\le\alpha} \binom{\alpha}{\nu} \sum_{h=\tilde{h}_0}^{h_0+N_0} \|D^\beta_x \phi(2^{-\frac{\delta h}{M}} D) D^{\alpha-\nu}_\xi a(\cdot,\xi)\|_{\mathcal{F}L^1} |D^\nu_\xi \varphi_h(\xi)|, \end{aligned} \tag{132}$$

where, for every $\xi \in \mathbb{R}^n$, the integers $N_0 > 0$ (independent of ξ), $h_0 = h_0(\xi) \ge -1$ and $\tilde{h}_0 = \tilde{h}_0(\xi)$ are the same as considered in (50), (56).

On the other hand, because $r > \frac{r}{\mu_* q}$, applying to $u = D^{\alpha-\nu}_\xi a(\cdot,\xi)$ the inequalities $(122)_{1,2}$ with $\varepsilon = 2^{-h\delta}$ and using estimates (12) and (64), we get for $h \ge -1$ and $\xi \in \mathcal{C}^{M,K}_h$

$$
\begin{aligned}
&\|D_x^\beta \phi(2^{-\frac{\delta h}{M}}D)D_\xi^{\alpha-\nu}a(\cdot,\xi)\|_{\mathcal{F}L^1} \le C_{r,\beta}2^{h\delta(\langle\beta,1/M\rangle-\kappa)_+}\|D_\xi^{\alpha-\nu}a(\cdot,\xi)\|_{\mathcal{F}L^p_{M,r}}\\
&\qquad\le C_{r,\alpha,\beta,\nu}\langle\xi\rangle_M^{m-\langle\alpha-\nu,1/M\rangle+\delta(\langle\beta,1/M\rangle-\kappa)_+}, \quad \text{if } \langle\beta,1/M\rangle\ne\kappa,\\
&\|D_x^\beta \phi(2^{-\frac{\delta h}{M}}D)D_\xi^{\alpha-\nu}a(\cdot,\xi)\|_{\mathcal{F}L^1} \le C_r\log^{1/q}(1+2^{h\delta})\|D_\xi^{\alpha-\nu}a(\cdot,\xi)\|_{\mathcal{F}L^p_{M,r}}\\
&\qquad\le C_{r,\alpha,\nu}\log^{1/q}(1+\langle\xi\rangle_M^\delta)\langle\xi\rangle_M^{m-\langle\alpha-\nu,1/M\rangle}\\
&\qquad\le C_{r,\alpha,\nu}\log(1+\langle\xi\rangle_M^\delta)\langle\xi\rangle_M^{m-\langle\alpha-\nu,1/M\rangle}, \quad \text{if } \langle\beta,1/M\rangle=\kappa,
\end{aligned}
$$

and

$$
|D^\nu\varphi_h(\xi)|\le C_\nu\langle\xi\rangle_M^{-\langle\nu,1/M\rangle},
$$

with suitable positive constants $C_{r,\beta}$, $C_{r,\alpha,\beta,\nu}$, C_r, $C_{r,\alpha,\nu}$, C_ν independent of h. Then summing the above inequalities over all h's such that $\tilde{h}_0\le h\le h_0+N_0$, from (132) it follows that

$$
\begin{aligned}
&\|D_x^\beta D_\xi^\alpha a^\#(\cdot,\xi)\|_{\mathcal{F}L^1}\le C_{\alpha,\beta}\langle\xi\rangle_M^{m-\langle\alpha,1/M\rangle+\delta(\langle\beta,1/M\rangle-\kappa)_+}, \quad \text{if } \langle\beta,1/M\rangle\ne\kappa,\\
&\|D_x^\beta D_\xi^\alpha a^\#(\cdot,\xi)\|_{\mathcal{F}L^1}\le C_{\alpha,\beta}\langle\xi\rangle_M^{m-\langle\alpha,1/M\rangle}\log(1+\langle\xi\rangle_M^\delta), \quad \text{if } \langle\beta,1/M\rangle=\kappa,
\end{aligned}
\tag{133}
$$

from which $a^\#(x,\xi)\in S^m_{M,\delta,\kappa}$ follows at once, recalling that $\mathcal{F}L^1$ is imbedded in the space of bounded continuous functions in $\mathbb{R}^n$.

As regards to symbol $a^\natural(x,\xi)$ defined in (131), applying inequalities (121) with $t=0$, together with estimates (12) and (64), and using similar arguments as above, for all integers $h\ge -1$ and $\xi\in\mathcal{C}_h^{M,K}$ we find

$$
\begin{aligned}
&\|D_\xi^\alpha a^\natural(\cdot,\xi)\|_{\mathcal{F}L^p_{r,M}}\\
&\qquad\le\sum_{\nu\le\alpha}\sum_{h=\tilde{h}_0}^{h_0+N_0}\binom{\alpha}{\nu}\|(I-\phi(2^{-h\delta/M}D))(D_\xi^{\alpha-\nu}a(\cdot,\xi))\|_{\mathcal{F}L^p_{r,M}}|D^\nu\varphi_h(\xi)|\\
&\qquad\le\sum_{\nu\le\alpha}\sum_{h=\tilde{h}_0}^{h_0+N_0}C_{\alpha,\nu}\|D_\xi^{\alpha-\nu}a(\cdot,\xi)\|_{\mathcal{F}L^p_{r,M}}|D^\nu\varphi_h(\xi)|\\
&\qquad\le\sum_{\nu\le\alpha}\sum_{h=\tilde{h}_0}^{h_0+N_0}C'_{\alpha,\nu}\langle\xi\rangle_M^{m-\langle\alpha-\nu,1/M\rangle}\langle\xi\rangle_M^{-\langle\nu,1/M\rangle}\le C_\alpha\langle\xi\rangle_M^{m-\langle\alpha,1/M\rangle},
\end{aligned}
$$

with positive constants $C_{\alpha,\nu}$, $C'_{\alpha,\nu}$, C_α independent of h; similarly, replacing (12) with (11) and (121) with $(122)_3$ (with $\varepsilon=2^{-h\delta}$) in the above estimates, we find

$$
\begin{aligned}
&\|D_\xi^\alpha a^\natural(\cdot,\xi)\|_{\mathcal{F}L^1}\\
&\quad\le \sum_{\nu\le\alpha}\sum_{h=\tilde{h}_0}^{h_0+N_0}\binom{\alpha}{\nu}\|(I-\phi(2^{-h\delta/M}D))(D_\xi^{\alpha-\nu}a(\cdot,\xi))\|_{\mathcal{F}L^1}|D^\nu\varphi_h(\xi)|\\
&\quad\le \sum_{\nu\le\alpha}\sum_{h=\tilde{h}_0}^{h_0+N_0} C_{\alpha,\nu}2^{-h\delta\left(r-\frac{n}{\mu*q}\right)}\|D_\xi^{\alpha-\nu}a(\cdot,\xi)\|_{\mathcal{F}L^p_{r,M}}|D^\nu\varphi_h(\xi)|\\
&\quad\le \sum_{\nu\le\alpha}\sum_{h=\tilde{h}_0}^{h_0+N_0} C'_{\alpha,\nu}\langle\xi\rangle_M^{-\delta\left(r-\frac{n}{\mu*q}\right)}\langle\xi\rangle_M^{m-\langle\alpha-\nu,1/M\rangle}\langle\xi\rangle_M^{-\langle\nu,1/M\rangle}\\
&\quad\le C_\alpha\langle\xi\rangle_M^{m-\delta\left(r-\frac{n}{\mu*q}\right)-\langle\alpha,1/M\rangle},
\end{aligned}
$$

where the numerical constants involved above are independent of h. The above inequalities yields $a^\natural(x,\xi)\in\mathcal{F}L^p_{r,M}S_{M,\delta}^{m-\left(r-\frac{n}{\mu*q}\right)}$, because of the arbitrariness of h and that the $\mathcal{C}_h^{M,K}$'s cover $\mathbb{R}^n$. □

7 Microlocal properties

In order to study the microlocal propagation of weighted Fourier–Lebesgue singularities for PDEs, this section is devoted to define local/microlocal versions of M-Fourier–Lebesgue spaces as well as M-homogeneous smooth symbols previously introduced in Sects. 3, 5, and to collect some basic tools and a few results needed at this purpose.

7.1 Local and microlocal function spaces

While the main focus of this paper is on M-homogeneous Fourier–Lebesgue spaces, in this section we define general scales of function spaces, where the microlocal propagation of singularities of pseudodifferential operators with M-homogeneous symbols, as defined in Sect. 5, will be then studied.

Let us consider a one-parameter family $\{\mathcal{X}_s\}_{s\in\mathbb{R}}$ of Banach spaces $\mathcal{X}_s$, $s\in\mathbb{R}$, such that

$$\mathcal{S}(\mathbb{R}^n)\subset\mathcal{X}_t\subset\mathcal{X}_s\subset\mathcal{S}'(\mathbb{R}^n),\quad \text{with continuous embedding,} \tag{134}$$

for arbitrary $s<t$. Following Taylor [23], we say that $\{\mathcal{X}_s\}_{s\in\mathbb{R}}$ is a *M-microlocal scale* provided that there exists a constant $\kappa_0>0$ such that for all $m\in\mathbb{R}$, $\delta\in[0,1[$, $\kappa>\kappa_0$ and $a(x,\xi)\in S^m_{M,\delta,\kappa}$ satisfying (19) for some compact $\mathcal{K}\subset\mathbb{R}^n$, the pseudodifferential operator $a(x,D)$ extends to a linear bounded operator

$$a(x,D):\mathcal{X}_{s+m}\to\mathcal{X}_s,\quad \forall\, s\in\mathbb{R}. \tag{135}$$

In view of Theorem 2, it is clear that for every $p \in [1, +\infty]$ the M-homogeneous Fourier–Lebesgue spaces $\{\mathcal{F}L^p_{s,M}\}_{s\in\mathbb{R}}$ constitute a M-microlocal scale, according to definition above; in this case the threshold κ_0 considered above is given by $\kappa_0 = [n/\mu_*]+1$. Other examples of M-microlocal spaces are provided by *M-homogeneous Sobolev* and *Hölder* spaces studied in [10].[4]

In order to allow the microlocal analysis performed in subsequent Sect. 7.2, the following local and microlocal counterparts of spaces $\mathcal{X}_s$, $s \in \mathbb{R}$, are given.

Definition 8 Let $s \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$ and $\xi^0 \in \mathbb{R}^n\backslash\{0\}$. We say that a distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ belongs to the local space $\mathcal{X}_{s,\mathrm{loc}}(x_0)$ if there exists a function $\phi \in C_0^\infty(\mathbb{R}^n)$, satisfying $\phi(x_0) \neq 0$, such that

$$\phi u \in \mathcal{X}_s. \tag{136}$$

We say that $u \in \mathcal{S}'(\mathbb{R}^n)$ belongs to the microlocal space $\mathcal{X}_{s,\mathrm{mcl}}(x_0, \xi^0)$ provided that there exist a function $\phi \in C_0^\infty(\mathbb{R}^n)$, satisfying $\phi(x_0) \neq 0$, and a symbol $\psi(\xi) \in S^0_M$, satisfying $\psi(\xi) \equiv 1$ on $\Gamma_M \cap \{|\xi|_M > \varepsilon_0\}$ for suitable M-conic neighborhood $\Gamma_M \subset \mathbb{R}^n\backslash\{0\}$ of ξ^0 and $0 < \varepsilon_0 < |\xi^0|_M$, such that

$$\psi(D)(\phi u) \in \mathcal{X}_s. \tag{137}$$

Under the same assumptions as above, we also write

$$x_0 \notin \mathcal{X}_s - \mathrm{singsupp}\,(u) \quad \text{and} \quad (x_0, \xi^0) \notin WF_{\mathcal{X}_s}(u) \tag{138}$$

respectively.

In the case $\mathcal{X}^s \equiv \mathcal{F}L^p_{s,M}$, it is clear that Definition 8 reduces to Definition 5.

It can be easily proved that $\mathcal{X}_s - \mathrm{singsupp}\,(u)$ is a closed subset of $\mathbb{R}^n$ and is called the *$\mathcal{X}_s$−singular support* of the distribution u, whereas $WF_{\mathcal{X}_s}(u)$ is a closed subset of $T^\circ\mathbb{R}^n$, $M - conic$ with respect to the ξ variable, and is called the *$\mathcal{X}_s$−wave front set* of u. The previous notions are natural generalizations of the classical notions of singular support and wave front set of a distribution introduced by Hörmander [15], see also [16].

Let π_1 be the canonical projection of $T^\circ\mathbb{R}^n$ onto $\mathbb{R}^n$, that is $\pi_1(x, \xi) = x$. Arguing as in the classical case, one can prove the following.

Proposition 9 *if $u \in \mathcal{X}_{s,\mathrm{mcl}}(x_0, \xi^0)$, with $(x_0, \xi^0) \in T^\circ\mathbb{R}^n$, then, for any $\varphi \in C_0^\infty(\mathbb{R}^n)$, such that $\varphi(x_0) \neq 0$, $\varphi u \in \mathrm{mcl}\mathcal{X}_{s,\mathrm{mcl}}(x_0, \xi^0)$. Moreover, we have:*

$$\mathcal{X}_s - \mathrm{singsupp}(u) = \pi_1(WF_{\mathcal{X}_s}(u)).$$

7.2 Microlocal symbol classes

We introduce now microlocal counterparts of the smooth symbol classes given in Definitions 3, 4 and studied in Sect. 5.

[4] Actually for M-homogeneous Sobolev and Hölder spaces, the continuity property (135) is extended to all pseudodifferential operators with symbol in $S^m_{M,\delta}$, without the need of the more restrictive decay conditions in Definition 4 and of the *locality* condition (19), see [10, Theorem 3.3 and Corollary 3.4].

Definition 9 let U be an open subset of $\mathbb{R}^n$ and $\Gamma_M \subset \mathbb{R}^n \setminus \{0\}$ an open M-conic set. For $m \in \mathbb{R}$, $\delta \in [0, 1]$ and $\kappa > 0$; we say that $a \in S'(\mathbb{R}^{2n})$ belongs to $S^m_{M,\delta}$ (resp. to $S_{M,\delta,\kappa}$) microlocally on $U \times \Gamma_M$ if $a_{|\, U \times \Gamma_M} \in C^\infty(U \times \Gamma_M)$ and for all $\alpha, \beta \in \mathbb{Z}^n_+$ there exists $C_{\alpha,\beta} > 0$ such that (14) (resp. (16), (17)) holds true for all $(x, \xi) \in U \times \Gamma_M$. We will write in this case $a \in mclS^m_{M,\delta}(U \times \Gamma_M)$ (resp. $a \in mclS^m_{M,\delta,\kappa}(U \times \Gamma_M)$). For $(x_0, \xi^0) \in T^\circ\mathbb{R}^n$, we set

$$mclS^m_{M,\delta\,(,\kappa)}(x_0, \xi^0) := \bigcup_{U,\,\Gamma_M} mclS^m_{M,\delta\,(,\kappa)}(U \times \Gamma_M), \tag{139}$$

where the union in the right-hand side is taken over all of the open neighborhoods $U \subset \mathbb{R}^n$ of x_0 and the open M-conic neighborhoods $\Gamma_M \subset \mathbb{R}^n \setminus \{0\}$ of ξ^0.

With the above stated notation, we say that $a \in \mathcal{S}'(\mathbb{R}^n)$ is *microlocally regularizing* on $U \times \Gamma_M$ if $a_{|\, U \times \Gamma_M} \in C^\infty(U \times \Gamma_M)$ and for every $\mu > 0$ and all $\alpha, \beta \in \mathbb{Z}^n_+$ a positive constant $C_{\mu,\alpha,\beta} > 0$ is found in such a way that:

$$|\partial^\alpha_\xi \partial^\beta_x a(x, \xi)| \le C_{\mu,\alpha,\beta}(1 + |\xi|)^{-\mu}, \quad \forall\, (x, \xi) \in U \times \Gamma_M. \tag{140}$$

Let us denote by $mclS^{-\infty}(U \times \Gamma_M)$ the set of all *microlocally regularizing symbols* on $U \times \Gamma_M$. For $(x_0, \xi^0) \in T^\circ\mathbb{R}^n$, we set:

$$mclS^{-\infty}(x_0, \xi^0) := \bigcup_{U,\,\Gamma_M} mclS^{-\infty}(U \times \Gamma_M)\,; \tag{141}$$

it is easily seen that $mclS^{-\infty}(U \times \Gamma_M) = \bigcap_{m>0} mclS^{-m}_{M,\delta}(U \times \Gamma_M)$ for all $\delta \in [0, 1]$ and $M \in \mathbb{N}^n$, and a similar identity holds for $mclS^{-\infty}(x_0, \xi^0)$.

It is immediate to check that symbols in $mclS^m_{M,\delta}(U \times \Gamma_M)$, $mclS^m_{M,\delta}(x_0, \xi^0)$ behave according to the same rules of "global" symbols, collected in Proposition 5. Moreover $S^m_{M,\delta\,(,\kappa)} \subset mclS^m_{M,\delta\,(,\kappa)}(U \times \Gamma_M) \subset mclS^m_{M,\delta\,(,\kappa)}(x_0, \xi^0)$ hold true, whenever $(x_0, \xi^0) \in T^\circ\mathbb{R}^n$, U is an open neighborhood of x_0 and Γ_M is an open M-conic neighborhood of ξ^0. A slight modification of the arguments used to prove Proposition 6, see also [10, Proposition 4.4], leads to the following microlocal counterpart.

Proposition 10 (Microlocal parametrix) *Assume that* $0 \le \delta < \mu_*/\mu^*$ *and* $\kappa > 0$ *and let* $a(x, \xi) \in S^m_{M,\delta,\kappa}$ *be microlocally M-elliptic at* $(x_0, \xi^0) \in T^\circ\mathbb{R}^n$. *Then there exist symbols* $b(x, \xi), c(x, \xi) \in S^{-m}_{M,\delta,\kappa}$ *such that*

$$c(x, D)a(x, D) = I + l(x, D) \quad \textit{and} \quad a(x, D)b(x, D) = I + r(x, D), \tag{142}$$

and $l(x, \xi), r(x, \xi) \in \mathrm{mcl}S^{-\infty}(x_0, \xi^0)$.

The notion of microlocal M-ellipticity, as well as the characteristic set, see Definition 6, can be readily extended to non-smooth M-homogeneous symbols (as, in principle, it only needs that the symbol $a(x, \xi)$ be a continuous function, at least for sufficiently large ξ); in particular, microlocally M-elliptic symbols in $\mathcal{F}L^p_{r,M}S^m_M$, with sufficiently large $r > 0$, must be considered later on. For a symbol $a(x, \xi) \in \mathcal{F}L^p_{r,M}S^m_M$, with $r > \frac{n}{\mu_* q}$, $p \in [1, +\infty]$ and $\frac{1}{p} + \frac{1}{q} = 1$, for every $0 < \delta \le 1$ let the symbol $a^{\#}(x, \xi)$ and $a^{\natural}(x, \xi)$ be defined as in (130), (131).

The following result can be proved along the same lines of the proof of [10, Proposition 7.3].

Proposition 11 *If $a(x, \xi) \in \mathcal{F}L^p_{r,M}S^m_M$, $m \in \mathbb{R}$, is microlocally M-elliptic at $(x_0, \xi^0) \in T^\circ\mathbb{R}^n$, then $a^{\#}(x, \xi) \in S^m_{M,\delta,\kappa}$ (with κ as in the statement of Proposition 8) is also microlocally M-elliptic at (x_0, ξ^0) for any $0 < \delta \le 1$.*

7.3 Microlocal continuity and regularity results

Let $\{\mathcal{X}_s\}_{s\in\mathbb{R}}$ be a M-microlocal scale as defined in Sect. 7.1. The following microlocal counterpart of the boundedness property (135) and microlocal $\mathcal{X}_s-$regularity follow along the same lines of the proof of [10, Theorem 5.4 and Theorem 6.1].

Proposition 12 *For $0 \le \delta < \mu_*/\mu^*$, $\kappa > \kappa_0$, $m \in \mathbb{R}$ and $(x_0, \xi^0) \in T^\circ\mathbb{R}^n$, assume that $a(x, \xi) \in S^\infty_{M,\delta} \cap \mathrm{mcl}S^m_{M,\delta,\kappa}(x_0, \xi^0)$. Then for all $s \in \mathbb{R}$*

$$u \in \mathcal{X}_{s+m,\mathrm{mcl}}(x_0, \xi^0) \quad \Rightarrow \quad a(x, D)u \in \mathcal{X}_{s,\mathrm{mcl}}(x_0, \xi^0). \tag{143}$$

Proposition 13 *For $0 \le \delta < \mu_*/\mu^*$, $\kappa > \kappa_0$, $m \in \mathbb{R}$, let $a(x, \xi) \in S^m_{M,\delta,\kappa}$ be microlocally M-elliptic at $(x_0, \xi^0) \in T^\circ\mathbb{R}^n$. Then for all $s \in \mathbb{R}$*

$$u \in \mathcal{S}'(\mathbb{R}^n) \ \text{ and } \ a(x, D)u \in \mathcal{X}_{s,\mathrm{mcl}}(x_0, \xi^0) \quad \Rightarrow \quad u \in \mathcal{X}_{s+m,\mathrm{mcl}}(x_0, \xi^0). \tag{144}$$

Resorting on the notions of M-homogeneous wave front set of a distribution and characteristic set of a symbol, the results of the above propositions can be also restated in the following

Corollary 2 *For $0 \le \delta < \mu_*/\mu^*$, $\kappa > \kappa_0$, $m \in \mathbb{R}$, $a(x, \xi) \in S^m_{M,\delta,\kappa}$ and $u \in \mathcal{S}'(\mathbb{R}^n)$, the following inclusions*

$$WF_{\mathcal{X}_s}(a(x, D)u) \subset WF_{\mathcal{X}_{s+m}}(u) \subset WF_{\mathcal{X}_s}(a(x, D)u) \cup \mathrm{Char}(a)$$

hold true for every $s \in \mathbb{R}$.

As particular case of Corollary 2 we obtain the result in Theorem 3

7.4 Proof of Theorem 4

This section is devoted to the proof of Theorem 4 concerning the microlocal propagation of Fourier–Lebesgue singularities of the linear PDE (26). As it will be seen below, the statement of Theorem 4 can be deduced as an immediate consequence of a more general result concerning a suitable class of pseudodifferential operators.

Since the coefficients c_α in the equation (26) belong to $\mathcal{F}L^p_{r,M,\mathrm{loc}}(x_0)$, it follows that the *localized symbol* $a_\phi(x,\xi) := \phi(x)a(x,\xi)$ belongs to the symbol class $\mathcal{F}L^p_{r,M}S^1_M := \mathcal{F}L^p_{r,M}S^1_{M,0}$, for some function $\phi \in C_0^\infty(\mathbb{R}^n)$ supported on a sufficiently small compact neighborhood of x_0 and satisfying $\phi(x_0) \neq 0$ (see Definition 2); moreover, by exploiting the M-homogeneity in ξ of the M-principal part of $a(x,\xi)$, the localized symbol $a_\phi(x,\xi)$ amounts to be microlocally M-elliptic at (x_0,ξ^0) according to Definition 6.

It is also clear that, by a locality argument, for any $u \in \mathcal{S}'(\mathbb{R}^n)$

$$a_\phi(x,D)u = a_\phi(x,D)(\psi u), \tag{145}$$

where $\psi \in C_0^\infty(\mathbb{R}^n)$ is some cut-off function, depending only on ϕ, that satisfies

$$0 \le \psi \le 1, \qquad \text{and} \qquad \psi \equiv 1, \quad \text{on } \operatorname{supp}\phi. \tag{146}$$

It tends out that only the identity (145) will be really exploited in the subsequent analysis; thus the symbol of a differential operator of the type considered in (26), with *point-wise local* M-homogeneous Fourier–Lebesgue coefficients, can be replaced with any symbol $a(x,\xi)$ of positive order m and local Fourier–Lebesgue coefficients at some point x_0, namely

$$a_\phi(x,\xi) \in \mathcal{F}L^p_{r,M}S^m_M, \quad \text{for some } \phi \in C_0^\infty(\mathbb{R}^n) \text{ satisfying } \phi(x_0) \neq 0, \tag{147}$$

so that the related pseudodifferential operator $a(x,D)$ be *properly supported*: while locality does not hold for a general symbol in $\mathcal{F}L^p_{r,M}S^m_M$ (unless it is a polynomial in ξ variable), identity (145) is still true whenever $a(x,D)$ is properly supported (see [1] for the definition and properties of a properly supported operator). For shortness here below we write $a(x,\xi) \in \mathcal{F}L^p_{r,M}S^m_M(x_0)$ to mean that condition (147) is satisfied by $a(x,\xi)$.

Theorem 7 *For $(x_0,\xi^0) \in T^\circ\mathbb{R}^n$, $p \in [1,+\infty]$ and $r > \frac{n}{\mu_* q} + \left[\frac{n}{\mu_*}\right] + 1$, where q is the conjugate exponent of p, let $a(x,\xi) \in \mathcal{F}L^p_{r,M}S^m_M(x_0)$ be, microlocally M-elliptic at (x_0,ξ^0) with positive order m, such that $a(x,D)$ is properly supported. For all $0 < \delta < \mu_*/\mu^*$ and $m + (\delta - 1)\left(r - \frac{n}{\mu_* q}\right) < s \le r + m$ we have*

$$\begin{aligned} &u \in \mathcal{F}L^p_{s-\delta\left(r-\frac{n}{\mu_* q}\right),M,\mathrm{loc}}(x_0), \\ &\text{and} \quad a(x,D)u \in \mathcal{F}L^p_{s-m,M,\mathrm{mcl}}(x_0,\xi^0) \end{aligned} \quad \Rightarrow \quad u \in \mathcal{F}L^p_{s,M,\mathrm{mcl}}(x_0,\xi^0). \tag{148}$$

Proof Let us set $f := a(x,D)u$ for $u \in \mathcal{F}L^p_{s-\delta\left(r-\frac{n}{\mu_* q}\right),M,\mathrm{loc}}(x_0)$. Since $a(x,D)$ is properly supported, suitable smooth functions $\phi \in C_0^\infty(\mathbb{R}^n)$ and ψ satisfying (145) and (146) can be found, supported on such a sufficiently small neighborhood of x_0 that $\psi u \in FL^p_{s-\delta\left(r-\frac{n}{\mu_* q}\right),M}$ and $a_\phi(x,\xi) \in \mathcal{F}L^p_{r,M}S^m_M$, cf. Definition 8 and (147).

Following the decomposition method illustrated in Sect. 6, for $0 < \delta < \mu_*/\mu^*$ let $a^{\#}_\phi(x,\xi) \in S^m_{M,\delta,\kappa}$ and $a^{\natural}_\phi(x,\xi) \in \mathcal{F}L^p_{r,M} S^{m-\delta\left(r-\frac{n}{\mu_* q}\right)}_{M,\delta}$ be defined as in (130), (131), with a_ϕ instead of a and where $\kappa = r - \frac{n}{\mu_* q}$, hence u satisfies the equation

$$a^{\#}_\phi(x,D)(\psi u) = \phi f - a^{\natural}_\phi(x,D)(\psi u).$$

Because $a^{\natural}_\phi(x,\xi) \in \mathcal{F}L^p_{r,M} S^{m-\delta\left(r-\frac{n}{\mu_* q}\right)}_{M,\delta}$, $\psi u \in \mathcal{F}L^p_{s-\delta\left(r-\frac{n}{\mu_* q}\right),M}$, whereas f (so also ϕf) belongs to $\mathcal{F}L^p_{s-m,M,\mathrm{mcl}}(x_0,\xi_0)$ (cf. Proposition 9), for the range of s belonging as prescribed in the statement of Theorem 7 (notice in particular that from $0 < \delta < \mu_*/\mu^* \le 1$ even the endpoint $s = r + m$ is allowed), one can apply Theorem 1 to find

$$a^{\#}_\phi(x,D)(\psi u) \in \mathcal{F}L^p_{s-m,M,\mathrm{mcl}}(x_0,\xi^0)\,;$$

hence, because $\kappa > [n/\mu_*]+1$, applying Theorem 3 to $a^{\#}_\phi(x,\xi)$ yields that ψu, hence u, belongs to $\mathcal{F}L^p_{s,M,\mathrm{mcl}}(x_0,\xi^0)$, which ends the proof. □

It is worth noticing that the result of Theorem 7 can be restated in terms of characteristic set of a symbol and Fourier–Lebesgue wave front set of a distribution as in the next result.

Proposition 14 *Let r, m, p, s and δ satisfy the same conditions as in Theorem 7. Then for $a(x,\xi) \in \mathcal{F}L^p_{r,M} S^m_M$ and $u \in \mathcal{F}L^p_{s-\delta\left(r-\frac{n}{\mu_* q}\right),M}$ we have*

$$WF_{\mathcal{F}L^p_{s,M}}(u) \subset WF_{\mathcal{F}L^p_{s-m,M}}(a(x,D)u) \cup \mathrm{Char}(a).$$

The statement of Theorem 7, as well as Proposition 14, applies in particular to the linear PDE (26) considered at the beginning of this section, thus Theorem 4 is proved.

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