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Multilinear pseudo-differential operators on product of Local Hardy spaces with variable exponents

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Abstract
In this paper, we establish a new atomic decomposition theory for Local Hardy spaces with variable exponents via local grand maximal characterization. By applying the refined atomic decomposition result, we prove that multilinear pseudo-differential operators are bounded on product of local Hardy spaces with variable exponents.

Keywords Multilinear pseudo-differential operators · Local Hardy spaces · Variable exponents · Atomic decomposition · Littlewood–Paley–Stein theory

Mathematics Subject Classification Primary 42B20; Secondary 42B30 · 46E30

1 Introduction

The study of Hardy spaces began in the early 1900s in the context of Fourier series and complex analysis in one variable. It was not until 1960 when the groundbreaking work in Hardy space theory in $\mathbb{R}^n$ came from Stein, Weiss, Coifman and C. Fefferman in [2,4,9]. While they are well suited as functional spaces for their applications to PDE's with constant coefficients, the Hardy spaces are not stable under multiplication by Schwartz class, a fact that seriously hinders their role when it comes to PDE's with variable coefficient. Thus, the theory of local Hardy space h^p plays an important role

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in various fields of analysis and partial differential equations. In particular, pseudo-differential operators are bounded on local Hardy spaces h^p for $0 < p < 1$, but they are not bounded on Hardy spaces H^p for $0 < p < 1$ (see [10]).

On the other hand, atomic decomposition is a significant tool in harmonic analysis and wavelet analysis for the study of function spaces and the operators acting on these spaces (see Meyer [24] and Coifman and Meyer [3]). Atomic decomposition was first introduced by Coifman [2] in one dimension in 1974 and later was extended to higher dimensions by Latter [22]. In 1979, Goldberg [10] introduced the atomic decomposition of local Hardy spaces.

Another stage in the progress of the theory of Hardy spaces was done by Nakai and Sawano [26] and Cruz-Uribe and Wang [7] recently when they independently considered Hardy spaces with variable exponents. However, it is quiet different to obtain the boundedness of operators on Hardy spaces with variable exponents. It is not sufficient to show the $H^{p(\cdot)}$-boundedness merely by checking the action of the operators on $H^{p(\cdot)}$-atoms. In the linear theory, the boundedness of some operators on variable Hardy spaces and some variable Hardy-type spaces have been established in [7,16,26,32,35] as applications of the corresponding atomic decompositions theories.

In more recent years, the study of multilinear operators on Hardy space theory has received increasing attention by many authors, see for example [12,17,18]. While some multilinear operators worked well on the product of local Hardy spaces, it is surprising that these similar results in the setting of variable exponents are still unknown. The boundedness of some multilinear operators on products of classical Hardy spaces was investigated by Grafakos and Kalton [12] and Li et al. [23]. The boundedness of bilinear pseudo-differential operators T_σ $(\sigma \in BS^0_{1,0})$ on classical Lebesgue spaces is proved by Bényi and Torres in [1]. Xiao et al. [34], established the boundedness of the bilinear pseudo-differential operator and the bilinear singular integral operators on product of local Hardy spaces. Very recently, Koezuka and Tomita [20] considered bilinear pseudo−differential operators with symbols in the bilinear Hörmander symbol class $BS^m_{1,1}$ on Triebel–Lizorkin spaces. Tan et al. [33], studied some multilinear operators are bounded on variable Lebesgue spaces $L^{p(\cdot)}$. However, there are some subtle difficulties in proving the boundedness results when we deal with the $h^{p(\cdot)}$-norm. The goal of this article is to show that multilinear pseudo-differential operators are bounded from product of local Hardy spaces with variable exponents into Lebesgue spaces with variable exponents via the elegant atomic decompositions theory of local Hardy spaces with variable exponents.

First we recall the definition of Lebesgue spaces with variable exponent. Note that the variable exponent spaces, such as the variable Lebesgue spaces and the variable Sobolev spaces, were studied by a substantial number of researchers (see, for instance, [6,21]. For any Lebesgue measurable function $p(\cdot) : \mathbb{R}^n \to (0, \infty]$ and for any measurable subset $E \subset \mathbb{R}^n$, we denote $p^-(E) = \inf_{x\in E} p(x)$ and $p^+(E) = \sup_{x\in E} p(x)$. Especially, we denote $p^- = p^-(\mathbb{R}^n)$, $p^+ = p^+(\mathbb{R}^n)$ and $p_- = \min\{p^-, 1\}$. Let $p(\cdot)$: $\mathbb{R}^n \to (0, \infty)$ be a measurable function with $0 < p^- \le p^+ < \infty$ and $\mathcal{P}^0$ be the set of all these $p(\cdot)$. Let $\mathcal{P}$ denote the set of all measurable functions $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ such that $1 < p^- \le p^+ < \infty$.

Definition 1.1 Let $p(\cdot) : \mathbb{R}^n \to (0, \infty]$ be a Lebesgue measurable function. The variable Lebesgue space $L^{p(\cdot)}$ consisits of all Lebesgue measurable functions f, for which the quantity $\int_{\mathbb{R}^n} |\varepsilon f(x)|^{p(x)} dx$ is finite for some $\varepsilon > 0$ and

$$\|f\|_{L^{p(\cdot)}} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \le 1 \right\}.$$

The variable Lebesgue spaces were first established by Orlicz [29] in 1931. Two decades later, Nakano [28] first systematically studied modular function spaces which include the variable Lebesgue spaces as specific examples. However, the modern development started with the paper [21] of Kováčik and Rákosník in 1991. As a special case of the theory of Nakano and Luxemberg, we see that $L^{p(\cdot)}$ is a quasi-normed space. Especially, when $p^- \ge 1$, $L^{p(\cdot)}$ is a Banach space.

We also recall the following class of exponent function, which can be found in [8]. Let $\mathcal{B}$ be the set of $p(\cdot) \in \mathcal{P}$ such that the Hardy-littlewood maximal operator M is bounded on $L^{p(\cdot)}$. An important subset of $\mathcal{B}$ is LH condition.

In the study of variable exponent function spaces it is common to assume that the exponent function $p(\cdot)$ satisfies LH condition. We say that $p(\cdot) \in LH$, if $p(\cdot)$ satisfies

$$|p(x) - p(y)| \le \frac{C}{-\log(|x-y|)}, \quad |x-y| \le 1/2$$

and

$$|p(x) - p(y)| \le \frac{C}{\log|x| + e}, \quad |y| \ge |x|.$$

It is well known that $p(\cdot) \in \mathcal{B}$ if $p(\cdot) \in \mathcal{P} \cap LH$. Moreover, example shows that the above LH conditions are necessary in certain sense, see Pick and Růžička [31] for more details. Next we also recall the definition of Local Hardy spaces with variable exponents $h^{p(\cdot)}$ as follows.

Definition 1.2 [26] Let $f \in \mathcal{S}'$, $p(\cdot) \in \mathcal{P}^0 \cap LH$ and $\varphi_t(x) = t^{-n}\varphi(t^{-1}x)$, $x \in \mathbb{R}^n$. Denote by $\mathcal{M}$ the grand maximal operator given by $\mathcal{M}_{loc} f(x) = \sup\{|\varphi_t * f(x)| : 0 < t < 1, \varphi \in \mathcal{F}_N\}$ for any fixed large integer N, where $\mathcal{F}_N = \{\varphi \in \mathcal{S} : \int \varphi(x)dx = 1, \sum_{|\alpha| \le N} \sup(1+|x|)^N |\partial^\alpha \varphi(x)| \le 1\}$. The local Hardy space with variable exponent $h^{p(\cdot)}$ is the set of all $f \in \mathcal{S}'$ for which the quantity

$$\|f\|_{h^{p(\cdot)}} = \|\mathcal{M}_{loc} f\|_{L^{p(\cdot)}} < \infty.$$

The main goal of this paper is to prove the following result:

Theorem 1.1 *Let $\sigma \in MB^0_{1,0}$ and let $p_1(\cdot), \ldots, p_m(\cdot) \in LH \cap \mathcal{P}^0$ and $p(\cdot) \in \mathcal{P}^0$ be Lebesgue measure functions satisfying*

$$\frac{1}{p_1(x)} + \cdots + \frac{1}{p_m(x)} = \frac{1}{p(x)}, \quad x \in \mathbb{R}^n. \tag{1.1}$$

Then T_σ extends to a bounded operator from $\prod_{j=1}^m h^{p_j(\cdot)}$ into $L^{p(\cdot)}$.

Throughout this paper, C or c will denote a positive constant that may vary at each occurrence but is independent to the essential variables, and $A \sim B$ means that there are constants $C_1 > 0$ and $C_2 > 0$ independent of the essential variables such that $C_1 B \le A \le C_2 B$. Given a measurable set $S \subset \mathbb{R}^n$, $|S|$ denotes the Lebesgue measure and χ_S means the characteristic function. For a cube Q, let Q^* denote with the same center and $2\sqrt{n}$ its side length, i.e. $l(Q^*) = 100\sqrt{n}l(Q)$. The symbols $\mathcal{S}$ and $\mathcal{S}'$ denote the class of Schwartz functions and tempered functions, respectively. As usual, for a function ψ on $\mathbb{R}^n$ and $\psi_t(x) = t^{-n}\psi(t^{-1}x)$. We also use the notations $j \wedge j' = \min\{j, j'\}$ and $j \vee j' = \max\{j, j'\}$. We write $\mathbb{N} = \{0, 1, 2, \ldots\}$. For $\varphi \in \mathcal{S}$ and $j \in \mathbb{Z}$ we write

$$\varphi_j(\xi) \equiv \varphi(2^{-j}\xi), \quad \varphi^j(x) \equiv 2^{jn}\varphi(2^j x).$$

We adopt the following definition of the Fourier transform and its inverse:

$$\mathcal{F}f(\xi) \equiv \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx, \quad \mathcal{F}^{-1}f(\xi) \equiv \int_{\mathbb{R}^n} f(\xi)e^{2\pi i x\cdot\xi}d\xi,$$

for $f \in L^1$. Using the definition of Fourier transform and its inverse, we also define

$$\varphi(D)f(x) \equiv \mathcal{F}^{-1}[\varphi \cdot \mathcal{F}f](x)$$

for $f \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$.

2 Boundedness of multilinear pseudo-differential operators

In this section, we will discuss the boundedness of multilinear pseudo−differential operators on product of Local Hardy spaces with variable exponents by using the atomic decomposition theory.

2.1 Atomic decomposition of local Hardy spaces with variable exponents

The atomic decomposition of Hardy spaces with variable exponents was first established independently in [7,26]. By using local grand maximal characterization we will establish the new atomic decompositions for local Hardy spaces with variable exponents $h^{p(\cdot)}$. In what follows, we give the definitions of local $(p(\cdot), q)$-atom and $(p(\cdot), q)-$ block for $h^{p(\cdot)}$.

Definition 2.1 Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$, $p(\cdot) \in \mathcal{P}^0$ and $1 < q \le \infty$. Fix an integer $d \ge d_{p(\cdot)} \equiv \min\{d \in \mathbb{N} : p^-(n + d + 1) > n\}$. Define a local $(p(\cdot), q)$-atom of

$h^{p(\cdot)}$ to be a function a of compact support which has the additional properties that $\|a\|_{L^q} \le \frac{|Q|^{1/q}}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}}$ and $\int_{\mathbb{R}^n} a(x)x^\alpha dx = 0$ for all $|\alpha| \le d$ and $|Q| \le 1$, where Q is the smallest cube containing the support of a.

Definition 2.2 Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$, $p(\cdot) \in \mathcal{P}^0$ and $1 < q \le \infty$. Define a $(p(\cdot), q)$-block of $h^{p(\cdot)}$ to be a function b of compact support which has the additional properties that $\|b\|_{L^q} \le \frac{|P|^{1/q}}{\|\chi_P\|_{L^{p(\cdot)}(\mathbb{R}^n)}}$ and $|P| > 1$, where P is the smallest cube containing the support of b.

For convenience, the set of all such pairs (a, Q) will be denoted by $\mathcal{A}(p(\cdot), q)$ and the set of all such pairs (b, P) will be denoted by $\mathcal{B}(p(\cdot), q)$.

For sequences of scalars $\{\lambda_j\}$ and cubes $\{Q_j\}$, define that

$$\mathcal{A}_s(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}) = \left\| \left\{ \sum_j \left(\frac{|\lambda_j|\chi_{Q_j}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}}} \right)^s \right\}^{\frac{1}{s}} \right\|_{L^{p(\cdot)}},$$

and for sequences of scalars $\{\kappa_j\}$ and cubes $\{P_j\}$,

$$\mathcal{B}_s(\{\kappa_j\}_{j=1}^{\infty}, \{P_j\}_{j=1}^{\infty}) = \left\| \left\{ \sum_j \left(\frac{|\kappa_j|\chi_{P_j}}{\|\chi_{P_j}\|_{L^{p(\cdot)}}} \right)^s \right\}^{\frac{1}{s}} \right\|_{L^{p(\cdot)}}.$$

When $s = p^-$, we denote

$$\mathcal{A}_{p^-}(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}) = \mathcal{A}(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty})$$

and

$$\mathcal{B}_{p^-}(\{\kappa_j\}_{j=1}^{\infty}, \{P_j\}_{j=1}^{\infty}) = \mathcal{B}(\{\kappa_j\}_{j=1}^{\infty}, \{P_j\}_{j=1}^{\infty}).$$

Now we give the definition of atomic local Hardy space with variable exponent $h_{atom}^{p(\cdot),q}$.

Definition 2.3 Let $1 < q \le \infty$ and $p(\cdot) \in \mathcal{P}^0 \cap LH$. The function space $h_{atom}^{p(\cdot),q}$ is defined to be the set of all distributions $f \in \mathcal{S}'$ which can be written as $f = \sum_j \lambda_j a_j + \sum_j \kappa_j b_j$ in $\mathcal{S}'$, where $\{a_j, Q_j\} \subset \mathcal{A}(p(\cdot), q)$ and $\{b_j, P_j\} \subset \mathcal{B}(p(\cdot), q)$ with the quantities

$$\mathcal{A}(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}) + \mathcal{B}(\{\kappa_j\}_{j=1}^{\infty}, \{P_j\}_{j=1}^{\infty}) < \infty.$$

One define

$$\|f\|_{h_{atom}^{p(\cdot),q}} \equiv \mathcal{A}(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}) + \mathcal{B}(\{\kappa_j\}_{j=1}^{\infty}, \{P_j\}_{j=1}^{\infty}).$$

Next we establish the atomic decomposition for localized Hardy spaces with variable exponents via local grand maximal characterization.

Theorem 2.1 *Let* $1 < q \leq \infty$ *and* $p(\cdot) \in \mathcal{P}^0 \cap LH$. *Then*

$$h^{p(\cdot)} = h^{p(\cdot),\infty}_{atom}.$$

Theorem 2.1 is a direct result of the following two theorems.

Theorem 2.2 *Let* $p(\cdot) \in \mathcal{P}^0 \cap LH$ *and* $0 < s < \infty$. *If* $f \in h^{p(\cdot)}$, *there are* $\{a_j, Q_j\} \subset \mathcal{A}(p(\cdot, q))$ *and* $\{b_j, P_j\} \subset \mathcal{B}(p(\cdot, q))$ *with*

$$\mathcal{A}_s(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}) + \mathcal{B}_s(\{\kappa_j\}_{j=1}^{\infty}, \{P_j\}_{j=1}^{\infty}) \leq C\|f\|_{h^{p(\cdot)}},$$

such that $f = \sum_j \lambda_j a_j + \sum_j \kappa_j b_j$, *where the series converges to* f *in both* $h^{p(\cdot)}$ *and* L^q *norms.*

Proof We follow Goldberg's and Nakai−Sawano's ideas from [10,26,27]. To prove this theorem we first recall the following lemma which connects Hardy spaces with variable exponents and local Hardy spaces with variable exponents.

Lemma 2.3 [26] *Suppose that* $p(\cdot) \in LH \cap \mathcal{P}^0$. *Let* $\psi \in \mathcal{S}$ *be a bump function satisfying* $\chi_{Q(0,1)} \leq \psi \leq \chi_{Q(0,2)}$. *Then we have the following norm equivalence:*

$$\|f\|_{h^{p(\cdot)}} \sim \|(1-\psi(D))f\|_{H^{p(\cdot)}} + \|\psi(D)f\|_{L^{p(\cdot)}}$$

Let $f \in h^{p(\cdot)}$. Then we can decompose $f = \psi(D)f + (1-\psi(D))f$. By Lemma 2.3,

$$\|\psi(D)f\|_{L^{p(\cdot)}} \leq C\|f\|_{h^{p(\cdot)}}, \quad \|(1-\psi(D))f\|_{H^{p(\cdot)}} \leq C\|f\|_{h^{p(\cdot)}}.$$

For $j \in \mathbb{Z}^n$, f can be expanded into the series $f = \sum_{j=1}^{\infty} \kappa_j b_j$, where

$$\kappa_j = \|\chi_{j+[0,1]^n}\|_{L^{p(\cdot)}} \sup_{x \in j+[0,1]^n} |\psi(D)f|$$

and

$$b_j(x) = \begin{cases} \dfrac{1}{\kappa_j}\chi_{m+[0,1]^n}(x)\psi(D)f(x), & \text{if} \quad \kappa_j \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then we get that b_j is supported on $j + [0, 1]^n$ and

$$\|b_j\|_{L^\infty} = \frac{1}{\kappa}\|\chi_{j+[0,1]^n}(x)\psi(D)f(x)\|_{L^\infty} = \frac{1}{\|\chi_{j+[0,1]^n}\|_{L^{p(\cdot)}}}.$$

We denote that $P_j = j + [0,1]^n$. Hence,

$$\{b_j, P_j\} \subset \mathcal{B}(p(\cdot, q))$$

and

$$\mathcal{B}_s(\{\kappa_j\}_{j=1}^{\infty}, \{P_j\}_{j=1}^{\infty}) \leq C\|f\|_{h^{p(\cdot)}},$$

which the inequality follows from

$$\begin{aligned}
&\mathcal{B}_s(\{\kappa_j\}_{j=1}^{\infty}, \{P_j\}_{j=1}^{\infty}) \\
&= \left\| \left\{ \sum_j \left(\frac{|\kappa_j|\chi_{P_j}}{\|\chi_{P_j}\|_{L^{p(\cdot)}}} \right)^s \right\}^{\frac{1}{s}} \right\|_{L^{p(\cdot)}} \\
&= \left\| \sum_j \left(\frac{|\kappa_j|\chi_{j+[0,1]^n}}{\|\chi_{j+[0,1]^n}\|_{L^{p(\cdot)}}} \right) \right\|_{L^{p(\cdot)}} \\
&\leq \|(M|\psi(D)f|^{\eta})^{\frac{1}{\eta}}\|_{L^{p(\cdot)}} \\
&\leq \|\psi(D)f\|_{L^{p(\cdot)}} \\
&\leq \|f\|_{h^{p(\cdot)}}.
\end{aligned}$$

for sufficient small η by the Plancherel–Polya–Nikols'kij inequality and the boundedness of the maximal operator M. On the other hand, we apply [26, Theorem 4.6] and [32, Theorem 1.1] to $(1-\psi(D))f$. Then we can obtain the decomposition of $(1-\psi(D))f$,

$$(1-\psi(D))f = \sum_j \lambda_j a_j \quad \text{for} \quad \{a_j, Q_j\} \subset \mathcal{A}_s(p(\cdot, q))$$

and

$$\mathcal{A}_s(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}) \leq C\|(1-\psi(D))f\|_{H^{p(\cdot)}} \leq C\|f\|_{h^{p(\cdot)}}.$$

Therefore, we have completed the proof of Theorem 2.2. □

Theorem 2.4 *Let $p(\cdot) \in \mathcal{P}^0 \cap LH$. For any $\{a_j, Q_j\} \subset \mathcal{A}(p(\cdot, q))$ and $\{b_j, P_j\} \subset \mathcal{B}(p(\cdot, q))$ satisfying*

$$\mathcal{A}(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}) + \mathcal{B}(\{\kappa_j\}_{j=1}^{\infty}, \{P_j\}_{j=1}^{\infty}) < \infty,$$

the series $\sum_j \lambda_j a_j + \sum_j \kappa_j b_j$ converges in $\mathcal{S}'$, belongs to $h^{p(\cdot)}$ and satisfies

$$\left\| \sum_j \lambda_j a_j \right\|_{h^{p(\cdot)}} \leq C\mathcal{A}(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty})$$

and

$$\Big\| \sum_j \kappa_j b_j \Big\|_{h^{p(\cdot)}} \le C\mathcal{B}(\{\kappa_j\}_{j=1}^{\infty}, \{P_j\}_{j=1}^{\infty})$$

Proof By [26, Theorem 4.6], we have

$$\Big\| \sum_j \lambda_j a_j \Big\|_{h^{p(\cdot)}} \le C\mathcal{A}(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}).$$

Suppose that $\psi \in C_c^{\infty}$ be a nonnegative function that equals 1 near the neighborhood of 0 and supported on $Q(0, 1)$. Observe that

$$\sup_{0<t\le 1} \left| t^{-n}\psi(t^{-1}\cdot) * \left[\sum_{j=1}^{\infty} \lambda_j b_j \right](x) \right| \le C \left(\sum_{j=1}^{\infty} (|\lambda_j| \chi_{3P_j(x)} M b_j(x))^{p_-} \right)^{1/p_-}.$$

By following the similar argument in [26, Theorem 4.6], then we can get the desired result. □

2.2 Some known results on multilinear pseudo-differential operators

We recall Hörmander class of pseudo-differential operators in [19]. Let T_σ be a classical pseudo-differential operators of the form

$$T_\sigma(f)(x) = \int_{\mathbb{R}^n} \sigma(x, \xi)\hat{f}(\xi)e^{ix\cdot\xi} d\xi, \quad f \in \mathcal{S},$$

where $\sigma \in S^m_{\rho,\sigma}$, that is, $\sigma(x, \xi)$ is smooth for $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi)| \le C(1 + |\xi|)^{m-\rho|\beta|+\sigma|\alpha|}.$$

Now we consider the multilinear pseudo-differential operators of the form

$$T_\sigma(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \sigma(x, \vec{\xi}) \prod_{j=1}^{m} \hat{f}_j(\xi_j) e^{2\pi i x\cdot\left(\sum_{j=1}^m \xi_j\right)} d\vec{\xi}, \quad \text{for} \quad f_j \in \mathcal{S}, j = 1, \dots, m,$$

where the symbol $\sigma(x, \vec{\xi}) \in MS^m_{\rho,\delta}$, for $m \in \mathbb{R}$ and $\rho, \delta \in [0, 1]$, that is, $\sigma(x, \vec{\xi})$ is smooth for $(x, \vec{\xi}) \in (\mathbb{R}^n)^m$ and

$$|\partial_x^\alpha \partial_{\xi_1}^{\beta_1} \dots \partial_{\xi_m}^{\beta_m} \sigma(x, \vec{\xi})| \le C \left(1 + \sum_{j=1}^{m} |\xi_j| \right)^{m-\rho\left(\sum_{j=1}^m |\beta_j|\right)+\sigma|\alpha|},$$

for all multi-indices α, β_j, $j = 1, \ldots, m$. We denote $\sigma(x, \vec{\xi}) \in BS^m_{\rho,\delta}$ for the symbol of bilinear pseudo-differential operators.

In this paper, we will consider the symbol $\sigma(x, \vec{\xi}) \in MS^0_{1,0}$, that is,

$$|\partial_x^\alpha \partial_{\xi_1}^{\beta_1} \ldots \partial_{\xi_m}^{\beta_m} \sigma(x, \vec{\xi})| \leq C \left(1 + \sum_{j=1}^m |\xi_j|\right)^{-\sum_{j=1}^m |\beta_j|},$$

for all multi-indices α, β_j, $j = 1, \ldots, m$.

If we define $k(x, \vec{\xi})$ is the inverse Fourier transform in the every $\xi_j -$ variable of the function $\sigma(x, \vec{\xi})$, then

$$\begin{aligned} T_\sigma(\vec{f})(x) &= \int_{(\mathbb{R}^n)^m} k(x, \vec{y}) \prod_{j=1}^m f_j(x - y_j) d\vec{y} \\ &= \int_{(\mathbb{R}^n)^m} K(x, \vec{y}) \prod_{j=1}^m f_j(y_j) d\vec{y}, \quad \text{for} \quad x \notin \operatorname{supp}(f) \cap \operatorname{supp}(g), \end{aligned}$$

where $K(x, y, z) = k(x, x - y, x - z)$.

The following refined estimates for $K(x, \vec{y})$ will play very important role in the proof of our main results.

Lemma 2.5 *Suppose that $\sigma \in MB^0_{1,0}$, the the kernel*

$$K(x, \vec{y}) \in C^\infty((\mathbb{R}^n)^m \backslash \{(x, \vec{y}) : x = y_1 = \cdots = y_m\})$$

and satisfies

$$|\partial_x^\alpha \partial_{\xi_1}^{\beta_1} \ldots \partial_{\xi_m}^{\beta_m} K(x, \vec{y})| \leq C \left(\sum_{j=1}^m |x - y_j|\right)^{-\left(2n + \sum_{j=1}^m |\beta_j| + M\right)},$$

for all multi-indices α, β_j, $j = 1, \ldots, m$ and all $M \geq 0$.

We remark that this lemma has been proved in [34] for bilinear symbol $BS^0_{1,0}$ and that this fact is essential used in [14,25]. Repeating the same argument in the proof of [34, Theorem 2.2], we can get the desired lemma.

2.3 Proof of Theorem 1.1

To prove Theorem 1.1, we need some necessary notations and requisite lemmas. The following generalized Hölder inequality on variable Lebesgue spaces can be found in in [5] or [33].

Lemma 2.6 *Given exponent function $p_i(\cdot) \in \mathcal{P}^0$, define $p(\cdot) \in \mathcal{P}^0$ by*

$$\frac{1}{p(x)} = \sum_{i=1}^{m} \frac{1}{p_i(x)},$$

where $i = 1, \ldots, m$. Then for all $f_i \in L^{p_i(\cdot)}$ and $f_1 \ldots f_m \in L^{p(\cdot)}$ and

$$\left\| \prod_{i=1}^{m} f_i \right\|_{p(\cdot)} \leq C \prod_{i=1}^{m} \|f_i\|_{p_i(\cdot)}.$$

We also need the following boundedness of the vector-valued maximal operator M, whose proof can be found in [6].

Lemma 2.7 *Let $p(\cdot) \in \mathcal{P}^0 \cap LH$. Then for any $q > 1$, $f = \{f_i\}_{i \in \mathbb{Z}}$, $f_i \in L_{loc}$, $i \in \mathbb{Z}$*

$$\| \|\mathbb{M}(f)\|_{l^q} \|_{L^{p(\cdot)}} \leq C \| \|f||_{l^q} \|_{L^{p(\cdot)}},$$

where $\mathbb{M}(f) = \{M(f_i)\}_{i \in \mathbb{Z}}$.

Lemma 2.8 [21] *Let $p(\cdot) \in \mathcal{P}$, $f \in L^{p(\cdot)}$ and $g \in L^{p'(\cdot)}$, then fg is integrable on $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq r_p \|f\|_{L^{p(\cdot)}} \|g\|_{L^{p'(\cdot)}},$$

where $r_p = 1 + 1/p^- - 1/p^+$. Moreover, for all $g \in L^{p'(\cdot)}$ such that $\|g\|_{L^{p'(\cdot)}} \leq 1$ we get that

$$\|f\|_{L^{p(\cdot)}} \leq \sup_g \left| \int_{\mathbb{R}^n} f(x)g(x)dx \right| \leq r_p \|f\|_{L^{p(\cdot)}}.$$

Proof By applying the atomic decomposition of local Hardy space $h^{p(\cdot)}$ in Theorem 2.2, for each $f_j \in h^{p_j(\cdot)}$, $j = 1, \ldots, m$, $0 < s < \infty$, f_j admits an atomic decomposition: There exists a sequence of nonnegative numbers η_{j,k_j}, κ_{j,k_j}, cubes Q_{j,k_j} satisfying

$$\mathcal{A}_s(\{\eta_{j,k_j}\}_{j=1}^{\infty}, \{Q_{j,k_j}\}_{j=1}^{\infty}) + \mathcal{B}_s(\{\kappa_{j,k_j}\}_{j=1}^{\infty}, \{Q_{j,k_j}\}_{j=1}^{\infty}) \leq C \|f_j\|_{h^{p_j(\cdot)}},$$

for $\{a_j, Q_j\} \subset \mathcal{A}(p(\cdot, q))$ and $\{b_j, Q_j\} \subset \mathcal{B}(p(\cdot, q))$, and f_j can be decomposed as

$$f_j = \sum_{k_j \in \mathbb{N}} \eta_{j,k_j} a_{j,k_j} + \sum_{k_j \in \mathbb{N}} \kappa_{j,k_j} b_{j,k_j} =: \sum_{k_j \in \mathbb{N}} \lambda_{j,k_j} c_{j,k_j} \quad \text{in} \quad h^{p_j(\cdot)} \cap L^{p^+ + 1},$$

where $\lambda_{j,k_j} = \eta_{j,k_j}$ and $c_{j,k_j} = a_{j,k_j}$ for $|Q_{j,k_j}| \leq 1$, and $\lambda_{j,k_j} = \kappa_{j,k_j}$ and $c_{j,k_j} = b_{j,k_j}$ for $|Q_{j,k_j}| > 1$.

Then by repeating the similar argument in [11], we can obtain

$$|T_\sigma(\vec{f})(x)| \leq \sum_{k_1} \cdots \sum_{k_m} |\lambda_{1,k_1}| \cdots |\lambda_{m,k_m}| |T_\sigma(c_{1,k_1}, \ldots, c_{m,k_m})(x)|. \quad (2.1)$$

For $x \in \mathbb{R}^n$, we can split (2.1) into two terms, that is,

$$\begin{aligned}
&|T_\sigma(\vec{f})(x)| \\
&\leq \sum_{k_1} \cdots \sum_{k_m} |\lambda_{1,k_1}| \cdots |\eta_{m,k_m}| |T_\sigma(c_{1,k_1}, \ldots, c_{m,k_m})(x)| \chi_{Q^*_{1,k_1} \cap \cdots \cap Q^*_{m,k_m}}(x) \\
&\quad + \sum_{k_1} \cdots \sum_{k_m} |\lambda_{1,k_1}| \cdots |\eta_{m,k_m}| |T_\sigma(c_{1,k_1}, \ldots, c_{m,k_m})(x)| \chi_{Q^{*,c}_{1,k_1} \cup \cdots \cup Q^{*,c}_{m,k_m}}(x) \\
&=: I + II.
\end{aligned}$$

First we will show that

$$\|I\|_{L^{p(\cdot)}} \leq C \prod_{j=1}^m \|f_j\|_{h^{p_j(\cdot)}}. \quad (2.2)$$

Now fix atoms $c_{1,k_1}, \ldots, c_{m,k_m}$ supported in cubes $Q_{1,k_1}, \ldots, Q_{m,k_m}$ respectively. Assume that $Q^*_{1,k_1} \cap \cdots \cap Q^*_{m,k_m} \neq \varnothing$, otherwise there is nothing to prove. Without loss of generality, assume that Q_{1,k_1} has the smallest size among all these cubes. Since $Q^*_{1,k_1} \cap \cdots \cap Q^*_{m,k_m} \neq \varnothing$, we can pick a cube $R_{k_1,\ldots,k_m}$ such that

$$Q^*_{1,k_1} \cap \cdots \cap Q^*_{m,k_m} \subset R_{k_1,\ldots,k_m} \subset R^*_{k_1,\ldots,k_m} \subset Q^{**}_{1,k_1} \cap \cdots \cap Q^{**}_{m,k_m} \quad (2.3)$$

and

$$|Q_{1,k_1}| \leq C|R_{k_1,\ldots,k_m}|.$$

Since T_σ has a bounded extension from $\prod_{j=1}^m L^{q_j}$ into L^q, for all $1 < q, q_j \leq \infty$, $\sum_{j=1}^m \frac{1}{q_j} = \frac{1}{q}$, as in [14]. By using the Hölder's inequality with exponents q and q', we have

$$\begin{aligned}
&\frac{1}{|R_{k_1,\ldots,k_m}|} \int_{R_{k_1,\ldots,k_m}} |T_\sigma(c_{1,k_1}, \ldots, c_{m,k_m})(x)| dx \\
&\leq \frac{1}{|R_{k_1,\ldots,k_m}|^q} \|T_\sigma(c_{1,k_1}, \ldots, c_{m,k_m})\|_{L^q} \\
&\leq C \frac{1}{|R_{k_1,\ldots,k_m}|^q} \prod_{j=1}^m \|c_{j,k_j}\|_{L^{q_j}} \\
&\leq C \frac{1}{|R_{k_1,\ldots,k_m}|^q} \prod_{j=1}^m \frac{|Q_j, k_j|^{\frac{1}{q_j}}}{\|\chi_{Q_{j,k_j}}\|_{p_j(\cdot)}}
\end{aligned}$$

$$\leq C \frac{1}{|Q_{1,k_1}|^q} \prod_{j=1}^{m} \frac{|Q_j, k_j|^{\frac{1}{q_j}}}{\|\chi_{Q_{j,k_j}}\|_{p_j(\cdot)}}. \tag{2.4}$$

If we choose $q_1 = q, q_2 = \cdots = q_m = \infty$, then the last inequality of (2.4) can be controlled by $C \prod_{j=1}^{m} \frac{1}{\|\chi_{Q_{j,k_j}}\|_{p_j(\cdot)}}$.

In order to prove (2.2), we need to introduce another lemma:

Lemma 2.9 *Let $p(\cdot) \in LH \cap \mathcal{P}^0$. Suppose that we are given a sequence of cubes $\{Q_j\}_{j=1}^{\infty}$ and a sequence of non-negative L^1–functions $\{F_j\}_{j=1}^{\infty}$. Then for $\sum_{j=1}^{\infty} \chi_{Q_j} F_j \in L^{p(\cdot)}$ we have*

$$\left\| \sum_{j=1}^{\infty} \chi_{Q_j} F_j \right\|_{L^{p(\cdot)}} \leq C \left\| \sum_{j=1}^{\infty} \left(\frac{1}{|Q_j|} \int_{Q_j} F_j(y) dy \right) \chi_{Q_j} \right\|_{L^{p(\cdot)}}. \tag{2.5}$$

The proof of this lemma can be obtained by the extrapolation theorem on variable Lebesgue spaces. For $w \in A_{p_0}$, repeating the similar argument in [12] for $0 < p_0 \leq 1$ and in [13,32] for $1 < p_0 < \infty$ we can get the weighted norm inequality by substituting L^{p_0} norm by $L^{p_0}_w$ as below:

$$\left\| \sum_{j=1}^{\infty} \chi_{Q_j} F_j \right\|_{L^{p_0}_w} \leq C \left\| \sum_{j=1}^{\infty} \left(\frac{1}{|Q_j|} \int_{Q_j} F_j(y) dy \right) \chi_{Q_j} \right\|_{L^{p_0}_w}.$$

Observe that $p(\cdot) \in LH \bigcap \mathcal{P}^0$, by Lemma 2.3, for

$$\left(\sum_{j=1}^{\infty} \chi_{Q_j} F_j, \sum_{j=1}^{\infty} \left(\frac{1}{|Q_j|} \int_{Q_j} F_j(y) dy \right) \chi_{Q_j} \right) \in \mathcal{F}$$

and $\sum_{j=1}^{\infty} \chi_{Q_j} F_j \in L^{p(\cdot)}$, then we get (2.5).

Applying Lemma 2.9, we obtain that

$$\begin{aligned}
&\|I\|_{L^{p(\cdot)}} \\
&\leq \left\| \sum_{k_1} \cdots \sum_{k_m} \prod_{j=1}^{m} |\lambda_{j,k_j}| |T_\sigma(c_{1,k_1}, \ldots, c_{m,k_m})| \chi_{R_{k_1,\cdots,k_m}} \right\|_{L^{p(\cdot)}} \\
&\leq \left\| \sum_{k_1} \cdots \sum_{k_m} \prod_{j=1}^{m} |\lambda_{j,k_j}| \left| \frac{1}{|R_{k_1,\ldots,k_m}|} \int_{R_{k_1,\ldots,k_m}} T_\sigma(c_{1,k_1}, \ldots, c_{m,k_m})(y) dy \right| \right. \\
&\quad \left. \times \chi_{R_{k_1,\ldots,k_m}} \right\|_{L^{p(\cdot)}}
\end{aligned}$$

$$\leq \left\| \sum_{k_1} \cdots \sum_{k_m} \prod_{j=1}^{m} |\lambda_{j,k_j}| \prod_{j=1}^{m} \frac{1}{\|\chi_{Q_{j,k_j}}\|_{p_j(\cdot)}} \prod_{i=1}^{m} \chi_{Q_{j,k_j}^{**}} \right\|_{L^{p(\cdot)}}$$
$$\leq \left\| \prod_{j=1}^{m} \left(\sum_{k_j} \frac{|\lambda_{j,k_j}|}{\|\chi_{Q_{j,k_j}}\|_{p_j(\cdot)}} \chi_{Q_{j,k_j}^{**}} \right) \right\|_{L^{p(\cdot)}}.$$

We observe that we only need to consider the case $p^- \leq 1$. The other case $p^- > 1$ is easier because of $h^{p(\cdot)} \sim L^{p(\cdot)}$ when $p^- > 1$. Applying Lemmas 2.6 and 2.7 yields that

$$\left\| \prod_{j=1}^{m} \left(\sum_{k_j} |\lambda_{j,k_j}| \|\chi_{Q_{j,k_j}}\|_{L^{p_j(\cdot)}}^{-1} \chi_{Q_{j,k_j}^{**}} \right) \right\|_{L^{p(\cdot)}}$$
$$\leq C \prod_{j=1}^{m} \left\| \left(\sum_{k_j} |\lambda_{j,k_j}|^{p^-} \|\chi_{Q_{j,k_j}}\|_{L^{p_j(\cdot)}}^{-p^-} \chi_{Q_{j,k_j}^{**}} \right)^{1/p^-} \right\|_{L^{p_j(\cdot)}}$$
$$\leq C \prod_{j=1}^{m} \left\| \left(\sum_{k_j} |\lambda_{j,k_j}|^{p^-} \|\chi_{Q_{j,k_j}}\|_{L^{p_j(\cdot)}}^{-p^-} M\chi_{Q_{j,k_j}} \right)^{1/p^-} \right\|_{L^{p_j(\cdot)}}$$
$$\leq C \prod_{j=1}^{m} \left\| \left(\sum_{k_j} |\lambda_{j,k_j}|^{p^-} \|\chi_{Q_{j,k_j}}\|_{L^{p_j(\cdot)}}^{-p^-} \chi_{Q_{j,k_j}} \right)^{1/p^-} \right\|_{L^{p_j(\cdot)}}$$
$$= C \prod_{j=1}^{m} \left\| \left(\sum_{k_j} \left(|\lambda_{j,k_j}| \frac{\chi_{Q_{j,k_j}}}{\|\chi_{Q_{j,k_j}}\|_{L^{p_j(\cdot)}}} \right)^{p^-} \right)^{\frac{1}{p^-}} \right\|_{L^{p_j(\cdot)}}$$
$$\leq C \prod_{j=1}^{m} \|f_j\|_{H^{p_j(\cdot)}}.$$

Next we will consider the estimate of II. Let A be a nonempty subset of $\{1, \ldots, m\}$, and we denote the cardinality of A by $|A|$, then $1 \leq |A| \leq m$. Let $A^c = \{1, \ldots, m\} \backslash A$. If $A = \{1, \ldots, m\}$, we define $(\cap_{j \in A} Q_{j,k_j}^{*,c}) \cap (\cap_{j \in A^c} Q_{i,k_j}^{*,c}) = \cap_{j \in A} Q_{j,k_j}^{*,c}$, then

$$Q_{1,k_1}^{*,c} \cup \cdots \cup Q_{m,k_m}^{*,c} = \cup_{A \subset \{1,\ldots,m\}} ((\cap_{j \in A} Q_{j,k_j}^{*,c}) \cap (\cap_{j \in A^c} Q_{j,k_j}^{*})).$$

Set $E_A = (\cap_{j \in A} Q_{j,k_j}^{*,c}) \cap (\cap_{j \in A^c} Q_{j,k_j}^{*})$. For fixed A, assume that $Q_{\tilde{j},k_{\tilde{j}}}$ is the smallest cubes in the set of cubes Q_{j,k_j}, $j \in A$. Let $z_{\tilde{j},k_{\tilde{j}}}$ be the center of the cube $Q_{\tilde{j},k_{\tilde{j}}}$ and $d = \max\{d_{p_1}, \ldots, d_{p_m}\}$. When $|Q_{j,k_j}| \leq 1$, $c_{\tilde{j},k_{\tilde{j}}}$ is an $(p(\cdot), q)$-atom. Then $c_{\tilde{j},k_{\tilde{j}}}$ has zero vanishing moment up to order d. By using the Taylor expansion we get

$$
\begin{aligned}
&T_\sigma(c_{1,k_1},\ldots,c_{m,k_m})(x)\\
&\quad\le \int_{(\mathbb{R}^n)^m} K(x,y_1,\ldots,y_m)c_{1,k_1}(y_1)\ldots c_{m,k_m}(y_m)d\vec{y}\\
&\quad= \int_{(\mathbb{R}^n)^{m-1}} \prod_{j\neq\tilde{j}} c_{j,k_j}(y_j)\int_{\mathbb{R}^n}\left[K(x,y_1,\ldots,y_m)-P^d_{z_{\tilde{j},k_{\tilde{j}}}}(x,y_1,\ldots,y_m)\right]c_{\tilde{j},k_{\tilde{j}}}d\vec{y}\\
&\quad= \int_{(\mathbb{R}^n)^{m-1}} \prod_{j\neq\tilde{j}} c_{j,k_j}(y_j)\\
&\qquad\times\int_{\mathbb{R}^n}\sum_{|\gamma|=d+1}\left(\partial^\gamma_{y_{\tilde{j}}}K\right)(x,y_1,\ldots,\xi,\ldots,y_2,y_m)\frac{\left(y_{\tilde{j}}-z_{\tilde{j},k_{\tilde{j}}}\right)^\gamma}{\gamma!}c_{\tilde{j}}(y_{\tilde{j}})d\vec{y}
\end{aligned}
$$

for some ξ on the line segment joining $y_{\tilde{j}}$ to $z_{\tilde{j},k_{\tilde{j}}}$, where $P^d_{z_{\tilde{j},k_{\tilde{j}}}}(x,y_1,\ldots,y_m)$ is the Taylor polynomial of $K(x,y_1,\ldots,y_m)$. Since $x\in(Q^*_{\tilde{j},k_{\tilde{j}}})^c$, we can easily obtain that $|x-\xi|\ge\frac{1}{2}|x-z_{\tilde{j},k_{\tilde{j}}}|$. Similarly, $|x-y_j|\ge\frac{1}{2}|x-z_{j,k_j}|$ for $y_j\in Q_{j,k_j}$, $j\in A\backslash\{\tilde{j}\}$.

By using the estimate for the kernel K in Lemma 2.5 and the size estimates for the atoms or blocks, we get that

$$
\begin{aligned}
&\int_{(\mathbb{R}^n)^{m-1}} \prod_{j\neq\tilde{j}} |c_{j,k_j}(y_j)|\\
&\quad\times\int_{\mathbb{R}^n}\sum_{|\gamma|=d+1}|\left(\partial^\gamma_{y_{\tilde{j}}}K\right)(x,y_1,\ldots,\xi,\ldots,y_2,y_m)|\frac{|y_{\tilde{j}}-z_{\tilde{j},k_{\tilde{j}}}|^\gamma}{\gamma!}|a_{\tilde{j}}(y_{\tilde{j}})|d\vec{y}\\
&\le C\int_{(\mathbb{R}^n)^{|A|}}\prod_{j\in A}|c_{j,k_j}(y_j)|\\
&\quad\times\int_{(\mathbb{R}^n)^{m-|A|}}\frac{|y_{\tilde{j}}-z_{\tilde{j},k_{\tilde{j}}}|^{d+1}}{\left(|x-\xi|+\sum_{j\neq\tilde{j}}|x-y_j|\right)^{mn+d+1}}\prod_{j\in A^c}|c_{j,k_j}(y_{\tilde{j}})|d\vec{y}\\
&\le C\Bigg(\prod_{j\in A}\|c_{j,k_j}\|_{L^1}\Bigg)\Bigg(\prod_{j\in A^c}\|c_{j,k_j}\|_{L^\infty}\Bigg)\\
&\quad\times\int_{(\mathbb{R}^n)^{m-|A|}}\frac{|y_{\tilde{j}}-z_{\tilde{j},k_{\tilde{j}}}|^{d+1}}{\left(|x-\xi|+\sum_{j\neq\tilde{j}}|x-y_j|\right)^{mn+d+1}}d\vec{y}_{A^c}\\
&\le C\Bigg(\prod_{j\in A}\|c_{j,k_j}\|_{L^1}\Bigg)\Bigg(\prod_{j\in A^c}\|c_{j,k_j}\|_{L^\infty}\Bigg)\\
&\quad\times\frac{|y_{\tilde{j}}-z_{\tilde{j},k_{\tilde{j}}}|^{d+1}}{\left(\sum_{j\in A}|x-z_{j,k_j}|\right)^{mn+d+1-n(m-|A|)}}
\end{aligned}
$$

$$\leq C\Big(\prod_{j\in A}\frac{|Q_{j,k_j}|}{\|\chi_{Q_{j,k_j}}\|_{L^{p_j(\cdot)}}}\Big)\Big(\prod_{j\in A^c}\frac{1}{\|\chi_{Q_{j,k_j}}\|_{L^{p_j(\cdot)}}}\Big)$$
$$\times\ \frac{|Q_{\tilde{j},k_{\tilde{j}}}|^{(d+1)/n}}{\big(\sum_{j\in A}|x-z_{j,k_j}|\big)^{mn+d+1-n(m-|A|)}}$$
$$\leq C\Big(\prod_{j\in A}\frac{|Q_{j,k_j}|}{\|\chi_{Q_{j,k_j}}\|_{L^{p_j(\cdot)}}}\Big)\Big(\prod_{j\in A^c}\frac{1}{\|\chi_{Q_{j,k_j}}\|_{L^{p_j(\cdot)}}}\Big)$$
$$\times\ \frac{|Q_{\tilde{j},k_{\tilde{j}}}|^{(d+1)/n}}{\big(\sum_{j\in A}|x-z_{j,k_j}|\big)^{mn+d+1-n(m-|A|)}}.$$

Observe that $x\in\cap_{j\in A}Q^{*,c}_{j,k_j}$, then we can find a constant C such that $|x-z_{j,k_j}|\geq C(|x-z_{j,k_j}|+l(Q_{j,k_j}))$. On the other hand, using the fact that $x\in\cap_{j\in A^c}Q^*_{j,k_j}$ yields that there exists a constant C such that $|x-z_{j,k_j}|\leq Cl(Q_{j,k_j})$ for $j\in A^c$. Then we have that

$$\frac{|Q_{j,k_j}|^{1+\frac{d+1}{n|A|}}}{\big(|x-z_{j,k_j}|+l(Q_{j,k_j})\big)^{n+\frac{d+1}{|A|}}}\geq C,\quad \text{for}\quad j\in A^c.$$

Moreover, since $Q_{\tilde{j},k_{\tilde{j}}}$ is the smallest cube among $\{Q_{j,l_j}\}_{j\in A}$, we have that $|Q_{\tilde{j},k_{\tilde{j}}}|\leq\prod_{j\in A}|Q_{j,l_j}|^{\frac{1}{|A|}}$. Thus,

$$|T(c_{1,k_1},\ldots,c_{m,k_m})(x)|$$
$$\leq C\Big(\prod_{j\in A}\frac{|Q_{j,k_j}|^{1+\frac{d+1}{n|A|}}}{\|\chi_{Q_{j,k_j}}\|_{L^{p_j(\cdot)}}\big(|x-z_{j,k_j}|+l(Q_{j,k_j})\big)^{n+\frac{d+1}{|A|}}}\Big)\Big(\prod_{j\in A^c}\frac{1}{\|\chi_{Q_{j,k_j}}\|_{L^{p_j(\cdot)}}}\Big)$$
$$\leq C\prod_{j=1}^{m}\frac{|Q_{j,k_j}|^{1+\frac{d+1}{n|A|}}}{\|\chi_{Q_{j,k_j}}\|_{L^{p_j(\cdot)}}\big(|x-z_{j,k_j}|+l(Q_{j,k_j})\big)^{n+\frac{d+1}{|A|}}} \tag{2.6}$$

for all $x\in E_A$.

When $|Q_{j,k_j}|>1$, $c_{\tilde{j},k_{\tilde{j}}}$ is an $(p(\cdot),q)-$block. Using the estimate for the kernel K in Lemma 2.5 and a computation similar to the above, we obtain

$$T_\sigma(c_{1,k_1},\ldots,c_{m,k_m})(x)$$
$$\leq\int_{(\mathbb{R}^n)^m}K(x,y_1,\ldots,y_m)c_{1,k_1}(y_1)\cdots c_{m,k_m}(y_m)d\vec{y}$$
$$=\int_{(\mathbb{R}^n)^{m-1}}\prod_{j\neq\tilde{j}}c_{j,k_j}(y_j)\int_{\mathbb{R}^n}K(x,y_1,\ldots,y_m)c_{\tilde{j},k_{\tilde{j}}}d\vec{y}$$

$$\leq C\int_{(\mathbb{R}^n)^{|A|}}\prod_{j\in A}|c_{j,k_j}(y_j)|\int_{(\mathbb{R}^n)^{m-|A|}}\frac{1}{\sum_{j=1}^m|x-y_j|^{mn+M}}\prod_{j\in A^c}|c_{j,k_j}(y_{\tilde{j}})|d\vec{y}$$
$$\leq C\Big(\prod_{j\in A}\|c_{j,k_j}\|_{L^1}\Big)\Big(\prod_{j\in A^c}\|c_{j,k_j}\|_{L^\infty}\Big)\int_{(\mathbb{R}^n)^{m-|A|}}\frac{1}{\sum_{j=1}^m|x-y_j|^{mn+M}}d\vec{y}_{A^c}$$
$$\leq C\Big(\prod_{j\in A}\frac{|Q_{j,k_j}|}{\|\chi_{Q_{j,k_j}}\|_{L^{p_j(\cdot)}}}\Big)\Big(\prod_{j\in A^c}\frac{1}{\|\chi_{Q_{j,k_j}}\|_{L^{p_j(\cdot)}}}\Big)$$
$$\times\frac{1}{\big(\sum_{j\in A}|x-z_{j,k_j}|\big)^{mn+M-n(m-|A|)}}$$
$$\leq C\prod_{j=1}^m\frac{|Q_{j,k_j}|^{1+\frac{M}{n|A|}}}{\|\chi_{Q_{j,k_j}}\|_{L^{p_j(\cdot)}}\big(|x-z_{j,k_j}|+l(Q_{j,k_j})\big)^{n+\frac{M}{|A|}}}$$

for all $M\geq 0$.

For convenience, we choose $M=d+1$. Then we obtain

$$\|II\|_{L^{p(\cdot)}}$$
$$\leq C\sum_{A\subset\{1,\ldots,m\}}$$
$$\times\Bigg\|\prod_{j=1}^m\sum_{k_j}|\lambda_{j,k_j}|\frac{|Q_{j,k_j}|^{1+\frac{d+1}{n|A|}}}{\|\chi_{Q_{j,k_j}}\|_{L^{p_j(\cdot)}}\big(|x-z_{j,k_j}|+l(Q_{j,k_j})\big)^{n+\frac{d+1}{|A|}}}\chi_{E_A}\Bigg\|_{L^{p(\cdot)}}$$
$$\leq C\sum_{A\subset\{1,\ldots,m\}}\prod_{j=1}^m$$
$$\times\Bigg\|\sum_{k_j}|\lambda_{j,k_j}|\frac{|l(Q_{j,k_j})|^{n+\frac{d+1}{|A|}}}{\|\chi_{Q_{j,k_j}}\|_{L^{p_j(\cdot)}}\big(|x-z_{j,k_j}|+l(Q_{j,k_j})\big)^{n+\frac{d+1}{|A|}}}\chi_{E_A}\Bigg\|_{L^{p_j(\cdot)}}.$$

Denote $\theta=\frac{n+\frac{d+1}{|A|}}{n}$ and we can choose d such that $\theta p_j^->1$. Therefore, we get that

$$\|II\|_{L^{p(\cdot)}}\leq C\prod_{j=1}^m\Bigg\|\sum_{k_j}|\lambda_{j,k_j}|\frac{(M\chi_{Q_{j,k_j}})^\theta}{\|\chi_{Q_{j,k_j}}\|_{L^{p_j(\cdot)}}}\Bigg\|_{L^{p_j(\cdot)}}$$
$$\leq\prod_{j=1}^m\Bigg\|\Bigg(\sum_{k_j}|\lambda_{j,k_j}|\frac{\chi_{Q_{j,k_j}}}{\|\chi_{Q_{j,k_j}}\|_{L^{p_j(\cdot)}}}\Bigg)^{\frac{1}{\theta}}\Bigg\|_{L^{\theta p_j(\cdot)}}^\theta\leq C\prod_{j=1}^m\|f_j\|_{H^{p_j(\cdot)}}.$$

Therefore, we complete the proof of Theorem 1.1. □

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