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On the Weinstein–Wigner transform and Weinstein–Weyl transform

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Abstract
In this paper, we define and study the Weinstein–Wigner transform and we prove its inversion formula. Next, we introduce and study the Weinstein–Weyl transform $\mathcal{W}_\sigma$ with symbol σ and we give an integral relation between it and the Weinstein–Wigner transform. At last, we give criteria in terms of σ for boundedness and compactness of the transform $\mathcal{W}_\sigma$.

Keywords Weinstein–Wigner transform · Weinstein–Weyl transform · Inversion formula · Boundedness and compactness

1 Introduction

The Weinstein operator $\Delta^d_{W,\alpha}$ defined on $\mathbb{R}^{d+1}_+ = \mathbb{R}^d \times (0, \infty)$, by

$$\Delta^d_{W,\alpha} = \sum_{j=1}^{d+1} \frac{\partial^2}{\partial x_j^2} + \frac{2\alpha+1}{x_{d+1}} \frac{\partial}{\partial x_{d+1}} = \Delta_d + L_\alpha, \quad \alpha > -1/2,$$

where Δ_d is the Laplacian operator for the d first variables and L_α is the Bessel operator for the last variable defined on $(0, \infty)$ by

$$L_\alpha u = \frac{\partial^2 u}{\partial x_{d+1}^2} + \frac{2\alpha+1}{x_{d+1}} \frac{\partial u}{\partial x_{d+1}}.$$

The Weinstein operator $\Delta^d_{W,\alpha}$ has several applications in pure and applied mathematics, especially in fluid mechanics [1,22].

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Very recently, many authors have been investigating the behaviour of the Weinstein transform (2.5) with respect to several problems already studied for the classical Fourier transform. For instance, Heisenberg-type inequalities [16], Paley–Wiener theorem [9], Uncertainty principles [11,17,19], multiplier Weinstein operator [18], continuous wavelet transform [10], and so forth.

In the classical case, the Fourier–Wigner transform and the Weyl transform were studied by Weyl [23] and Wong [24]. Recently, many authors have been investigating the behaviour of this operators in many setting [5,6,8,12,15]. These Fourier-like transformations are generalized in the context of differential-differences operators [2–4]. This paper is an attempt to fill this gap by generalizing these operators to the Weinstein transform.

Using the harmonic analysis associated with the Weinstein operator (generalized translation operators, generalized convolution, Weinstein transform, ...) and the same idea as for the classical case, we define and study in this paper the Wigner transforms and the Weyl transform associated with the Weinstein operator which we call Weinstein–Wigner transform and Weinstein–Weyl transform.

This paper is organized as follows. In Sect. 2, we recall some properties of harmonic analysis for the Weinstein operators. In Sect. 3, we define the Fourier–Wigner transform $\mathcal{V}$ in the Weinstein setting, and we have established for it an inversion formula. In last Section, we introduce and study the Weinstein–Weyl transforms $\mathcal{W}_\sigma$ for $\sigma \in \mathcal{S}_*(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$ and we prove they are compact operators from $L^2_\alpha(\mathbb{R}^{d+1}_+)$ into itself. Next, we define $\mathcal{W}_\sigma$ for $\sigma \in L^2_\alpha(d\mu_\alpha \otimes d\mu_\alpha)$, with $1 \leq p \leq 2$ and we prove the boundedness and compactness of these transforms on these spaces. In the last, we define $\mathcal{W}_\sigma$ for $\sigma \in \mathcal{S}'_*(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$.

2 Preliminaires

For all $\lambda = (\lambda_1, \ldots, \lambda_{d+1}) \in \mathbb{C}^{d+1}$, the system

$$\begin{aligned} &\frac{\partial^2 u}{\partial x_j^2}(x) = -\lambda_j^2 u(x), \quad \text{if } 1 \leq j \leq d \\ &L_\alpha u(x) = -\lambda_{d+1}^2 u(x), \\ &u(0) = 1, \quad \frac{\partial u}{\partial x_{d+1}}(0) = 0, \quad \frac{\partial u}{\partial x_j}(0) = -i\lambda_j, \quad \text{if } 1 \leq j \leq d \end{aligned} \tag{2.1}$$

has a unique solution denoted by $\Lambda^d_\alpha(\lambda, .)$, and given by

$$\Lambda^d_\alpha(\lambda, x) = e^{-i<x',\lambda'>} j_\alpha(x_{d+1}\lambda_{d+1}) \tag{2.2}$$

where $x = (x', x_{d+1})$, $x'_d = (x_1, x_2, \ldots, x_d)$, $\lambda = (\lambda', \lambda_{d+1})$, $\lambda'_d = (\lambda_1, \lambda_2, \ldots, \lambda_d)$ and j_α is the normalized Bessel function of index α defined by

$$j_\alpha(x) = \Gamma(\alpha+1)\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{2^k k!\Gamma(\alpha+k+1)}.$$

The function $(\lambda, x) \mapsto \Lambda_\alpha^d(\lambda, x)$ is called the Weinstein kernel and has a unique extension to $\mathbb{C}^{d+1} \times \mathbb{C}^{d+1}$, and satisfied the following properties.

(i) For all $(\lambda, x) \in \mathbb{C}^{d+1} \times \mathbb{C}^{d+1}$ we have

$$\Lambda_\alpha^d(\lambda, x) = \Lambda_\alpha^d(x, \lambda).$$

(ii) For all $(\lambda, x) \in \mathbb{C}^{d+1} \times \mathbb{C}^{d+1}$ we have

$$\Lambda_\alpha^d(\lambda, -x) = \Lambda_\alpha^d(-\lambda, x).$$

(iii) For all $(\lambda, x) \in \mathbb{C}^{d+1} \times \mathbb{C}^{d+1}$ we get

$$\Lambda_\alpha^d(\lambda, 0) = 1.$$

(iv) For all $\nu \in \mathbb{N}^{d+1}$, $x \in \mathbb{R}^{d+1}$ and $\lambda \in \mathbb{C}^{d+1}$ we have

$$\left|D_\lambda^\nu \Lambda_\alpha^d(\lambda, x)\right| \le \|x\|^{|\nu|}\, e^{\|x\|\|\mathrm{Im}\,\lambda\|}$$

where $D_\lambda^\nu = \partial^\nu/(\partial\lambda_1^{\nu_1}\ldots\partial\lambda_{d+1}^{\nu_{d+1}})$ and $|\nu| = \nu_1 + \ldots + \nu_{d+1}$. In particular, for all $(\lambda, x) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$, we have

$$\left|\Lambda_\alpha^d(\lambda, x)\right| \le 1. \tag{2.3}$$

In the following we denote by

(i) $-\lambda = (-\lambda', \lambda_{d+1})$
(ii) $C_*(\mathbb{R}^{d+1})$, the space of continuous functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
(iii) $S_*(\mathbb{R}^{d+1})$, the space of the C^∞ functions, even with respect to the last variable, and rapidly decreasing together with their derivatives.
(iv) $\mathcal{S}_*(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$, the Schwartz space of rapidly decreasing functions on $\mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$ even with respect to the last two variables.
(v) $L_\alpha^p(\mathbb{R}_+^{d+1})$, $1 \le p \le \infty$, the space of measurable functions f on $\mathbb{R}_+^{d+1}$ such that

$$\|f\|_{\alpha,p} = \left(\int_{\mathbb{R}_+^{d+1}} |f(x)|^p\, d\mu_\alpha(x)\right)^{1/p} < \infty, \quad p \in [1, \infty),$$
$$\|f\|_{\alpha,\infty} = \operatorname{ess}\sup_{x\in\mathbb{R}_+^{d+1}} |f(x)| < \infty,$$

where $d\mu_\alpha(x)$ is the measure on $\mathbb{R}_+^{d+1} = \mathbb{R}^d \times (0,\infty)$ given by

$$d\mu_\alpha(x) = \frac{x_{d+1}^{2\alpha+1}}{(2\pi)^d 2^{2\alpha}\Gamma^2(\alpha+1)}dx.$$

For a radial function $\varphi \in L^1_\alpha(\mathbb{R}_+^{d+1})$ the function $\tilde{\varphi}$ defined on $\mathbb{R}_+$ such that $\varphi(x) = \tilde{\varphi}(|x|)$, for all $x \in \mathbb{R}_+^{d+1}$, is integrable with respect to the measure $r^{2\alpha+d+1}dr$, and we have

$$\int_{\mathbb{R}_+^{d+1}} \varphi(x)d\mu_\alpha(x) = a_\alpha \int_0^\infty \tilde{\varphi}(r)r^{2\alpha+d+1}dr, \tag{2.4}$$

where

$$a_\alpha = \frac{1}{2^{\alpha+\frac{d}{2}}\Gamma(\alpha+\frac{d}{2}+1)}.$$

The Weinstein transform generalizing the usual Fourier transform, is given for $\varphi \in L^1_\alpha(\mathbb{R}_+^{d+1})$ and $\lambda \in \mathbb{R}_+^{d+1}$, by

$$\mathcal{F}_{W,\alpha}(\varphi)(\lambda) = \int_{\mathbb{R}_+^{d+1}} \varphi(x)\Lambda_\alpha^d(x,\lambda)d\mu_\alpha(x), \tag{2.5}$$

We list some known basic properties of the Weinstein transform are as follows. For the proofs, we refer [13,14].

(i) For all $\varphi \in L^1_\alpha(\mathbb{R}_+^{d+1})$, the function $\mathcal{F}_{W,\alpha}(\varphi)$ is continuous on $\mathbb{R}_+^{d+1}$ and we have

$$\left\|\mathcal{F}_{W,\alpha}\varphi\right\|_{\alpha,\infty} \le \|\varphi\|_{\alpha,1}. \tag{2.6}$$

(ii) The Weinstein transform is a topological isomorphism from $\mathcal{S}_*(\mathbb{R}^{d+1})$ onto itself. The inverse transform is given by

$$\mathcal{F}^{-1}_{W,\alpha}\varphi(\lambda) = \mathcal{F}_{W,\alpha}\varphi(-\lambda), \text{ for all } \lambda \in \mathbb{R}_+^{d+1}. \tag{2.7}$$

(iii) Parseval formula: For all $\varphi, \phi \in \mathcal{S}_*(\mathbb{R}^{d+1})$, we have

$$\int_{\mathbb{R}_+^{d+1}} \varphi(x)\overline{\phi(x)}d\mu_\alpha(x) = \int_{\mathbb{R}_+^{d+1}} \mathcal{F}_{W,\alpha}(\varphi)(x)\overline{\mathcal{F}_{W,\alpha}(\phi)(x)}d\mu_\alpha(x).$$

(v) Plancherel formula: For all $\varphi \in L^2_\alpha(\mathbb{R}_+^{d+1})$, we have

$$\left\|\mathcal{F}_{W,\alpha}\varphi\right\|_{\alpha,2} = \|\varphi\|_{\alpha,2}. \tag{2.8}$$

(vi) Plancherel Theorem: The Weinstein transform $\mathcal{F}_{W,\alpha}$ extends uniquely to an isometric isomorphism on $L^2_\alpha(\mathbb{R}_+^{d+1})$.

(vii) Inversion formula: Let $\varphi \in L^1_\alpha(\mathbb{R}^{d+1}_+)$ such that $\mathcal{F}_{W,\alpha}\varphi \in L^1_\alpha(\mathbb{R}^{d+1}_+)$, then we have

$$\varphi(\lambda) = \int_{\mathbb{R}^{d+1}_+} \mathcal{F}_{W,\alpha}\varphi(x)\Lambda^d_\alpha(-\lambda, x)d\mu_\alpha(x), \text{ a.e. } \lambda \in \mathbb{R}^{d+1}_+. \tag{2.9}$$

Using relations (2.6) and (2.8) with Marcinkiewicz's interpolation theorem [21] we deduce that for every $\varphi \in L^p_\alpha(\mathbb{R}^{d+1}_+)$ for all $1 \leq p \leq 2$, the function $\mathcal{F}_{W,\alpha}(\varphi) \in L^q_\alpha(\mathbb{R}^{d+1}_+)$, $q = p/(p-1)$, and

$$\left\| \mathcal{F}_{W,\alpha}\varphi \right\|_{\alpha,q} \leq \|\varphi\|_{\alpha,p} . \tag{2.10}$$

By using the Weinstein kernel, we can also define a generalized translation, for a function $\varphi \in \mathcal{S}_*(\mathbb{R}^{d+1})$ and $y \in \mathbb{R}^{d+1}_+$ the generalized translation $\tau^\alpha_x\varphi$ is defined by the following relation

$$\mathcal{F}_{W,\alpha}(\tau^\alpha_x\varphi)(y) = \Lambda^d_\alpha(x, y)\mathcal{F}_{W,\alpha}(\varphi)(y). \tag{2.11}$$

Note that for $\varphi \in L^p_\alpha(\mathbb{R}^{d+1}_+)$, $1 \leq p \leq \infty$ and $x \in \mathbb{R}^{d+1}_+$. Then $\tau^\alpha_x\varphi$ belongs to $L^p_\alpha(\mathbb{R}^{d+1}_+)$ and we have

$$\left\| \tau^\alpha_x\varphi \right\|_{\alpha,p} \leq \|\varphi\|_{\alpha,p} . \tag{2.12}$$

Proposition 2.1 *Let $\varphi \in L^1_\alpha(\mathbb{R}^{d+1}_+)$. Then for all $x \in \mathbb{R}^{d+1}_+$,*

$$\int_{\mathbb{R}^{d+1}_+} \tau^\alpha_x\varphi(y)d\mu_\alpha(y) = \int_{\mathbb{R}^{d+1}_+} \varphi(y)d\mu_\alpha(y). \tag{2.13}$$

Proof The result comes from combination identities (2.9) and (2.11). □

3 The Weinstein–Wigner transform

The Weinstein–Wigner transform is the mapping $\mathcal{V}$ defined on $\mathcal{S}_*(\mathbb{R}^{d+1}) \times \mathcal{S}_*(\mathbb{R}^{d+1})$ by

$$\mathcal{V}(\varphi, \psi)(x, y) = \int_{\mathbb{R}^{d+1}_+} \varphi(\lambda)\tau^\alpha_x\psi(-\lambda)\Lambda^d_\alpha(y, \lambda)d\mu_\alpha(\lambda). \tag{3.1}$$

Also, we can write $\mathcal{V}$ in terms of Weinstein transform of the product of φ and $\tau^\alpha_x\psi$ as follow

$$\mathcal{V}(\varphi, \psi)(x, y) = \mathcal{F}_{W,\alpha}(\widetilde{\varphi\tau^\alpha_x\psi})(y). \tag{3.2}$$

Proposition 3.1 (i) *The Weinstein–Wigner transform* $\mathcal{V}$ *is a bilinear, continuous mapping from* $\mathcal{S}_*(\mathbb{R}^{d+1}) \times \mathcal{S}_*(\mathbb{R}^{d+1})$ *onto* $\mathcal{S}_*(\mathbb{R}^{\mathbf{d+1}} \times \mathbb{R}^{\mathbf{d+1}})$.

(ii) *For all* $\varphi, \psi \in L^2_\alpha(\mathbb{R}^{d+1}_+)$, *then* $\mathcal{V}(\varphi, \psi)$ *belongs to* $L^2_\alpha \cap L^\infty_\alpha(\mu_\alpha \otimes \mu_\alpha)$, *and we have*

$$\|\mathcal{V}(\varphi, \psi)\|_{L^\infty_\alpha(\mu_\alpha \otimes \mu_\alpha)} \leq \|\varphi\|_{\alpha,2} \, \|\psi\|_{\alpha,2} \, , \tag{3.3}$$

$$\|\mathcal{V}(\varphi, \psi)\|_{L^2_\alpha(\mu_\alpha \otimes \mu_\alpha)} \leq \|\varphi\|_{\alpha,2} \, \|\psi\|_{\alpha,2} \, . \tag{3.4}$$

(iii) *Let* $p \in [1, 2]$ *and* q *such that* $\frac{1}{p} + \frac{1}{q} = 1$. *If* $(\varphi, \psi) \in L^q_\alpha(\mathbb{R}^{d+1}_+) \times L^p_\alpha(\mathbb{R}^{d+1}_+)$, *then* $\mathcal{V}(\varphi, \psi) \in L^\infty_\alpha(\mathbb{R}^{d+1}_+) \times L^\infty_\alpha(\mathbb{R}^{d+1}_+)$, *and we have*

$$\|\mathcal{V}(\varphi, \psi)\|_{L^\infty_\alpha(\mu_\alpha \otimes \mu_\alpha)} \leq \|\varphi\|_{\alpha,2} \, \|\psi\|_{\alpha,2} \, . \tag{3.5}$$

Proof (i) Let $\varphi, \psi \in \mathcal{S}_*(\mathbb{R}^{d+1})$. Consider the function Φ defined on $\mathbb{R}^{d+1}_+ \times \mathbb{R}^{d+1}_+$ by

$$\Phi(x, y) = \varphi(y)\tau^\alpha_x \psi(-y).$$

Then the Weinstein–Wigner transform $\mathcal{V}$ can be written as follow

$$\mathcal{V}(\varphi, \psi)(x, y) = (Id \otimes \mathcal{F}_{W,\alpha})(\Phi)(x, y),$$

where Id is the identity operator. Since the Weinstein transform is a topological isomorphism from $\mathcal{S}_*(\mathbb{R}^{d+1})$ onto itself, therefore, we get the result.

(ii) According to the definition of the Weinstein–Wigner transform (3.1), Hölder's inequality and (2.12), we obtain the estimate (3.3).
The inequality (3.4) holds from the identity (3.2), the Plancherel formula (2.8), Minkowski's inequality for integrals [7, p. 186] and (2.12).

(iii) The result follows from the definition of the Weinstein–Wigner transform (3.1), Hölder's inequality and the inequality (2.12).

□

Proposition 3.2 *Let* $\varphi, \psi \in \mathcal{S}_*(\mathbb{R}^{d+1})$. *Then for every* $\xi, \lambda \in \mathbb{R}^{d+1}_+$, *we have*

$$\mathcal{F}_{W,\alpha} \otimes \mathcal{F}^{-1}_{W,\alpha}(\mathcal{V}(\varphi, \psi))(\xi, \lambda) = \Lambda^d_\alpha(\lambda, \xi)\varphi(\lambda)\mathcal{F}_{W,\alpha}(\psi)(\xi).$$

Proof Put $\varphi, \psi \in \mathcal{S}_*(\mathbb{R}^{d+1})$. From the definition of the Weinstein–Wigner transform (3.1) and (3.2) and according to Fubini's theorem, we get

$$\begin{aligned}
\mathcal{F}_{W,\alpha} \otimes \mathcal{F}^{-1}_{W,\alpha}(\mathcal{V}(\varphi, \psi))(\xi, \lambda) &= \int_{\mathbb{R}^{d+1}_+} \int_{\mathbb{R}^{d+1}_+} \mathcal{V}(\varphi, \psi)(x, y)\Lambda^d_\alpha(\xi, x)\Lambda^d_\alpha(-\lambda, y)d\mu_\alpha(x)d\mu_\alpha(y) \\
&= \int_{\mathbb{R}^{d+1}_+} \left(\int_{\mathbb{R}^{d+1}_+} \mathcal{F}_{W,\alpha}(\varphi\widetilde{\tau^\alpha_x \psi})(y)\Lambda^d_\alpha(-\lambda, y)d\mu_\alpha(y) \right) \Lambda^d_\alpha(\xi, x)d\mu_\alpha(x) \\
&= \varphi(\lambda) \int_{\mathbb{R}^{d+1}_+} \tau^\alpha_x \psi(-\lambda)\Lambda^d_\alpha(\xi, x)d\mu_\alpha(x).
\end{aligned}$$

Finally, by the expression of the generalized translation $\tau_x^\alpha \varphi$ in term of the Weinstein kernel (2.11), we obtain

$$\mathcal{F}_{W,\alpha} \otimes \mathcal{F}_{W,\alpha}^{-1}(\mathcal{V}(\varphi,\psi))(\xi,\lambda) = \varphi(\lambda)\mathcal{F}_{W,\alpha}(\tau_{-\lambda}^\alpha \psi)(\xi) = \Lambda_\alpha^d(\lambda,\xi)\varphi(\lambda)\mathcal{F}_{W,\alpha}(\psi)(\xi).$$

□

Corollary 3.3 *Let $\varphi, \psi \in \mathcal{S}_*(\mathbb{R}^{d+1})$. Then we have for all $\xi, \lambda \in \mathbb{R}_+^{d+1}$*

$$\int_{\mathbb{R}_+^{d+1}} \mathcal{F}_{W,\alpha} \otimes \mathcal{F}_{W,\alpha}^{-1}(\mathcal{V}(\varphi,\psi))(\xi,\lambda) d\mu_\alpha(\lambda) = \mathcal{F}_{W,\alpha}(\varphi)(\xi)\mathcal{F}_{W,\alpha}(\psi)(\xi).$$

$$\int_{\mathbb{R}_+^{d+1}} \mathcal{F}_{W,\alpha} \otimes \mathcal{F}_{W,\alpha}^{-1}(\mathcal{V}(\varphi,\psi))(\xi,\lambda) d\mu_\alpha(\xi) = \varphi(\lambda)\psi(-\lambda).$$

Theorem 3.4 *Let $\psi \in L_\alpha^1(\mathbb{R}_+^{d+1}) \cap L_\alpha^2(\mathbb{R}_+^{d+1})$ such that $c = \int_{\mathbb{R}_+^{d+1}} \psi(x) d\mu_\alpha(x) \neq 0$. Then, we have for all $\varphi \in L_\alpha^1(\mathbb{R}_+^{d+1}) \cap L_\alpha^2(\mathbb{R}_+^{d+1})$*

$$\mathcal{F}_{W,\alpha}(\varphi)(y) = \frac{1}{c} \int_{\mathbb{R}_+^{d+1}} \mathcal{V}(\varphi,\psi)(x,y) d\mu_\alpha(x).$$

Proof According to the definition of the Weinstein–Wigner transform (3.1), Fubini's theorem and relation (2.13), we get

$$\begin{aligned}\int_{\mathbb{R}_+^{d+1}} \mathcal{V}(\varphi,\psi)(x,y) d\mu_\alpha(x) &= \int_{\mathbb{R}_+^{d+1}} \Lambda_\alpha^d(y,t)\varphi(t) \left(\int_{\mathbb{R}_+^{d+1}} \tau_x^\alpha \psi(-t) d\mu_\alpha(x) \right) d\mu_\alpha(t) \\ &= c\mathcal{F}_{W,\alpha}(\varphi)(y),\end{aligned}$$

which completes the proof. □

Corollary 3.5 *Let $\psi \in L_\alpha^1(\mathbb{R}_+^{d+1}) \cap L_\alpha^2(\mathbb{R}_+^{d+1})$ such that $c = \int_{\mathbb{R}_+^{d+1}} \psi(x) d\mu_\alpha(x) \neq 0$. Then, we have*

(i) *For all $\varphi \in L_\alpha^1(\mathbb{R}_+^{d+1}) \cap L_\alpha^2(\mathbb{R}_+^{d+1})$ such that $\mathcal{F}_{W,\alpha}(\varphi) \in L_\alpha^1(\mathbb{R}_+^{d+1})$,*

$$\varphi(z) = \frac{1}{c} \int_{\mathbb{R}_+^{d+1}} \Lambda_\alpha^d(-y,z) \left(\int_{\mathbb{R}_+^{d+1}} \mathcal{V}(\varphi,\psi)(x,y) d\mu_\alpha(x) \right) d\mu_\alpha(y).$$

(ii) *For all $\varphi \in L_\alpha^1(\mathbb{R}_+^{d+1}) \cap L_\alpha^2(\mathbb{R}_+^{d+1})$,*

$$\|\varphi\|_{\alpha,2}^2 = \frac{1}{c^2} \int_{\mathbb{R}_+^{d+1}} \left| \int_{\mathbb{R}_+^{d+1}} \mathcal{V}(\varphi,\psi)(x,y) d\mu_\alpha(x) \right|^2 d\mu_\alpha(y).$$

4 The Weinstein–Weyl transform

In this section, we introduce and study the Weinstein–Weyl transform.

4.1 Case I: $\sigma \in \mathcal{S}_*(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$

Let $\sigma \in \mathcal{S}_*(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$, we define the Weyl transform $\mathcal{W}_\sigma$ associated to the Weinstein operators on $\mathcal{S}_*(\mathbb{R}^{d+1})$, by

$$\mathcal{W}_\sigma(\varphi)(x) = \int_{\mathbb{R}_+^{d+1}} \int_{\mathbb{R}_+^{d+1}} \sigma(y,z)\Lambda_\alpha^d(x,z)\tau_x^\alpha\varphi(-y)d\mu_\alpha(y)d\mu_\alpha(z), \quad x \in \mathbb{R}_+^{d+1}. \tag{4.1}$$

Theorem 4.1 *Let $\sigma \in \mathcal{S}_*(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$. Then the Weinstein–Weyl transform is continuous from $\mathcal{S}_*(\mathbb{R}^{d+1})$ into itself.*

Proof Let $\varphi \in \mathcal{S}_*(\mathbb{R}^{d+1})$. Since the Weinstein transform is a topological isomorphism from $\mathcal{S}_*(\mathbb{R}^{d+1})$ onto itself, so according to relation (2.7), we have

$$\tau_x^\alpha\varphi(-y) = \int_{\mathbb{R}_+^{d+1}} \Lambda_\alpha^d(-x,t)\Lambda_\alpha^d(y,t)\mathcal{F}_{W,\alpha}(\varphi)(t)d\mu_\alpha(t), \quad x, y \in \mathbb{R}_+^{d+1}.$$

Then, by (4.1) and Fubini's theorem, we get

$$\begin{aligned}
\mathcal{W}_\sigma(\varphi)(x) &= \int_{\mathbb{R}_+^{d+1}} \Lambda_\alpha^d(x,z)\Bigg\{\int_{\mathbb{R}_+^{d+1}} \Lambda_\alpha^d(-x,t)\mathcal{F}_{W,\alpha}(\varphi)(t) \\
&\quad \left(\int_{\mathbb{R}_+^{d+1}} \sigma(y,z)\Lambda_\alpha^d(y,t)d\mu_\alpha(y)\right) d\mu_\alpha(t)\Bigg\}d\mu_\alpha(z) \\
&= \int_{\mathbb{R}_+^{d+1}} \Lambda_\alpha^d(x,z)\Bigg\{\int_{\mathbb{R}_+^{d+1}} \Lambda_\alpha^d(-x,t)\mathcal{F}_{W,\alpha}(\varphi)(t) \\
&\quad \mathcal{F}_{W,\alpha}(\sigma(.,z))(t)d\mu_\alpha(t)\Bigg\}d\mu_\alpha(z).
\end{aligned}$$

Taking into account that the function $(t,z) \to \mathcal{F}_{W,\alpha}(\sigma(.,z))(t)$ belongs to $\mathcal{S}_*(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$ and that the mapping $\varphi \to H_\varphi$ given by,

$$H_\varphi(t,z) = \mathcal{F}_{W,\alpha}(\varphi)(t)\mathcal{F}_{W,\alpha}(\sigma(.,z))(t), \quad t, z \in \mathbb{R}_+^{d+1},$$

is continuous from $\mathcal{S}_*(\mathbb{R}^{d+1})$ to $\mathcal{S}_*(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$, and we have for all $x \in \mathbb{R}_+^{d+1}$,

$$\begin{aligned}
\mathcal{W}_\sigma(\varphi)(x) &= \int_{\mathbb{R}_+^{d+1}} \Lambda_\alpha^d(x,z)\left(\int_{\mathbb{R}_+^{d+1}} \Lambda_\alpha^d(-x,t)H_\varphi(t,z)d\mu_\alpha(t)\right) d\mu_\alpha(z) \\
&= \mathcal{F}_{W,\alpha}^{-1} \otimes \mathcal{F}_{W,\alpha}(H_\varphi)(x,x).
\end{aligned}$$

The result comes from the fact $\mathcal{F}_{W,\alpha}^{-1} \otimes \mathcal{F}_{W,\alpha}$ is an isomorphism from $\mathcal{S}_*(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$ onto itself. □

Lemma 4.2 *Let $\sigma \in \mathcal{S}_*(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$. Then, the function*

$$h(x, y) = \int_{\mathbb{R}_+^{d+1}} \Lambda_\alpha^d(x, z)\tau_x^\alpha\left(\sigma(., z)\right)(-y)d\mu_\alpha(z) \tag{4.2}$$

is defined on $\mathbb{R}_+^{d+1} \times \mathbb{R}_+^{d+1}$ and belongs to $\mathcal{S}_*(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$.

Proof We put the function H defined on $\mathbb{R}_+^{d+1} \times \mathbb{R}_+^{d+1}$ by

$$H(t, x) = \int_{\mathbb{R}_+^{d+1}} \Lambda_\alpha^d(x, z)\mathcal{F}_{W,\alpha}\left(\sigma(., z)\right)(t)d\mu_\alpha(z).$$

Then the function h can be written in the form

$$h(x, y) = \tau_x^\alpha\left[\mathcal{F}_{W,\alpha}^{-1}(H(., x))\right](-y) = \tau_x^\alpha\left[(Id \otimes \mathcal{F}_{W,\alpha}^{-1})(H)(., x)\right](-y).$$

Since the function $(t, x) \to H(t, x)$ belongs to $\mathcal{S}_*(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$, so by the fact, that the Weinstein transform is a topological isomorphism from $\mathcal{S}_*(\mathbb{R}^{d+1})$ onto itself, we deduce that the function $(Id \otimes \mathcal{F}_{W,\alpha}^{-1})(H)$ belongs to $\mathcal{S}_*(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$. In the end, the result comes from the fact that for all $f \in \mathcal{S}_*(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$, the function $(x, y) \to \tau_x^\alpha\left[f(., x)\right](-y)$ belongs to $\mathcal{S}_*(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$. □

Theorem 4.3 *Let $\sigma \in \mathcal{S}_*(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$, then we have the following results*

(i) *For all $\varphi \in \mathcal{S}_*(\mathbb{R}^{d+1})$,*

$$\mathcal{W}_\sigma(\varphi)(x) = \int_{\mathbb{R}_+^{d+1}} h(x, y)\varphi(y)d\mu_\alpha(y).$$

where $h(x, y)$ is the kernel given by (4.2).

(ii) *For all $\varphi \in \mathcal{S}_*(\mathbb{R}^{d+1})$ and $1 \leq p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$,*

$$\|\mathcal{W}_\sigma(\varphi)\|_{\alpha,q} \leq \|h\|_{L_\alpha^q(\mu_\alpha \otimes \mu_\alpha)} \|\varphi\|_{\alpha,p}.$$

(iii) *For $1 \leq p, q < \infty$ such that $q = \frac{p}{p-1}$, the operator $\mathcal{W}_\sigma$ can be extended to a bounded operator from $L_\alpha^p(\mathbb{R}_+^{d+1})$ into $L_\alpha^q(\mathbb{R}_+^{d+1})$. In particular $\mathcal{W}_\sigma$ is a compact Hilbert–Schmidt operator from $L_\alpha^2(\mathbb{R}_+^{d+1})$ onto itself.*

Proof (i) Let $\varphi \in \mathcal{S}_*(\mathbb{R}^{d+1})$. The Weyl–Weinstein transform can be written as follow

$$\mathcal{W}_\sigma(\varphi)(x) = \int_{\mathbb{R}_+^{d+1}} \Lambda_\alpha^d(x, z)\left(\int_{\mathbb{R}_+^{d+1}} \tau_x^\alpha\varphi(-y)\sigma(y, z)d\mu_\alpha(y)\right)d\mu_\alpha(z).$$

Using Fubini's theorem, and the following equality

$$\int_{\mathbb{R}_+^{d+1}} \tau_x^\alpha \varphi(-y)\sigma(y,z)d\mu_\alpha(y) = \int_{\mathbb{R}_+^{d+1}} \varphi(y)\tau_x^\alpha\left(\sigma(.,z)\right)(-y)d\mu_\alpha(y),$$

we deduce that

$$\mathcal{W}_\sigma(\varphi)(x) = \int_{\mathbb{R}_+^{d+1}} h(x,y)\varphi(y)d\mu_\alpha(y).$$

(ii) Follows from (i) and the combination of Hölder's inequality, and Lemma 4.2.
(iii) We deduce, from (ii) and the fact that the space $\mathcal{S}_*(\mathbb{R}^{d+1})$ is dense in $L^p_\alpha(\mathbb{R}_+^{d+1})$, for all $1 \leq p < \infty$ that $\mathcal{W}_\sigma$ can be extended to a bounded operator from $L^p_\alpha(\mathbb{R}_+^{d+1})$ into $L^q_\alpha(\mathbb{R}_+^{d+1})$. Finally, we deduce by Lemma 4.2, that the kernel h belongs to $L^2_\alpha(\mu_\alpha \otimes \mu_\alpha)$, hence $\mathcal{W}_\sigma(\varphi)$ is a compact Hilbert–Schmidt operator. □

4.2 Case II: $\sigma \in L^p_\alpha(\mu_\alpha \otimes \mu_\alpha)$, $p \in [1,2]$

In this section, we show that the Weyl–Weinstein transform with symbol $\sigma \in L^p_\alpha(\mu_\alpha \otimes \mu_\alpha)$, $p \in [1,2]$, is a compact operator. We denote by $\mathcal{B}(L^2_\alpha(\mathbb{R}_+^{d+1}))$ the $\mathbb{C}^*$-algebra of bounded operators Ψ from $L^2_\alpha(\mathbb{R}_+^{d+1})$ into itself, equipped with the norm

$$\|\Psi\| := \sup_{\|\varphi\|_{\alpha,2}=1} \|\Psi(\varphi)\|_{\alpha,2}.$$

Let $\sigma \in \mathcal{S}_*(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$. We define the operator K_σ on $\mathcal{S}_*(\mathbb{R}^{d+1}) \times \mathcal{S}_*(\mathbb{R}^{d+1})$, by

$$K_\sigma(\varphi,\psi)(z) := \int_{\mathbb{R}_+^{d+1}} \int_{\mathbb{R}_+^{d+1}} \sigma(x,y)\Lambda_\alpha^d(z,y)\mathcal{V}(\varphi,\psi)(x,y)d\mu_\alpha(x)d\mu_\alpha(y), \quad z \in \mathbb{R}_+^{d+1}. \tag{4.3}$$

Lemma 4.4 *Let $\sigma \in \mathcal{S}_*(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$. For all $\varphi, \psi \in \mathcal{S}_*(\mathbb{R}^{d+1})$, we have*

$$K_\sigma(\varphi,\psi)(0) := \langle \mathcal{W}_\sigma(\psi), \overline{\varphi}\rangle_{\alpha,2},$$

where $\langle .,. \rangle_{\alpha,2}$ is the inner product of $L^2_\alpha(\mathbb{R}_+^{d+1})$.

Proof According to the definition of the Weinstein–Wigner transform (3.1) and (4.3), we have

$$K_\sigma(\varphi,\psi)(0) = \int_{\mathbb{R}_+^{d+1}}\int_{\mathbb{R}_+^{d+1}} \sigma(x,y)\mathcal{V}(\varphi,\psi)(x,y)d\mu_\alpha(x)d\mu_\alpha(y)$$
$$= \int_{\mathbb{R}_+^{d+1}}\int_{\mathbb{R}_+^{d+1}} \sigma(x,y)\left(\int_{\mathbb{R}_+^{d+1}} \varphi(t)\tau_x^\alpha\psi(-t)\Lambda_\alpha^d(y,t)d\mu_\alpha(t)\right)d\mu_\alpha(x)d\mu_\alpha(y).$$

Then, applying Fubini's theorem, we get

$$K_\sigma(\varphi,\psi)(0)=\int_{\mathbb{R}_+^{d+1}} \varphi(t)\left(\int_{\mathbb{R}_+^{d+1}}\int_{\mathbb{R}_+^{d+1}} \sigma(x,y)\tau_x^\alpha\psi(-t)\Lambda_\alpha^d(y,t)d\mu_\alpha(x)d\mu_\alpha(y)\right)d\mu_\alpha(t).$$

Using the fact $\tau_x^\alpha\psi(-t) = \tau_t^\alpha\psi(-x)$, then by the definition of the Weyl–Weinstein transform (4.1), we obtain

$$K_\sigma(\varphi,\psi)(0) = \int_{\mathbb{R}_+^{d+1}} \varphi(t)\mathcal{W}_\sigma(\psi)(t)d\mu_\alpha(t) = \langle \mathcal{W}_\sigma(\psi),\overline{\varphi}\rangle_{\alpha,2}.$$

□

Theorem 4.5 *For $1 \le p \le 2$, there exists a unique bounded operator*

$$T : L_\alpha^p(\mu_\alpha\otimes\mu_\alpha) \longrightarrow \mathcal{B}(L_\alpha^2(\mathbb{R}_+^{d+1}))$$
$$\sigma \longmapsto T_\sigma$$

such that for all $\varphi,\psi \in \mathcal{S}_*(\mathbb{R}^{d+1})$,

$$\langle T_\sigma(\psi),\overline{\varphi}\rangle_{\alpha,2} = \int_{\mathbb{R}_+^{d+1}}\int_{\mathbb{R}_+^{d+1}} \sigma(x,y)\mathcal{V}(\varphi,\psi)(x,y)d\mu_\alpha(x)d\mu_\alpha(y),$$

with

$$\|T_\sigma\| \le \|\sigma\|_{L_\alpha^p(\mu_\alpha\otimes\mu_\alpha)}.$$

Proof (i) First case: $p = 2$. Let $\sigma \in \mathcal{S}_*(\mathbb{R}^{d+1}\times\mathbb{R}^{d+1})$. For $\psi \in \mathcal{S}_*(\mathbb{R}^{d+1})$, we put

$$T_\sigma(\psi) = \mathcal{W}_\sigma(\psi). \tag{4.4}$$

By Lemma 4.4, we obtain

$$\langle T_\sigma(\psi),\overline{\varphi}\rangle_{\alpha,2} = \langle \mathcal{W}_\sigma(\psi),\overline{\varphi}\rangle_{\alpha,2} = K_\sigma(\varphi,\psi)(0)$$
$$= \int_{\mathbb{R}_+^{d+1}}\int_{\mathbb{R}_+^{d+1}} \sigma(x,y)\mathcal{V}(\varphi,\psi)(x,y)d\mu_\alpha(x)d\mu_\alpha(y).$$

Moreover, according to Hölder's inequality and (3.4), we get

$$\langle T_\sigma(\psi),\overline{\varphi}\rangle_{\alpha,2} \le \|\sigma\|_{L_\alpha^2(\mu_\alpha\otimes\mu_\alpha)}\|\varphi\|_{\alpha,2}\|\psi\|_{\alpha,2},$$

which implies that $T_\sigma \in \mathcal{B}(L^2_\alpha(\mathbb{R}^{d+1}_+))$ and

$$\|T_\sigma\| \leq \|\sigma\|_{L^2_\alpha(\mu_\alpha\otimes\mu_\alpha)}. \tag{4.5}$$

Next, we consider $L^2_\alpha(\mu_\alpha \otimes \mu_\alpha)$ and $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence in $\sigma \in \mathcal{S}_*(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$ such that $\|\sigma_n - \sigma\|_{L^2_\alpha(\mu_\alpha\otimes\mu_\alpha)} \to 0$ as $n \to 0$. According to inequality (4.5) we have, for all $m, n \in \mathbb{N}$,

$$\begin{aligned}\|T_{\sigma_m} - T_{\sigma_n}\| &\leq \|\sigma_m - \sigma_n\|_{L^2_\alpha(\mu_\alpha\otimes\mu_\alpha)}\\ &\leq \|\sigma_m - \sigma\|_{L^2_\alpha(\mu_\alpha\otimes\mu_\alpha)} + \|\sigma_n - \sigma\|_{L^2_\alpha(\mu_\alpha\otimes\mu_\alpha)}.\end{aligned}$$

Consequently $(T_{\sigma_n})_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{B}(L^2_\alpha(\mathbb{R}^{d+1}_+))$ which converges to T_σ, moreover, by relation (4.5) the limit T_σ is independent of the choice of $(\sigma_n)_{n\in\mathbb{N}}$ and

$$\|T_\sigma\| = \lim_{n\to\infty} \|T_{\sigma_n}\| \leq \lim_{n\to\infty} \|\sigma_n\|_{L^2_\alpha(\mu_\alpha\otimes\mu_\alpha)} \leq \|\sigma\|_{L^2_\alpha(\mu_\alpha\otimes\mu_\alpha)}.$$

On the other hand, for all $\varphi, \psi \in \mathcal{S}_*(\mathbb{R}^{d+1})$, we have

$$\begin{aligned}\langle T_\sigma(\psi), \overline{\varphi}\rangle_{\alpha,2} &= \lim_{n\to\infty} \langle T_{\sigma_n}(\psi), \overline{\varphi}\rangle_{\alpha,2}\\ &= \lim_{n\to\infty} \int_{\mathbb{R}^{d+1}_+}\int_{\mathbb{R}^{d+1}_+} \sigma_n(x,y)\mathcal{V}(\varphi,\psi)(x,y)d\mu_\alpha(x)d\mu_\alpha(y)\\ &= \int_{\mathbb{R}^{d+1}_+}\int_{\mathbb{R}^{d+1}_+} \sigma(x,y)\mathcal{V}(\varphi,\psi)(x,y)d\mu_\alpha(x)d\mu_\alpha(y).\end{aligned}$$

(ii) Second case $p = 1$. Let $\sigma \in \mathcal{S}_*(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$ and consider the the operator T_σ defined by (4.4). Then, according to Hölder's inequality and the estimate (3.3), we have for all $\varphi, \psi \in \mathcal{S}_*(\mathbb{R}^{d+1})$,

$$\begin{aligned}|\langle T_\sigma(\psi), \overline{\varphi}\rangle_{\alpha,2}| &\leq \|\sigma\|_{L^1_\alpha(\mu_\alpha\otimes\mu_\alpha)} \|\mathcal{V}(\varphi,\psi)\|_{L^\infty_\alpha(\mu_\alpha\otimes\mu_\alpha)}\\ &\leq \|\sigma\|_{L^1_\alpha(\mu_\alpha\otimes\mu_\alpha)} \|\varphi\|_{\alpha,2}\|\psi\|_{\alpha,2}.\end{aligned}$$

which implies that $T_\sigma \in \mathcal{B}(L^2_\alpha(\mathbb{R}^{d+1}_+))$ and

$$\|T_\sigma\| \leq \|\sigma\|_{L^1_\alpha(\mu_\alpha\otimes\mu_\alpha)}.$$

Next, in a similar way to the proof of (i), we obtain the result for $p = 1$.

(iii) For $1 < p < 2$, the result comes from the cases $p = 1, 2$ and the Riesz–Thorin theorem [20], which complete the proof.

□

Theorem 4.6 *Let $p \in [1, 2]$ and $\sigma \in L^p_\alpha(\mu_\alpha \otimes \mu_\alpha)$. Then T_σ is a compact operator from $L^2_\alpha(\mathbb{R}^{d+1}_+)$ into itself.*

Proof Let $p \in [1,2]$, $\sigma \in L^p_\alpha(\mu_\alpha \otimes \mu_\alpha)$ and $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence in $\sigma \in \mathcal{S}_*(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$ such that $\lim_{n\to 0} \|\sigma_n - \sigma\|_{L^2_\alpha(\mu_\alpha\otimes\mu_\alpha)} = 0$. From Theorem 4.5, we have $\|T_{\sigma_m} - T_{\sigma_n}\| \leq \|\sigma_m - \sigma_n\|_{L^2_\alpha(\mu_\alpha\otimes\mu_\alpha)}$. Hence, T_{σ_n} converges to T_σ in $\mathcal{B}(L^2_\alpha(\mathbb{R}^{d+1}_+))$ and the result comes by Theorem 4.3 (iii). □

4.3 Case III: $\sigma \in \mathcal{S}'_*(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$

In this section, we denote by $\mathcal{S}'_*(\mathbb{R}^{d+1})$ the topological dual of the space of $\mathcal{S}_*(\mathbb{R}^{d+1})$ and $\mathcal{S}'_*(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$ the topological dual of the space of $\mathcal{S}_*(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$.

For $\sigma \in \mathcal{S}'_*(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$ and $\psi \in \mathcal{S}_*(\mathbb{R}^{d+1})$, we define the Weinstein–Weyl transform on $\mathcal{S}_*(\mathbb{R}^{d+1})$, by

$$(\mathcal{W}_\sigma(\psi))(\varphi) = \sigma\left(\mathcal{V}(\varphi,\psi)\right), \quad \varphi \in \mathcal{S}_*(\mathbb{R}^{d+1}), \tag{4.6}$$

where $\mathcal{V}$ is the Weinstein–Wigner transform given by (3.1).

It follows from Proposition 3.1 (i) that $\mathcal{W}_\sigma$ defined in (4.6) belongs to $\mathcal{S}_*(\mathbb{R}^{d+1})$.

Now, we denote by σ_h the element of $\mathcal{S}'_*(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$ defined by

$$\sigma_h(G) = \int_{\mathbb{R}^{d+1}_+}\int_{\mathbb{R}^{d+1}_+} G(x,y)h(x,y)d\mu_\alpha(x)d\mu_\alpha(y), \tag{4.7}$$

where h is a slowly increasing function defined on $\mathbb{R}^{d+1}_+ \times \mathbb{R}^{d+1}_+$.

Proposition 4.7 *Let $\sigma_1 \in \mathcal{S}'_*(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$, given by the function equal to 1 and ψ a function in $\mathcal{S}_*(\mathbb{R}^{d+1})$. Then, we have*

$$\mathcal{W}_{\sigma_1}(\psi) = c\delta,$$

where c is the constant given in Theorem 3.4 and δ is the Dirac distribution at 0.

Proof According to relations (4.6) and (4.7), we have for all $\varphi \in \mathcal{S}_*(\mathbb{R}^{d+1})$,

$$\left(\mathcal{W}_{\sigma_1}(\psi)\right)(\varphi) = \sigma_1\left(\mathcal{V}(\varphi,\psi)\right) = \int_{\mathbb{R}^{d+1}_+}\left(\int_{\mathbb{R}^{d+1}_+} \mathcal{V}(\varphi,\psi)(x,y)d\mu_\alpha(x)\right)d\mu_\alpha(y),$$

then, by Theorem 3.4, we obtain

$$\left(\mathcal{W}_{\sigma_1}(\psi)\right)(\varphi) = \sigma_1\left(\mathcal{V}(\varphi,\psi)\right) = \int_{\mathbb{R}^{d+1}_+} \mathcal{F}_{W,\alpha}(\varphi)(y)d\mu_\alpha(y).$$

Applying the inversion formula of the Weinstein transform (2.9), we finish the proof. □

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