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On a Riesz basis of exponentials related to a family of analytic operators and application

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Abstract
In this paper, we are interested by the perturbed operator

$$T(\varepsilon) := T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \cdots + \varepsilon^k T_k + \cdots$$

where $\varepsilon \in \mathbb{C}$, T_0 is a closed densely defined linear operator on a separable Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(T_0)$ having isolated eigenvalues with multiplicity one whereas $T_1, T_2, \ldots$ are linear operators on $\mathcal{H}$ having the same domain $\mathcal{D} \supset \mathcal{D}(T_0)$ and satisfying a specific growing inequality. The basic idea here is to investigate under sufficient conditions the existence of Riesz bases of exponentials, where the exponents corresponding as a sequence of eigenvalues of $T(\varepsilon)$, can be developed as entire series of ε. An application to a nonself-adjoint problem describing the radiation of a vibrating structure in a light fluid is presented.

Keywords Eigenvalues · Elastic membrane · Families of exponentials · Isolated point · Riesz bases

1 Introduction and main results

The concept of nonharmonic exponentials are originated from the celebrated work of Paley and Wiener [14] where the authors studied the stability the trigonometric system $\{e^{int}\}_{n\in\mathbb{Z}}$ under small perturbations of the integers. Much improvement has been made subsequently by many mathematicians such as Pavlov in [15]. In fact, his famous theorem on the Riesz basis property of exponential family open up many problems as the problem of radiation of a vibrating structure in a light fluid initially motivated by Filippi et al. [8]. This problem has been extensively studied in literature

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[2,5–7,10,11]. Mainly in [2], the authors proved the existence of a sequence of complex numbers $(\varepsilon_n)_n$ such that the family of exponentials associated to the eigenvalues of the operator $(I+\varepsilon_n K)^{-1}\dfrac{d^2}{dx^2}$ forms a Riesz basis in $L^2\big(0,T\big)$, for some $T>0$. Here K is the integral operator with kernel the Hankel function of the first kind and order 0.

It is clear here that the Riesz basis of exponentials given in [2] depends on the sequence $(\varepsilon_n)_n$ and is not related to the exact eigenvalue problem considered in [8].

It is along this line of thoughts that we try to give some supplements to the results developed in [2] in order to give a Riesz basis of exponentials where the exponents coincide with the eigenvalues of the integro-differential operator $(I+\varepsilon K)^{-1}\dfrac{d^2}{dx^2}$, for a fixed ε.

To this interest, we consider the following operator introduced by Sz-Nagy [13]:

$$T(\varepsilon) := T_0+\varepsilon T_1+\varepsilon^2 T_2+\cdots+\varepsilon^k T_k+\cdots,$$

verifying the following hypotheses:

(**H1**) T_0 is closed densely defined linear operator on a separable Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(T_0)\subset\mathcal{H}$.
(H2) The eigenvalues $(\lambda_n)_{n\in\mathbb{N}^*}$ of T_0 are isolated and with multiplicity one.
(H3) The family of exponentials $\{e^{i\lambda_n t}\}_1^\infty$ forms a Riesz basis in $L^2(0,T)$, for some $T>0$.
Let $T_1,T_2,T_3,\ldots$ be linear operators on $\mathcal{H}$ having the same domain $\mathcal{D}$ and satisfying the hypothesis:
(H4) $\mathcal{D}\supset\mathcal{D}(T_0)$ and there exist $a,b,q>0$ and $\beta\in]0,1]$ such that for all $k\geq 1$

$$\|T_k\varphi\|\leq q^{k-1}(a\|\varphi\|+b\|T_0\varphi\|^\beta\|\varphi\|^{1-\beta}) \text{ for all } \varphi\in\mathcal{D}(T_0). \tag{1.1}$$

Before stating our main results, it is interesting to note that in order to prove the stability of many problems and to show the existence of such bases, many authors such as Feki et al. [6], Jeribi [9] and Sz-Nagy [13] studied the asymptotic behavior of the spectrum and developed many approaches on the comportment of the eigenvalues. Among this direction, we recall the following theorems developed in [6].

Theorem 1.1 [6, Theorem 2.1] *Assume that the assumptions* $(H1)$ *and* $(H4)$ *hold. Then for* $|\varepsilon|<\frac{1}{q}$, *the series*

$$\sum_{k\geq 0}\varepsilon^k T_k\varphi$$

converges for all $\varphi\in\mathcal{D}(T_0)$. *If* $T(\varepsilon)\varphi$ *denotes its limit, then* $T(\varepsilon)$ *is a linear operator with domain* $\mathcal{D}(T_0)$ *and for* $|\varepsilon|<\frac{1}{q+\beta b}$, *the operator* $T(\varepsilon)$ *is closed.* □

Let $n\in\mathbb{N}^*$. We denote by λ_n the eigenvalue number n of the operator T_0, $d_n=d(\lambda_n,\sigma(T_0)\setminus\{\lambda_n\})$: the distance between λ_n and $\sigma(T_0)\setminus\{\lambda_n\}$ and $\mathcal{C}_n=\mathcal{C}(\lambda_n,r_n)$: the circle with center λ_n and with radii $r_n = \frac{d_n}{2}$. Since $(T_0 - zI)^{-1}$ is an analytic function of z, $\|(T_0 - zI)^{-1}\|$ is a continuous function of z. So, we denote by:

$$M_n := \max_{z \in \mathcal{C}_n} \|(T_0 - zI)^{-1}\|$$

and

$$N_n := \max_{z \in \mathcal{C}_n} \|T_0(T_0 - zI)^{-1}\| = \max_{z \in \mathcal{C}_n} \|I + z(T_0 - zI)^{-1}\|.$$

Theorem 1.2 [6, Theorem 3.1] *Assume that hypotheses* $(H1)$, $(H2)$ *and* $(H4)$ *hold. Let* φ_n (*respectively* φ_n^*) *be an eigenvector of* T_0 (*respectively* T_0^* : *the adjoint of* T_0) *associated to the eigenvalue* λ_n (*respectively* $\overline{\lambda_n}$) *such that* $\|\varphi_n\| = \|\varphi_n^*\|$ *and* $\varphi_n^*(\varphi_n) = 1$. *Then:*

(i) *For* $|\varepsilon| < \frac{1}{q+\alpha_n+r_n M_n \alpha_n}$, $T(\varepsilon)$ *will have a unique point of the spectrum in the neighborhood of* λ_n, *and this point* $\lambda_n(\varepsilon)$ *will be also with multiplicity one.*

(ii) *For* $|\varepsilon| < \frac{1}{q+\alpha_n+\omega_n^2 r_n M_n \alpha_n}$, $\lambda_n(\varepsilon)$ *and the corresponding eigenvector* $\varphi_n(\varepsilon)$ *can be developed into an entire series of* ε:

$$\lambda_n(\varepsilon) = \lambda_n + \varepsilon\lambda_{n,1} + \varepsilon^2\lambda_{n,2} + \cdots$$
$$\varphi_n(\varepsilon) = \varphi_n + \varepsilon\varphi_{n,1} + \varepsilon^2\varphi_{n,2} + \cdots$$

and we have

$$|\lambda_{n,i}| \leq \omega_n^2 r_n^2 M_n \alpha_n (q + \alpha_n + \omega_n^2 r_n M_n \alpha_n)^{i-1} \quad for\ all \quad i \geq 1$$

and

$$\|\varphi_{n,i}\| \leq \omega_n r_n M_n (q + \alpha_n + \omega_n^2 r_n M_n \alpha_n)^i \ \ for\ all\ \ i \geq 1,$$

where $\omega_n = \|\varphi_n\|$ *and* $\alpha_n := aM_n + bN_n^{\beta} M_n^{1-\beta}$. □

Based on the fact that in application to numerical analysis the basis is truncated, we prove the existence of Riesz bases of exponentials in $L^2(0, T)$ *having the forms* $\{e^{i\lambda_n(\varepsilon)t}\}_1^N \cup \{e^{i\lambda_n(\varepsilon_n)t}\}_{N+1}^{\infty}$ *and* $\{e^{i\lambda_n(\varepsilon)t}\}_1^N \cup \{e^{i\lambda_n t}\}_{N+1}^{\infty}$ *for* $|\varepsilon|$ *enough small,* $N \in \mathbb{N}^*$ *and* $t \in (0, T)$.

More precisely, we prove that:

Theorem 1.3 *Suppose that the hypotheses* $(H1)$–$(H4)$ *are satisfied. Then, there exist a constant* $C_N > 0$ $(N \geq 1)$, *a sequence of complex numbers* $(\varepsilon_n)_{n\in\mathbb{N}^*}$ *and two sequences of eigenvalues* $\{\lambda_n(\varepsilon)\}_{n\in\mathbb{N}^*}$ *and* $\{\lambda_n(\varepsilon_n)\}_{n\in\mathbb{N}^*}$ *having the form*

$$\lambda_n(\varepsilon) = \lambda_n + \varepsilon\lambda_{n,1} + \varepsilon^2\lambda_{n,2} + \cdots$$
$$\lambda_n(\varepsilon_n) = \lambda_n + \varepsilon_n\lambda_{n,1} + \varepsilon_n^2\lambda_{n,2} + \cdots$$

such that for $|\varepsilon| \in]0, C_N[$*, the systems*

(i) $\{e^{i\lambda_n(\varepsilon)t}\}_1^N \cup \{e^{i\lambda_n(\varepsilon_n)t}\}_{N+1}^\infty$
(ii) $\{e^{i\lambda_n(\varepsilon)t}\}_1^N \cup \{e^{i\lambda_n t}\}_{N+1}^\infty$

form Riesz bases in $L^2(0, T)$. □

We point out here that Theorem 1.3 ameliorates Theorem 3.2 stated in [2]. In fact, the first N vectors in the two bases are not depending on a sequence of complex numbers $(\varepsilon_n)_n$. Consequently, in application to numerical analysis after truncating the basis, the exponential families given in Theorem 1.3 are related to the operator $T(\varepsilon)$ for a fixed ε; while, in [2, Theorem 3.2], the obtained Riesz basis is associated to the eigenvalues of a sequence of operators $(T(\varepsilon_n))_{n\in\mathbb{N}^*}$ depending on the sequence $(\varepsilon_n)_{n\in\mathbb{N}^*}$.

In some applications, the verification of the hypothesis $(H3)$ is quite hard. As a tentative approach to grapple with such difficulty, we had the idea to generalize Theorem 1.3 by assuming, instead of $(H3)$, the following assumption:

(**H3**′) The family $\{e^{if(\lambda_n)t}\}_1^\infty$ forms a Riesz basis in $L^2(0, T)$ where $T > 0$ and f is a H-lipschitz function, i.e.,

$$\exists\, H > 0 \text{ such that } \forall x, y \in [c, +\infty[,\ c > 0,\ |f(x) - f(y)| \le H|x - y|.$$

So by an analogous reasoning, we get the following result:

Theorem 1.4 *Suppose that the hypotheses* $(H1)$, $(H2)$, $(H3')$ *and* $(H4)$ *are satisfied. Then, there exist a positive constant* $C_N > 0$ $(N \ge 1)$, *a sequence of complex numbers* $(\varepsilon_n)_{n\in\mathbb{N}^*}$ *and two sequences of eigenvalues* $\{\lambda_n(\varepsilon)\}_{n\in\mathbb{N}^*}$ *and* $\{\lambda_n(\varepsilon_n)\}_{n\in\mathbb{N}^*}$ *having the form*

$$\lambda_n(\varepsilon) = \lambda_n + \varepsilon\lambda_{n,1} + \varepsilon^2\lambda_{n,2} + \cdots$$
$$\lambda_n(\varepsilon_n) = \lambda_n + \varepsilon_n\lambda_{n,1} + \varepsilon_n^2\lambda_{n,2} + \cdots$$

such that for $|\varepsilon| \in]0, C_N[$*, the systems*

(i) $\{e^{if(\lambda_n(\varepsilon))t}\}_1^N \cup \{e^{if(\lambda_n(\varepsilon_n))t}\}_{N+1}^\infty$
(ii) $\{e^{if(\lambda_n(\varepsilon))t}\}_1^N \cup \{e^{if(\lambda_n)t}\}_{N+1}^\infty$

form Riesz bases in $L^2(0, T)$. □

Notice here that in [2] the authors proved the existence of a sequence of complex numbers $(\varepsilon_n)_{n\in\mathbb{N}^*}$ such that the system $\{e^{if(\lambda_n(\varepsilon_n))t}\}_1^\infty$ forms a Riesz basis in $L^2(0, T)$; whereas, in Theorem 1.4 the given bases are associated to the eigenvalues of $T(\varepsilon)$ for a fixed ε since in application to numerical analysis the basis is truncated.

To show the importance of our results, we consider the problem of radiation of a vibrating structure in a light fluid motivated by Filippi et al. in [8]. More precisely, we consider the following operators

$$\begin{cases} T_0 : \mathcal{D}(T_0) \subset L^2\big(]-L, L[\big) \longrightarrow L^2\big(]-L, L[\big) \\ \varphi \longrightarrow T_0\varphi(x) = \dfrac{d^4\varphi}{dx^4} \\ \mathcal{D}(T_0) = H_0^2\big(]-L, L[\big) \cap H^4\big(]-L, L[\big), \end{cases}$$

$$\begin{cases} B : \mathcal{D}(B) \subset L^2\big(]-L, L[\big) \longrightarrow L^2\big(]-L, L[\big) \\ \varphi \longrightarrow B\varphi(x) = \dfrac{d^2\varphi}{dx^2} \\ \mathcal{D}(B) = H_0^2\big(]-L, L[\big) \cap H^4\big(]-L, L[\big) \end{cases}$$

and

$$\begin{cases} K : L^2\big(]-L, L[\big) \longrightarrow L^2\big(]-L, L[\big) \\ \varphi \longrightarrow K\varphi(x) = \dfrac{i}{2}\displaystyle\int_{-L}^{L} H_0(k|x - x'|)\varphi(x')dx', \end{cases}$$

where H_0 is the Hankel function of the first kind and order 0 and the following eigenvalue problem: Find the values $\lambda \in \mathbb{C}$ for which there is a solution $u \in H_0^2\big(]-L, L[\big) \cap H^4\big(]-L, L[\big)$, $u \neq 0$ for the equation

$$T_0u + \varepsilon K(T_0 - B) = \lambda(I + \varepsilon K)u$$

where $\lambda = \frac{\omega^2\rho_1}{T_1}$, $\varepsilon = \frac{2\rho_0}{\rho_1}$.

The contents of this paper are organized as follow: Sect. 2 is devoted to introduce some basic definitions about Riesz basis and present its fundamental properties. In Sect. 3, we prove that the families of exponentials associated to some eigenvalues of the perturbed operator $T(\varepsilon)$ form Riesz bases. In the last section, we apply the results of Sect. 3 to a problem of radiation of a vibrating structure in a light fluid.

2 Preliminary results

In this section, we introduce some definitions and preliminary results that we will need in the sequel to characterize the notion of basis, Riesz basis and Riesz basis of exponential family.

Definition 2.1 [16] A sequence of vectors $\{\varphi_n\}_{n\in\mathbb{N}^*}$ in a separable Hilbert space $\mathcal{H}$ is said to be a basis for $\mathcal{H}$ if to each vector $x \in \mathcal{H}$ there corresponds a unique sequence of scalars $\{c_n\}_{n\in\mathbb{N}^*}$ such that

$$x = \sum_{n=1}^{\infty} c_n\varphi_n$$

converges for the norm of $\mathcal{H}$. □

Definition 2.2 [16] A basis $\{\varphi_n\}_{n\in\mathbb{N}^*}$ in a separable Hilbert space $\mathcal{H}$ is said to be Riesz basis for $\mathcal{H}$, if it is equivalent to an orthonormal basis; i.e., $\varphi_n = Ge_n$ for all $n \in \mathbb{N}^*$ where $\{e_n\}_{n\in\mathbb{N}^*}$ is an orthonormal basis for $\mathcal{H}$ and G is a bounded invertible operator on $\mathcal{H}$. □

Definition 2.3 A set of vectors $\{\varphi_n\}_{n\in\mathbb{N}^*}$ is said to be ω-linearly independent if

$$\sum_{n=1}^{\infty} c_n\varphi_n = 0 \Longrightarrow c_n = 0 \text{ for all } n.$$

□

Proposition 2.1 [3, Theorem 2.13] *The two statements below are equivalent:*

(i) $\{\varphi_n\}_{n\in\mathbb{N}^*}$ *is a Riesz basis for a separable Hilbert space* $\mathcal{H}$.
(ii) $\{\varphi_n\}_{n\in\mathbb{N}^*}$ *is* ω*-linearly independent and there exist numbers* $A, B > 0$ *such that*

$$A\|\varphi\|^2 \leq \sum_{n=1}^{\infty} |\langle\varphi, \varphi_n\rangle|^2 \leq B\|\varphi\|^2 \text{ for all } \varphi \in \mathcal{H}.$$

The constants A and B are called lower and upper bounds of the Riesz basis.

□

The following perturbation result for Riesz basis due to O. Christensen (see [4, Corollary 22.1.5]) will play a crucial role in our considerations.

Theorem 2.1 [4, Corollary 22.1.5] *Let* $\{\varphi_n\}_{n\in\mathbb{N}^*}$ *be a Riesz basis of a separable Hilbert space* $\mathcal{H}$ *with lower bound A and let* $\{g_n\}_{n\in\mathbb{N}^*}$ *be a collection of vectors in* $\mathcal{H}$. *If there exists a positive constant* $R < A$ *such that*

$$\sum_{n=1}^{\infty} |\langle f, \varphi_n - g_n\rangle|^2 \leq R\|f\|^2, \quad \forall f \in \mathcal{H},$$

then $\{g_n\}_{n\in\mathbb{N}^*}$ *is a Riesz basis for* $\mathcal{H}$. □

Now, we state some basic definitions that we will need to derive a precise description of the concept of Riesz basis family of exponential developed by Pavlov [15].

Definition 2.4 [1] An entire function $f(z)$ is said to be of exponential type if the inequality

$$|f(z)| \leq Ae^{B|z|}, \quad \forall z \in \mathbb{C} \tag{2.1}$$

holds for some positive constants A and B. The smallest of constants B such that (2.1) holds is said to be exponential type of f. □

Definition 2.5 [1] The growth indicator of an exponential type function f, is a 2π-periodic function on $\mathbb{R}$, defined by the equality

$$h_f(\phi) = \lim_{r\to\infty} \sup \frac{1}{r} \ln |f(re^{i\phi})|, \quad \phi \in [-\pi, \pi].$$

The indicator diagram of f is a convex set G_f such that

$$h_f(\phi) = \sup_{k\in G_f} Re(ke^{-i\phi}), \quad \phi \in [-\pi, \pi].$$

□

Definition 2.6 [1] An entire function f of exponential type is said to be a function of the Cartwright class if

$$\int_{\mathbb{R}} \frac{\max(\ln |f(x)|, 0)}{1+x^2} dx < +\infty.$$

In particular, the function f of exponential type satisfying the condition

$$\int_{\mathbb{R}} \frac{|f(x)|^2}{1+x^2} dx < \infty$$

belongs to the Cartwright class. □

Remark 2.1 The indicator diagram of a Cartwright class function is an interval $[i\alpha, i\beta]$, $\alpha \le \beta$, of the imaginary axis. Its length is the width of indicator diagram (see [1, p. 59–60]). □

Definition 2.7 [1, p. 101] An entire function of exponential type with simple zeros $\{\lambda_n\}_1^\infty$ and with the width of indicator diagram T is called a generating function of exponential family $\{e^{i\lambda_n t}\}_1^\infty$ in $L^2(0, T)$. □

Definition 2.8 [1, 1.27] An entire function of exponential type is said to be of sine type if

(i) the zeros of f lie in a strip $\{z \in \mathbb{C} \; such \; that \; |Imz| \le H\}$ for some $H > 0$.
(ii) there exist $h \in \mathbb{R}$ and positive constants c_1, c_2 such that

$$c_1 \le |f(x+ih)| \le c_2, \quad \forall x \in \mathbb{R}.$$

□

We close this section by the following theorem obtained by Pavlov [15].

Theorem 2.2 [15] *Let $\Lambda := \{\lambda_n\}_1^\infty$ be a countable set of complex numbers.*
The family $\{e^{i\lambda_n t}\}_1^\infty$ forms a Riesz basis in $L^2(0, T)$ if and only if the following conditions are satisfied:

(i) *The sequence* $\{\lambda_n\}_1^\infty$ *lies in a strip parallel to the real axis:*

$$\sup_{n\geq 1} |Im\lambda_n| < \infty.$$

(ii) *The family* $\{\lambda_n\}_1^\infty$ *is separated, i.e.,*

$$\delta(\Lambda) := \inf_{n\neq m} |\lambda_n - \lambda_m| > 0.$$

(iii) *The generating function of the family* $\{e^{i\lambda_n t}\}_1^\infty$ *on the interval* $(0, T)$ *satisfies the Muckenhoupt condition*

$$\sup_{I\in J} \left\{ \frac{1}{|I|^2} \int_I |f(x+ih)|^2 dx \int_I |f(x+ih)|^{-2} dx \right\} < \infty$$

for some $h \in \mathbb{R}$, *where* J *is the set of all intervals of the real axis.* □

3 Proof of main results

The aim of this section is to prove that the families of exponentials form Riesz basis in $L^2(0, T)$, where the exponents coincide with some eigenvalues of the operator $T(\varepsilon)$.

The proof of Theorem 1.3 is as follow:

Proof of Theorem 1.3 (i) Let $n \in \mathbb{N}^*$, $N \geq 1$ and λ_n the eigenvalue number n of T_0. We have,

$$\begin{aligned} |e^{i\lambda_n(\varepsilon)t} - e^{i\lambda_n t}| &= |e^{i\lambda_n t}(e^{i(\lambda_n(\varepsilon)-\lambda_n)t} - 1)| \\ &= |e^{i\lambda_n t}||e^{i(\lambda_n(\varepsilon)-\lambda_n)t} - 1| \\ &\leq 2e^{|Im\lambda_n|t} e^{-Im((\lambda_n(\varepsilon)-\lambda_n)\frac{t}{2})} \left|\sin\left((\lambda_n(\varepsilon) - \lambda_n)\,\frac{t}{2}\right)\right|. \end{aligned}$$

As the family $\{e^{i\lambda_n t}\}_1^\infty$ forms a Riesz basis in $L^2(0, T)$, then Theorem 2.2 implies that the sequence $\{\lambda_n\}_1^\infty$ lies in a strip parallel to the real axis. Thus, there exists a positive constant h such that

$$\forall n \geq 1, \quad |Im\lambda_n| \leq h \quad \text{where} \quad h := \sup_n |Im\lambda_n|.$$

Consequently, we obtain

$$
\begin{aligned}
|e^{i\lambda_n(\varepsilon)t} - e^{i\lambda_n t}| &\leq 2e^{ht}e^{\left|Im\left((\lambda_n(\varepsilon)-\lambda_n)\frac{t}{2}\right)\right|}\left[\sin^2\left(Re\left((\lambda_n(\varepsilon)-\lambda_n)\frac{t}{2}\right)\right)\right.\\
&\quad \left. + \sinh^2\left(Im\left((\lambda_n(\varepsilon)-\lambda_n)\frac{t}{2}\right)\right)\right]^{\frac{1}{2}}\\
&\leq 2e^{ht}e^{\left|Im\left((\lambda_n(\varepsilon)-\lambda_n)\frac{t}{2}\right)\right|}\left[\ \left|\sin\left(Re\left((\lambda_n(\varepsilon)-\lambda_n)\frac{t}{2}\right)\right)\right|\right.\\
&\quad \left. + \left|\sinh\left(Im\left((\lambda_n(\varepsilon)-\lambda_n)\frac{t}{2}\right)\right)\right|\ \right]\\
&\leq 2e^{ht}e^{\left|Im((\lambda_n(\varepsilon)-\lambda_n)\frac{t}{2})\right|}\left[\ \left|Re\left((\lambda_n(\varepsilon)-\lambda_n)\frac{t}{2}\right)\right|\right.\\
&\quad \left. + \sinh\left|Im\left((\lambda_n(\varepsilon)-\lambda_n)\frac{t}{2}\right)\right|\ \right]\\
&\leq 2e^{ht}e^{\left|Im((\lambda_n(\varepsilon)-\lambda_n)\frac{t}{2})\right|}\left[\ \left|Re\left((\lambda_n(\varepsilon)-\lambda_n)\frac{t}{2}\right)\right|\right.\\
&\quad \left. + \left|Im\left((\lambda_n(\varepsilon)-\lambda_n)\frac{t}{2}\right)\right| e^{\left|Im((\lambda_n(\varepsilon)-\lambda_n)\frac{t}{2})\right|}\right]\\
&\leq 2e^{ht}e^{|Im((\lambda_n(\varepsilon)-\lambda_n)t)|}\,|(\lambda_n(\varepsilon)-\lambda_n)t|\,.
\end{aligned}
\tag{3.1}
$$

Furthermore, according to hypothesis $(H2)$ the family $\{e^{i\lambda_n t}\}_1^\infty$ forms a Riesz basis in $L^2(0, T)$. Hence, Proposition 2.1 yields the existence of numbers $A, B > 0$ such that

$$A\|u\|^2 \leq \sum_{n=1}^{\infty}|\langle u, e^{i\lambda_n t}\rangle|^2 \leq B\|u\|^2 \quad \forall u \in L^2(0, T).$$

We set

$$C_N = \min_{n\in[1,N]} \frac{\sqrt{A}}{\eta\omega_n^2 r_n^2 M_n\alpha_n n\sqrt{T}te^{tr_{1,n}} + \sqrt{A}(q + \alpha_n + \omega_n^2 r_n M_n\alpha_n)}$$

where $\eta^2 = \sum_{k=1}^{\infty}\frac{4}{k^2}$ and $r_{1,n} = r_n + h$.

Let $n \in [1, N]$. It is easy to see that if $|\varepsilon| \in]0, C_N[$, we have

$$|\varepsilon| < \frac{1}{q + \alpha_n + \omega_n^2 r_n M_n\alpha_n},$$

then it follows from Theorem 1.2 that the operator $T(\varepsilon)$ admits a unique eigenvalue with multiplicity one, denoted $\lambda_n(\varepsilon)$, inside the circle $\mathcal{C}_n$ and we obtain

$$e^{|Im((\lambda_n(\varepsilon)-\lambda_n)t)|} \leq e^{r_n t}. \tag{3.2}$$

Hence, Eqs. (3.1) and (3.2) yield

$$|e^{i\lambda_n(\varepsilon)t} - e^{i\lambda_n t}| \leq 2te^{r_{1,n}t}\,|\lambda_n(\varepsilon) - \lambda_n|\,. \tag{3.3}$$

Moreover, since $|\varepsilon| < \frac{1}{q+\alpha_n+\omega_n^2 r_n M_n \alpha_n}$ Theorem 1.2 implies that $\lambda_n(\varepsilon)$ can be developed as entire series of ε and we have

$$\lambda_n(\varepsilon) = \lambda_n + \varepsilon\lambda_{n,1} + \varepsilon^2\lambda_{n,2} + \cdots \tag{3.4}$$

where

$$|\lambda_{n,i}| \leq \omega_n^2 r_n^2 M_n \alpha_n (q + \alpha_n + \omega_n^2 r_n M_n \alpha_n)^{i-1} \quad \forall i \geq 1. \tag{3.5}$$

So, using Eqs. (3.3), (3.4) and (3.5), we get

$$\begin{aligned}
\|e^{i\lambda_n(\varepsilon)t} - e^{i\lambda_n t}\|^2 &= \int_0^T |e^{i\lambda_n(\varepsilon)t} - e^{i\lambda_n t}|^2 dt \\
&\leq \int_0^T \left(2te^{tr_{1,n}} \sum_{i=1}^{\infty} |\varepsilon|^i |\lambda_{n,i}|\right)^2 dt \\
&\leq \int_0^T \left(2te^{tr_{1,n}} \sum_{i=1}^{\infty} |\varepsilon|^i \omega_n^2 r_n^2 M_n \alpha_n (q + \alpha_n + \omega_n^2 r_n M_n \alpha_n)^{i-1}\right)^2 dt \\
&\leq \int_0^T \left(2te^{tr_{1,n}} \omega_n^2 r_n^2 M_n \alpha_n |\varepsilon| \sum_{i=0}^{\infty} \left(|\varepsilon| \left(q + \alpha_n + \omega_n^2 r_n M_n \alpha_n\right)\right)^i\right)^2 dt \\
&\leq \int_0^T \left(\frac{2te^{tr_{1,n}} \omega_n^2 r_n^2 M_n \alpha_n |\varepsilon|}{1 - |\varepsilon|(q + \alpha_n + \omega_n^2 r_n M_n \alpha_n)}\right)^2 dt \\
&< \int_0^T \frac{4A}{\eta^2 n^2 T} dt \\
&\leq \frac{4A}{\eta^2 n^2}.
\end{aligned}$$

On the other hand, for each eigenvalue λ_n of T_0, we fix $\varepsilon_n \in \mathbb{C}$ such that

$$|\varepsilon_n| \in \left]0, \frac{\sqrt{A}}{\eta\omega_n^2 r_n^2 M_n \alpha_n n \sqrt{T} te^{tr_{1,n}} + \sqrt{A}(q + \alpha_n + \omega_n^2 r_n M_n \alpha_n)}\right[,$$

where $\eta^2 = \sum_{k=1}^{\infty} \frac{4}{k^2}$ and $r_{1,n} = r_n + h$.

As for all $n \geq 1$ we have

$$|\varepsilon_n| < \frac{1}{q + \alpha_n + \omega_n^2 r_n M_n \alpha_n},$$

so making the same reasoning as the above, we get

$$\|e^{i\lambda_n(\varepsilon_n)t} - e^{i\lambda_n t}\|^2 < \frac{4A}{\eta^2 n^2}, \quad \text{for all} \quad n \geq N+1.$$

If we set

$$R := \sum_{n=1}^{\infty} \|e^{i\lambda_n(\varepsilon)t} - e^{i\lambda_n t}\|^2,$$

we can easily see that $R < A$. Now, let $f \in L^2(0,T)$ and $f_n \in \{e^{i\lambda_n(\varepsilon)t}\}_1^N \cup \{e^{i\lambda_n(\varepsilon_n)t}\}_{N+1}^{\infty}$. Then, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \left|\left\langle f, e^{i\lambda_n t} - f_n\right\rangle\right|^2 &= \sum_{n=1}^{N} \left|\left\langle f, e^{i\lambda_n t} - e^{i\lambda_n(\varepsilon)t}\right\rangle\right|^2 + \sum_{n=N+1}^{\infty} \left|\left\langle f, e^{i\lambda_n t} - e^{i\lambda_n(\varepsilon_n)t}\right\rangle\right|^2 \\
&= \sum_{n=1}^{N} \left|\int_0^T f(t)\left(e^{i\lambda_n t} - e^{i\lambda_n(\varepsilon)t}\right) dt\right|^2 \\
&\quad + \sum_{n=N+1}^{\infty} \left|\int_0^T f(t)\left(e^{i\lambda_n t} - e^{i\lambda_n(\varepsilon_n)t}\right) dt\right|^2 \\
&\leq \sum_{n=1}^{N} \left(\int_0^T |f(t)| \left|e^{i\lambda_n t} - e^{i\lambda_n(\varepsilon)t}\right| dt\right)^2 \\
&\quad + \sum_{n=N+1}^{\infty} \left(\int_0^T |f(t)| \left|e^{i\lambda_n t} - e^{i\lambda_n(\varepsilon_n)t}\right| dt\right)^2 \\
&\leq \sum_{n=1}^{N} \int_0^T |f(t)|^2\, dt \int_0^T \left|e^{i\lambda_n t} - e^{i\lambda_n(\varepsilon)t}\right|^2 dt \\
&\quad + \sum_{n=N+1}^{\infty} \int_0^T |f(t)|^2\, dt \int_0^T \left|e^{i\lambda_n t} - e^{i\lambda_n(\varepsilon_n)t}\right|^2 dt \\
&\leq \|f\|^2 \left(\sum_{n=1}^{N} \left\|e^{i\lambda_n(\varepsilon)t} - e^{i\lambda_n t}\right\|^2 + \sum_{n=N+1}^{\infty} \left\|e^{i\lambda_n(\varepsilon_n)t} - e^{i\lambda_n t}\right\|^2\right) \\
&= R\|f\|^2.
\end{aligned}$$

Consequently, using Theorem 2.1, the system $\{f_n\}_{n\in\mathbb{N}^*}$ forms a Riesz basis in $L^2(0,T)$. Hence, the family $\{e^{i\lambda_n(\varepsilon)t}\}_1^N \cup \{e^{i\lambda_n(\varepsilon_n)t}\}_{N+1}^{\infty}$ forms a Riesz basis in $L^2(0,T)$. This achieves the proof of the first item.

(ii) To prove the second item, it suffices to choose the same constant C_N and the result follows immediately from Theorem 2.1. □

We end this section with the following proof of Theorem 1.4.

Proof of Theorem 1.4 (i) Let $n \in \mathbb{N}^*$, $N \geq 1$ and λ_n the eigenvalue number n of T_0. We have

$$|e^{if(\lambda_n(\varepsilon))t} - e^{if(\lambda_n)t}| = |e^{if(\lambda_n)t}(e^{i(f(\lambda_n(\varepsilon))-f(\lambda_n))t} - 1)|.$$

On the other hand, since f is a H-lipschitz function, we obtain

$$\begin{aligned}|e^{if(\lambda_n(\varepsilon))t} - e^{if(\lambda_n)t}| &\leq t\,|f(\lambda_n(\varepsilon)) - f(\lambda_n)| \\ &\leq tH\,|\lambda_n(\varepsilon) - \lambda_n|\,.\end{aligned}$$

To prove the first item, we set

$$C_N = \min_{n\in[1,N]} \frac{\sqrt{A}}{\eta\omega_n^2 r_n^2 M_n \alpha_n n\sqrt{T}tH + \sqrt{A}(q + \alpha_n + \omega_n^2 r_n M_n \alpha_n)}$$

where $\eta^2 = \sum_{k=1}^{\infty} \frac{1}{k^2}$ and let $|\varepsilon| \in]0, C_N[$.

So making the same reasoning as the one in Theorem 1.3 with

$$|\varepsilon_n| \in \left]0, \frac{\sqrt{A}}{\eta\omega_n^2 r_n^2 M_n \alpha_n n\sqrt{T}tH + \sqrt{A}(q + \alpha_n + \omega_n^2 r_n M_n \alpha_n)}\right[\quad \textit{for all } n \geq 1,$$

we get the desired result.

(ii) To prove (ii), it suffices to choose the same constant C_N and to apply Theorem 2.1. □

4 Application to a problem of radiation of a vibrating structure in a light fluid

We consider an elastic membrane lying in the domain $-L < x < L$ of the plane $z = 0$. The two half-spaces $z < 0$ and $z > 0$ are filled with gas. The membrane is excited by a harmonic force $F(x)e^{-i\omega t}$.

Now, let us consider the following boundary value problem:

$$\left(\frac{d^4}{dx^4} - \frac{m\omega^2}{D}\right)u(x) - i\rho_0 \int_{-L}^{L} H_0(k|x - x'|)\left(\frac{\omega^2}{D} - \frac{1}{m}\left(\frac{d^4}{dx^4} - \frac{d^2}{dx^2}\right)\right)u(x')dx' = \frac{F(x)}{D}, \tag{4.1}$$

for all $x \in]-L, L[$ where u designates the displacement of the membrane such that $u(x) = \frac{\partial u(x)}{\partial x} = 0$ for $x = -L$ and $x = L$ and H_0 is the Hankel function of the first kind and order 0; while the mechanical parameters of the membrane are E the Young modulus, ν the Poisson ratio, m the surface density, h the thickness of the membrane and $D := \frac{Eh^3}{12(1-\nu^2)}$ the rigidity and the fluid is characterized by ρ_0 the density, c the sound speed and $k := \frac{\omega}{c}$ the wave number.

The problem (4.1) satisfy the following system:

$$\left(\frac{d^4}{dx^4} - \frac{m\omega^2}{D}\right) u(x) = \frac{1}{D}(F(x) - P(x)) \text{ for all } x \in]-L, L[,$$

where

$$u(x) = \frac{\partial u(x)}{\partial x} = 0 \text{ for } x = -L \text{ and } x = L,$$
$$P(x) = \lim_{\eta \to 0^+} (p(x, \eta) - p(x, -\eta))$$

and

$$p(x, z) = -sgn\, zi \frac{\rho_0}{2} \int_{-L}^{L} H_0(k\sqrt{(x - x')^2 + z^2}) \left(\omega^2 - \frac{D}{m}\left(\frac{d^4}{dx^4} - \frac{d^2}{dx^2}\right)\right) u(x')dx',$$

for $z < 0$ or $z > 0$, where p denotes the acoustic pressure in the fluid.

Among this direction, we consider the following operator:

$$\begin{cases} T_0 : \mathcal{D}(T_0) \subset L^2\big(]-L, L[\big) \longrightarrow L^2\big(]-L, L[\big) \\ \qquad\qquad \varphi \longrightarrow T_0\varphi(x) = \dfrac{d^4\varphi}{dx^4} \\ \mathcal{D}(T_0) = H_0^2\big(]-L, L[\big) \cap H^4\big(]-L, L[\big). \end{cases}$$

Now, we state a straightforward but useful result from [11]:

Lemma 4.1 [11, Lemmas 3.1, 3.2 and 3.4] *The following assertions hold:*

(i) *T_0 is a self-adjoint operator with dense domain.*
(ii) *The injection from $\mathcal{D}(T_0)$ into $L^2\big(]-L, L[\big)$ is compact.*
(iii) *The resolvent set of T_0 is not empty. In fact, $0 \in \rho(T_0)$.*
(iv) *The spectrum of T_0 is constituted only of point spectrums which are positive, denumerable and of which the multiplicity is one and which have no finite limit points and satisfies*

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to +\infty.$$

Further,

$$\left(\frac{(2n+1)\pi}{4L}\right)^4 \leq \lambda_n \leq \left(\frac{(2n+3)\pi}{4L}\right)^4,$$

i.e.,

$$\lambda_n \sim_{+\infty} \left(\frac{n\pi}{2L}\right)^4.$$

□

Remark 4.1 It follows from Lemma 4.1(ii) and (iii) that the operator T_0 has a compact resolvent. Since T_0 is self-adjoint, then let

$$T_0\varphi = \sum_{n=1}^{\infty} \lambda_n \langle \varphi, \varphi_n \rangle \varphi_n$$

be its spectral decomposition, where $\lambda_n = \alpha n^4$ is the n^{th} eigenvalue of T_0 associated to the eigenvector $\varphi_n(x) = \mu e^{\sqrt[4]{\lambda_n}x} + \eta e^{-\sqrt[4]{\lambda_n}x} + \theta e^{i\sqrt[4]{\lambda_n}x} + \delta e^{-i\sqrt[4]{\lambda_n}x}$. □

Consequently, we define the operator B by:

$$\begin{cases} B = T_0^{\frac{1}{2}} : \mathcal{D}(B) \subset L^2\big(]-L, L[\big) \longrightarrow L^2\big(]-L, L[\big) \\ \qquad\qquad\qquad\qquad \varphi \longrightarrow B\varphi(x) = \dfrac{d^2\varphi}{dx^2} \\ \mathcal{D}(B) = \left\{ \varphi \in L^2\big(]-L, L[\big) \text{ such that } \displaystyle\sum_{n=1}^{\infty} \lambda_n |\langle \varphi, \varphi_n \rangle|^2 < \infty \right\}. \end{cases}$$

In the remaining part of this section, we consider the following operator:

$$\begin{cases} K : L^2\big(]-L, L[\big) \longrightarrow L^2\big(]-L, L[\big) \\ \qquad\qquad \varphi \longrightarrow K\varphi(x) = \dfrac{i}{2} \displaystyle\int_{-L}^{L} H_0(k|x - x'|)\varphi(x')dx', \end{cases}$$

and the following eigenvalue problem:

Find the values $\lambda(\varepsilon) \in \mathbb{C}$ for which there is a solution $\varphi \in H_0^2\big(]-L, L[\big) \cap H^4\big(]-L, L[\big)$, $\varphi \neq 0$ for the equation

$$T_0\varphi + \varepsilon K(T_0 - B)\varphi = \lambda(\varepsilon)(I + \varepsilon K)\varphi \tag{4.2}$$

where $\lambda = \frac{m\omega^2}{D}$ and $\varepsilon = \frac{2\rho_0}{m}$.

In view of [12, chapter 9, section 4], λ is the eigenvalue and φ is the eigenvector.

Note that λ and φ each depend on the value of ε. So, we designate by $\lambda := \lambda(\varepsilon)$ and $\varphi := \varphi(\varepsilon)$.

For $|\varepsilon| < \frac{1}{\|K\|}$, the operator $I + \varepsilon K$ is invertible. Then, the problem (4.2) become:

Find the values $\lambda(\varepsilon) \in \mathbb{C}$ for which there is a solution $\varphi \in H_0^2\big(]-L, L[\big) \cap H^4\big(]-L, L[\big)$, $\varphi \neq 0$ for the equation

$$(I + \varepsilon K)^{-1}T_0\varphi + \varepsilon(I + \varepsilon K)^{-1}K(T_0 - B)\varphi = \lambda(\varepsilon)\varphi. \tag{4.3}$$

The problem (4.3) is equivalent to:

Find the values $\lambda(\varepsilon) \in \mathbb{C}$ for which there is a solution $\varphi \in H_0^2(]-L, L[) \cap H^4(]-L, L[)$, $\varphi \neq 0$ for the equation

$$\left(\frac{d^4}{dx^4} - \varepsilon K \frac{d^2}{dx^2} + \varepsilon^2 K^2 \frac{d^2}{dx^2} + \cdots + (-1)^n \varepsilon^n K^n \frac{d^2}{dx^2} + \cdots\right)\varphi = \lambda(\varepsilon)\varphi.$$

We denote by $T_n := (-1)^n K^n \frac{d^2}{dx^2}$ where $\mathcal{D}(T_n) = H^2(]-L, L[)$, for all $n \geq 1$.

In the sequel we shall need the following results:

Proposition 4.1 [5, Proposition 4.1]

(i) *There exist positive constants $a, b, q > 0$ and $\beta \in \left[\frac{1}{2}, 1\right]$ such that for all $\varphi \in \mathcal{D}(T_0)$ and for all $k \geq 1$, we have*

$$\|T_k\varphi\| \leq q^{k-1}(a\|\varphi\| + b\|T_0\varphi\|^{\beta}\|\varphi\|^{1-\beta}).$$

(ii) *For $|\varepsilon| < \frac{1}{\|K\|}$, the series $\sum_{k\geq 0} \varepsilon^k T_k\varphi$ converges for all $\varphi \in \mathcal{D}(T_0)$. If we designate its sum by $T(\varepsilon)\varphi$, we define a linear operator $T(\varepsilon)$ with domain $\mathcal{D}(T_0)$. For $|\varepsilon| < \frac{1}{\|K\|(1+\beta)}$, the operator $T(\varepsilon)$ is closed.*

□

Proposition 4.2 *The family $\{e^{i\sqrt[4]{\lambda_n}t}\}_n$ forms a Riesz basis in $L^2(]0, 4L[)$.* □

Proof It follows from [2, Theorem 4.1] that the family $\{e^{i\sqrt{\mu_n}t}\}_1^{\infty}$ forms a Riesz basis in $L^2(]0, 4L[)$. Further, as $B = T_0^{\frac{1}{2}}$ then $\mu_n = \sqrt{\lambda_n}$ is the nth eigenvalue of B. Consequently, the system $\{e^{i\sqrt[4]{\lambda_n}t}\}_1^{\infty}$ forms a Riesz basis in $L^2(]0, 4L[)$. □

The main result of this section is formulated as follow:

Theorem 4.1 *There exist a positive constant $C_N > 0$ $(N \geq 1)$, a sequence of complex numbers $(\varepsilon_n)_{n\in N^*}$ and two sequences of eigenvalues $\{\lambda_n(\varepsilon_n)\}_{n\in\mathbb{N}^*}$ and $\{\lambda_n(\varepsilon)\}_{n\in\mathbb{N}^*}$ of $T(\varepsilon)$ having the form*

$$\lambda_n(\varepsilon_n) = \lambda_n + \varepsilon_n\lambda_{n,1} + \varepsilon_n^2\lambda_{n,2} + \cdots$$
$$\lambda_n(\varepsilon) = \lambda_n + \varepsilon\lambda_{n,1} + \varepsilon^2\lambda_{n,2} + \cdots$$

such that for $|\varepsilon| \in]0, C_N[$, the systems

(i) $\{e^{i\sqrt[4]{|\lambda_n(\varepsilon)|}t}\}_1^N \cup \{e^{i\sqrt[4]{|\lambda_n(\varepsilon_n)|}t}\}_{N+1}^{\infty}$

(ii) $\{e^{i\sqrt[4]{|\lambda_n(\varepsilon)|}t}\}_1^N \cup \{e^{i\sqrt[4]{\lambda_n}t}\}_{N+1}^{\infty}$

form Riesz bases in $L^2(]0, 4L[)$. □

Proof The result follows immediately from Theorem 1.4, Lemma 4.1 and Propositions 4.1 and 4.2. □

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