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Asymptotic behavior of solutions for nonlinear parabolic operators with natural growth term and measure data

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Abstract

We are interested in the asymptotic behavior, as t tends to $+\infty$, of finite energy solutions and entropy solutions u_n of nonlinear parabolic problems whose model is

$$\begin{cases} u_t - \Delta_p u + g(u)|\nabla u|^p = \mu & \text{in } (0,T)\times\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ u(t,x) = 0 & \text{on } (0,T)\times\partial\Omega \end{cases} \tag{0.1}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded open set, $N \geq 3$, $u_0 \in L^1(\Omega)$ is a nonnegative initial data, while $g : \mathbb{R} \mapsto \mathbb{R}$ is a real function in $C^1(\mathbb{R})$ which satisfies sign condition with positive derivative and μ is a nonnegative measure independent on time which does not charge sets of null p-capacity.

Keywords Asymptotic behavior · Natural growth term · Nonlinear parabolic operators · Measure data · p-capacity

Mathematics Subject Classification 28A12 · 35B40 · 35B51 · 35R06

Résumé

Comportement asymptotique des solutions pour des opérateurs paraboliques non linéaire avec un terme de croissance naturelle et une donnée mesure. Nous somme interessés au comportement asymptotique, quant t tend vers $+\infty$, des solutions énergétiques finies et des solutions entropiques u_n des problèmes paraboliques non linéaires dont le modèle est (0.1) où $\Omega \subseteq \mathbb{R}^N$ est un ouvert borné, $N \geq 3$, $u_0 \in L^1(\Omega)$ est une donnée initiale non négative, tandis que $g : \mathbb{R} \mapsto \mathbb{R}$ est une fonction réelle de classe $C^1(\mathbb{R})$ qui satisfait la condition du signe avec une dérivée positive et μ est une

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mesure non négative indépendante du temps qui ne prend pas en charge les parties de p-capacité nulle.

Contents

1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set, $N \geq 3$, this paper is devoted to the study of the asymptotic behaviour of solutions for nonlinear parabolic equations (1.1) under Leray–Lions assumptions and in the context of Entropy/Renormalized solutions with measures: for a review on classical results, see for instance [1,55,98] (see also [36,39, 51,54,61,68,75] for more details). More recently, asymptotic behaviour of weak finite energy solutions (weak solutions at least) was studied in [64] for a general class of quasilinear parabolic problems with lower order term $u|\nabla u|^2$ (depending on $|\nabla u|$) and with L^1-data, in [2,77] for Radon measure data without any natural growth term and in [4,79,80,82] for existence and nonexistence results when μ is a general, possibly singular, Radon measure. Let emphasize that the study of the asymptotic behavior of solutions is strictly related to several comparison principles between subsolutions and supersolutions inspired by [56], applied for Cauchy problems [75] and developed for elliptic and parabolic viscous Hamilton–Jacobi equations in [28]. Moreover, if we consider μ is a Radon measure which does not depend on time, we shall prove that, under suitable assumptions, the entropy solutions, which exist and are unique as in [48], of such problems converge to the stationary solution. In other words, we investigate the asymptotic behavior, with respect to the time variable t, of finite energy solutions and of entropy solutions for nonlinear parabolic equations

$$\begin{cases} u_t - \Delta_p u + g(u)|\nabla u|^p = \mu & \text{in } (0,T)\times\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ u(t,x) = 0 & \text{in } (0,T)\times\partial\Omega \end{cases} \tag{1.1}$$

where μ does not depend on time ($\mu \in \mathcal{M}_0(\Omega)$) and $T > 0$. The dynamics of the solutions of (1.1) is governed by two completing effects, namely those resulting from the "p-Laplace" term $-\Delta_p u$ and those corresponding to the "natural growth" term $g(u)|\nabla u|^p$. Our aim is to figure out whether one of the two terms effects rules the asymptotic behavior, according to the bounded Radon measure $\mu \in \mathcal{M}_0(\Omega)$ when the initial data $u_0 \in L^1(\Omega)$. Since the nonlinear term $g(u)|\nabla u|^p$ belongs to $L^1(Q)$, it acts as an absorption term for nonnegative solutions. It is already known that Gagliardo–Nirenberg estimates are used to get compactness results and large time behaviour on the solutions u and its gradients ∇u for $q < q_c = p - \frac{N}{N+1}$, and that the parabolic term u_t becomes element of $L^{s'}(0, T; W^{-1,s'}(\Omega))$ for $s' = \frac{q}{p-1}$ (see [51,54,77], we also refer to [4,79] for more precise informations). In particular, if $p \in (1, 2 - \frac{1}{N+1})$ and the initial data decays sufficiently regular like L^2 or $L^1(\Omega)$, the existence and uniqueness of solutions can't be established since, if $p \leq 2 - \frac{1}{N+1}$ then $\frac{Np+p-N}{N+1} \leq 1$, this implies that, in order to obtain the existence of a solution for p close to 1, it is necessary to go out of the framework of classical Sobolev spaces (see [47, Example 2.16] that can be easily extended to the parabolic case). For the very singular measure (resp. the initial singular data), there are two effective parts $\mathcal{M}_0(Q)$ and $\mathcal{M}_s(Q)$ (respectively, $\mathcal{M}_0(\Omega)$ and $\mathcal{M}_s(\Omega)$), as already noticed in [34,53], and the picture is more complicated. In this paper, we consider just elements μ in $\mathcal{M}_0(\Omega)$ (absolutely continuous measures which does not depend on time), more precisely, if we consider a measure μ without time variable t, i.e., there exits a bounded Radon measure ν on Ω such that, for every Borel set $B \subseteq \Omega$ and $0 < t_0, t_1 < T$, we have $\mu(B \times (t_0, t_1)) = (t_1 - t_0)(B)$ (see Theorems 2.2 and 2.3). Recall that problems with local quadratic term with respect to the gradient (of the type $g(u)|\nabla u|^2$) and for sufficiently smooth data ($\mu \in L^1(Q)$) and nonnegative initial datum, asymptotic behavior results have been proved in [64], the case when μ is absolutely continuous with respect to the p-capacity and $g = 0$ is investigated in [81] and the case where μ is general, under extra linear conditions on the operator and using duality solution u in $L^2(0, T; H_0^1(\Omega))$, is treated in [82]. Finally in [83] problem (1.1) is studied under some assumptions adopted in the present paper with $g = 0$, u_0 lies in $L^1(\Omega)$ and for changing-sign measure data (without sign condition on μ). As already pointed out in this last paper, the extension to general measure data and natural growth term seems to be not always possible. In order to get compactness results stated in [2, Proposition 5.2], we assume for instance that $u(t, x) \in C(0, T; L^1(\Omega)) \cap L^q(0, T; W_0^{1,q}(\Omega))$, in this case solutions $u(t, x)$ satisfies the following estimates

$$
\begin{aligned}
&\|u(t, x)\|_{L^m(Q)} \leq C, \quad \forall 1 \leq m < p - 1 + \frac{p}{N}, \quad \forall (t, x) \in Q, \\
&\|\nabla u(t, x)\|_{L^s(Q)} \leq C, \quad \forall 1 \leq s < p - \frac{N}{N+1}, \quad \forall (t, x) \in Q.
\end{aligned} \tag{1.2}
$$

Thus the aim of our work is to investigate the link between capacities, asymptotic behavior of entropy solutions $u(t, x)$ as t tends to infinity and the term measure $\mu \in \mathcal{M}_0(\Omega)$ which allows, or which is needed, to have solutions in some appropriate sense. In fact, the main point is the relationship between the possibility to find a limit function of the entropy solutions of problems (1.1) using stability properties of these sequences, as they naturally arise, when one tries to solve (1.1) by comparing the solutions $u(t, x)$ with subsolutions and supersolutions. For example, letting (f_n) and u_0^n be the standard approximations of f and u_0 constructed by convolution, and consider the approximating problems

$$\begin{cases} (u_n)_t - \Delta_p u_n + g(u_n)|\nabla u_n|^p = \mu_n \text{ in } (0,T)\times\Omega, \\ u_n(0,x) = u_0^n(x) \text{ in } \Omega, \quad u_n(t,x) = 0 \text{ in } (0,T)\times\Omega, \end{cases} \tag{1.3}$$

where $\mu_n = f_n - \text{div}(G)$ with $G \in L^{p'}(\Omega)^N$, we have to study the possibility to find a solution of the corresponding elliptic problem as limit of a subsequence (u_n) of solutions of (1.1) as t, n tend to infinity. We are going to see that, regardless of any other assumption of $g(s)$ except for (2.14)–(2.15), a compactness result on sequences (u_n) is always available. On the other hand, if may happen that $\underline{u}$ and $\overline{u}$ are, respectively, a subsolution and supersolution of (1.1), then $\underline{u} \leq u \leq \overline{u}$. Precisely, we prove that if u_0 is assumed to be in $L^1(\Omega)$, a necessary and sufficient condition to pass to the limit in (1.3) and to get a solution of (1.1) is the sign condition of $g(s)$ and the positiveness of $g'(s)$. In particular, if $\mu_0 \in \mathcal{M}_0(\Omega)$ the assumption $g'(s) > 0$ implies that if u_1, u_2 are, respectively, the subsolution and supersolution, then $u_1 \leq u_2$ and the whole sequence $u(T,x)$ converges to v in $L^1(\Omega)$ where v is the solution of the corresponding elliptic (stationary) problem. Let us recall that this kind of comparison results of solutions plays a crucial role in the existence theory for nonlinear equations with integrable and measure data. As for elliptic initial boundary value problems, these calculations are studied in [30] in the case of dual data $\mu \in H^{-1}(\Omega)$ (recall that $\mu \in H^{-1}(\Omega)$ if and only if $f = \text{div}(G)$ where $G \in L^2(\Omega)^N$) and natural term with quadratic growth condition, these techniques have been adapted by [63] for parabolic problems. Here we generalize these results to the case of measures under the assumptions that the sequence (μ_n) converges to μ in what is called the narrow topology of measures and satisfies a sort of decomposition result with respect to the elliptic (parabolic) p-capacity, loosely speaking, the first part of μ is an element of $L^1(\Omega)$ and the second part is a divergentiel form of vector fields of $L^{p'}(\Omega)^N$. These requirements are satisfied, for instance, by elements of $\mathcal{M}_0(\Omega)$ and also by approximations μ_n constructed through convolution. Moreover, we present, in *Appendix*, an asymptotic result using a specific approximation on the data, based on the notion of G-convergence of operators and measure data in duality form, and we prove that regularity of solutions and assumptions on a, g and μ are actually necessary in some sense to get an asymptotic result since the G-convergence of the data may be false if u_n is only assumed to converge to u in $L^p(0,T; W_0^{1,p}(\Omega))$ without other convergence of the momenta. As consequence of these remarks, we are led to the problem of finding a suitable definition of solutions of (1.1) which may provide asymptotic behavior and stability properties at the same time, and this is why we choose to set our results in the framework of the so-called entropy solutions. Let us recall that the definition of entropy solutions was given in [38] in the context of elliptic equations and then adapted to the parabolic problems in [94], while in the theory of boundary value problems with measure data it has often used in order to get existence and uniqueness of solutions, see [24–27] for Sobolev spaces and [16,70] for generalized Sobolev spaces (Orlicz spaces). Finally, an extension of this framework has been given in [48] in the case of diffuse measure data and in [4,79] for singular measures. We follow the approach of this last papers, using this notion of solutions, when dealing with $\mu \in \mathcal{M}_0(\Omega)$ and showing how the entropy formulation emphasize the asymptotic behavior properties mentioned above by selecting sub, super and stable entropy solutions. We would like to emphasize that the asymptotic results obtained in the present paper could be classified in the blowing-up, fractional or variational inequality theories, see [5,8,86].

The paper is organized as follows. In Sect. 2 we state some basic notions and tools. Section 3 is devoted to the parabolic problem in finite energy sense with a smooth right-hand side and with absorbing lower order term, where in this case we show comparison results with respect to the weak formulation. These results allow us to obtain in, Theorem 3.1, an asymptotic behaviour result in $L^1(\Omega)$ with finite energy solutions for right-hand measures. In Sect. 4, we perform the analysis of the parabolic problem (1.1) in entropy sense with again appropriate sub and super-entropy solutions in determined spaces. By Comparison lemma and Potential analysis ("heat Kernel techniques") we deduce in, Theorem 4.4, a general asymptotic result of entropy solutions for parabolic equations (1.1). Section 5 (*Appendix*) ends the paper with the treatment of a model example of monotone operators with a specific "homogeneity" by using G-convergence properties.

2 Preliminaries

As already outlined, the asymptotic behaviour of solutions to (1.1) is determined not only by the exponent p of the nonlinear terms $-\Delta_p u$ and $|\nabla u|^p$ but also by the sign condition of g, size N and shape of the initial and source conditions. In the present section, we attempt to describe this variety of different effects of this terms, imposing particular Leray–Lions assumptions on a. In order to present our results in the most transparent form, we divide this section into subsections.

2.1 Notations

Given a real Banach space V, we will denote by $C^\infty(\mathbb{R}; V)$ the space of functions $u : \mathbb{R} \mapsto V$ which are infinitely many times differentiable (according to the definition of Fréchet differentiability in Banach spaces) and by $C_c^\infty(\mathbb{R}; V)$ the space of functions in $C^\infty(\mathbb{R}; V)$ having compact support. For a, b in $\mathbb{R}$, $C_c^\infty([a, b]; V)$ will be the space of restrictions to $[a, b]$ of functions of $C_c^\infty(\mathbb{R}; V)$ and $C([a, b]; V)$ the space of continuous functions from $[a, b]$ into V. Then for $1 \le p < +\infty$, $L^p(a, b; V)$ is the space of measurable functions $u : [a, b] \to V$ such that

$$\|u\|_{L^p(a,b;V)} = \left(\int_a^b \|u\|_V^p dt \right)^{\frac{1}{p}} < +\infty,$$

while $L^\infty(a, b; V)$ is the space of measurable functions such that

$$\|u\|_{L^\infty(a,b;V)} = \operatorname*{sup-ess}_{[a,b]} \|u\|_V < +\infty.$$

Of course both spaces are meant to be quotiented, as usual, with respect to the almost everywhere equivalence. The reader can find a presentation of these topics in [46]. Let us recall that, for $1 \le p \le \infty$, $L^p(a, b; V)$ is a Banach space, moreover if $1 \le p < \infty$

and V' (the dual space of V) is separable, then the dual space of $L^p(a, b; V)$ can be identified with $L^{p'}(a, b; V')$. Now, given a function u in $L^p(a, b; V)$, it is possible to define a time derivative of u in the space of vector valued distributions $\mathcal{D}'(a, b; V)$, which is the space of linear continuous functions from $C_c^\infty(a, b)$ into V (see [97]). In fact, the definition is the following

$$\langle u_t, \psi\rangle = -\int_a^b u\psi_t \, dt \qquad \forall\psi \in C_c^\infty(a, b)$$

where the equality is meant in V. If u belongs to $C^1(a, b; V)$ this definition clearly coincides with the Fréchet-derivative of u. In the following, when u_t is said to belong to the space $L^q(a, b; \widetilde{V})$ ($\widetilde{V}$ being a Banach space) this means that there exists a function z in $L^q(a, b; \widetilde{V}) \cap \mathcal{D}'(a, b; V)$ such that

$$\langle u_t, \psi\rangle = -\int_a^b u\psi_t \, dt = \langle z, \psi\rangle \qquad \forall\psi \in C_c^\infty(a, b).$$

Let $u \in L^p(a, b; V)$ be such that u_t belongs to $L^{p'}(a, b; V')$, then u belongs to $C([a, b]; V)$. This result also allows to deduce, for functions u and v enjoying these properties, the integration by parts formula

$$\int_a^b \langle v, u_t\rangle dt + \int_a^b \langle u, v_t\rangle dt = (u(b), v(b)) - (u(a), v(a)) \tag{2.1}$$

where here $\langle\cdot,\cdot\rangle$ is the duality between V and V'. Note that (2.1) makes sense, its proof relies on the fact that $C_c^\infty([a, b]; V)$ is dense in the space of functions $u \in L^p(a, b; V)$ such that u_t belongs to $L^{p'}(a, b; V')$ endowed with the norm $\|u\| = \|u\|_{L^p(a,b;V)} + \|u_t\|_{L^{p'}(a,b;V')}$ together with the fact that (2.1) is true for u, v in $C_c^\infty([a, b]; V)$. We will see in the next section a possible extensions of these spaces in the context of parabolic initial boundary value problems with generalized divergence form operators. Finally, for a real number r, we denote by $r^+ := \max\{r, 0\}$ its positive part and by $r^- := \max\{-r, 0\}$ its negative part. The letter C will denote generic positive constants, which do not depend on t and may vary from line to line during computations.

2.2 Capacity

The concept of capacity is indispensable to an understanding of local behavior properties of functions in a Sobolev space. In a sense, capacity takes the place of measure in Egorov and Lusin type theorems for Sobolev functions. Various capacity estimates also play a decisive role in studies of partial differential equations. We develop here the variational connection with the elliptic (resp. parabolic) Sobolev spaces, in particular, we assume that u is p-admissible with respect to the measure μ if obtained by $d\mu(x) = w(x)dx$ (resp. $d\mu(t, x) = w(t, x)dxdt$) (we refer to [54,100] for elliptic (p, μ)-capacity and [49,84] for parabolic (p, μ)-capacity).

Definition 2.1 Suppose that K is a compact subset of Ω. Let

$$W_0(K,\Omega)=\{u\in C_c^\infty(\Omega):u\geq 1 \text{ on } K\}$$

and define

$$\text{cap}_{p,\mu}(K,\Omega)=\inf_{u\in W_0(K,\Omega)}\int_\Omega |\nabla u|^p d\mu.$$

Further, if $U\subset\Omega$ is an open set

$$\text{cap}_{p,\mu}(U,\Omega)=\sup_{\substack{K\subset U\subset\Omega\\ K \text{ compact}}}\text{cap}_{p,\mu}(K,\Omega),$$

and finally, for an arbitrary set $E\subset\Omega$

$$\text{cap}_{p,\mu}(E,\Omega)=\inf_{\substack{E\subset U\subset\Omega\\ U \text{ open}}}\text{cap}_{p,\mu}(U,\Omega).$$

The number $\text{cap}_{p,\mu}(E,\Omega)\nu[0,\infty]$ is called the (variational) (p,μ)-capacity of the condenser (E,Ω), clearly $\text{cap}_{p,\mu}(E,\Omega)<+\infty$ if $E\subset\Omega$. There is no ambiguity in having two different definitions for the (p,μ)-capacity of a condenser (K,Ω) when K is compact that they give the same number $\text{cap}_{p,\mu}(K,\Omega)$, see [54, Theorem 2.2]. Using an extension in time, we observe that the set $W_0(K,\Omega)$ in the definition above can be replaced by the larger sets

$$\begin{cases} W_0(K,Q)=\{u\in C_c^\infty(Q):u\geq 1 \text{ a.e. in } K\},\\ W(K,Q)\ \ =\{u\in W:u\geq 1 \text{ a.e. in } K\}\end{cases} \tag{2.2}$$

where $W=\{u\in L^p(0,T;W_0^{1,p}(\Omega;\mu)\cap L^2(\Omega)),\ u_t\in L^{p'}(0,T;(W_0^{1,p}(\Omega;\mu)\cap L^2(\Omega))')\}$ endowed with its natural norm $\|u\|_W=\|u\|_{L^p\left(0,T;W_0^{1,p}(\Omega;\mu)\right)}+\|u_t\|_{L^{p'}(0,T;(W_0^{1,p}(\Omega;\mu)\cap L^2(\Omega))')}$ without affecting $\text{cap}_{p,\mu}(K,Q)$. Indeed, let $u\in W(K,Q)$; we may clearly assume that $\inf\emptyset=+\infty$. Then choosing characteristic function χ_K such that $\chi_K=0$ on $Q\backslash K$ and $\chi_K=1$ in neighborhood of K, if $\varphi\in C_0^\infty(Q)$ is a sequence of smooth functions with compact support on Q such that $\varphi\geq\chi_K$, then

$$\text{cap}_{p,\mu}(K,Q)\leq\int_Q\|\varphi\|_{W(K,Q)}. \tag{2.3}$$

It is also useful to observe that if $U\subseteq Q$ is an open set, the parabolic (p,μ)-capacity is such that

$$\text{cap}_{p,\mu}(U,Q)=\inf\{\|u\|_{W(K,Q)}\}.$$

Then for any Borel set $B \subseteq Q$, we can define

$$\text{cap}_{p,\mu}(B, Q) = \inf \{\text{cap}_{p,\mu}(U),\ B \subseteq U \subseteq Q\}.$$

Let us state some further results about parabolic p-capacity; the first one is the characterization of the relationship between sets of zero parabolic capacity and sections of the parabolic cylinder with both zero $\mathcal{L}^N$-measure sets and zero parabolic (p, μ)-capacity sets, while the second one shows that any function in $W(K, Q)$ admits a cap_p-quasi continuous representative. Let us recall that a function u is called $\text{cap}_{p,\mu}$-quasi continuous if for every $\epsilon > 0$ there exists an open set F_ϵ, with $\text{cap}_p(F_\epsilon) \leq \epsilon$ and such that $u_{|Q\setminus F_\epsilon}$ (the restriction of u to $Q \setminus F_\epsilon$) is continuous in $Q \setminus F_\epsilon$. As usual, a property will be said to hold $\text{cap}_{p,\mu}$-quasi everywhere if it holds everywhere except on a set of zero capacity. The third one states that measures in $\mathcal{M}_0(Q)$ which does not depend on time coincides with measures in Ω.

Theorem 2.2 *Let B be a Borel set in Ω and let $t_0 \in (0, T)$. One has*

$$cap_{p,\mu}(\{t_0\} \times B) = 0 \quad \textit{if and only if } \text{meas}_{\,\Omega}(B) = 0. \tag{2.4}$$

Proof See [49, Theorem 2.15]. □

Notice that, by virtue of Theorem 2.2, if a measure is concentrated on a section $\{t_0\} \times \Omega$, it does not charge sets of zero parabolic capacity if and only if it belongs to $L^1(\Omega)$.

Theorem 2.3 *Let $B \subset \Omega$ be a Borel set, and $0 \leq t_0 < t_1 \leq T$. Then we have*

$$cap_{p,\mu}((t_0, t_1) \times B) = 0 \quad \textit{if and only if } cap_{p,\mu}(B) = 0. \tag{2.5}$$

Proof See [49, Theorem 2.16]. □

Hence from now on, we shall identify measures in $\mathcal{M}_0(Q)$ and $\mathcal{M}_0(\Omega)$ and for more simplicity we will denotes cap_p instead of $\text{cap}_{p,\mu}$ and W instead of $W(K, Q)$.

2.3 Leray–Lions operators

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, with smooth boundary, we will denote by $\partial\Omega$ its lateral surface, p and p' be two real numbers with $\frac{1}{p} + \frac{1}{p'} = 1$. In what follows, $|\zeta|$ and $\zeta \cdot \zeta'$ will denote respectively the Euclidean norm of a vector $\zeta \in \mathbb{R}^N$ and the scalar product between ζ and $\zeta' \in \mathbb{R}^N$. Let then $a : \Omega \times \mathbb{R}^N \mapsto \mathbb{R}^N$ be a Carathéodory function (i.e. measurable with respect to x for every fixed ζ in $\mathbb{R}^N$ and continuous with respect to ζ for almost every fixed x in Ω) such that the following assumptions hold true

$$a(x, \zeta) \cdot \zeta \geq \alpha|\zeta|^p \quad \alpha > 0, \tag{2.6}$$

$$|a(x, \zeta)| \leq \beta|\zeta|^{p-1} \quad \beta > 0, \tag{2.7}$$

$$(a(x,\zeta)-a(x,\eta))\cdot(\zeta-\eta)>0 \tag{2.8}$$

for every ζ,η in $\mathbb{R}^N$ ($\zeta \neq \eta$) and almost every x in Ω. Thanks to (2.6)–(2.8), it is possible to define on the space $W_0^{1,p}(\Omega)$ and $L^p(0,T;W_0^{1,p}(\Omega))$ the operator $A(u) = -\text{div}(a(x,\nabla u))$, which then maps $W_0^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega))$ and $L^p(0,T;W_0^{1,p}(\Omega)$ into $L^{p'}(0,T;W^{-1,p'}(\Omega)$ where A is bounded and coercive. Given f in $L^{p'}(0,T;W^{-1,p'}(\Omega))$ and u_0 in $L^2(\Omega)$, by a weak solution of

$$\begin{cases} u_t - \text{div}(a(x,\nabla u)) = f & \text{in } Q := (0,T)\times\Omega, \\ u = 0 & \text{on } \Sigma := (0,T)\times\partial\Omega, \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{2.9}$$

we mean a function u in $L^p(0,T;W^{1,p}(\Omega))$ which satisfies the equation (2.9) in the sense of distributions, that is

$$-\int_Q u\psi_t\varphi\,dxdt + \int_Q a(x,\nabla u)\cdot\nabla\varphi\psi\,dxdt = \int_0^T \langle f,\varphi\rangle\psi\,dt \quad \forall\psi\in C_c^\infty(0,T),\ \forall\varphi\in C_c^\infty(\Omega) \tag{2.10}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$. As a consequence of equation (2.10) we deduce that u_t (which initially only belongs to $\mathcal{D}'(0,T;W_0^{1,p}(\Omega))$) in fact belongs to $L^{p'}(0,T;W^{-1,p'}(\Omega))$ and it follows that

$$\int_0^T \langle u_t,v\rangle dt + \int_Q a(x,\nabla u)\cdot\nabla v\,dxdt = \int_0^T \langle f,v\rangle dt \quad \forall v\in L^p(0,T;W_0^{1,p}(\Omega)).$$

Moreover from the injection result previously mentioned, if $p \geq \frac{2N}{N+2}$ then u belongs to $C([0,T];L^2(\Omega))$, which gives a meaning to the initial condition $u(0)$ (i.e. $u(0)=u_0$ in $L^2(\Omega)$). Nevertheless, even if $p < \frac{2N}{N+2}$, it is possible to find a weak solution u of (2.9) which belongs to $C([0,T];L^2(\Omega))$ as stated in the following classical result by J. Leray and J.-Louis Lions.

Theorem 2.4 *Let (2.6)–(2.8) hold true, and let f be in $L^{p'}(0,T;W^{-1,p'}(\Omega))$. Then there exists a weak solution u in $L^p(0,T;W_0^{1,p}(\Omega))\cap C([0,T];L^2(\Omega))$ of problem (2.9).*

Proof See [61,62]. □

Remark 2.5 The equation appearing in (2.9) can be considered both in the space of vector valued distributions, as we said before in (2.10), and in the space of distributions in Q, that is

$$-\int_Q u\frac{\partial\zeta}{\partial t}\,dxdt + \int_Q a(x,\nabla u)\cdot\nabla\zeta\,dxdt = \int_0^T \langle f,\zeta\rangle \quad \forall\zeta\in C_c^\infty((0,T)\times\Omega). \tag{2.11}$$

2.4 Entropy solutions

In order to obtain existence and asymptotic results, an entropy formulation is proposed, it is very close to the one which has been introduced for the elliptic case in [38]. In the case where $a(x, \nabla u)$ does not depend on t, existence and uniqueness of entropy solutions have been proved, using semigroup theory in [17], this formulation gives a solution for problem (2.9) when $f \in L^1(Q)$.

Definition 2.6 For $f \in L^1(Q)$, $u_0 \in L^1(\Omega)$ and Ω an open bounded set of $\mathbb{R}^N$, we define an entropy solution of (2.9) as a function $u \in C(0, T; L^1(\Omega))$ such that for all $k > 0$, $T_k(u) \in L^p(0, T; W_0^{1,p}(\Omega))$ and

$$\begin{aligned} &\int_\Omega \Theta_k(u-\varphi)(T)\,dx - \int_\Omega \Theta_k(u_0-\varphi(0))\,dx + \int_0^T \langle \varphi_t, T_k(u-\varphi)\rangle\,dt \\ &\quad + \int_Q a(x,\nabla u)\cdot\nabla T_k(u-\varphi)\,dxdt \le \int_Q f T_k(u-\varphi)\,dxdt \end{aligned} \tag{2.12}$$

where $\Theta_k(s) = \int_0^s T_k(\tau)d\tau$ is the primitive of the truncation function $T_k(s)$ for all $k > 0$ and for every $\varphi \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q) \cap C([0, T]; L^1(\Omega))$ such that $\varphi_t \in L^{p'}(0, T; W^{-1,p'}(\Omega))$.

Then we have the following result.

Theorem 2.7 *Let Ω be an open bounded set of $\mathbb{R}^N$, $f \in L^1(Q)$, $u_0 \in L^1(\Omega)$ and a satisfies* (2.6)–(2.8)*, then there exists one entropy solution of problem* (2.9).

We consider now the nonlinear equation (1.1) with initial condition u_0 in $L^1(\Omega)$ and right-hand side as a smooth measure μ on Ω which is absolutely continuous with respect to the p-capacity associated with the operator $-\mathrm{div}(a(x, \nabla u))$. We extend the previous notion of entropy solution, which is a generalization of Definition 2.6, given in [94]. To this end, we define

$$\begin{aligned} E = \{&\varphi \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q) \\ &\text{such that } \varphi_t \in L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^1(Q)\}. \end{aligned} \tag{2.13}$$

According to [85], one has $E \subset C([0, T]; L^1(\Omega))$.

Since we would like to have $g(u)|\nabla u|^p$ belongs to $L^1(Q)$ and $|\nabla u| \in L^{p'}(Q)$, we thus say that (1.1) is a nonlinear parabolic problem with lower order term depending on $|\nabla u|$ (with power-like nonlinearity with respect to $|\nabla u|$), we shall consider the following assumptions on the real C^1-function $g(s)$

$$g(s)s \ge 0 \quad \forall s \in \mathbb{R}, \tag{2.14}$$

$$g'(s) > 0 \quad \forall s \in \mathbb{R}, \tag{2.15}$$

and we denote by $u = u(t, x)$ the corresponding solutions to the parabolic problem (1.1).

Definition 2.8 Under hypothesis (2.6)–(2.8) and (2.14)–(2.15), if $u_0 \in L^1(\Omega)$, $\mu \in \mathcal{M}_0(\Omega)$ and $(f, -\mathrm{div}(G))$ is a decomposition of μ according to the p-capacity, an entropy solution of (1.1) is a measurable function u such that

$$g(u)|\nabla u|^p \in L^1(Q) \ \text{ for all } p > 1, \tag{2.16}$$

$$T_k(u) \in L^p(0, T; W_0^{1,p}(\Omega)) \ \text{ for all } k \geq 0, \tag{2.17}$$

$$t \in [0, T] \mapsto \int_\Omega \Theta_k(u - \varphi)(t, x)\, dx \tag{2.18}$$

is (a.e. equal to) a continuous function, and for all $k \geq 0$ and every $\varphi \in E$

$$\begin{aligned} &\int_\Omega \Theta_k(u - \varphi)(T, x)dx - \int_\Omega \Theta_k(u_0(x) - \varphi(0, x))dx + \int_0^T \langle \varphi_t, T_k(u - \varphi)\rangle dt \\ &\quad + \int_Q a(x, \nabla u) \cdot \nabla T_k(u - \varphi)dxdt + \int_Q g(u)|\nabla u|^p T_k(u - \varphi)dxdt \\ &\leq \int_Q f T_k(u - \varphi)dxdt + \int_Q G \cdot \nabla T_k(u - \varphi)dxdt. \end{aligned} \tag{2.19}$$

Remark that in (2.19), we denote by $\langle \cdot, \cdot \rangle$ the duality product between $W^{-1,p'}(\Omega) + L^1(\Omega)$ and $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and the definition chosen of entropy solutions uses an inequality instead of an equality, this is a standard choice for entropy solutions because it's sufficient to obtain the uniqueness (in the case when a does not depend on u and $\mu \in L^1(Q)$ for example, see [94]), and makes the proof of existence quite easier (there is no need to prove the strong convergence of gradient of the approximate solutions).

3 Case of finite energy solutions (Results and comments)

In this section we state a comparison principle result for the first energy solutions and an asymptotic behaviour theorem which allows us to reconstruct a limit function from the knowledge of its parabolic extension. The proofs are sketched, since they are partially identical to those given in [64,81] when the cases $L^{p'}(0, T; W^{-1,p'}(\Omega))$, $L^1(Q)$ and measure as data are treated. Indeed as we said in the previous section all relevant properties (for regular measures) can be extended to general (singular) measures, see [82]. We begin with a potential theorem which will be used in the proof of the main result, let us recall that $p^\star$ denotes the Sobolev conjugate exponent of p, that is $p^\star = \frac{pN}{N-p}$, and the limit function of functions $u(t, x)$ in $L^1(\Omega)$ will be denoted as usual by $v(x)$.

Theorem 3.1 *Let $\mu \in \mathcal{M}_0(\Omega)$ be a nonnegative measure independent of the variable t, $u_0 \in L^1(\Omega)$ be a nonnegative initial function, $u(t, x)$ be the weak solution of problem*

(1.1) *and* $v(x)$ *the weak solution of the corresponding elliptic problem*

$$\begin{cases} -\operatorname{div}(a(x,\nabla v)) + g(v)|\nabla v|^p = \mu & in\ \Omega, \\ v = 0 & on\ \partial\Omega. \end{cases} \tag{3.1}$$

Then

$$\lim_{T\to+\infty} u(T,x) = v(x)\ in\ L^1(\Omega). \tag{3.2}$$

Moreover, if $\mu \in \mathcal{M}_0(\Omega)$ *be independent on* t *and* $u_0 \in L^1(\Omega)$ *"without sign" assumptions, the result holds true, at least, in* $L^{p^\star}(\Omega)$.

Remark 3.2 (i) Recall that v is an entropy solution of the boundary value problem (3.1) if v is finite a.e., its truncated function $T_k(v) \in W_0^{1,p}(\Omega)$ and it holds

$$\int_\Omega a(x,\nabla v)\cdot\nabla T_k(v-\varphi)dx + \int_\Omega g(v)|\nabla v|^p dx \le \int_\Omega T_k(v-\varphi)d\mu \tag{3.3}$$

for every $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and for all $k>0$.
(ii) Observe that $v(x)$ is a solution of (1.1) with itself as initial data, being $v(x)$ independent of t.
(iii) The convergence of solutions $u(t,x)$ to $v(x)$ could be stronger than the one obtained under the assumptions of Theorem 3.1.
(iv) Thanks to the sign condition of μ and g we have both $u(t,x)$ and $v(x)$ are nonnegative (it's sufficient to take $T_k(u^-)$ as test function in (1.1) (resp. in (3.1)) and using the sign of u_0 to deduce that $u^-(t,x)=0$ a.e. in Q (resp. $v^-(x)=0$ a.e. in Ω)).
(v) Depending on the regularity of the data, the convergence in norm to the stationary solution can be improved, for example if $\mu \in L^q(\Omega)$ with $q > \frac{N}{p}$, the convergence of $u(t,x)$ in (3.2) is at least $*$-weakly in $L^\infty(\Omega)$ and a.e. in Ω.

Now, let us state the following definition of subsolutions and supersolutions of problem (1.1) that will be useful in the sequel to prove the comparison result cited above.

Definition 3.3 We say that $z \in L^p(0,T;W_0^{1,p}(\Omega))$ is a subsolution (resp. $w \in L^p(0,T;W_0^{1,p}(\Omega))$ is a supersolution) of problem (1.1) if $g(z)|\nabla z|^p \in L^1(Q)$ (resp. $g(w)|\nabla w|^p \in L^1(Q)$) and

$$\begin{cases} z_t - \Delta_p z + g(z)|\nabla z|^p \le \mu \text{ in } Q=(0,T)\times\Omega, \\ z(0,x) \le u_0(x) \text{ in } \Omega, \quad z(t,x)\le 0 \text{ on } (0,T)\times\partial\Omega, \end{cases} \tag{3.4}$$

$$\begin{cases} w_t - \Delta_p w + g(w)|\nabla w|^p \ge \mu \text{ in } Q=(0,T)\times\Omega, \\ w(0,x) \ge u_0(x) \text{ in } \Omega, \quad w(t,x)\ge 0 \text{ on } (0,T)\times\partial\Omega, \end{cases} \tag{3.5}$$

in the weak sense, i.e., $z(t, x)$ (resp. $w(t, x)$) satisfies

$$\int_0^T \langle z_t, \varphi \rangle dt + \int_Q \nabla z \cdot \nabla \varphi dx dt + \int_Q g(z)|\nabla z|^p \varphi dx dt \leq \int_Q \varphi d\mu \tag{3.6}$$

$$\left(\text{resp.} \int_0^T \langle w_t, \varphi \rangle dt + \int_Q \nabla w \cdot \nabla \varphi dx dt + \int_Q g(w)|\nabla w|^p \varphi dx dt \geq \int_Q \varphi d\mu \right) \tag{3.7}$$

for every $\varphi \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ with $\varphi \geq 0$ a.e. in Q.

According with results in literature, we are able to state and prove an (elliptic-parabolic versions) of comparison lemma that will play a key role in the proof of Theorem 3.1. It concerns the particular case for entropy sub and super solutions with restriction on the sign of the data (nonnegative measures) and that imply the uniqueness of solutions for problem (1.1) (i.e. solutions of (1.1) turn out to be, respectively, subsolutions and supersolutions in the sense of Definition 3.3). Recall that results about comparison principle for weak sub\super solutions of nonlinear elliptic–parabolic problems with lower order terms was mainly devoted to cases in which solutions are smooth (say for instance continuous) with lower order terms which has, at most, a power growth with respect to the gradient; more specially for Hamiltonian Carathéodory functions $H(x, \zeta) : \Omega \times \mathbb{R}^N \mapsto \mathbb{R}$ (resp., $H(t, x, \zeta)$ with respect to the time variable t). Let us mention (in the elliptic framework) the pioneering work of Barles and Murat [30, Theorem 1.1, Theorem 2.3 and Theorem 3.1] where general structure conditions on the lower order term were given to ensure that the comparison principle holds. Moreover, as it was observed on that paper, the method (called "linearized" approach) relies on a change of unknown to reduce the problem to the former good situation (transformed problem). Recall also that, in [30], see also [20,35], the authors considers $\mu = 0$ or $\mu \in H^{-1}(\Omega)$, in this cases, the proof consists in making a change of variable $u = \varphi(v)$ in the model equation where φ is a C^2-function in $\mathbb{R}$ with $\varphi' > 0$ and proving that the transformed equation satisfies the structure condition. This ideas was refined in [21] and a slightly improvement of the condition of [30] is proved for data small enough. A different kind of comparison principle is proved in [18] for lower order term with quadratic growth with respect the last variable of the form $g(u)|\nabla u|^2$ for some nonnegative continuous function g in $(0, +\infty)$. In that paper, the authors imposed an integrability condition at zero (this result also handles the case that g is singular); however, their techniques require strongly that the lower order function and the differential operator do not depend on x and some further extensions where done in [9–12]. We stress that, even when $p = 2$ and for sufficiently smooth solutions, the comparison principle is not trivial without assuming suitable assumptions, see [7,57] for $1 < p < 2$. In the case $p > 2$, the comparison principle is more difficult since the operator turns out to be nonlinear and degenerate. There are a little results in the literature giving comparison principles when $p > 2$ (except, the new papers of T. Leonori and his co-authors in [63,65–67], and the paper of A. Porretta [87] where some model examples are contained when the lower order term has precisely the growth as $|\nabla u|^p$). A rather general structure conditions on the lower order functions which imply the comparison principle for weak solutions, when $p > 2$, was alternatively used either the approach of Barles and Murat [30], based on a linearization principle coupled with change of variable $v = \varphi(u)$ with φ is a C^3-monotone function such that $\varphi' \neq 0$, this approach mostly relying on the linearization of the operator (it suggests to use the fundamental work [50] to get a weight estimates on $|\nabla u|^{p-2}$ for a useful linearization, and this is a key-point in order to get the comparison principle). We refer the reader to [67] for a totally description of this approach under general regularity conditions and for different issues. In this last paper, different choices of lower order terms are considered which allow to observe interesting phenomena compared with other possible choice of data, see [67, Theorem 1.1, Theorem 1.2, Theorem 1.3]. Observe that, based on a "convexity" argument, the result of [67, Theorem 1.3] applies in particular to the stationary case of (1.1) when $\mu \geq 0$. Then, it is remarkable that conditions of [67] extends our limiting case $|\nabla u|^p$ to the case $|\nabla u|^q$ with $1 \leq q < p-1$, $p-1 \leq q < p$ and $q \geq p$. The proof of [67, Theorem 1.1] follows the approach of [30] and can be applied, with care, to our problem, on the other hand [67, Theorem 1.3] uses a "convexity" approach needed to prove the stationary comparison Lemma 3.5 of problem (3.1). Finally, in order to deal with problem (1.1), we are devoted a special attention to handle the elliptic case, so the extension of time-dependent comparison principle can be easily deduced in the context of parabolic operators. Note that, in this last framework, it is necessary to modify the structure conditions given in [67, Sections 1,2] by using the ideas already mentioned and suggested by Barles and Murat at the end of their work [30, Section 3.3], especially, the change of the structure condition using the conjugate exponent p' and the specific choice of "exponential-type" test functions in Taylor formula for both a and the lower order term, and then it is easy to get similar results for problems like (1.1). According to the stationary results, we stress that a version of comparison (and a uniqueness as a byproduct) result specially devoted to the model problem

$$-\Delta u + g(u)|\nabla u|^2 = f \text{ in } \Omega \tag{3.8}$$

where f belongs to $H^{-1}(\Omega)$ and $g : \mathbb{R} \mapsto \mathbb{R}$ is C^1-function such that $g(0) = 0$, $g'(u) > 0$ for all $u \in \mathbb{R}$, see [30, Theorem 2.6 and Remark 2.7], was proved for (weak) solutions $u \in H^1(\Omega)$ such that $g(u)|\nabla u|^2 \in L^1(\Omega)$ using the convexity of the function $\int_0^t g(x)dx$ and a specific structure condition of the form

$$g'(u)|\chi|^2 - \frac{1}{2n}(2g(u)|\chi| - z(u)\chi|^2 > 0 \text{ a.e. } x \in \Omega, \tag{3.9}$$

for all $u \in \mathbb{R}$, for every $\chi \in \mathbb{R}^n$ with $\chi \neq 0$, for some $n > 0$ and some continuous function $z : \mathbb{R} \mapsto \mathbb{R}$ such that $\exp\left(-\frac{1}{n}\int_0^t z(s)ds\right) \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ (these conditions are satisfied, for instance, with $z = 2g$). By using this argument, an elliptic comparison principle result has been proved by Leonori & Petitta, [64, Lemma 2.2], for unbounded entropy sub and supersolutions with no restrictions on the sign of the datum and adapted to the parabolic case with general nonnegative data in [64, Lemma 2.3] for problems of the form

$$u_t - \Delta u + u|\nabla u|^2 = f \text{ in } \Omega \tag{3.10}$$

with L^1-data that do not depend on time, while the lower order term satisfies a structure condition as in (3.9). Their result is proved by looking for an inequality solved by the difference between the bounded parts of the sub and supersolutions and by using the linearization of the lower order term inspired by the ideas of [30]. Let us recall that this is a typical approach for singular operators (i.e. $p \le 2$); however, it does not seem that the case of general Leray–Lions operators (i.e. *degenerate*) can be treated in such a nice way since the invertibility does not make sense for nonlinear operators. An interesting particular case where a similar result may proved is the case where a does not depend on the solution and satisfies

$$(a(x,\zeta) - a(x,\eta)) \cdot (\zeta - \eta) \ge \gamma |\zeta - \eta| \quad \gamma > 0 \tag{3.11}$$

for almost every $x \in \Omega$ and for all ζ, η in $\mathbb{R}^N$ (this is, for example, the case of the p-Laplace operator for $p \ge 2$), which can be extended to the model example

$$-\operatorname{div}(a(x,\nabla u)) + g(u)|\nabla u|^p = f(x) \text{ in } \Omega \tag{3.12}$$

where g satisfies the assumptions (2.14)–(2.15). In this case the maximum principle holds in $W^{1,p}(\Omega) \cap L^\infty(\Omega)$ provided that for some $n > 0$

$$\frac{\partial b}{\partial v} - \frac{1}{n}\left|\frac{\partial b}{\partial \zeta}\right|^{p'} > 0 \text{ a.e. } x \in \Omega,\ \forall v \in \mathbb{R},\ \forall \chi \in \mathbb{R}^N \tag{3.13}$$

where

$$\begin{cases} a(x,\zeta) = \varphi'(v) a(x, \varphi'(v)\nabla v), \\ b(x,\zeta) = -\dfrac{\varphi''}{\varphi'}(v) a(x,\varphi'(v)\nabla v)\cdot \nabla v + \dfrac{1}{\varphi'(v)} g(\varphi(v))|\varphi'(v)\nabla v|^p - f(x) \end{cases} \tag{3.14}$$

and v is the solution of the transformed equation obtained, for convenient, by change of function $u = \varphi(v)$ satisfying a structure condition, this equation has the form

$$-\operatorname{div}\left(a(x,\varphi'(v)\nabla v)\right) + b(x,\varphi'(v)\nabla v) = f \text{ in } \Omega. \tag{3.15}$$

In this case the computations follows along the lines of those of [64, Theorem 1.2], we just specify the choice of a specific test function using $S(\varphi) = \exp(-\alpha\varphi^{-k})$ for some constants $\alpha > 0$ and $k > 0$ where k depends only on p. This result allows us to propose a different approach to prove Lemma 3.5 and to deduce a natural extension to the parabolic case. We give now the comparison principle, proved in [67] as a main result and stated here as a Lemma, which concern general Hamiltonian equations, namely

$$-\Delta_p u + H(x,u,\nabla u) = F \in W^{-1,p'}(\Omega) \text{ in } \Omega. \tag{3.16}$$

Lemma 3.4 *Assume that H is a C^1-function which satisfies growth conditions and that F belongs to $W^{-1,p'}(\Omega)$. Assume moreover that, for every $\epsilon > 0$, there exists C_ϵ which satisfies*

$$H(x,s,\zeta) - (1-\epsilon)^{p-1} H\left(x, \frac{t+\epsilon k}{1-\epsilon}, \frac{\eta}{1-\epsilon}\right) \\ \leq C_\epsilon |\zeta - \eta|^{p-1} (1 + |\zeta - \eta|) \quad \forall s \leq t \tag{3.17}$$

for $k > 0$, $x \in \Omega$, $\zeta, \eta \in \mathbb{R}^N$ and for $\{s, \frac{t+\epsilon k}{1-\epsilon}\} \in [-M, M]$ where $M = \max\left(\|u_1\|_{L^\infty(\Omega)}, \|u_2\|_{L^\infty(\Omega)}\right)$. If u_1 and u_2 belong to $W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and are, respectively, a subsolution and a supersolution of (3.16) such that $(u_1 - u_2)^+ \in W_0^{1,p}(\Omega)$, then

$$u_1 \leq u_2 \text{ a.e. in } \Omega. \tag{3.18}$$

Proof See [67, Theorem 1.3]. □

Observe that condition (3.17) implies that $H(x,s,\zeta)$ is nondecreasing in s; on the other hand if H is independent of s, it implies that $H(x,0) \leq 0$. This later case applies in particular to the model problem (3.12) when $f \in L^1(\Omega)$ and f is nonnegative. More generality, Lemma 3.4 can be applied to the equation $-\Delta_p u + g(u)|\nabla u|^q = \mu$ where g is nondecreasing, $\mu \in \mathcal{M}_0(\Omega)$ with $\mu \geq 0$ and $q \in [p-1, p]$ (see [67, Corollary 3.1], the limiting case $q = p-1$ is quite delicate but it is admitted since the properties of the character g may give a contribution for this limit case). This choice, that has been mainly inspired by [95], uses both an argument via linearization and a method that exploits a sort of convexity of the Hamiltonian term with respect to the gradient. Recall that the two approaches (i.e. [63] and [67]) are, in some sense, complementary since the first one ("the linearization") works in the case $1 < p \leq 2$ while the second one (the "convexity") deals with $p \geq 2$. Of course, the only case in which both of them are in force is when $p = 2$. The proof of the following lemma, except for its very beginning and the use of a measure as data instead of just Lebesgue function is similar to the proof of [67, Theorem 1.3] by assuming that $g(u) \geq 0$ and there is no loss in assumption the positivity of g. In fact, let us consider $s_0 := \inf\{s \in \mathbb{R} : g(s) = 0\}$; if $s_0 > -\infty$, then s_0 is a subsolution since $\mu \geq 0$ and so $\widetilde{u}_1 := \max(u_1, s_0)$ is still a subsolution. Since $g(u_2) \geq 0$ implies $u_2 \geq s_0$, it would still hold that $\widetilde{u}_1 \leq u_2$ at $\partial\Omega$. Therefore, we could replace u_1 with $\widetilde{u}_1$ for which $g(\widetilde{u}_1) \geq 0$ and, by proving $u_2 \geq \widetilde{u}_1$, we still deduce $u_2 \geq u_1$. Thus, in the following, we can and we will assume that $g(u) \geq 0$ a.e. in Ω.

Lemma 3.5 *Assume that $\mu \in \mathcal{M}_0(\Omega)$ be a nonnegative measure. If u_1, u_2 belong to $W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and are, respectively, the entropy subsolution and supersolution of problem*

$$-\operatorname{div}(a(x,\nabla u)) + g(u)|\nabla u|^p = \mu \text{ in } \Omega$$

where g is a continuous nondecreasing function such that (2.14)–(2.15) holds and such that $(u_1 - u_2)^+ \in W_0^{1,p}(\Omega)$. Then

$$u_1 \leq u_2 \text{ a.e. in } \Omega. \tag{3.19}$$

In particular, (3.19) has at most one (entropy) solution in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Proof 1st *approach (Convexity method).* Let us now explain how the proof of [63] has to be modified in the present framework by following the ideas of [67, Theorem 1.3]. In order to do it, we recall the definitions of u_1 and u_2 in (3.1) and we focus on the one related to the entropy subsolution u_1, we consider $u_1^\epsilon = (1-\epsilon)u_1 - \epsilon k$ with $\epsilon \in (0,1)$ and $k \geq \|u_1^-\|_{L^\infty(\Omega)}$ (in order to get $u_1^\epsilon \leq u_1$) and we multiply its inequality by $(1-\epsilon)^{p-1}$. By taking into account its difference with the second inequality solved by u_2, we get

$$\begin{aligned} -\mathrm{div}(a(x, \nabla u_1^\epsilon)) + \mathrm{div}(a(x, \nabla u_2)) &\leq g(u_2)|\nabla u_2|^p \\ &-(1-\epsilon)^{p-1} g\left(\frac{u_1^\epsilon + \epsilon k}{1-\epsilon}\right)\left|\frac{\nabla u_1^\epsilon}{1-\epsilon}\right|^p, \end{aligned} \tag{3.20}$$

recalling this inequality and defining $m_\epsilon = \text{ess-sup}_\Omega(u_1^\epsilon - u_2)$. We want to conclude that $(u_1^\epsilon - u_2 - k)^+ = 0$, then, we suppose, by contradiction, that $m_\epsilon > 0$. Since $w = (u_1^\epsilon - u_2 - k)^+$ belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $(u_1^\epsilon - u_2 - k)^+$ can be used as a test function in the above inequality, with $k \in (0, m_\epsilon)$, obtaining

$$\begin{aligned} &\int_\Omega \big(a(x, \nabla u_1^\epsilon) - a(x, \nabla u_2)\big) \cdot \nabla(u_1^\epsilon - u_2 - k)^+ dx \\ &\quad \leq \int_\Omega \left(g(u_2)|\nabla u_2|^p - (1-\epsilon)^{p-1} g\left(\frac{u_1^\epsilon + \epsilon k}{1-\epsilon}\right)\left|\frac{|\nabla u_1^\epsilon|}{1-\epsilon}\right|^p\right)(u_1^\epsilon - u_2 - k)^+ dx. \end{aligned} \tag{3.21}$$

Now, let us fix $k > M = \max(\|u_1\|_{L^\infty(\Omega)}, \|u_2\|_{L^\infty(\Omega)})$. Then, for every $s \leq t$ we get $t \geq -M$, and so $t < \frac{t+\epsilon k}{1-\epsilon} \in [-M, M]$. We assume g is nondecreasing and, without loss of generality, to be positive. So that, for every $\epsilon > 0$ and for any $s \leq t$, we get

$$g(s) \leq g(t) < g\left(\frac{t+\epsilon k}{1-\epsilon}\right) \tag{3.22}$$

because $-M \leq t < \frac{t+\epsilon k}{1-\epsilon}$ (due to $k > M$). Then, we deduce that

$$\begin{aligned} &g(u_2)|\nabla u_2|^p - (1-\epsilon)^{p-1} g\left(\frac{t+\epsilon k}{1-\epsilon}\right)\left(\frac{|\nabla u_1^\epsilon|}{1-\epsilon}\right)^p \\ &\quad \leq g\left(\frac{t+\epsilon k}{1-\epsilon}\right)\left[|\nabla u_2|^p - \frac{|\nabla u_1^\epsilon|^p}{1-\epsilon}\right]. \end{aligned} \tag{3.23}$$

Using a convexity argument (recall that $p > 2$), we obtain

$$|\nabla u_2|^p \leq (1-\delta)\left(\frac{|\nabla u_1^\epsilon|}{1-\delta}\right)^p + \delta\left(\frac{|\nabla u_2 - \nabla u_1^\epsilon|}{\delta}\right)^p \quad \forall \delta \in (0,1). \tag{3.24}$$

Then, choosing $\delta > 0$ such that

$$(1-\delta)^{p-1} = \frac{1}{1-\epsilon}$$

and using the fact that $g(s) \geq 0$ for all $s \in \mathbb{R}$, we finally obtain

$$\begin{aligned} g\left(\frac{t+\epsilon k}{1-\epsilon}\right)\left[|\nabla u_2|^p - \frac{|\nabla u_1^\epsilon|^p}{1-\epsilon}\right] &\leq C_\epsilon |\nabla u_2 - \nabla u_1^\epsilon|^p \\ &\leq C_\epsilon |\nabla u_2 - \nabla u_1^\epsilon|^{p-1}(1 + |u_2 - u_1^\epsilon|). \end{aligned} \tag{3.25}$$

In order to estimate the right hand side, we have

$$\begin{aligned} &\int_\Omega C_\epsilon |\nabla u_2 - \nabla u_1^\epsilon|^p dx \\ &\leq \int_\Omega C_\epsilon \left(|\nabla u_2|^{p-2}\nabla u_2 - |\nabla u_1^\epsilon|^{p-2}\nabla u_1^\epsilon\right)\cdot(\nabla u_2 - \nabla u_1^\epsilon)dx \\ &\leq \int_\Omega C_\epsilon \left(|\nabla u_2|^{p-2}\nabla u_2 - |\nabla u_1^\epsilon|^{p-2}\nabla u_1^\epsilon\right)\cdot \nabla(u_1^\epsilon - u_2 - k)^+ dx \\ &\leq C_\epsilon \int_\Omega |\nabla(u_1^\epsilon - u_2)|^{p-1}(u_1^\epsilon - u_2 - k)^+ dx \\ &\quad + C_\epsilon \int_\Omega |\nabla(u_1^\epsilon - u_2|^p (u_1^\epsilon - u_2 - \epsilon)^+ dx. \end{aligned} \tag{3.26}$$

Using a classical inequality and young inequalities, we get

$$\frac{1}{2}\int_\Omega |\nabla w|^p dx \leq C_\epsilon \int_{A_k} |w|^p dx + C_\epsilon \int_\Omega |\nabla w|^p w dx \tag{3.27}$$

where $A_k = \{x \in \Omega : u_1^\epsilon - u_2 \geq k\} \cap \{x \in \Omega : \nabla u_1^\epsilon \neq \nabla u_2\}$ and we still denote by C_ϵ possibly different constants depending on ϵ. Since $w \leq m_\epsilon - k$, by choosing k sufficiently close to m_ϵ, the last term can be absorbed in the left-hand side and we deduce

$$\frac{1}{4}\int_\Omega |\nabla w|^p dx \leq C_\epsilon \int_{A_\epsilon} |w|^p dx \tag{3.28}$$

which implies, by Poincaré-Sobolev inequality (where $p^\star$ denotes the Sobolev conjugate of p if $p < N$ or any number greater than p if $p \geq N$)

$$\|w\|^p_{L^{p^\star}(\Omega)} \leq C_\epsilon \int_{A_k} |w|^p dx \leq C_\epsilon \|w\|^p_{L^{p^\star}(\Omega)} |A_k|^{1-\frac{p}{p^\star}}. \tag{3.29}$$

Since $|A_k| \to 0$ as $k \to m_\epsilon$, we conclude that $(u_1^\epsilon - u_2 - k)^+ = 0$ for some $k < m_\epsilon$, getting a contradiction with the definition of m_ϵ. This concludes the verification of the proof of [67, Theorem 1.3] with $q = p$.

2nd *approach (Linearized method).* The idea of the proof is to make the change of function $u = \varphi(v)$ where φ is a C^2-function in $\mathbb{R}$ with $\varphi' > 0$. Using $\frac{y}{\varphi'(v)}$ where $y \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ as a test function in the variational formulation of (3.1) yields the equation

$$\begin{aligned}
&-\mathrm{div}(\varphi'(v)a(x, \varphi'(v)\nabla v)) - \frac{\varphi''(v)}{\varphi'(v)} a(x, \varphi'(v)\nabla v) \cdot \nabla v \\
&\quad + \frac{1}{\varphi'(v)} g(\varphi(v))|\varphi'(v)\nabla v|^p = 0.
\end{aligned} \tag{3.30}$$

We now have to choose a function φ such that (3.30) is true and satisfying the structure condition of Lemma 3.4. Since

$$\begin{cases} a(x, \zeta) = \varphi'(v)a(x, \varphi'(v)\zeta), \\ b(x, \zeta) = -\dfrac{\varphi''(v)}{\varphi'(v)} a(x, \varphi'(v)\zeta) \cdot \zeta + \dfrac{1}{\varphi'(v)} g(\varphi(v))|\varphi'(v)\zeta|^p. \end{cases} \tag{3.31}$$

It is sufficient to prove that for some $n > 0$

$$\frac{\partial b}{\partial v} - \frac{1}{n}\left|\frac{\partial b}{\partial \zeta}\right|^{p'} > 0 \quad \text{with } \frac{1}{p} + \frac{1}{p'} = 1. \tag{3.32}$$

We actually use the function φ defined by

$$\varphi(v) = -\frac{1}{A}\log\left(e^{-kAv} + \frac{1}{k}\right) \tag{3.33}$$

where we first fix $A > 0$ and then choose $k > 0$ large enough (recall that u_1 and u_2 are assumed to be bounded). Thus we only need the range of φ to cover $[-M, M]$ with $M = \max\left(\|u_1\|_{L^\infty(\Omega)}, \|u_2\|_{L^\infty(\Omega)}\right)$. This is the case if k large enough, and more precisely if $k > e^{MA}$. We compute $\frac{\partial b}{\partial v}$, by setting $\chi = \varphi'(v)\zeta$, to get

$$\begin{cases} a(x, \zeta) = \varphi'(v)a(x, \chi), \\ b(x, \zeta) = -\dfrac{\varphi''(v)}{\varphi'(v)} a(x, \chi) \cdot \zeta + \dfrac{1}{\varphi'(v)} g(\varphi(v))|\chi|^p. \end{cases} \tag{3.34}$$

Then, we have

$$\begin{aligned}\frac{\partial b}{\partial v}(x,\zeta) &= -\left(\frac{\varphi''}{\varphi'}\right)'(v)a(x,\chi)\cdot\zeta - \frac{\varphi''(v)}{\varphi'(v)^2}g(\varphi(v))|\chi|^p \\ &\quad + g'(\varphi(v))|\chi|^p + p\frac{\varphi''}{\varphi'}\zeta g(\varphi(v))|\chi|^{p-1} \\ &= -\left(\frac{\varphi''}{\varphi'}\right)(v)a(x,\chi)\cdot\zeta - \frac{\varphi''(v)}{\varphi'(v)^2}g(\varphi(v))|\chi|^p \\ &\quad + p\chi\frac{\varphi''}{\varphi'(v)^2}g(\varphi(v))|\chi|^{p-1} \\ &\quad - \left(\frac{\varphi''}{\varphi'}\right)(v)a(x,\chi)\zeta + g'(\varphi(v))|\chi|^p \\ &\quad + \frac{\varphi''(v)}{\varphi'(v)^2}\Big[p\chi g(\varphi(v))|\chi|^{p-1} - g(\varphi(v))|\chi|^p\Big]. \end{aligned} \tag{3.35}$$

In order to obtain an expression in the "old" variable $u = \varphi(v)$, it is convenient to introduce the function ω defined by

$$\omega = \varphi' \circ \varphi^{-1}, \quad \text{i.e.,} \quad \omega(u) = \omega(\varphi(v)) = \varphi'(v). \tag{3.36}$$

After some straightforward computations, see [30, page 86], this formula becomes

$$\begin{aligned}\frac{\partial b}{\partial v}(x,\zeta) &= \frac{1}{\omega(u)}\Big\{-\omega''(u)a(x,\chi)\cdot\chi \\ &\quad + \omega'(u)\Big[p\chi g(u)|\chi|^{p-1} - g(u)|\chi|^p\Big]\Big\} + g'(u)|\chi|^p. \end{aligned} \tag{3.37}$$

An analogous computation yields

$$\frac{\partial b}{\partial \zeta}(x,\zeta) = pg(u)|\chi|^{p-1} - 2\frac{\omega'(u)}{\omega(u)}\chi. \tag{3.38}$$

Since $u = \varphi(v) = -\frac{1}{A}\log\left(e^{-kAv} + \frac{1}{k}\right)$ and $\omega(u) = \varphi'(v)$, we have

$$\omega(u) = k - e^{Au}. \tag{3.39}$$

In order to compare u_1 and u_2, which both belong to $L^\infty(\Omega)$ with $\|u_i\|_{L^\infty(\Omega)} \le M$ for $i = 1, 2$, it is enough to prove that, for some $n > 0$

$$\frac{\partial b}{\partial v} - \left|\frac{\partial b}{\partial \zeta}\right|^{p'} > 0 \text{ a.e. } x \in \Omega,\ \forall u \in \mathbb{R},\ |u| \le M,\ \forall \chi \in \mathbb{R}^N. \tag{3.40}$$

Again using the fact that u_1 and u_2 belong to $L^\infty(\Omega)$, we get the estimate

$$\left[p\chi g(u)|\chi|^{p-1} - g(u)|\chi|^p\right] = pg(u)|\chi|^p - g(u)|\chi|^p = (p-1)g(u)|\chi|^p > 0 \tag{3.41}$$

if $\chi \neq 0$, $g > 0$ and $p > 1$, this coincides with conditions of the result of [67] for $p > 1$ but with g nonincreasing (in order to use the convexity property of g). Therefore assumption of [63, Theorem 1.2] is fulfilled and the proof is complete. □

Finally, recall that, in order to prove the comparison principle in Lemmas 3.5, different techniques have been developed. Let us mention, among the others, the results that have been proved by using the monotone "rearrangement" technique (see for instance [23] and references cited therein) or by means of "viscosity" solutions (see [22,40] and their references). Our second problem, that have been mainly inspired by [74,88], see also [90], and used in [64,87–89] to get some regularity results on elliptic and parabolic problems with absorption, uses an argument via "Hopf" transformation. Recall that in that papers, which extends the results for "absolutely continuous" measures to more general "singular" measures, the authors give an alternative proof of existence and uniqueness for large/explosive solutions, i.e., solutions which blow up uniformly at the boundary, as well as to solve nonlinear equations (with absorption) in unbounded domains using the notion of elliptic–parabolic capacity. This kind of phenomena due to absorption terms has been investigated for semilinear evolution problems of the form

$$\begin{cases} u_t - \Delta u + |u|^{r-1}u = \mu \text{ in } (0,T)\times\Omega, \\ u(t,x) = 0 \text{ on } (0,T)\times\partial\Omega, \quad u(t,x) = u_0 \text{ in } \Omega, \end{cases}$$

which does not always have a solution for any measure μ on Q and any initial data $u_0 \in L^1(\Omega)$ or in $\mathcal{M}_b(\Omega)$, see [90]. Again, under the assumption that $g \in L^1(\mathbb{R})$ it is possible to prove the existence of a solution for any measure u_0 as singular initial data, more precisely

$$\begin{cases} u_t - \Delta_p u + g(u)|\nabla u|^p = f \text{ in } (0,T)\times\Omega, \\ u(t,x) = 0 \text{ on } (0,T)\times\partial\Omega, \quad u(t,x) = u_0 \text{ in } \Omega, \end{cases}$$

where $u_0 = u_0^r + u_0^s$ with $u_0^r \in L^1(\Omega)$ and u_0^s is concentrated on a set of zero p-capacity (i.e. $u_0^s = u_0 \llcorner E$ with meas(E)=0), for example, μ_s can be considered as $\delta_{x_0}(x)$ in space dimension $N \geq 2$, see [90, Section 6], but the result should be justified rigorously. The case where the p-Laplace operator is replaced by a nonlinear divergence type operator has been proved in [34] (see also references therein for more details about these topics), recall that in that papers the authors interested just on compactness results but not on asymptotic results. Based on these ideas and on the comparison result of Lemma 3.5, we will show that the same change of variable cannot be done, but it can be replaced with the use of "*exponential-type*" test functions, whose

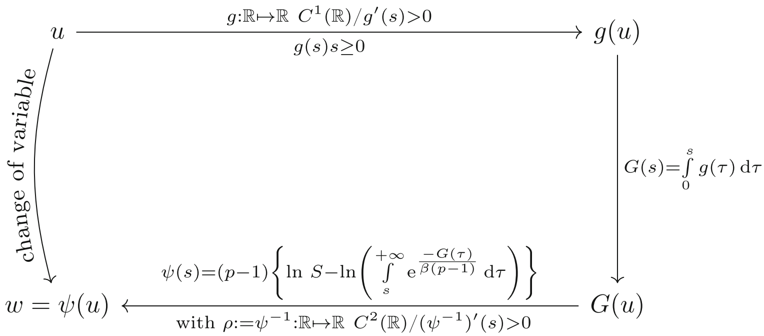

Fig. 1 The diagram of the change of unknown

role is again to get rid of the natural growth term. Now, if μ is a nonnegative measure in $\mathcal{M}_0(\Omega)$ and $g : \mathbb{R} \mapsto \mathbb{R}$ is a continuous nonincreasing function, non identically zero, satisfying (2.14)–(2.15), the last proof of Lemma 3.5 can be properly adapted to the case of nondegenerate parabolic equations (1.1) set in W and the computations follow along the lines of those above. Here we propose another method to prove the comparison principle for parabolic problems (1.1) showing how the entropy solutions emphasize a (determined) exponential formulation by performing the "Hopf-Cole" transformation of (1.1) by means of "exponential" test functions and a "standard" change of unknown. More precisely, we will show that $w_1 \le w_2$ a.e. in Q where $w_i = \psi(u_i)$, $i = 1, 2$, obtained by change of variable function ψ that we will specify later.

Lemma 3.6 *Let $u_0^1, u_0^2 \in L^1(\Omega)$ such that $0 \le u_0^1 \le u_0^2$ and let $\mu \in \mathcal{M}_0(\Omega)$ be a nonnegative measure, g satisfies (2.14)–(2.15). If u_1 and u_2 belong to $W \cap L^\infty(Q)$ and are, respectively, the subsolution and supersolution of problem (1.1). Then*

$$u_1 \le u_2 \text{ a.e. in } Q \quad \forall t \in (0, T). \tag{3.42}$$

Proof We start by considering the change of unknown $w = \psi(u)$ where $\psi : \mathbb{R} \mapsto \mathbb{R}$ is defined by

$$\psi(s) = (p-1)\left[\ln S - \ln\left(\int_s^{+\infty} e^{\frac{-G(\tau)}{\beta(p-1)}} d\tau\right)\right] \text{ with}$$
$$S = \int_0^{+\infty} e^{-\frac{G(\tau)}{\beta(p-1)}} d\tau,\ G(s) = \int_0^s g(\tau) d\tau. \tag{3.43}$$

Observe that ψ is a C^1-function satisfying

$$\begin{cases} (p-1)\psi''(s) = \psi'(s)^2 - \dfrac{\psi'(s)g(s)}{\beta}, \\ \psi(0) = 0 \text{ and } \psi'(0) = \dfrac{p-1}{S}. \end{cases} \tag{3.44}$$

We show that elementary computations lead to the comparison principle result for solutions of (1.1), the main tool we are going to use is the existence of a transformed problem satisfied by $w = \psi(u)$. Precisely, let

$$\widetilde{a}(x,s,\zeta) = \psi'(\rho(s))^{p-1} a\left(x, \frac{\zeta}{\psi'(\rho(s))}\right) \text{ where}$$
$$\rho = \psi^{-1} \text{ is a continuous increasing function.}$$

Since ψ' and ρ are continuous, $\widetilde{a}$ is a Carathéodory function on $\Omega \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}^N$ and satisfies

$$\begin{aligned}\widetilde{a}(x,s,\zeta) &= \psi'(\rho(s))^{p-1} a\left(x, \frac{\zeta}{\psi'(\rho(s))}\right) \cdot \zeta \\ &= \psi'(\rho(s))^{p} a\left(x, \frac{\zeta}{\psi'(\rho(s))}\right) \cdot \frac{\zeta}{\psi'(\rho(s))} \\ &\geq \alpha \psi'(\rho(s))^{p} \left|\frac{\zeta}{\psi'(\rho(s))}\right|^{p} = \alpha|\zeta|^p, \end{aligned} \tag{3.45}$$

and

$$|\widetilde{a}(x,s,\zeta)| = \left|\psi'(\rho(s))^{p-1} a\left(x, \frac{\zeta}{\psi'(\rho(s))}\right)\right| \leq \beta|\zeta|^{p-1}. \tag{3.46}$$

Now, remark that by assumption (2.8)

$$\begin{aligned}&(\widetilde{a}(x,s,\zeta) - \widetilde{a}(x,s,\eta))(\zeta - \eta) \\ &= \left(\psi'(\rho(s))^{p-1} a\left(x, \frac{\zeta}{\psi'(\rho(s))}\right)\right. \\ &\quad \left. -\psi'(\rho(s))^{p-1} a\left(x, \frac{\eta}{\psi'(\rho(s))}\right)\right)(\zeta - \eta) \\ &= \psi'(\rho(s))^{p} \left(a\left(x, \frac{\zeta}{\psi'(\rho(s))}\right)\right. \\ &\quad \left. -a\left(x, \frac{\eta}{\psi'(\rho(s))}\right)\right)\left(\frac{\zeta}{\psi'(\rho(s))} - \frac{\eta}{\psi'(\rho(s))}\right) > 0\end{aligned}$$

since $\psi'(\rho(s))^p > 0$. This means that $\widetilde{a}$ satisfies assumptions (2.6)–(2.8) with the same constants α and β. Let us look for the function $\widetilde{H}$ of the form

$$\widetilde{H}(x,s,\zeta) = \widetilde{a}(x,s,\zeta) \cdot \zeta \left[1 - \frac{g(\rho(s))}{\beta \psi'(\rho(s))}\right] + \psi'(\rho(s))^{p-1} g(\rho(s)) \left|\frac{\zeta}{\psi'(\rho(s))}\right|^p,$$

observe that

$$\widetilde{H}(x,s,\zeta) \geq \alpha|\zeta|^p + \psi'(\rho(s))^{p-1} g(\rho(s)) \left|\frac{\zeta}{\psi'(\rho(s))}\right|^p - \frac{g(\rho(s))}{\psi'(\rho(s))}|\zeta|^p = \alpha|\zeta|^p. \tag{3.47}$$

Being $u = \rho(w)$ and $\nabla u = \frac{\nabla w}{\psi'(\rho(s))}$, it transforms into the following relation between a and $\widetilde{a}$

$$\widetilde{a}(x, w, \nabla w) = \psi'(u)^{p-1} a(x, \nabla u), \tag{3.48}$$

this means (in the sense of distributions) that

$$\begin{aligned}
&-\mathrm{div}(\widetilde{a}(x, w, \nabla w)) \\
&\quad = -\mathrm{div}(\psi'(u)^{p-1} a(x, \nabla u)) \\
&\quad = -\psi'(u)^{p-1}\mathrm{div}(a(x, \nabla u)) - (p-1)\psi'(u)^{p-2}\psi''(u) a(x, \nabla u)\cdot\nabla u \\
&\quad = \psi(u)^{p-1}\left[\mu - u_t - g(u)|\nabla u|^p\right] - \psi'(u)^p a(x, \nabla u)\cdot\nabla u \\
&\qquad + \frac{\psi'(u)^{p-1} g(u)}{\beta} a(x, \nabla u)\cdot\nabla u \\
&\quad = \mu\psi'(u)^{p-1} - \psi'(u)^{p-1}u_t - \psi'(u)^{p-1} g(u)|\nabla u|^p \\
&\qquad - \widetilde{a}(x, w, \nabla w)\cdot\nabla w\left[1 - \frac{g(u)}{\beta\psi'(u)}\right] \\
&\quad = \mu\psi'(u)^{p-1} - \psi'(u)^{p-1}u_t - \widetilde{H}(x, w, \nabla w) \\
&\quad = \mu\psi'(\rho(w))^{p-1} - \psi'(\rho(w))^{p-2}\psi'(u)u_t - \widetilde{H}(x, w, \nabla w) \\
&\quad = \mu\psi'(\rho(w))^{p-1} - \psi'(\rho(w))^{p-2} w_t - \widetilde{H}(x, w, \nabla w).
\end{aligned} \tag{3.49}$$

We found that w is a solution in $W \cap L^\infty(Q)$ solving

$$\begin{cases} \psi'(\rho(w))^{p-2} w_t - \mathrm{div}(\widetilde{a}(x, w, \nabla w)) + \widetilde{H}(x, w, \nabla w) \\ = \mu\psi'(\rho(w))^{p-1} \text{ in } (0, T)\times\Omega, \\ w_n(t, x) = 0 \text{ on } (0, T)\times\partial\Omega, \quad w_n(0, x) = \psi'(u_0(x)) = w_0(x) \text{ in } \Omega. \end{cases} \tag{3.50}$$

Indeed, one can fix $\eta \in W \cap L^\infty(Q)$ and use $\frac{\eta}{\psi'(u)^{p-1}} = \frac{\eta}{\psi'(\rho(w))^{p-1}}$ as test function to find that

$$\begin{aligned}
&\int_0^t \left\langle \psi'(\rho(w))^{p-2} w_t, \frac{\eta}{\psi'(\rho(w))^{p-1}} \right\rangle dt && (A) \\
&\quad + \int_Q \frac{\widetilde{a}(x, w, \nabla w)\cdot\nabla\eta}{\psi'(\rho(w))^{p-1}} dxdt && (B) \\
&\quad - (p-1)\int_Q \frac{\widetilde{a}(x, w, \nabla w)\cdot\nabla w}{[\psi'(\rho(w))^{p-1}]^2}\psi'(\rho(w))^{p-2}\psi''(\rho(w))\rho'(w)\eta dxdt && (C) \\
&\quad + \int_Q \frac{\widetilde{H}(x, w, \nabla w)}{\psi'(\rho(w))^{p-1}}\eta dxdt && (D) \\
&= \int_Q \mu\eta dxdt. && (E)
\end{aligned}$$

Precisely, the term $[(C)+(D)]$ is equal to (were we recall the definition of $\widetilde{H}$)

$$\begin{aligned}&\frac{\eta}{[\psi'(\rho(w))^{p-1}]^2\psi'(\rho(w))}\\&\quad\times\left[\widetilde{a}(x,w,\nabla w)\cdot\nabla w\left[-\psi'(\rho(w))^p+\frac{g(\rho(w))\psi'(\rho(w))^{p-1}}{\beta}\right]\right.\\&\quad\left.+\psi'(\rho(w))^{p-1}\psi'(\rho(w))\widetilde{H}(x,w,\nabla w)\right]\end{aligned}$$

and after simplification of equal terms

$$\begin{aligned}&[(C)+(D)]\\&=\frac{\eta}{[\psi'(\rho(w))^{p-1}]^2\psi'(\rho(w))}\left[\psi'(\rho(w))^{2p-1}g(\rho(w))\left|\frac{\nabla w}{\psi'(\rho(w))}\right|^p\right.\\&\quad+\psi'(\rho(w))\widetilde{a}(x,w,\nabla w)\cdot\nabla w\\&\quad\left.+\psi'(\rho(w))^p g(\rho(w))\left|\frac{\nabla w}{\psi'(\rho(w))}\right|^p-\frac{g(\rho(w))\widetilde{a}(x,w,\nabla w)\cdot\nabla w}{\beta}\right].\end{aligned}$$

By assumptions on $\widetilde{a}$, the term in square brackets is nonnegative, this implies that

$$\text{the term } [(C)+(D)] \text{ is nonnegative if } \eta \text{ is nonnegative.}$$

We want to show that $u_1\le u_2$ a.e. in Q when u_1, u_2 are defined as in Figure 1. Thus, as just proved, we are able to choose $\eta=(u_1-u_2)^+$ (which is nonnegative) as a test function in the difference between the weak formulations solved by w_1 and w_2 to get

$$\begin{aligned}&\int_0^t\left\langle\frac{(w_1)_t}{\psi'(\rho(w_1))},(u_1-u_2)^+\right\rangle dt+\int_Q\left(\frac{\widetilde{a}(x,w_1,\nabla w_1)}{\psi'(\rho(w_1))^{p-1}}-\frac{\widetilde{a}(x,w_2,\nabla w_2)}{\psi'(\rho(w_2))^{p-1}}\right)\\&\quad\cdot\nabla(u_1-u_2)^+dxdt\\&\le\int_Q\mu(u_1-u_2)^+dxdt,\end{aligned}\tag{3.51}$$

this implies (recall the definition of $\widetilde{a}$ and the fact that $u_t=\frac{w_t}{\psi'(u)}=\frac{w_t}{\psi'(\rho(w))}$)

$$\begin{aligned}&\int_0^t\langle(u_1-u_2)_t,(u_1-u_2)^+\rangle dt+\int_Q(a(x,\nabla u_1)-a(x,\nabla u_2))\cdot\nabla(u_1-u_2)^+dxdt\\&\le\int_Q\mu(u_1-u_2)^+dxdt.\end{aligned}\tag{3.52}$$

Now, if μ is sufficiently small, we apply the integration by parts and (2.8) obtaining

$$\int_\Omega\frac{|(u_1-u_2)^+|^2(t)}{2}dx\le\int_\Omega\frac{|(u_1-u_2)^+|^2(0)}{2}dx\tag{3.53}$$

and so, since $(u_1 - u_2)^+(0) = 0$, it follows that for every fixed $0 \le t \le T$

$$u_1(t, x) \le u_2(t, x) \text{ a.e. in } Q. \tag{3.54}$$

□

Proof of Theorem 3.1 We check that the result is true in the case where $\mu \in \mathcal{M}_0(\Omega)$ and u_0 smooth; let us fix p, $p^\star$ such that $p > 1$ and $p^* = \frac{Np}{N-p}$, and let us consider, as in the previous proof, the structure of the change of variable

$$w(t, x) = \psi(u(t, x)) = (p-1)\left[\ln S - \ln \int_s^{+\infty} e^{-\frac{G(\tau)}{\beta(p-1)}} d\tau\right]$$
$$\text{with } G(s) = \int_0^s g(\tau) d\tau \tag{3.55}$$

where $\psi^{-1} : \mathbb{R} \mapsto \mathbb{R}$ is a continuous function defined on $\mathbb{R}$ and satisfying $(\psi^{-1})'(s) > 0$ for any $s \in \mathbb{R}$. Let w be the solution of the following associated problem

$$\begin{cases} \psi'(\rho(w))^{p-2} w_t - \operatorname{div}(\widetilde{a}(x, w, \nabla w)) & \\ +\widetilde{H}(x, w, \nabla w) = \mu \psi'(\rho(w))^{p-1} & \text{in } (0, T) \times \Omega, \\ w(0, x) = \psi(u_0(x)) & \text{in } \Omega, \\ w(t, x) = 0 & \text{on } (0, T) \times \partial\Omega. \end{cases} \tag{3.56}$$

Since g is increasing, then $g \notin L^1(\mathbb{R})$ and so $e^{-\frac{G(s)}{\beta(p-1)}} \in L^1(\mathbb{R})$, therefore the functional function $s \mapsto \psi(s)$ is an increasing continuous function on $\mathbb{R}$; in particular it is a bijection over the real axis and it is well-defined. So by Leray–Lions theorem there exists a unique variational solution $w \in L^p(0, T; W_0^{1,p}(\Omega)) \cap C(0, T; L^2(\Omega))$ such that $w_t \in L^p(0, T; W^{-1,p'}(\Omega))$ and $w(t, x) = 0$, that is

$$\int_0^T \langle \psi'(\rho(w))^{p-2} w_t, \varphi \rangle dt + \int_Q \widetilde{a}(x, w, \nabla w) \cdot \nabla \varphi dx dt$$
$$+ \int_Q \widetilde{H}(x, w, \nabla w) \varphi = \int_Q \psi'(\rho(w))^{p-1} \varphi d\mu$$

for all $\varphi \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q) \cap C(0, T; L^1(\Omega))$ with $\varphi_t \in L^{p'}(0, T; W^{-1,p'}(\Omega))$. Let us define u_n and w_n as solutions of the following initial boundary value problems

$$\begin{cases} (u_n)_t - \operatorname{div}(a(x, u_n)) + g(u_n)|\nabla u_n|^p = \mu & \text{in } (0, T) \times \Omega, \\ u_n(x, 0) = u_0^n(x) = \min(nv(x), u_0) & \text{in } \Omega, \\ u_n(t, x) = 0 & \text{on } (0, T) \times \partial\Omega, \end{cases} \tag{3.57}$$

$$\begin{cases} \psi'(\rho(w_n))^{p-2}(w_n)_t - \operatorname{div}(\widetilde{a}(x, w_n, \nabla w_n)) \\ +\widetilde{H}(x, w_n, \nabla w_n) = \mu'(\rho(w_n))^{p-1} & \text{in } (0,T)\times\Omega, \\ w_n(0,x) = \psi(u_0^n(x)) & \text{in } \Omega, \\ w_n(t,x) = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{3.58}$$

Recall that, since u_n and w_n are in $C(0,T; L^1(\Omega))$, then $u_n(0,x)$ and $w_n(0,x)$ belong to $L^1(\Omega)$ and are well defined. Moreover by virtue of the asymptotic results of [64,81] and the decomposition of μ, i.e., $\mu = f - \operatorname{div}(G)$ where $f \in L^1(\Omega)$ and $G \in L^{p'}(\Omega)^N$ we have $u_n(t,x)$ converges, as t tends to $+\infty$, to $v(x) \in L^{p^\star}(\Omega)$. Now, using properties of ψ (recalling that ψ is a Lipschitz continuous function), we get

$$\lim_{t\to+\infty} \psi(u_n(t,x)) = \psi(v(x)) \text{ in } L^{p^\star}(\Omega). \tag{3.59}$$

Up to subsequences

$$w_n(t,x) \to w(x) \text{ a.e. in } Q \tag{3.60}$$

being $w(x)$ solution of the corresponding elliptic problem of (3.58). Now, since $u_n(t,x) \le u(t,x)$ and $w_n(t,x) \le w(t,x)$ then by using the proof of Lemma 3.6 with test functions $\varphi = (w - w_n)$, we deduce

$$\int_\Omega |(w - w_n)|^2(t)dx \le \int_\Omega |(w_0 - w_0^n)|^2(0)dx, \tag{3.61}$$

then there exists $\epsilon > 0$ such that for n large enough

$$\begin{cases} \|w - w_n\|_{L^2(\Omega)} \le \dfrac{\epsilon}{2}, \\ \|w - \psi(v(x))\|_{L^2(\Omega)} \le \dfrac{\epsilon}{2} + \|\psi(u_n) - \psi(v(x))\|_{L^2(\Omega)} \le \epsilon, \end{cases} \tag{3.62}$$

and then up to subsequences and using the fact that φ is continuous

$$\psi(u_n(t,x)) \to \psi(v(x)) \text{ a.e. in } Q, \quad u_n(t,x) \to v(x) \text{ a.e. in } Q \tag{3.63}$$

then $u(t,x) = v(x)$ a.e. in Q. Thanks to the classical results of Potential theory, see [41,81], and under suitable assumptions on the data, that is, $0 \le u(t,x) \le \widetilde{u}(t,x)$ where $\widetilde{u}(t,x)$ is the heat potential (which coincides with supersolutions) associated to the problem

$$\begin{cases} \widetilde{u}_t - \operatorname{div}(a(x,\widetilde{u})) + g(\widetilde{u})|\nabla\widetilde{u}|^p = \mu & \text{in } \mathbb{R}^N \times (0,T), \\ \widetilde{u}(0,x) = \widetilde{u}_0(x) & \text{in } \mathbb{R}^N \end{cases} \tag{3.64}$$

where $\widetilde{u}_0$ is the trivial extension of u_0 at 0 outside Ω, provided that $\widetilde{u}(t,x)$ converges to $\widetilde{v}(x)$ in $L^1(\Omega)$ where $\widetilde{v}(x)$ is the solution of the corresponding elliptic heat equation.

Due to the Vitali's lemma and the unicity of the limit, we deduce that $u(t, x)$ converges to $v(x)$ in $L^1(\Omega)$. □

4 Case of entropy solutions (Main result and proof)

In this section, we will prove an asymptotic behaviour result which concern entropy solutions of problems (1.1) in the case where the term measure does not depend on time. First observe that entropy solutions of problems (1.1) are solutions of the corresponding elliptic problem.

Proposition 4.1 *Let $\mu \in \mathcal{M}_0(\Omega)$ be independent on time and let v be the entropy solution of the elliptic problem (3.1). Then v is the unique solution of the parabolic problem (1.1) with $u_0 = v$, in the entropy sense introduced in Definition 2.8, for any fixed $T > 0$.*

Proof First of all, let us suppose that $u_0, v \in L^2(\Omega)$; we have to check that v is an entropy solution of problem (1.1), to do that we can use the variational formulation (2.19) and choosing $T_k(v - \varphi)$ as test function to obtain

$$\int_0^T \langle v_t, T_k(v-\varphi)\rangle dt + \int_Q a(x, \nabla v) \cdot \nabla T_k(v-\varphi) dxdt \\ + \int_Q g(v)|\nabla v|^p T_k(v-\varphi) dxdt \leq \int_Q T_k(v-\varphi) d\mu. \tag{4.1}$$

Through the integration by parts, we have

$$\int_0^T \langle v_t, T_k(v-\varphi)\rangle dt \\ = \int_\Omega \Theta_k(v-\varphi)(T) dx - \int_\Omega (u_0 - \varphi(0)) dx + \int_0^T \langle \varphi_t, T_k(v-\varphi)\rangle dt \\ = \int_\Omega (u_0 - \varphi(0)) dx + \omega(k) \tag{4.2}$$

where $\Theta_k(s)$ indicates the primitive function of $T_k(s)$ and $\omega(k)$ denotes a nonnegative quantity which vanishes as k diverges, while

$$\int_Q T_k(v-\varphi) d\mu = \int_Q (v-\varphi) d\mu + \omega(k). \tag{4.3}$$

Using [47, Theorem 2.33, Theorem 10.1], we deduce

$$\int_Q a(x, \nabla v) \cdot \nabla T_k(v-\varphi) dxdt + \int_Q g(v)|\nabla v|^p T_k(v-\varphi) dxdt \\ = \int_Q v d\lambda_k(x) dt \tag{4.4}$$

where λ_k is a diffuse measure in $\mathcal{M}_0(\Omega)$ which converges to μ in the narrow topology of measures, see [91,92]. Thus, recalling that v is bounded continuous, and using the dominated convergence theorem, we get

$$\int_Q a(x, \nabla v) \cdot \nabla T_k(v-\varphi)dxdt + \int_Q g(v)|\nabla v|^p T_k(v-\varphi)dxdt \\ = \int_Q v d\mu + \omega(k). \tag{4.5}$$

Gathering together all these facts, we have that v is an entropy solution of (1.1) having itself as initial data. □

Proposition 4.1 allows us to deduce that the entropy solution u of problem (1.1) belongs to $C(0,T;L^1(\Omega))$ for any fixed $T>0$. Indeed $w=u-v$ uniquely solves problem (1.1) with u_0-v as initial datum and $\mu=0$ in the entropy sense, and so $w\in C(0,T;L^1(\Omega))$, this is due to a result of [85] since w turns out to be an entropy solution in the sense of the definition given in [94]. Therefore, as we said before, for fixed μ and $u_0\in L^1(\Omega)$ one can uniquely determine u and v, solutions of the above problems defined for any time $T>0$. Let us give the following definition relative to the sub/super solutions formulated as in Definition 2.8.

Definition 4.2 A function $\underline{u}\in C(0,T;L^1(\Omega))$ is an entropy supersolution of (1.1) if $g(\underline{u})|\nabla\underline{u}|^p\in L^1(Q)$ for every $p>1$, $T_k(\underline{u})\in L^p(0,T;W_0^{1,p}(\Omega))$ for all $k>0$ and if

$$\int_\Omega \Theta_k(\underline{u}-\varphi)(T)dx - \int_\Omega \Theta_k(\underline{u}_0-\varphi(0))dx \\ + \int_0^T \langle \varphi_t, T_k(\underline{u}-\varphi)\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}dt \\ + \int_Q a(x,\nabla\underline{u})\cdot\nabla T_k(\underline{u}-\varphi)dxdt + \int_Q g(\underline{u})|\nabla\underline{u}|^p T_k(\underline{u}-\varphi)dxdt \\ \geq \int_Q T_k(\underline{u}-\varphi)d\mu \tag{4.6}$$

for any $\varphi\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q)\cap C(0,T;L^1(\Omega))$ with $\varphi_t\in L^{p'}(0,T;W^{-1,p'}(\Omega))$, while $\overline{u}$ is an entropy subsolution if $-\overline{u}$ is an entropy supersolution solution.

By Lemma 3.6, we easily deduce the following result.

Lemma 4.3 *Let $\underline{u}$ and $\overline{u}$ be, separately, an entropy supersolution and an entropy subsolution of problem (1.1) and let u be the unique entropy solution of the same problem. Then $\underline{u}\leq u\leq\overline{u}$. In addition, if $\underline{u}$ and $\overline{u}$ are continuous with values in $L^1(\Omega)$, then we have that $\underline{u}(t,x)\leq u(t,x)\leq\overline{u}(t,x)$ a.e. in Ω for every fixed t.*

Proof Observe that, if the function $\underline{u}$ (resp. $\overline{u}$) is a limit of regular solutions of approximating problems (1.3) with smooth data $\mu_n = f_n - \operatorname{div}(G)$, where f_n

is a sequence of smooth functions that converges to f in $L^1(\Omega)$, and initial data $u_n(0,x) = \widetilde{u}_0^n = \min(u_0^n, \underline{u})$ (resp. $u_n(0,x) = \widetilde{u}_0^n = \max(u_0^n, \overline{u})$). Thanks to the stability results of [48,94], we have $\underline{u} \le \widetilde{u}$ (resp. $\widetilde{u} \le \overline{u}$) where $\widetilde{u}$ is an entropy solution of problem (1.1) with $u(0,x) = \widetilde{u}(0) = \underline{u}_0$ (resp. $u(0,x) = \widetilde{u}(0) = \overline{u}_0$) as initial data. Therefore, by Lemma 3.6, we get $\underline{u}(t,x) \le u(t,x) \le \overline{u}(t,x)$ for every fixed $t > 0$. □

Now we can state our main result.

Theorem 4.4 *Let $\mu \in \mathcal{M}_0(Q)$ be independent on time t, let $u(t,x)$ be an entropy solution of problem (1.1) with $u_0 \in L^1(\Omega)$ as initial data, and let $v(x)$ be the entropy solution of the corresponding elliptic problem (3.1). Then*

$$\lim_{T\to+\infty} u(T,x) = v(x) \text{ in } L^1(\Omega). \tag{4.7}$$

Remark 4.5 It is also expected that $u(T,x)$ converges toward a renormalized solution with a correction in the form of extra definition resulting from μ, this remark is supported by what is already known from renormalized solutions in the theory of nonlinear parabolic problems with measures, see [2,3,47,71,77,78] and references therein.

Proof of Theorem 4.4 We split the proof in few steps.

Step 1. The case $u_0 = 0$ and $\mu \ge 0$. It can be easily seen that, for a parameter $\tau > 0$, both $u(t,x)$ and $u^\tau(t,x) = u(t+\tau,x)$ are entropy solutions of problem (1.1) with, respectively, 0 and $u(\tau,x) \ge 0$ as initial datum. Moreover from Lemma 4.3 we deduce that $u(t+\tau,x) \ge u(t,x)$ for $t,\tau > 0$. On the other hand, recall that u is a monotone nondecreasing function in t and so it converges to a function $\widetilde{v}$ almost everywhere in Ω and in $L^1(\Omega)$, using also Proposition 4.1 we get $u(t,x) \le v(x)$. Recalling that u is obtained as limit of regular solutions with smooth data μ_n, we can define u_n^τ as the solutions of

$$\begin{cases} u_n^\tau(t,x)_t - \operatorname{div}(a(x,\nabla u_n^\tau)) + g(u_n^\tau)|\nabla u_n^\tau|^p = \mu_n & \text{in } (0,1)\times\Omega, \\ u_n^\tau(0,x) = u_n(\tau,x) & \text{in } \Omega, \\ u_n^\tau = 0 & \text{on } (0,1)\times\partial\Omega. \end{cases} \tag{4.8}$$

We take advantage of the change of variable $\tau = T - t$ to deduce that u solves a similar nonlinear parabolic problem. In particular if $\mu \ge 0$, g satisfies (2.14)–(2.15) and by classical comparison results one has that $u(t,x)$ is decreasing in time. Moreover, by comparison principle, we have that u^n is increasing with respect to n and, again by the comparison result of Lemma 4.3, we have that, for fixed $t \in (0,1)$

$$u^\tau(0,x) \le u^\tau(t,x) = u(\tau+t,x) \le u(\tau+1,x) = u^{\tau+1}(0,x), \tag{4.9}$$

$$u^\tau(1,x) \le u^\tau(t,x) = u(\tau+t,x) \le u(\tau,x) = u^{\tau-1}(1,x), \tag{4.10}$$

and so its limit $\widetilde{u}$ does not depend on time and is solution of elliptic problem (3.1). This concludes using a similar argument that the limit of u^τ does not depend on time.

Thus, using u_n^τ (which also does not depend on time) in (4.8) and integrating by parts, we obtain

$$\int_\Omega \Theta_n(u^n - \varphi)(1)dx - \int_\Omega \Theta_n(u^n - \varphi)(0)dx$$
$$+ \int_0^1 \langle \varphi_t, T_k(u_n - \varphi)\rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} dt$$
$$+ \int_Q a(x, \nabla u_n) \cdot \nabla T_k(u_n - \varphi)dxdt$$
$$+ \int_Q g(u_n)|\nabla u_n|^p T_k(u_n - \varphi)dxdt \le \int_Q T_k(u_n - \varphi)d\mu. \tag{4.11}$$

It follows from the passage to the limit on n, using monotone convergence theorem that

$$\int_\Omega \Theta_n(w(x) - \varphi(1))dx - \int_\Omega \Theta_k(w(x) - \varphi(0))dx$$
$$+ \int_0^1 \langle \varphi_t, T_k(w - \varphi)\rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} dt$$
$$+ \int_0^1 \langle w_t, T_k(w - \varphi)\rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} dt = 0 \tag{4.12}$$

and so $w(x) = v(x)$. If μ has no sign we can reason separately with μ^+ and μ^- obtaining (4.12) and then using the formulation (4.2) to conclude (see *Step.4*). If v is the entropy solution of problem (3.1), we proved in Proposition 4.1 that v is also the entropy solution of the initial boundary value problem (1.1) with v itself as initial datum. Therefore, by comparison Lemma 4.3, if $0 \le u_0 \le v$, we have that the solution $u(t, x)$ of (1.1) converges to v in $L^1(\Omega)$ as t tends to infinity; using the fact that $u(t, x)$ is an entropy solution for parabolic problem with homogeneous initial data, while v is a nonnegative entropy solution with itself as initial data.

Step 2. The case $u_0 = \lambda v$, $\lambda > 1$, *and* $\mu \ge 0$. Now, let us take $u_\lambda(t, x)$ the solution of problem (1.1) with $u_0 = \lambda v$ as initial datum for some $\lambda > 1$ and again $\mu \ge 0$. Hence, since λv does not depend on time, we have that it is an entropy solution of the parabolic problem (1.1). Using the fact that v is a subsolution of the same problem, we found that $v(x) \le u_\lambda(t, x) \le \lambda v(x)$ a.e. in Ω for all positive time t (applying again the comparison Lemma 4.3). In order to get $u_\lambda(t + \tau, x) \le u_\lambda(t, x)$ for all $t, \tau > 0$ a.e. in Ω, we use the fact that the datum μ does not depend on time and we apply the comparison result also between $u_\lambda(t + \tau, x)$, which is the solution with $u_0 = u_\lambda(\tau, x)$ where τ is a positive parameter, and $u_\lambda(t, x)$ the solution with $u_0 = \lambda v$ as initial data. Which yields, by virtue of this final monotonicity result, that there exists a function $\overline{v} \ge v$ such that $u_\lambda(t, x)$ converges to $\overline{v}$ a.e. in Ω as t tends to infinity. Clearly $\overline{v}$ does not depend on t and we can develop the same argument used before to prove the passage to the limit in the approximating entropy formulation, and so, by uniqueness, we can obtain that $\overline{v} = v$. So, we have proved that the result holds for the solution starting from $u_0 = \lambda v$

as initial datum with $\lambda > 1$ and $\mu \geq 0$. Since we proved before that the result holds true also for the solution starting from $u_0 = 0$, then, again applying the comparison argument, we can conclude in the same way that the convergence to v holds true for solutions starting from u_0 such that $0 \leq u_0 \leq \lambda v$ as initial datum for fixed $\lambda > 1$.

Step 3. The case $0 \leq u_0 \in L^1(\Omega)$ *and* $\mu \geq 0$. In order to deal with this case, we adapt a suitable idea of [99] (called Harnack inequality). Namely, if $\mu \neq 0$, then $v > 0$ (which implies that λv tends to $+\infty$ in Ω as λ diverges). Without loss of generality we can suppose $\mu \neq 0$ (the case $\mu \equiv 0$ is the easier one and it can be proved as in [81]). Let us denote, henceforth, by $u_{0,\lambda}$ the monotone nondecreasing family (with respect to λ) of functions such that $u_{0,\lambda} = \min(u_0, \lambda v)$. Indeed, we have, as before for every fixed $\lambda > 1$, $u_\lambda(t, x)$, the entropy solution of problem (1.1) with $u_{0,\lambda}$ as initial datum, converges to v a.e. in Ω as t tends to infinity. As a consequence of the standard compactness argument, we also have that $T_k(u_\lambda(t, x))$ converges to $T_k(v)$ weakly in $W_0^{1,p}(\Omega)$ as t diverges and for every fixed $k > 0$. Therefore, thanks to Lebesgue theorem, we can easily check that $u_{0,\lambda}$ converges to u_0 in $L^1(\Omega)$ as λ tends to infinity. We claim that, using a stability result for entropy/renormalized solutions of the nonlinear problem (1.1), see [3,78], that $T_k(u_\lambda(t, x))$ converges to $T_k(u)(t, x)$ strongly in $L^p(0, T; W_0^{1,p}(\Omega))$ as λ tends to infinity. Because $z_\lambda = u - u_\lambda$ solves the problem (1.1) with $u_0 - u_{0,\lambda}$ as initial datum, then z_λ turns out, see [48,94], to be a renormalized/entropy solution of the same problem, so that

$$\int_\Omega \Theta_k(u - u_\lambda)(t)dx \leq \int_\Omega \Theta_k(u_0 - u_{0,\lambda})dx \tag{4.13}$$

for every $k, t > 0$. Dividing the above inequality by k and passing to the limit as k tends to 0, we obtain

$$\|u(t, x) - u_\lambda(t, x)\|_{L^1(\Omega)} \leq \|u_0(x) - u_{0,\lambda}(x)\|_{L^1(\Omega)} \tag{4.14}$$

for every $t > 0$. We then deduce

$$\|u(t, x) - v(x)\|_{L^1(\Omega)} \leq \|u(t, x) - u_\lambda(t, x)\|_{L^1(\Omega)} + \|u_\lambda(t, x) - v(x)\|_{L^1(\Omega)}. \tag{4.15}$$

Now, from the fact that the estimate in (4.14) is uniform in t for every fixed ϵ, we can choose $\overline{\lambda}$, large enough, and reads as follows

$$\|u(t, x) - u_{\overline{\lambda}(t,x)}\|_{L^1(\Omega)} \leq \frac{\epsilon}{2} \tag{4.16}$$

for every $t > 0$. Indeed, as a consequence of the result proved above, there exists $\overline{t}$ such that

$$\|u_{\overline{\lambda}}(t, x) - v(x)\|_{L^1(\Omega)} \leq \frac{\epsilon}{2} \text{ for every } t > \overline{t}, \tag{4.17}$$

which in turn concludes the proof of the result in the case of nonnegative data μ and initial datum u_0.

Step 4. The case where $\mu \in \mathcal{M}_0(\Omega)$ is independent on t and $u_0 \in L^1(\Omega)$ without sign assumptions. We consider again the function $z(t,x) = u(t,x) - v(x)$; thanks to Proposition 4.1 it turns out to solve problem (1.1) with $u_0 - v$ as initial data and $\mu = 0$, and so, if either $u_0 \le v$ or $u_0 \ge v$ then the result is true since $z(t,x)$ tends to zero in $L^{p^\star}(\Omega)$, as t diverges, thanks to what we proved above (the same does not seem to work in the case of convergence in $L^1(\Omega)$ because of technical difficulties that arise if trying to generalize comparison Lemma 3.6). Now, if $u^\oplus$ and $u^\ominus$ solve problem (1.1) with, respectively, $\max(u_0, v)$ and $\min(u_0, v)$ as initial data, then, by a consequence of the comparison result, we have $u^\ominus(t,x) \le u(t,x) \le u^\oplus(t,x)$ a.e. in Ω for any t, which concludes the proof, at least in $L^{p^\star}(\Omega)$, since the result holds true for both $u^\oplus$ and $u^\ominus$. □

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5 Appendix (In connection with *G*-convergence)

In this section, we shall study, following an idea of [81], the connection between the G-convergence of a sequence $(u_\tau)_{\tau>0}$ and the asymptotic behaviour of the corresponding solutions $u_\tau(t,x)$ relative to the operators $A : u \mapsto \operatorname{div}(a(x, u_\tau)_{\tau>0})$ when $u_\tau(0,x) = u_0^\tau(x) = 0$ in Ω. We prove a convergence result for entropy solutions of a nonlinear parabolic problem with nonnegative measure $\mu \in \mathcal{M}_0(\Omega)$ with $\mu \ne 0$ using the theory of G-convergence [6,19,42,72,76,81]. To this aim, let us consider the following initial boundary value problem

$$\begin{cases} (u_\tau)_t - \operatorname{div}(a(x, u_\tau)) + g(u_\tau)|u_\tau|^p = \mu_\tau \text{ in } Q = (0,T) \times \Omega, \\ u_\tau(0,x) = 0 \text{ in } \Omega, \quad u_\tau(t,x) = 0 \text{ on } (0,T) \times \partial\Omega \end{cases} \tag{5.1}$$

where $T > 0$ is any positive constant and $\mu \in \mathcal{M}_0(\Omega)$ ($\mu \ne 0$) is a Radon measure with bounded variation which does not charge the sets of zero p-capacity and which does not depend on the time variable t (in accordance with the definition given in Theorem 2.3).

Theorem 5.1 *Let $\mu_\tau \in \mathcal{M}_0(\Omega)$ be a measure such that $\mu_\tau \ne 0$. Let $u_\tau(t,x)$ ($\tau > 0$) be the entropy solution of parabolic problem (5.1) corresponding to μ_τ and $v_\tau(x)$ the entropy solution of the following corresponding elliptic problem*

$$\begin{cases} -\operatorname{div}(a(x, v_\tau)) + g(v_\tau)|v_\tau|^p = \mu_\tau & \text{in } \Omega, \\ u_\tau = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

Then, we have

$$\lim_{\tau\to\infty}\lim_{T\to\infty} u_\tau(t,x) = \lim_{\tau\to\infty} v_\tau(x) = +\infty \ a.e. \ in \ \Omega.$$

Proof First, note that $u_\tau(0,x) = u_0^\tau(x) = 0$ (this condition is essential in order to deal with some difficulties) and suppose that $\mu \in W^{-1,p'}(\Omega)$ (independent of time), we have

$$\mu \in W^{-1,p'}(\Omega) \quad \text{if and only if } \mu = -\text{div}(G) \text{ with } G \in L^{p'}(\Omega)^N. \tag{5.3}$$

Then for $\mu = \mu_\tau$ where

$$\mu_\tau = \begin{cases} \tau\mu & \text{if } f = 0, \\ \tau f - \text{div}(G) & \text{if } f = 0 \end{cases}$$

with $f \geq 0 \in L^1(\Omega)$ and $G \in L^{p'}(\Omega)^N$. We have

$$\int_0^T \langle (u_\tau)_t, \varphi\rangle dt + \int_Q a(x,\nabla u_\tau)\cdot\nabla\varphi dxdt + \int_Q g(u_\tau)|\nabla u_\tau|^p\varphi dxdt \\ = \tau\int_Q G\cdot\nabla\varphi dxdt \tag{5.4}$$

for every $\varphi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ such that $\varphi_t \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $\varphi(T,x) = 0$. Hence, for $\varphi = u_\tau$, (5.4) becomes

$$\int_0^T \langle (u_\tau)_t, u_\tau\rangle dt + \int_Q a(x,\nabla u_\tau)\cdot\nabla u_\tau dxdt + \int_Q g(u_\tau)|\nabla u_\tau|^p u_\tau dxdt \\ = \tau\int_Q G\cdot\nabla u_\tau dxdt. \tag{5.5}$$

Moreover, by (2.6) we have

$$\int_0^T \langle (u_\tau)_t, u_\tau\rangle dt + \alpha\int_Q |\nabla u_\tau|^p dxdt + \int_Q g(u_\tau)|\nabla u_\tau|^p u_\tau dxdt \\ \leq \tau\int_Q G\cdot\nabla u_\tau dxdt \leq \tau\|G\|_{L^{p'}(\Omega)^N}\|\nabla u_\tau\|_{L^p(Q)^N} \tag{5.6}$$

and then, using the integration by parts and assumptions (2.14) (recall that $u_0^\tau(0) = 0$) we get

$$\int_\Omega \frac{[u_\tau(T)]^2}{2}dx + \alpha\int_Q |\nabla u_\tau|^p dxdt \leq \tau\|G\|_{L^{p'}(\Omega)^N}\|\nabla u_\tau\|_{L^p(Q)^N}. \tag{5.7}$$

Now, since $u_\tau \neq 0$ and the fact that the first term is nonnegative, we can divide the above expression by $\tau \|\nabla u\|_{L^p(Q)^N}$, getting

$$\frac{1}{\tau}\left(\int_Q |\nabla u_\tau|^p dxdt\right)^{\frac{p-1}{p}} = \left(\int_Q \left|\nabla\left(\frac{u_\tau}{\tau^{\frac{1}{p-1}}}\right)\right|^p dxdt\right)^{\frac{p-1}{p}} \leq \|G\|_{L^{p'}(\Omega)^N}. \tag{5.8}$$

Therefore, we have that $\frac{u_\tau}{\tau^{\frac{1}{p-1}}}$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$, and so there exist a function $u \in L^p(0,T;W_0^{1,p}(\Omega))$ and a subsequence, such that, up to subsequences, $\frac{u_\tau}{\tau^{\frac{1}{p-1}}}$ weakly converges in $L^p(0,T;W_0^{1,p}(\Omega))$ (and then a.e.) to u as τ tends to infinity. So, it is enough to prove that $u > 0$ almost everywhere on Q to conclude the proof. To this aim, for every $\tau > 0$, let us define

$$a_\tau(x,\zeta) = \frac{1}{\tau} a\left(x, \tau^{\frac{1}{p-1}}\zeta\right) \text{ (for the } p\text{–Laplacian, we have } a_\tau \equiv a) \tag{5.9}$$

and then we can easily check that such an operator satisfies assumptions (2.6)–(2.8) (with the same constants α and β). Now, $\frac{u_\tau}{\tau^{\frac{1}{p-1}}}$ satisfies the parabolic problem

$$\begin{cases} \tau^{\frac{1}{p-1}}\left(\frac{u_\tau}{\tau^{\frac{1}{p-1}}}\right)_t - \operatorname{div}\left(a_\tau\left(x, \nabla(\frac{u_\tau}{\tau^{\frac{1}{p-1}}})\right)\right) & \\ +g\left(\tau^{\frac{1}{p-1}}\left(\frac{u_\tau}{\tau^{\frac{1}{p-1}}}\right)\right)\left|\tau^{\frac{1}{p-1}}\nabla\left(\frac{u_\tau}{\tau^{\frac{1}{p-1}}}\right)\right|^p = \mu & \text{in } (0,T)\times\Omega, \\ \frac{u_\tau}{\tau^{\frac{1}{p-1}}}(0,x) = 0 & \text{in } \Omega, \\ \frac{u_\tau}{\tau^{\frac{1}{p-1}}}(t,x) = 0 & \text{on } (0,T)\times\partial\Omega \end{cases} \tag{5.10}$$

in a variational sense. Indeed, for every $\varphi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ such that $\varphi_t \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $\varphi(T,x) = 0$, we have

$$\int_0^T \langle (u_\tau)_t, \varphi\rangle\, dt + \int_Q a_\tau\left(x, \nabla\left(\frac{u_\tau}{\tau^{\frac{1}{p-1}}}\right)\right)\cdot\nabla\varphi dxdt + \int_Q g(u_\tau)|\nabla u_\tau|^p \varphi dxdt$$
$$= \int_Q G\cdot\nabla\varphi dxdt. \tag{5.11}$$

Moreover, thanks to [6, Theorem 3.1], we have that the family of operators (a_τ) G-converges in the class of Leray–Lions operators, that is, there exists a Carathéodory function $\overline{a}$ satisfying assumptions (2.6)–(2.8), and a sequence of indices $\tau_k = \tau(k)$ (called τ again) such that

$$a_\tau(x,\nabla u_\tau) \underset{\text{G-converges}}{\longrightarrow} \overline{a}(x,\nabla u_\tau). \tag{5.12}$$

So, because of that, being u the weak limit of $\frac{u_\tau}{\tau^{\frac{1}{p-1}}}$ in $L^p(0,T;W_0^{1,p}(\Omega))$, we get that

$$a\left(x, \nabla\left(\frac{u_\tau}{\tau^{\frac{1}{p-1}}}\right)\right) \underset{\tau_k\to\infty}{\rightharpoonup} \overline{a}(x,\nabla u) \text{ weakly in } L^{p'}(Q)^N. \tag{5.13}$$

Therefore, using this result in (5.10), we have

$$\int_0^T \langle u_t, \varphi\rangle dt + \int_Q \overline{a}(x,\nabla u)\cdot\nabla\varphi dxdt + \int_Q g(u)|\nabla u|^p\varphi dxdt = \int_Q G\cdot\nabla\varphi dxdt \tag{5.14}$$

for every $\varphi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ such that $\varphi_t \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $\varphi(T,x)=0$; and so, u is a variational solution of problem

$$\begin{cases} u_t - \operatorname{div}(\overline{a}(x,\nabla u)) + g(u)|\nabla u|^p = \mu & \text{in } (0,T)\times\Omega, \\ u(0,x)=0 & \text{in } \Omega, \\ u(t,x)=0 & \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{5.15}$$

Then, recalling that $\mu \neq 0$ and using a suitable Harnack type result adapted to parabolic inequalities [99], we deduce that $u(t,x) > 0$ a.e. on Q. Now, if $\mu \in \mathcal{M}_0(\Omega)$, we have

$$\mu_\tau = \begin{cases} \tau\mu & \text{if } f=0, \\ \tau f - \operatorname{div}(G) & \text{if } f\neq 0 \end{cases}$$

where $f \in L^1(\Omega)$ a nonnegative function and $G \in L^{p'}(\Omega)^N$ (see [26]), we can suppose, without loss of generality, that $u_\tau = \tau\chi_E - \operatorname{div}(G)$ for a suitable set $E \subseteq \Omega$ of positive measure; indeed, f, being nonidentically zero, it turns out to be strictly bounded away from zero on a suitable $E \subseteq \Omega$, and so there exists a constant C such that $f \geq C\chi_E$, and then, once we proved our result for such a μ_τ, we can easily prove the statement by applying again a comparison argument. Now, reasoning analogously as above we deduce that $\frac{u_\tau}{\tau^{\frac{1}{p-1}}}$ solves the parabolic problem

$$\begin{cases} \tau^{\frac{1}{p-1}}\left(\frac{u_\tau}{\tau^{\frac{1}{p-1}}}\right)_t - \operatorname{div}\left(a_\tau\left(x, \nabla(\frac{u_\tau}{\tau^{\frac{1}{p-1}}})\right)\right) \\ +g\left(\tau^{\frac{1}{p-1}}\left(\frac{u_\tau}{\tau^{\frac{1}{p-1}}}\right)\right)\left|\tau^{\frac{1}{p-1}}\nabla\left(\frac{u_\tau}{\tau^{\frac{1}{p-1}}}\right)\right|^p = \chi_B - \frac{1}{\tau}\operatorname{div}(G) \text{ in } Q, \\ \frac{u_\tau}{\tau^{\frac{1}{p-1}}}(0,x)=0 \text{ in } \Omega, \\ \frac{u_\tau}{\tau^{\frac{1}{p-1}}}(t,x)=0 \text{ on } (0,T)\times\partial\Omega \end{cases} \tag{5.16}$$

Moreover

$$\chi_B - \frac{1}{\tau}\mathrm{div}(G) \longrightarrow \chi_B \text{ stronly in } W^{-1,p'}(\Omega) \text{ as } \tau \to \infty. \tag{5.17}$$

Therefore, since the G-convergence is stable under such type of convergence of data, we have, that the weak limit u of $\frac{u_\tau}{\tau^{\frac{1}{p-1}}}$ in $L^p(0, T; W_0^{1,p}(\Omega))$ solves

$$\begin{cases} u_t - \mathrm{div}(\overline{a}(x, \nabla u)) + g(u)|\nabla u|^p = \chi_B & \text{in } Q = (0, T) \times \Omega, \\ u(0, x) = 0 & \text{in } \Omega, \\ u(t, x) = 0 & \text{on } (0, T) \times \partial\Omega, \end{cases} \tag{5.18}$$

and so we deduce, as above, that $u(t, x) > 0$ a.e. on $Q = (0, T) \times \Omega$, which implies that u_τ goes to infinity as τ and T tends to infinity and then we conclude the result of Theorem 5.1. □

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