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Harmonic oscillator perturbed by a decreasing scalar potential

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Abstract
In this paper we study the perturbation $L = H + V$, where $H = -\frac{d^{2m}}{dx^{2m}} + x^{2m}$ on $\mathbb{R}$, $m \in \mathbb{N}^*$ and V is a decreasing scalar potential. Let λ_k be the k^{th} eigenvalue of H. We suppose that the eigenvalues of L around λ_k can be written in the form $\lambda_k + \mu_k$. The main result of the paper is an asymptotic formula for fluctuation $\{\mu_k\}$ which is given by a transformation of V. In the case $m = 1$ we recover a result on the harmonic oscillator.

Keywords Averaging method · Pseudo-differential operator · Perturbation theory · Spectrum · Eigenvalue asymptotics

Mathematics Subject Classification Primary 99Z99; Secondary 00A00

1 Introduction and main results

We consider in $\mathbb{R}$ the operator H defined by

$$H = -\frac{d^{2m}}{dx^{2m}} + x^{2m}, \quad m \in \mathbb{N}^* \tag{1.1}$$

We recall that H [1] is essentially self-adjoint in $C_0^\infty(\mathbb{R})$ with compact resolvent. Its spectrum is the increasing sequence $\{\lambda_k\}_{k\geq 0}$ of eigenvalues of finite multiplicity, such as there exists a positive integer k_0, for $k \geq k_0$, λ_k is simple and has the following asymptotic expansion

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$$\lambda_k^{\frac{1}{m}} = \frac{2\pi}{T}\left(k + \frac{1}{2}\right) + O(k^{-1}) \quad k \to +\infty \tag{1.2}$$

with

$$T = \int_{-1}^{1} (1 - u^{2m})^{\frac{1}{2m}-1} du = \frac{1}{m} B\left(\frac{1}{2m}, \frac{1}{2m}\right) \tag{1.3}$$

where B is the beta function. Let $V \in \mathbb{C}^{\infty}(\mathbb{R}, \mathbb{R})$ which satisfies the following estimate

$$|V^{(n)}(x)| \leq C_n(1 + x^2)^{-\frac{s}{2}},\ x \in \mathbb{R},\ n \in \mathbb{N},\ s \in \mathbb{R}_+^* - \{1\} \tag{1.4}$$

Along this article we set

$$\delta = \begin{cases} s & if\ 0 < s < 1 \\ 1 & if\ s > 1 \end{cases} \tag{1.5}$$

Remark 1.1 So as not to burden our work, the case "$s = 1$" will be treated in a different way, we will treat this case in another work.

The operator $L = H + V$ is essentially self-adjoint with compact resolvent [2]. The Min-Max theorem [3] shows that the spectrum of L around λ_k can be written in the form $\lambda_k + \mu_k$. Our goal is to study the asymptotic behavior of the fluctuation μ_k when $k \to +\infty$, by expressing it using a transformation of V. Our main result is

Theorem 1.2 (Main Theorem) *μ_k has the asymptotic expansion*

$$\mu_k = \frac{1}{T}\int_{-1}^{1} \frac{V(y\lambda_k^{\frac{1}{2m}})}{(1 - y^{2m})^{1-\frac{1}{2m}}} dy + O(\lambda_k^{\frac{-\delta-1}{2m}}), \quad \forall m \geq 2$$

For $m = 1$, the case of the Harmonic Oscillator, the asymptotic behavior of μ_k is given by

Theorem 1.3 $\mu_k = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} V\left(\sqrt{\lambda_k}\sin t\right)dt + O\left(\lambda_k^{-\delta+\eta}\right)$
where $\eta \in]0, \frac{\delta}{2}[$.

Many authors interested in this kind of problems, especially the case of the Harmonic Oscillator [4,5]. The case $m > 1$ seems to us as not to have been treated yet. Briefly recall the content of [4], the author studies the perturbation $L = A + B$, where

$$A = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2 - 1\right), \qquad B(x) \sim |x|^{-\alpha}\sum_m a_m \cos \omega_m x$$

and he proved that μ_k has the following asymptotic expansion

$$\mu_k \sim k^{-(\frac{\alpha}{2}+\frac{1}{4})}\widetilde{V}(\sqrt{2k}) + \frac{C}{\sqrt{2k}} \quad k \to +\infty$$

$\widetilde{V}$ represents the "Radon Transform" of V. In recent works, we find in [6] a study of

$$D = -\frac{d^2}{dx^2} + x^2 + q(x)$$

where real functions q, q' and $x \to \int_0^x q(s)ds$ are bounded. μ_k has the asymptotic expansion

$$\mu_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} q(\sqrt{\lambda_k} sin\theta)\, \mathrm{d}\theta + O(k^{\frac{-1}{3}}) \quad as\ k \to +\infty \tag{1.6}$$

We notice that, in the case $s < 1$, the expansion (1.6) has the same main part as shown in Theorem 1.3, even though we don't need to suppose that $x \to \int_0^x q(s)ds$ is bounded. In addition, if $s \in]\frac{2}{3}, 1[$, we get a better estimate and the same goes for $s > 1$ because $\eta \in]0, \frac{1}{2}[$. We can also mention Pushnitski [7], who studied the case where $q \in C_0^\infty(\mathbb{R})$. He proved that μ_k admits the next development in series

$$\mu_k = \sum_{j=1}^{+\infty} c_j \lambda_k^{\frac{-j}{2}} \quad \lambda_k \to +\infty \tag{1.7}$$

with some coefficients $c_j \in \mathbb{R}$, in particular $c_1 = \frac{1}{\pi} \int_{-\infty}^{+\infty} q(x)\, \mathrm{d}x$, and $c_2 = 0$.

Remark 1.4 We want to go further in studying the operator $H_{k,l} = -\frac{d^{2k}}{dx^{2k}} + x^{2l}$, where $k, l \in \mathbb{N}^*$, then by giving k the value of "1", we'll reach important results that have a lot off applications in the field of physics, especially the quartic oscillator.

Our main tool is the averaging Method of Weinstein [8,9], whose origins go back to the classical work on celestial mechanics [10]. Note that for $m \in \mathbb{N}^* - \{1\}$, this method cannot be used directly in this case because H, viewed as a Pseudo-differential operator (OPD), doesn't have a periodic flow, but the operator $H^{\frac{1}{m}}$ does have this property, so we start reducing ourselves to a perturbation (1.8) of the operator $H^{\frac{1}{m}}$

$$L_m = H^{\frac{1}{m}} + B, \quad m \in \mathbb{N}^* \tag{1.8}$$

where B is an operator to be determined. We apply the Averaging Method, firstly we replace B in perturbation (1.8) by the average

$$\overline{B} = \frac{1}{T} \int_0^T e^{-itH^{\frac{1}{m}}} B e^{itH^{\frac{1}{m}}} dt \tag{1.9}$$

where T is the period of the flow of $H^{\frac{1}{m}}$, T is given by (1.3) [11]. The main advantage of this method is that $\overline{B}$ is a compact operator and L_m, $\overline{L}_m = H^{\frac{1}{m}} + \overline{B}$ are almost unitary equivalent, that means it exists a unitary operator U such as $UL_mU^{-1} - \overline{L}_m$ is compact. Note that L_m and $\overline{L}_m$ are also with compact resolvent. Using min-max theorem, the parts of their spectrum around $\lambda_k^{\frac{1}{m}}$ are respectively of the form $\lambda_k^{\frac{1}{m}} + \upsilon_k$

and $\lambda_k^{\frac{1}{m}} + \overline{\upsilon}_k$. Then we study $\overline{\upsilon}_k$ by using a functional calculus of the operator H. We begin by establishing the link between υ_k and μ_k.

Proposition 1.5

$$\mu_k = m\lambda_k^{1-\frac{1}{m}}\upsilon_k + O(\lambda_k^{-1}), \quad m \in \mathbb{N}^*$$

Using a functional calculus for H, we obtain

Proposition 1.6

$$m\lambda_k^{1-\frac{1}{m}}\overline{\upsilon}_k = \frac{1}{T}\int_{-1}^{1} \frac{V(\lambda_k^{\frac{1}{2m}} y)}{(1-y^{2m})^{1-\frac{1}{2m}}}dy + O(\lambda_k^{\frac{-\delta-1}{2m}})$$

The following proposition gives the relation between υ_k and $\overline{\upsilon}_k$

Proposition 1.7

$$\upsilon_k = \overline{\upsilon}_k + O(\lambda_k^{\frac{-\delta+\eta}{m}+\frac{2}{m}-2})$$

where $\eta \in]0, \min(2, m-1+\frac{\delta}{2})[$

Remark 1.8 Note that for $m = 1$, H has a periodic flow of period π, so we can directly apply the Averaging Method to H.

This paper is organized as follows. The next section contains auxiliary fact concerning the proprieties of Weyl pseudo-differential operators which are the main tool in this article. In Sect. 3, we study the operator L_m and show the relation between μ_k and υ_k by proving Proposition 1.5. The Sect. 4 is devoted to the functional calculus for the operator H, and we establish Proposition 1.6. In the last section, we study the relation between the spectrum of L_m and $\overline{L}_m$ and we prove Proposition 1.7. Finally, we justify the asymptotic expansion given by Theorems 1.2 and 1.3

2 Weyl pseudo-differential operator

Let $\rho \in [0, 1]$, $q \in \mathbb{R}$. $\Gamma_\rho^q(\mathbb{R}\times\mathbb{R})$ denote the space symbols associated with the temperate weight function $\mathbb{R}^2 : (x, \xi) \to (1 + x^2 + \xi^2)^{\frac{q}{2}}$ [15]. Precisely the space of function $a \in C^\infty(\mathbb{R}^2)$ satisfies $\forall\alpha, \beta \in \mathbb{N}, \exists\, C_{\alpha,\beta} > 0$

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)\right| \le C_{\alpha,\beta}(1 + x^2 + \xi^2)^{\frac{q-\rho(\alpha+\beta)}{2}} \tag{2.1}$$

We will use the standard Weyl quantization of the symbols. To be precise, if $a \in \Gamma_\rho^q$, then for $u \in S(\mathbb{R})$ the operator associated is defined by :

$$op^w(a)u(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}\times\mathbb{R}} e^{i\langle x-y,\xi\rangle} a\left(\tfrac{x+y}{2}, \xi\right) u(y)dyd\xi \tag{2.2}$$

Let us now introduce the notion of asymptotic expansion.

Definition 2.1 Let $a_j \in \Gamma_\rho^{q_j} (j \in \mathbb{N}^*)$, we suppose that q_j is a decreasing sequence tending towards $-\infty$. We say that $a \in C^\infty(\mathbb{R} \times \mathbb{R})$ has an asymptotic expansion and we write

$$a \sim \sum_{j=1}^{\infty} a_j$$

if

$$a - \sum_{j=1}^{r-1} a_j \in \Gamma_\rho^{q_r} \quad \forall r \geq 2$$

We use the notation G_ρ^q for the set of operators $op^w(a)$ if $a \in \Gamma_\rho^q$. In order to prove our main results, we shall recall some well-known results [12–14].

Theorem 2.2 (Calderon-Vailloncourt theorem) *If $a \in \Gamma_0^0$ then the operator $op^w(a)$ is bounded.*

Proposition 2.3 (Compactness) *If $a \in \Gamma_\rho^q$ and $q < 0$, $\rho \in [0, 1]$, then the operator $op^w(a)$ is compact.*

We will need the following proposition for the composition of pseudo-differential operators.

Proposition 2.4 *Let $A \in G_\rho^p$, $B \in G_\rho^q$, $\rho \in]0, 1]$, p and $q \in \mathbb{R}$. Then the operator $AB \in G_\rho^{p+q}$. Its Weyl symbol admits the following asymptotic behavior*

$$c \sim \sum_{j \geq 0} c_j$$

In particular

$$c(x, \xi) - a(x, \xi).b(x, \xi) \in \Gamma_\rho^{p+q-2\rho}$$

where

$$c_j = \frac{1}{2^j} \sum_{\alpha+\beta=j} \frac{(-1)^{|\beta|}}{\alpha!\beta!} (\partial_\xi^\alpha \partial_x^\beta a)(\partial_x^\alpha \partial_\xi^\beta b) \tag{2.3}$$

a and b are respectively the Weyl symbol of A and B

In the next proposition we are giving an extension where the case "$\rho = 0$", we have the following result

Proposition 2.5 *If $A \in G_1^m$ and $(B_i)_{i \in \{1,\ldots p\}}$ is the set of operators such as $B_i \in G_0^{m_i}$ then:*

(i) *The operator $AB_1 \in G_0^{m+m_1}$, its Weyl symbol is giving by (2.3) where $c_j \in \Gamma_0^{m+m_1-j}$*
(ii) *The commutator $[A, B_1] \in G_0^{m+m_1-1}$*

(iii) *If A is elliptic and $m > 0$, then $B_1 \dots B_P A^{-\frac{m_1+\cdots+m_p}{m}}$ is bounded.*

Proposition 2.6 (Functional calculus) *Let A be an elliptic operator included in G_1^m, its Weyl symbol admits the development $a \sim \sum_{j \geq 0} a_j$. Then for any real number q we have $A^q \in G_1^{mq}$. Moreover, its weyl symbol admits the following asymptotic behavior*

$$\sigma_{A^q} \sim \sum_{j=0}^{+\infty} \sigma_{A^q,j}, \quad \sigma_{A^q,j} \in \Gamma_1^{mq-j}$$

where $\sigma_{A^q,0} = {a_0}^q$, $\sigma_{A^q,1} = q a_1 . a_0^{q-1}$.

3 Reduction to a perturbation of $H^{\frac{1}{m}}$

If we translate H by a strictly positive constant, we can always assume that L is positive and $\left\| H^{-1} V \right\| < 1$. We can reduce ourselves to a perturbation of $H^{\frac{1}{m}}$ by writing

$$(H + V)^{\frac{1}{m}} = H^{\frac{1}{m}} + W$$

consequently

$$W = B + H^{\frac{1}{m}} (H^{-1} V)^2 \sum_{k=0}^{+\infty} \alpha_{k+2} (H^{-1} V)^k$$

where

$$B = \frac{1}{m} H^{-(1-\frac{1}{m})} V, \quad \alpha_k = \frac{\frac{1}{m}(\frac{1}{m} - 1) \dots (\frac{1}{m} - k + 1)}{k!} \tag{3.1}$$

We can write

$$L^{\frac{1}{m}} - L_m = H^{\frac{1}{m}} (H^{-1} V)^2 \sum_{k=0}^{+\infty} \alpha_{k+2} (H^{-1} V)^k \tag{3.2}$$

Since $\left\| H^{-1} V \right\| < 1$, the operator $\sum_{k=0}^{+\infty} \alpha_{k+2} (H^{-1} V)^k$ is bounded in $L^2(\mathbb{R})$. Using the fact that, $H^{-1} \in G_1^{-2m}$, $V \in G_0^0$ and (iii) Proposition 2.5, we obtain that the operator $(L^{\frac{1}{m}} - L_m) H^{2-\frac{1}{m}}$ is bounded. We deduce that there exists a constant $c > 0$ such that

$$- c H^{-2+\frac{1}{m}} \leq L^{\frac{1}{m}} - L_m \leq c H^{-2+\frac{1}{m}} \tag{3.3}$$

According to the min-max theorem, we get

$$(\lambda_k + \mu_k)^{\frac{1}{m}} = \lambda_k^{\frac{1}{m}} + \upsilon_k + O\left(\lambda_k^{-2+\frac{1}{m}}\right) \tag{3.4}$$

Using the fact that $\{\mu_k\}$ is bounded and the Taylor's formula for the function $t \to \left(1 + \frac{\mu_k}{t}\right)^{\frac{1}{m}}$, we obtain the estimate

$$\mu_k = m\lambda_k^{1-\frac{1}{m}} \upsilon_k + O(\lambda_k^{-1}) \tag{3.5}$$

This completes the proof of Proposition 1.5.

4 Functional calculus for the operator *H* and the asymptotic behavior of $\overline{\upsilon_k}$

Recall that $\overline{L}_m$ is obtained by replacing B in L_m by $\overline{B}$. Putting

$$\overline{V} = \frac{1}{T}\int_0^T W(t)dt, \quad W(t) = e^{-itH^{\frac{1}{m}}} V e^{itH^{\frac{1}{m}}} \tag{4.1}$$

and From (3.1)

$$\overline{B} = \frac{1}{m} H^{-\left(1-\frac{1}{m}\right)} \overline{V} \tag{4.2}$$

We have the following proposition

Proposition 4.1 *$\overline{V} \in G_0^{-\delta}$, and its Weyl symbol checks*

$$\sigma_{\overline{V}} - \sigma_{\overline{V},0} \in \Gamma_0^{-\delta-1}$$

where

$$\sigma_{\overline{V},0}(x,\xi) = \frac{1}{T}\int_{-1}^{1} \frac{V(y.\sigma_H^{\frac{1}{2m}})}{(1-y^{2m})^{1-\frac{1}{2m}}} dy$$

and σ_H is the Weyl symbol of H.

Proof First we will study the Weyl symbol of the operator $W(t)$. In order to apply the Egorov's theorem (Theorem IV-10 [15]) checks his assumptions. Using Proposition 2.6, the operator $H^{\frac{1}{m}} \in G_1^2$ and $\sigma_{H^{\frac{1}{m}}}$ admits the following development

$$\sigma_{H^{\frac{1}{m}}} \sim \sum_{j=0}^{+\infty} \sigma_{H^{\frac{1}{m}},j} \quad \sigma_{H^{\frac{1}{m}},j} \in \Gamma_1^{2-j} \tag{4.3}$$

where $\sigma_{H^{\frac{1}{m}},0} = (\sigma_H)^{\frac{1}{m}}, \sigma_{H^{\frac{1}{m}},1} = 0$
An elementary calculation shows that

$$\partial_x^{\alpha}\partial_{\xi}^{\beta}\sigma_{H^{\frac{1}{m}},j} \in L^{\infty}(\mathbb{R}\times\mathbb{R}), \quad \alpha, \beta, j \in \mathbb{N}, \ \alpha+\beta+j > 2$$

and

$$\partial_x^\alpha \partial_\xi^\beta \varphi(t) \in L^\infty(\mathbb{R}\times\mathbb{R}), \quad \alpha+\beta \geq 1$$

uniformly with respect to t, where $\varphi(t)$ denotes the Hamiltonian flow of $\sigma_H^{\frac{1}{m}}$. We recall that $\varphi(t) = (x(t), \xi(t))$ is a solution of the system

$$(S) \quad \begin{cases} \frac{dx(t)}{dt} = \frac{\partial \sigma_H^{\frac{1}{m}}}{\partial \xi} = 2E^{\frac{1}{m}-1}\xi^{2m-1}(t) \\ \frac{d\xi(t)}{dt} = \frac{-\partial \sigma_H^{\frac{1}{m}}}{\partial x} = -2E^{\frac{1}{m}-1}x^{2m-1}(t) \\ x(0) = x, \quad \xi(0) = \xi \\ x^{2m}(t) + \xi^{2m}(t) = x^{2m} + \xi^{2m} = E \qquad (*) \end{cases}$$

$(*)$ is the first integral of this system

Now we apply the Egorov's theorem, this yields, for $t \in \mathbb{R}$, $W(t)$ is an (OPD), its Weyl symbol admits the following development $\sigma_{W(t)} \sim \sum_{j\geq 0} \sigma_{W(t),j}$

$$\sigma_{W(t),j} = \int_0^t i^{-1} \sum_{\substack{\alpha+\beta+l+k=j+1 \\ 0\leq l\leq j-1}} C_{\alpha,\beta}(\partial_\xi^\alpha \partial_x^\beta \sigma_{H^{1/m},k})(\partial_\xi^\beta \partial_x^\alpha \sigma_{W,l}(\tau)|\varphi^{t-\tau} d\tau \tag{4.4}$$

where

$$C_{\alpha,\beta} = (1-(-1)^{\alpha+\beta})\Gamma(\alpha,\beta)$$

in particular

$$\sigma_{W(t),0}(x,\xi) = Vox(t), \qquad \sigma_{W(t),1}(x,\xi) = 0 \tag{4.5}$$

We start by determining the class of $\sigma_{\overline{V},0} = \frac{1}{T}\int_0^T V(x(t))dt$. We can suppose that $x(0) > 0$, $\frac{dx(0)}{dt} > 0$ as initial conditions, other cases are treated in the same way, for now we must study the properties of the function $x(t)$ on $[0, T]$. From the system (S) we get

$$dt = \pm \frac{dx}{2E^{\frac{1}{m}-1}(E-x^{2m})^{1-\frac{1}{2m}}} \tag{4.6}$$

By combining the fact that $x(t)$ is a smooth periodic function of period T with (4.6), we can ensure that the function $x(t)$ reaches its maximum in t_0, $x(t_0) = E^{\frac{1}{2m}}$ and its minimum on t_1, $x(t_1) = -E^{\frac{1}{2m}}$.

For now we have

$$\sigma_{\bar{V},0} = \frac{1}{T}\left[\int_0^{t_0} V(x(t))dt + \int_{t_0}^{t_1} V(x(t))dt + \int_{t_1}^{T} V(x(t))dt\right]$$

We make the change of variable $x(t)=u$, obviously $x(t)$ is increasing on $[0,t_0]$, we get

$$\int_0^{t_0} V(x(t))dt = \frac{E^{1-\frac{1}{m}}}{2}\int_x^{E^{\frac{1}{2m}}} \frac{V(u)}{(E-u^{2m})^{1-\frac{1}{2m}}}du \tag{4.7}$$

After the same calculation on $[t_0,t_1]$ and $[t_1,T]$ we obtain

$$\sigma_{\overline{V},0} = \frac{E^{1-\frac{1}{m}}}{T}\int_{-E^{\frac{1}{2m}}}^{E^{\frac{1}{2m}}} \frac{V(u)}{(E-u^{2m})^{1-\frac{1}{2m}}}du \tag{4.8}$$

Now we apply the change of variable $y = uE^{\frac{-1}{2m}}$, we have

$$\sigma_{\overline{V},0} = \frac{1}{T}\int_{-1}^{1} \frac{V(y\sigma_H^{\frac{1}{2m}})}{(1-y^{2m})^{1-\frac{1}{2m}}}dy \tag{4.9}$$

Let's now determine the class to which $\sigma_{\overline{V},0}$ belongs. Using (1.4) we get

$$\left|\sigma_{\overline{V},0}\right| \le c\int_0^1 \frac{1}{(1+y\sigma_H^{\frac{1}{2m}})^s(1-y^{2m})^{1-\frac{1}{2m}}}dy \tag{4.10}$$

We have 2 cases
1st **case** $0<s<1$

$$\left|\sigma_{\bar{V},0}\right| \le c(1+\sigma_H^{\frac{1}{2m}})^{-s}\int_0^1 \frac{1}{y^s(1-y^{2m})^{1-\frac{1}{2m}}}dy \tag{4.11}$$

$$\le c(1+x^2+\xi^2)^{\frac{-s}{2}} \tag{4.12}$$

2nd **case** $1<s$
We set $\sigma_H^{\frac{1}{2m}}=a$ and we split the integral into two parts,

$$\sigma_{\bar{V},0} \le c\int_0^{\frac{\sqrt{2}}{2}} \left(\frac{1}{1+a^2y^2}\right)^{\frac{s}{2}} \frac{1}{(1-y^{2m})^{1-\frac{1}{2m}}}dy + c\int_{\frac{\sqrt{2}}{2}}^{1} \left(\frac{1}{1+a^2y^2}\right)^{\frac{s}{2}} \frac{1}{(1-y^{2m})^{1-\frac{1}{2m}}}dy \tag{4.13}$$

Since $s > 1$ we have

$$\int_0^{\frac{\sqrt{2}}{2}} \left(\frac{1}{1+a^2y^2}\right)^{\frac{s}{2}} \frac{1}{(1-y^{2m})^{1-\frac{1}{2m}}} dy \le c \int_0^{\frac{\sqrt{2}}{2}} \frac{1}{(1+a^2y^2)^{\frac{s}{2}}} dy$$

After applying change both of variables $y = \frac{u}{1+u}$, and $v = u\sqrt{1+a^2}$, we get

$$\int_0^{\frac{\sqrt{2}}{2}} \left(\frac{1}{1+a^2y^2}\right)^{\frac{s}{2}} \frac{1}{(1-y^{2m})^{1-\frac{1}{2m}}} dy \le \frac{c}{1+a}$$

On the other side we have

$$\begin{aligned}\int_{\frac{\sqrt{2}}{2}}^{1} \left(\frac{1}{1+a^2y^2}\right)^{\frac{s}{2}} \frac{1}{(1-y^{2m})^{1-\frac{1}{2m}}} dy &\le \frac{c}{(1+a)^s} \int_{\frac{\sqrt{2}}{2}}^{1} \frac{1}{(1-y^{2m})^{1-\frac{1}{2m}}} dy \\ &\le \frac{c}{(1+a)^s}\end{aligned}$$

We obtain the following estimate

$$\sigma_{\overline{V},0} \le \frac{c}{\left(1+\sigma_H^{\frac{1}{2m}}\right)} \le c(1+x^2+\xi^2)^{\frac{-1}{2}}. \tag{4.14}$$

The same estimates hold for $\partial_x^\alpha \partial_\xi^\beta \sigma_{\overline{V},0}(x,\xi)$, $\alpha, \beta \in \mathbb{N}$.

From (4.12) and (4.14) we have

$$\sigma_{\bar{V},0} \in \Gamma_0^{-\delta}, \quad \text{for } s \in \mathbb{R}_+^* - \{1\}$$

Combining (1.4), (4.4) and (4.5) we deduce that there exist $c > 0$ such that: for all $t \in [0, T]$

$$|\sigma_{W(t)} - Vox(t)| \le c\,(1+x^2+\xi^2)^{\frac{-1}{2}}(1+x^2(t))^{\frac{-s}{2}} \tag{4.15}$$

by integrating (4.15) along the interval $[0, T]$ and following the same previous calculation we have

$$|\sigma_{\overline{V}} - \frac{1}{T}\int_0^T Vox(t)\,dt| \le c(1+x^2+\xi^2)^{\frac{-1-\delta}{2}} \tag{4.16}$$

The same estimates hold for $\partial_x^\alpha \partial_\xi^\beta \sigma_{\overline{V}}(x,\xi)$, $\alpha, \beta \in \mathbb{N}$. Finally we conclude

$$\sigma_{\overline{V}} - \frac{1}{T}\int_0^T Vox(t)\,dt \in \Gamma_0^{-\delta-1} \tag{4.17}$$

□

In the following we will use a functional calculus for the operator H, this allows us to give the asymptotic behavior of $\overline{v}_k$. The functional calculus on (OPD) was studied in the case where the functions are in the Hörmander class S_1^r ($r \in \mathbb{R}$) see [15,16]. In our work we are dealing with the case of the operator H plus a function in the class $S^r_{1-\frac{1}{2m}}$. More precisely, the set of functions $f \in C^\infty(\mathbb{R})$ such that for all $k \in \mathbb{N}$, there exist $C_k \geq 0$ such that

$$\left| f^{(k)}(x) \right| \leq C_k (1 + |x|)^{r-(1-\frac{1}{2m})k} \tag{4.18}$$

We recall that the main symbol of $\overline{V}$ is written

$$\sigma_{\overline{V},0} = f(\sigma_H) \tag{4.19}$$

where

$$f(x) = \frac{1}{T} \int_{-1}^{1} \frac{V(yx^{\frac{1}{2m}})}{(1-y^{2m})^{1-\frac{1}{2m}}} dy$$

a direct calculation shows that $f \in S^{-\frac{\delta}{2m}}_{1-\frac{1}{2m}}$ for $s \in \mathbb{R}_+^* - \{1\}$.

The operator $f(H)$ is defined by a functional calculus of self-adjoint operators, then the spectrum of $f(H)$ is the sequence $\{f(\lambda_k)\}_k$. We have the following proposition

Proposition 4.2 *$f(H)$ is an OPD included in $G_0^{-\delta}$ and its Weyl symbol admits the following development*

$$\sigma_{f(H)} \sim \sum_{j \geq 0} \sigma_{f(H),2j}$$

$$\sigma_{f(H),2j} = \sum_{k=2}^{3j} \frac{d_{j,k}}{k!} f^{(k)}(\sigma_H) \quad \forall j \geq 1$$

where $d_{j,k} \in \Gamma_1^{2mk-4j}$ and $\sigma_{f(H),2j} \in \Gamma_0^{-\delta-j}$,
in particular

$$\sigma_{f(H),0} = f(\sigma_H) = \sigma_{\overline{V},0}, \quad \sigma_{f(H),1} = 0$$

Proof For studying $f(H)$ We follow the same strategy in [16] , we will use the Mellin transformation, this later consist of

(1) To Study the operator $(H - \lambda)^{-1}$
(2) To study the operator H^{-s} using its Cauchy's integral formula

$$H^{-s} = \frac{1}{2\pi i} \int_{\Delta} \lambda^{-s} (H - \lambda)^{-1} d\lambda$$

(3) Studying $f(H)$ using the representation formula

$$f(H) = \frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty} M[f](s)H^{-s}ds$$

where $r < 0$ and $\rho < -r$
we only change the construction of the $(H-\lambda)^{-1}$-parametrix. We prove by induction that the $(H-\lambda)^{-1}$ is an OPD and its Weyl symbol admits the development $b_\lambda \sim \sum b_{j,\lambda}$ where

$$\begin{cases} b_{0,\lambda} = (\sigma_H - \lambda)^{-1} \\ b_{2j+1,\lambda} = 0 \\ b_{2j,\lambda} = \sum\limits_{k=2}^{3j}(-1)^k d_{j,k}.b_{0,\lambda}^{k+1}, \quad d_{j,k} \in \Gamma_1^{2mk-4j} \end{cases}$$

□

Proof of Proposition 1.6 using Proposition 4.2 and (4.19) formula we conclude

$$\sigma_{f(H)} - \sigma_{\overline{V},0} \in \Gamma_0^{-\delta-1} \tag{4.20}$$

by combining (4.20) and Proposition 4.1 we get

$$\sigma_{\overline{V}} - \sigma_{f(H)} \in \Gamma_0^{-\delta-1} \tag{4.21}$$

From (4.21) and Proposition 2.5 (iii) we deduce that the operator $(\overline{V} - f(H))H^{\frac{\delta+1}{2m}}$ is bounded. We can write

$$\frac{1}{m}H^{\frac{1}{m}-1}(\overline{V} - f(H)) = \left[\bar{L}_m - (H^{\frac{1}{m}} + \tfrac{1}{m}H^{\frac{1}{m}-1}f(H))\right]$$

Finally we get that the operator $[\bar{L}_m - (H^{\frac{1}{m}} + \frac{1}{m}H^{\frac{1}{m}-1}f(H))]H^{1+\frac{\delta}{2m}-\frac{1}{2m}}$ is also bounded. According to min-max theorem we have

$$\overline{\upsilon}_k = \frac{1}{m}\lambda_k^{\frac{1}{m}-1}f(\lambda_k) + O\left(\lambda_k^{-\left(1+\frac{\delta}{2m}-\frac{1}{2m}\right)}\right) \tag{4.22}$$

Then

$$m\lambda_k^{1-\frac{1}{m}}\overline{\upsilon_k} = \frac{1}{T}\int_{-1}^{1}\frac{V\left(\lambda_k^{\frac{1}{2m}}y\right)}{(1-y^{2m})^{1-\frac{1}{2m}}}dy + O(\lambda_k^{\frac{-\delta-1}{2m}}) \tag{4.23}$$

□

Remark 4.3 We note that from (1.4) we have the following estimate

$$\lambda_k^{\frac{1}{m}-1}f(\lambda_k) = O(\lambda_k^{\frac{-\delta}{2m}+\frac{1}{m}-1})$$

5 The relation between the spectrum of L_m and $\overline{L}_m$

Proof of Proposition 1.7 To establish Proposition 1.7, we need to prove the next result

Proposition 5.1 *There exists a skew-symmetric operator $Q \in G_0^{-(2m-2+\delta)}$ such as the operator $(e^Q L_m e^{-Q} - \overline{L}_m) H^{\frac{\delta-\eta}{m}+2-\frac{2}{m}}$ is bounded, where $\eta \in]0, 2[$.*

Proof The Q operator is built using the Q_1 and Q_2 operators as follows

$$Q = Q_1 + Q_2 \tag{5.1}$$

where

$$Q_1 = \frac{i}{mT} H^{\frac{1}{m}-1} \int_0^T (T-t) W(t) dt$$

and

$$Q_2 = -\frac{1}{2T} \int_0^T (T-t) \int_0^t \left[\frac{1}{m} H^{\frac{1}{m}-1} W(t), \frac{1}{m} H^{\frac{1}{m}-1} W(r) \right] dr dt$$

Before starting the proof we could make sure that

$$\left[Q_1, H^{\frac{1}{m}} \right] = \frac{1}{m} H^{\frac{1}{m}-1} (\overline{V} - V) \quad \left[Q_2, H^{\frac{1}{m}} \right] = -\frac{1}{2} \left[Q_1, \frac{1}{m} H^{\frac{1}{m}-1} V \right] - \overline{\overline{V}} \tag{5.2}$$

where $\overline{\overline{V}} = \frac{1}{2Ti} \int_0^T \int_0^t \left[\frac{1}{m} H^{\frac{1}{m}-1} W(t), \frac{1}{m} H^{\frac{1}{m}-1} W(r) \right] dr dt$
We notice $Ad Q . L_m = [Q, L_m]$. The differential equation

$$\begin{cases} \frac{dX}{dt} = [Q, X] \\ X(0) = L_m \end{cases} \tag{5.3}$$

has a unique solution

$$X(t) = e^{tADQ} . L_m = e^{tQ} L_m e^{-tQ}$$

From (5.2) and (5.3) we get

$$\begin{aligned} e^Q L_m e^{-Q} - \overline{L}_m = & \left\{ -\overline{\overline{V}} + \frac{1}{2m} \left[Q_2, H^{\frac{1}{m}-1} V \right] \right\} \\ & + \frac{1}{2m} \left\{ \left[Q, H^{\frac{1}{m}-1} \overline{V} \right] + \frac{1}{2} \left[Q, \left[Q_1, H^{\frac{1}{m}-1} V \right] \right] \right\} \\ & + \frac{1}{2} \left\{ \left[Q, \left[Q_2, \frac{1}{m} H^{\frac{1}{m}-1} V \right] \right] - \left[Q, \overline{\overline{V}} \right] \right\} \\ & + \sum_{n \geq 2} \frac{(Ad Q)^n}{(n+1)!} \left[Q, H^{\frac{1}{m}} \right] + \sum_{n \geq 2} \frac{(Ad Q)^n}{(n+1)!} \left[Q, \frac{1}{m} H^{\frac{1}{m}-1} V \right] \end{aligned}$$

To complete the proof of proposition we need the following lemma

Lemma 5.2

$$Q_1 \in G_0^{-(2m+2-\delta)} \quad and \quad \overline{\overline{V}}, Q_2 \in G_0^{-(4m-4+2\delta-2\eta)}$$

where $\eta \in]0, 2[$

Proof Using Proposition 2.5 (i) we can prove in analog way of Proposition 4.1 that $Q_1 \in G_0^{-(2m+2-\delta)}$. Now let's determine the class of $\overline{\overline{V}}$, we can write

$$\overline{\overline{V}} = \frac{1}{2Ti} \int_0^T [S(t), B(t)]dt$$

where

$$S(t) = \frac{1}{m} H^{\frac{1}{m}-1} W(t), \qquad B(t) = \int_0^t S(r)dr$$

let's start by clarifying the class of the operator $\int_0^T S(t)B(t)dt$. For now we are interested in the operator $S(t)B(t)$, its Weyl symbol c_t is given in [15] by

$$c_t(x, \xi) = \frac{1}{\pi^2} \int e^{-2i(r\rho-\omega\tau)} \sigma_{S(t)}(x+\omega, \xi+\rho)\sigma_{B(t)}(x+r, \xi+\tau)d\rho d\omega d\tau dr. \quad (5.4)$$

We split the oscillator integral c_t into two parts $c_t^{(1)}$ and $c_t^{(2)}$, then we use the cutoff functions

$$\omega_{1,\varepsilon}(x, \xi, \omega, \tau, r, \rho) = \chi \left[\frac{\omega^2 + \rho^2 + r^2 + \tau^2}{\varepsilon(1 + x^2 + \xi^2)^{\frac{\eta}{2}}} \right] \quad \text{and} \quad \omega_{2,\varepsilon} = 1 - \omega_{1,\varepsilon}$$

where $\chi \in C_0^\infty(\mathbb{R})$, $\chi \equiv 1$ in $[-1, 1]$, $\chi \equiv 0$ in $\mathbb{R}\backslash]-2, 2[$, $R = \omega^2 + \rho^2 + r^2 + \tau^2$, $\varepsilon > 0$ and $\eta > 0$. Let's consider

$$d_j(x, \xi, \omega, \tau, r, \rho) = \omega_{j,\varepsilon}(x, \xi, \omega, \tau, r, \rho)\sigma_{S(t)}(x + \omega, \rho + \xi)\sigma_{B(t)}(x + r, \rho + \xi) \quad (5.5)$$

$c_t^{(1)}$ (resp $c_t^{(2)}$) the integral obtained in (5.4) by replacing the amplitude by d_1 (resp d_2)

Study of $c_t^{(2)}$

On the support of d_2 we have $R \geq \varepsilon(1 + x^2 + \xi^2)^{\frac{\eta}{2}}$. We make an integration by parts using the operator

$$M = \frac{1}{2iR}(-\rho\partial_r - r\partial_\rho + \tau\partial_\omega + \omega\partial_\tau)$$

We have for all $k \in \mathbb{N}$

$$c_t^{(2)} = \frac{1}{\pi^2} \int e^{-2i(r\rho-\omega\tau)} ({}^t M)^k d_2 \, d\rho \, d\omega \, d\tau \, dr$$

then we obtain for all $k > 0$

$$\left|c_t^{(2)}\right| \leq C_k(1 + x^2 + \xi^2)^{\frac{-\eta k}{4}}$$

Uniformly with respect to $t \in [0, T]$

Study of $c_t^{(1)}$

On the support of d_1 we have

$$c_t^{(1)}(x,\xi) = \frac{1}{\pi^2}\int_{R\leq 2\varepsilon(1+x^2+\xi^2)^{\frac{\eta}{2}}} e^{-2i(r\rho-\omega\tau)}\sigma_{S(t)}(x+\omega,\xi+\rho) \times \sigma_{B(t)}(x+r,\xi+\tau)\omega_{1,\varepsilon}d\rho d\omega d\tau dr \tag{5.6}$$

$$\int_0^T \left|c_t^{(1)}\right|dt \leq c\int_{R\leq 2\varepsilon(1+x^2+\xi^2)^{\frac{\eta}{2}}} d\rho d\omega d\tau dr \int_0^T \left|\sigma_{S(t)}(x+\omega,\xi+\rho)\right|dt \times \int_0^T \left|\sigma_{S(t)}(x+r,\xi+\tau)\right|dt \tag{5.7}$$

By using (4.4) and (1.4) we can deduce for all $\alpha, \beta \in \mathbb{N}$

$$\left|\partial_x^\alpha\partial_\xi^\beta\sigma_{S(t)}(x,\xi)\right| \leq c_{\alpha,\beta}(1+x^2+\xi^2)^{-(m-1)}(1+x^2(t))^{\frac{-s}{2}}$$

by integrating along the interval $[0, T]$ and following the same reasoning in Proposition 4.1 we get

$$\left|\partial_x^\alpha\partial_\xi^\beta\int_0^T \sigma_{S(t)}(x,\xi)dt\right| \leq c_{\alpha,\beta}(1+x^2+\xi^2)^{\frac{-(2m-2+\delta)}{2}} \tag{5.8}$$

From (5.7) and (5.8) we have

$$\int_0^T \left|c_t^{(1)}\right|dt \leq c\int_{R\leq 2\varepsilon(1+x^2+\xi^2)^{\frac{\eta}{2}}} (1+(x+\omega)^2+(\xi+\rho)^2)^{\frac{-(2m-2+\delta)}{2}} \times (1+(x+r)^2+(\xi+\tau)^2)^{\frac{-(2m-2+\delta)}{2}} d\rho d\omega d\tau dr$$

On the support of d_1, for ε small enough and since $\eta \in]0, 2[$, there are positive constants c, c', C, C' such that

$$\begin{cases} c(1+x^2+\xi^2)^{\frac{1}{2}} \leq (1+(x+\omega)^2+(\rho+\xi)^2)^{\frac{1}{2}} \leq C(1+x^2+\xi^2)^{\frac{1}{2}} \\ c'(1+x^2+\xi^2)^{\frac{1}{2}} \leq (1+(x+r)^2+(\tau+\xi)^2)^{\frac{1}{2}} \leq C'(1+x^2+\xi^2)^{\frac{1}{2}} \end{cases}$$

It follows that

$$\int_0^T c_t^{(1)}dt \leq C(1+x^2+\xi^2)^{-(2m-2+\delta)}\int_{R\leq 2\varepsilon(1+x^2+\xi^2)^{\frac{\eta}{2}}} d\rho d\omega d\tau dr \tag{5.9}$$

Finally

$$\int_0^T c_t^{(1)} dt \leq c(1 + x^2 + \xi^2)^{-(2m-2+\delta)+\eta} \tag{5.10}$$

At the end by denoting σ the Weyl symbol of the operator $\displaystyle\int_0^T S(t)B(t)dt$, we have

$$\begin{aligned}
|\sigma| &\leq \int_0^T \left|c_t^{(1)}\right| dt + \int_0^T \left|c_t^{(2)}\right| dt \\
&\leq C\left[(1 + x^2 + \xi^2)^{\frac{-\eta k}{4}} + (1 + x^2 + \xi^2)^{-(2m-2+\delta-\eta)}\right] \\
&\leq C(1 + x^2 + \xi^2)^{-(2m-2+\delta-\eta)}
\end{aligned}$$

We obtain the same estimates for $\partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi)$, we prove by the same way that $Q_2 \in G_0^{-(4m-4+\delta-2\eta)}$ □

We return to the proof of Proposition 5.1, since $V \in G_0^0, \overline{V} \in G_0^{-\delta}$, $Q_1, Q \in G_0^{-(2m-2+\delta)}$ and $\overline{\overline{V}}$, $Q_2 \in G_0^{-(4m-4+2\delta-2\eta)}$, by using Proposition 2.5 we have

$$\begin{cases}
\left\| \left\{ -\overline{\overline{V}} + \frac{1}{2m}\left[Q_2, H^{\frac{1}{m}-1}V\right]\right\} H^{2-\frac{2}{m}+\frac{\delta-\eta}{m}} \right\| \leq C \\
\left\| \left\{ \left[Q, H^{\frac{1}{m}-1}\overline{V}\right] + \frac{1}{2}\left[Q, \left[Q_1, H^{\frac{1}{m}-1}V\right]\right]\right\} H^{2-\frac{2}{m}+\frac{\delta}{m}} \right\| \leq C \\
\left\| \left\{ \frac{1}{2}\left[Q, \left[Q_2, H^{\frac{1}{m}-1}V\right]\right] - \left[Q, \overline{\overline{V}}\right]\right\} H^{4-\frac{4}{m}+\frac{2\delta-2\eta}{m}} \right\| \leq C \\
\left\| (ADQ)^n \left[Q, H^{\frac{1}{m}}\right] H^{2-\frac{2}{m}+\frac{\delta}{m}} \right\| \leq C\|Q\|^{n-2} \quad (n \geq 2) \\
\left\| (ADQ)^n \left[Q, H^{\frac{1}{m}-1}V\right] H^{2-\frac{2}{m}+\frac{\delta}{m}} \right\| \leq C\|Q\|^{n-2}
\end{cases}$$

From what precedes we deduce that $(e^Q L_m e^{-Q} - \overline{L}_m)H^{\frac{\delta-\eta}{m}+2-\frac{2}{m}}$ is bounded. □

Come back to the proof of Proposition 1.7. We deduce from Proposition 5.1 that there exists a constant $c > 0$ such that

$$-cH^{\frac{-\delta+\eta}{m}+\frac{2}{m}-2} \leq e^Q L_m e^{-Q} - \overline{L}_m \leq cH^{\frac{-\delta+\eta}{m}+\frac{2}{m}-2}$$

According to the min-max theorem

$$\upsilon_k = \overline{\upsilon}_k + \mathrm{O}(\lambda_k^{\frac{-\delta+\eta}{m}+\frac{2}{m}-2}) \tag{5.11}$$

To have a good estimate, let us specify the best choice of η. Combining Remark 4.3, (5.11) and Proposition 5.1, we choose

$$\eta \in]0, \min(2, m - 1 + \tfrac{\delta}{2})[\tag{5.12}$$

□

Now we prove Theorems 1.2 and 1.3. It is enough to combine Propositions 1.5, 1.6 and 1.7 we deduce

$$\mu_k = \frac{1}{T}\int_{-1}^{1} \frac{V(y\lambda_k^{\frac{1}{2m}})}{(1-y^{2m})^{1-\frac{1}{2m}}}dy + \mathrm{O}(\lambda_k^{\frac{-\delta-1}{2m}}), \quad \forall m \geq 2$$

and for $m = 1$ (harmonic oscillator case)

$$\mu_k = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} V\left(\sqrt{\lambda_k}\sin t\right)dt + O(\lambda_k^{-\delta+\eta}),$$

where $\eta \in]0, \frac{\delta}{2}[$.

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