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On the images of Dunkl–Sobolev spaces under the Schrödinger semigroup associated to Dunkl operators

C. Sivaramakrishnan[1] · **D. Sukumar**[1] · **D. Venku Naidu**[1]

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Abstract In this article, we consider the Schrödinger semigroup related to the Dunkl–Laplacian Δ_μ (associated to finite reflection group G) on $\mathbb{R}^n$. We characterize the image of $L^2(\mathbb{R}^n, e^{u^2} h_\mu(u)du)$ under the Schrödinger semigroup as a reproducing kernel Hilbert space. We define Dunkl–Sobolev space in $L^2(\mathbb{R}^n, e^{u^2} h_\mu(u)du)$ and characterize it's image under the Schrödinger semigroup associated to $G = \mathbb{Z}_2^n$ as a reproducing kernel Hilbert space up to equivalence of norms. Also we provide similar results for Schrödinger semigroup associated to Dunkl–Hermite operator.

Keywords Segal–Bargmann transform · Schrödinger semigroup · Weighted Bergman space · Dunkl–Sobolev space

Mathematics Subject Classification Primary 46E35; Secondary 46F12 · 47D06 · 35B65 · 35J10

1 Introduction

In [5], Dunkl introduced a differential operator associated to a finite reflection group G on $\mathbb{R}^n$ which is generated by fixed root system $\mathcal{R}$ and a non-negative multiplicity

✉ D. Venku Naidu
venku@iith.ac.in

C. Sivaramakrishnan
ma13p1001@iith.ac.in

D. Sukumar
suku@iith.ac.in

1 Department of Mathematics, Indian Institute of Technology Hyderabad, Hyderabad, Telangana 502285, India

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function μ on the root system. This differential operator is called as a Dunkl operator. The Dunkl operator is same as the directional derivative on $\mathbb{R}^n$ when $\mu = 0$.

During the last three decades Dunkl theory which is parallel to the theory of Fourier analysis has been developed by many authors. We refer to [4,9,24] and references therein for further details. Our study is related to the Segal–Bargmann analysis which was initiated by Segal [20] and Bargmann [1]. It deals with the problems related to some integral transforms on $L^2(\mathbb{R}^n, du)$ and characterizes the image as a weighted Bergman space. The study of several generalizations of classical Segal–Bargmann transform can be found in [6,7,10,11,14,15] and the Segal–Bargmann transform associated to the Dunkl–Laplacian can be found in [2,3,16,21–23].

In [17], Rosler considered the Dunkl–Laplacian Δ_μ and proved that it generates the Heat semigroup $e^{t\Delta_\mu}$ on $L^2(\mathbb{R}^n, h_\mu(u)du)$. He also proved, $e^{t\Delta_\mu}$ is an integral transform with the integral kernel $\Gamma_\mu(t, x, y)$. In [22], Sontz characterized the image of $L^2(\mathbb{R}, h_\mu(u)du)$ under the heat kernel semigroup $e^{t\Delta_\mu}$, as a direct sum of Fock type spaces. He also characterized in [23], the image of $L^2(\mathbb{R}^n, h_\mu(u)du)$ under the heat kernel semigroup as a reproducing kernel Hilbert space.

In this article, we study the image of certain function space under the Schrödinger semigroup $e^{it\Delta_\mu}$ associated to the Dunkl–Laplacian and Dunkl–Hermite operator. Using the heat kernel given in [17] we can see that, the Schrödinger semigroup is an integral transform on $L^2(\mathbb{R}^n, h_\mu(u)du)$ with the integral kernel $\Gamma_\mu(it, x, y)$. Since the Schrödinger semigroup $e^{it\Delta_\mu}$ is a unitary operator on $L^2(\mathbb{R}^n, h_\mu(u)du)$, the function $e^{it\Delta_\mu} f$ cannot be extended as an entire function for all $f \in L^2(\mathbb{R}^n, h_\mu(u)du)$. Hence we consider a suitable function space on which $e^{it\Delta_\mu} f$ can be extended as an entire function and characterize the corresponding image space as a reproducing kernel Hilbert space.

In the Context of Dunkl operator for $\mu = 0$, the above study for $n = 1$ is done by Hayashi and Saitoh in [8] and for the general case done in [13] by Parui et al.

This paper is organized as follows: In Sect. 2 we give an introduction to Dunkl operator, Dunkl transform and some results on the heat kernel transform which are necessary to prove our main results. In Sect. 3 we identify the image of $L^2(\mathbb{R}^n, e^{u^2} h_\mu(u)du)$ under Schrödinger semigroup $e^{it\Delta_\mu}$ as a reproducing kernel Hilbert space, by using the techniques given in [13,22,23]. Moreover, in the case of $G = \mathbb{Z}_2^n$, we identify the image of $L^2(\mathbb{R}^n, e^{u^2} h_\mu(u)du)$ under $e^{it\Delta_\mu}$ as a tensor product of Fock type spaces. In Sect. 4 we consider the Dunkl–Sobolev space defined by the Dunkl operator associated to the group $\mathbb{Z}_2^n$ and identify the image of Sobolev space under the semigroup $e^{it\Delta_\mu}$ as a reproducing kernel Hilbert space up to an equivalence of norms. In Sect. 5 we discuss the same kind of results as in Sects. 3 and 4 for the Schrödinger semigroup e^{-itH_μ} associated to the Dunkl–Hermite operator H_μ.

2 Preliminaries

Let $(\mathbb{R}^n, \langle , \rangle)$ be a standard Euclidean inner product space. For a non-zero vector $v \in \mathbb{R}^n$, define the reflection on the hyperspace $\{v\}^\perp$ by $\sigma_v(x) = x - 2\frac{\langle x, v\rangle}{||v||^2} v$ for $x \in \mathbb{R}^n$. A finite subset $\mathcal{R} \subset \mathbb{R}^n \backslash \{0\}$ is called a root system if $\sigma_v(\mathcal{R}) = \mathcal{R}$ for all $v \in \mathcal{R}$. For a given $\beta \in \mathbb{R}^n \backslash \cup_{v \in \mathcal{R}} \{v\}^\perp$ define a positive root system $\mathcal{R}_+ = \{v \in \mathcal{R} : \langle \beta, v\rangle > 0\}$.

The group G is generated by the reflections $\{\sigma_v : v \in \mathcal{R}\}$ is called a reflection group, which is a subgroup of the orthogonal group on $\mathbb{R}^n$. A function $\mu : \mathcal{R} \to [0, \infty)$ which is invariant under the action of G on the root system is called multiplicity function. The weight function $h_\mu(x)$, associated with the root system $\mathcal{R}$ and the multiplicity function μ, is defined by $h_\mu(x) = \prod_{v \in R_+} |\langle x, v\rangle|^{2\mu(v)},\ x \in \mathbb{R}^n$. For $0 \neq \xi \in \mathbb{R}^n$, Dunkl derivative is defined by

$$D_{\xi,\mu} f(x) = \partial_\xi f(x) + \sum_{v \in \mathcal{R}_+} \mu(v)\langle v, \xi\rangle \frac{f(x) - f(\sigma_v x)}{\langle x, v\rangle}, \tag{2.1}$$

where ∂_ξ is the directional derivative in the direction of ξ. Now consider the equation,

$$D_{\xi,\mu} f(x, y) = \langle \xi, y\rangle f(x, y) \quad \text{for } x, y, \xi \in \mathbb{R}^n.$$

In the above equation $D_{\xi,\mu} f$ is the Dunkl derivative of f with respect to the x-variable. The above equation has a unique real analytic solution $E_\mu : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and it can be extended as an analytic function $E_\mu : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$, which is called as a Dunkl kernel or generalized exponential kernel. In the remaining sections we use the following notations

- $D_{i,\mu} := D_{e_i,\mu}$ for $i = 1, \ldots n$, where $\{e_i : i = 1, 2, ..., n\}$ is a standard orthonormal basis for $\mathbb{R}^n$.
- $\mathbb{N}^n = \mathbb{N} \times \cdots \times \mathbb{N}$ (n-times), where $\mathbb{N} := \{0, 1, 2, \ldots\}$.
- For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$, define $|\alpha| := \alpha_1 + \cdots + \alpha_n$, $\alpha! := \alpha_1! \cdots \alpha_n!$.
- $z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ and $z^2 := z_1^2 + \cdots + z_n^2$, for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $\alpha \in \mathbb{N}^n$.

Definition 2.1 The Dunkl transform of a function $f \in L^1(\mathbb{R}^n, h_\mu(x))$ is defined by $\widehat{f}(y) = c_\mu^{-1} \int_{\mathbb{R}^n} f(x) E_\mu(-iy, x) h_\mu(x) dx,\ y \in \mathbb{R}^n$, where $c_\mu = \int_{\mathbb{R}^n} e^{-\frac{x^2}{2}} h_\mu(x) dx$, a constant.

Definition 2.2 The generalized translation (or Dunkl translation) of a function $f \in L^2(\mathbb{R}^n, h_\mu(u)du)$ is defined by

$$\tau_y^\mu f(x) = c_\mu^{-1} \int_{\mathbb{R}^n} \widehat{f}(\xi) E_\mu(ix, \xi) E_\mu(-iy, \xi) h_\mu(\xi) d\xi, \quad x, y \in \mathbb{R}^n. \tag{2.2}$$

Definition 2.3 Generalized convolution of $f, g \in L^2(\mathbb{R}^n, h_\mu(u)du)$ is given by

$$f *_\mu g(x) = \int_{\mathbb{R}^n} f(y) \tau_x^\mu \check{g}(y) h_\mu(y) dy, \tag{2.3}$$

where $\check{g}(u) = g(-u)$. Equivalently it can be written as

$$f *_\mu g(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) \widehat{g}(\xi) E_\mu(ix, \xi) h_\mu(\xi) d\xi.$$

Dunkl–Laplacian on $\mathbb{R}^n$ is defined by

$$\Delta_\mu := \sum_{k=1}^{n} D_{k,\mu}^2.$$

Dunkl–Laplacian generates a strongly continuous semigroup $\left(e^{t\Delta_\mu}\right)_{t\geq 0}$ on $L^2(\mathbb{R}^n, h_\mu(u)du)$. The fundamental solution or heat kernel of the heat equation $\partial_t u = \Delta_\mu u$ on $(0,\infty)\times\mathbb{R}^n$ is given by the function,

$$F_\mu(t,x) = \frac{M_\mu}{t^{\nu_\mu+\frac{n}{2}}} e^{-\frac{x^2}{4t}}, \quad \text{where } \nu_\mu = \sum_{v\in R_+} \mu(v) \quad \text{and} \quad M_\mu = c_\mu^{-1} 2^{-(\nu_\mu+\frac{n}{2})}.$$

It can be seen that, the generalized translation of heat kernel is given by

$$\Gamma_\mu(t,x,y) = \frac{M_\mu}{t^{\nu_\mu+\frac{n}{2}}} e^{-\frac{x^2+y^2}{4t}} E_\mu\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right), \quad x, y \in \mathbb{R}^n. \tag{2.4}$$

That is, $\Gamma_\mu(t,x,y) = \tau_y^\mu F_\mu(t,x)$. In [17], Rosler proved that the heat kernel transform associated to Dunkl–Laplacian Δ_μ is an integral transform and it is given by

$$e^{t\Delta_\mu} f(x) := \begin{cases} \int_{\mathbb{R}^n} f(u)\Gamma_\mu(t,x,u)h_\mu(u)du & \text{if } t > 0 \\ f & \text{if } t = 0. \end{cases}$$

Since $e^{t\Delta_\mu} f$ is a Dunkl convolution of f with the function F_μ, it can be extended as an entire function on $\mathbb{C}^n$. Hence we can treat $e^{t\Delta_\mu}$ as a linear operator from $L^2(\mathbb{R}^n, h_\mu(u)du)$ to $\mathcal{O}(\mathbb{C}^n)$, the space of all analytic functions on $\mathbb{C}^n$.

We need some notations to state the theorem related to the image characterization, for the case $n = 1$. For $z \in \mathbb{C}$, $t > 0$ and $\mu > 0$, define the weight functions

$$\nu_{e,\mu,t}(z) := \pi^{-1} 2^{\mu+\frac{1}{2}} (2t)^{\mu-\frac{1}{2}} e^{\frac{z^2+\bar{z}^2}{8t}} K_{\mu-\frac{1}{2}}\left(\left|\frac{z}{(4t)^{\frac{1}{2}}}\right|^2\right) \left|\frac{z}{(4t)^{\frac{1}{2}}}\right|^{2\mu+1} \tag{2.5}$$

and

$$\nu_{o,\mu,t}(z) := \pi^{-1} 2^{\mu+\frac{1}{2}} (2t)^{\mu-\frac{1}{2}} e^{\frac{z^2+\bar{z}^2}{8t}} K_{\mu+\frac{1}{2}}\left(\left|\frac{z}{(4t)^{\frac{1}{2}}}\right|^2\right) \left|\frac{z}{(4t)^{\frac{1}{2}}}\right|^{2\mu+1}, \tag{2.6}$$

where for $r \in \left\{\mu+\frac{1}{2}, \mu-\frac{1}{2}\right\}$, K_r is the Macdonald function of order r which can be found in [12]. Any function $f \in \mathcal{O}(\mathbb{C})$ can be written as $f = f_e + f_o$, a sum of even and odd functions defined by $f_e(z) = \frac{f(z)+f(-z)}{2}$ and $f_o(z) = \frac{f(z)-f(-z)}{2}$, respectively. Consider the space

$$\mathcal{C}_{\mu,t}(\mathbb{C}) := \left\{ f \in \mathcal{O}(\mathbb{C}) : f_e \in L^2(\mathbb{C}, \nu_{e,\mu,t}(z)dz) \text{ and } f_o \in L^2(\mathbb{C}, \nu_{o,\mu,t}(z)dz) \right\}.$$

The space $\mathcal{C}_{\mu,t}(\mathbb{C})$ is a Hilbert space with respect to the inner product

$$\langle f, g\rangle_{\mathcal{C}_{\mu,t}(\mathbb{C})} := \langle f_o, g_o\rangle_{L^2(\mathbb{C},\nu_{o,\mu,t}(z)dz)} + \langle f_e, g_e\rangle_{L^2(\mathbb{C},\nu_{e,\mu,t}(z)dz)} \quad \text{for } f, g \in \mathcal{C}_{\mu,t}(\mathbb{C}).$$

Theorem 2.4 [22] *For $t > 0$, the operator $e^{t\Delta_\mu} : L^2(\mathbb{R}, |u|^{2\mu}\,du) \to \mathcal{C}_{\mu,t}(\mathbb{C})$ is unitary.*

The general case was studied by Sontz in [23] and he identified the image of $L^2(\mathbb{R}^n, h_\mu(u)du)$ under $e^{t\Delta_\mu}$ as a reproducing kernel Hilbert space. This theorem can be stated as follows:

Theorem 2.5 *The operator $e^{t\Delta_\mu} : L^2(\mathbb{R}^n, h_\mu(u)du) \to \mathcal{C}_{\mu,t}(\mathbb{C}^n)$ is unitary, where $\mathcal{C}_{\mu,t}(\mathbb{C}^n)$ is the Hilbert space of analytic functions on $\mathbb{C}^n$ with reproducing kernel*

$$\mathbb{K}_{\mu,t}(z, w) := e^{-\left(\frac{z^2+\overline{w}^2}{8t}\right)} E_\mu\left(\frac{z}{2t^{\frac{1}{2}}}, \frac{\overline{w}}{2t^{\frac{1}{2}}}\right), \quad z, w \in \mathbb{C}^n. \tag{2.7}$$

Let us introduce some identities which will be useful to prove our main results. For $n = 1$ and $\mu > 0$, the Dunkl operator associated to the reflection group $\mathbb{Z}_2$ is denoted by $\mathcal{D}_\mu$ and it is given by

$$(D_\mu f)(x) = \frac{df}{dx}(x) + \frac{\mu}{x}(f(x) - f(-x)), \quad x \in \mathbb{R}. \tag{2.8}$$

For $k \in \mathbb{N} := \{0, 1, 2, \ldots\}$, the generalized factorial function is defined by,

$$\gamma_\mu(2k) = \frac{2^{2k}k!\Gamma(k + \mu + \frac{1}{2})}{\Gamma(\mu + \frac{1}{2})} \quad \text{and} \quad \gamma_\mu(2k + 1) = \frac{2^{2k+1}k!\Gamma(k + \mu + \frac{3}{2})}{\Gamma(\mu + \frac{1}{2})}.$$

Generalized factorial function has the following recursion formula:

$$\gamma_\mu(k + 1) = (k + 1 + 2\mu\theta_{k+1})\gamma_\mu(k), \quad k \in \mathbb{N} \tag{2.9}$$

where $\theta_{k+1} = 0$ if $k + 1$ is even and 1 if $k + 1$ is odd.

The generalized exponential kernel for one dimensional space is

$$e_\mu(xy) := E_\mu(x, y) = \sum_{k=0}^{\infty} \frac{(xy)^k}{\gamma_\mu(k)} \quad \text{for } x, y \in \mathbb{R}. \tag{2.10}$$

Generalized Hermite polynomials and inversion formula for Hermite polynomials are given by

$$H_k^\mu(x) = k! \sum_{j=0}^{[\frac{k}{2}]} \frac{(-1)^j (2x)^{k-2j}}{j!\gamma_\mu(k - 2j)} \quad \text{and} \quad \frac{(2x)^k}{\gamma_\mu(k)} = \sum_{j=0}^{[\frac{k}{2}]} \frac{H_{k-2j}(x)}{j!(k - 2j)!}. \tag{2.11}$$

Further we need the following identities:

$$D_\mu(e^{-x^2}H_k^\mu(x)) = -e^{-x^2}\frac{\gamma_\mu(k+1)}{(k+1)\gamma_\mu(k)}H_{k+1}^\mu(x). \tag{2.12}$$

$$2xH_k(x) = \frac{(k+1+2\mu\theta_{k+1})}{k+1}H_{k+1}^\mu(x) + 2kH_{k-1}^\mu(x). \tag{2.13}$$

$$\int_{\mathbb{R}} u^k e_\mu(-ixu)e^{-u^2}\,|u|^{2\mu}\,du = \left(-\frac{i}{2}\right)^k \frac{\Gamma(\mu+\frac{1}{2})\gamma_\mu(k)}{k!}e^{-\frac{x^2}{4}}H_k^\mu\left(\frac{x}{2}\right). \tag{2.14}$$

We refer [16] for further details.

Let $\rho(z)$ be a strictly positive continuous function on $\mathbb{C}^n$. We define the weighted Bergman space associated to ρ by

$$\mathcal{HL}_\rho^2 := \mathcal{HL}^2(\mathbb{C}^n, \rho(z)dz) = \left\{ f \in \mathcal{O}(\mathbb{C}^n) : \int_{\mathbb{C}^n} |f(z)|^2\,\rho(z)dx < \infty \right\}.$$

Inner product on $\mathcal{HL}^2(\mathbb{C}^n, \rho(z)dz)$ is given by,

$$\langle F, G\rangle_{\mathcal{HL}_\rho^2} := \int_{\mathbb{C}^n} F(z)\overline{G(z)}\rho(z)dz, \quad \text{where } F, G \in \mathcal{HL}^2(\mathbb{C}^n, \rho(z)dz).$$

It is known that $\mathcal{HL}^2(\mathbb{C}^n, \rho(z)dz)$ is a reproducing kernel Hilbert space.

3 Image of $L_\mu^2(\mathbb{R})$ under Schrödinger semigroup

It is well known that Δ_μ is self-adjoint, so the operator $i\Delta_\mu$ is skew-adjoint. By Stones theorem, $i\Delta_\mu$ generates a strongly continuous unitary semigroup $\left(e^{it\Delta_\mu}\right)_{t\geq 0}$ on $L^2(\mathbb{R}^n, h_\mu(u)du)$, where

$$e^{it\Delta_\mu}f := \begin{cases} \int_{\mathbb{R}^n} f(y)\Gamma_\mu(it, ., y)h_\mu(y)dy & \text{if } t > 0 \\ f & \text{if } t = 0. \end{cases}$$

Moreover, $e^{it\Delta_\mu}f$ solves

$$i\partial_t u = \Delta_\mu u;\ u(x,0) = f(x), \quad \text{where} \quad f \in L^2(\mathbb{R}^n, h_\mu(u)du),$$

the Schrödinger equation associated to the Dunkl–Laplacian Δ_μ on $\mathbb{R}^n$. In contrast to the heat semigroup, the Schrödinger semigroup is unitary on $L^2(\mathbb{R}^n, h_\mu(u)dx)$, so the solution of the Schrödinger equation cannot be extended as an entire function on $\mathbb{C}^n$. If we assume enough decay on f near infinity then it is expected that $e^{it\Delta_\mu}f$ can be extended as an analytic function on $\mathbb{C}^n$. To achieve this we consider the space $L^2_\mu(\mathbb{R}^n) := L^2(\mathbb{R}^n, e^{u^2}h_\mu(u)du)$ and for $f \in L^2_\mu(\mathbb{R}^n)$, using Morera's and Dominated convergence theorems, it is easy to see that $e^{it\Delta_\mu}f$ can be extended as an analytic function on $\mathbb{C}^n$. Hence $e^{it\Delta_\mu}(L^2_\mu(\mathbb{R}^n))$ is a subspace of space of all analytic functions on $\mathbb{C}^n$. It is clear that $e^{it\Delta_\mu} : L^2(\mathbb{R}^n, e^{u^2}h_\mu(u)du) \to e^{it\Delta_\mu}(L^2_\mu(\mathbb{R}^n))$ is linear and bijective. Since it is injective, $e^{it\Delta_\mu}(L^2_\mu(\mathbb{R}^n))$ is made into a Hilbert space simply by transferring the Hilbert space structure of $L^2(\mathbb{R}^n, e^{u^2}h_\mu(u)du)$ to $e^{it\Delta_\mu}(L^2_\mu(\mathbb{R}^n))$ so that the Schrödinger semigroup $e^{it\Delta}$ is an isometric isomorphism from $L^2(\mathbb{R}^n, e^{u^2}h_\mu(u)du)$ onto $e^{it\Delta_\mu}(L^2_\mu(\mathbb{R}^n))$. This means that

$$\left\langle e^{it\Delta_\mu}f, e^{it\Delta_\mu}g\right\rangle_{e^{it\Delta_\mu}(L^2_\mu(\mathbb{R}^n))} := \langle f, g\rangle_{L^2_\mu(\mathbb{R}^n)}, \quad f, g \in L^2_\mu(\mathbb{R}^n).$$

Our aim is to identify this space as a reproducing kernel Hilbert space. This kind of characterization associated to $\mu = 0$ can be found in [13]. In fact, it says the following: $e^{it\Delta} : L^2(\mathbb{R}^n, e^{u^2}du) \to \mathcal{HL}^2(\mathbb{C}^n, w_t(x+iy)dxdy)$ is unitary, where $w_t(x+iy) = \frac{1}{(2\sqrt{\pi t})^n}e^{\frac{xy}{t}-\frac{y^2}{4t^2}}$. To state our main result we need the following.

Define a linear map $\mathcal{G} : \mathcal{O}(\mathbb{C}^n) \to \mathcal{O}(\mathbb{C}^n)$ by

$$\mathcal{G}F(z) = (2it)^{\nu_\mu+\frac{n}{2}}\, e^{-itz^2}F(2tz),$$

and consider the space,

$$\mathcal{H}_{\mu,t}(\mathbb{C}^n) := \left\{F \in \mathcal{O}(\mathbb{C}^n) : \mathcal{G}(F) \in \mathcal{C}_{\mu,\frac{1}{2}}(\mathbb{C}^n)\right\}.$$

The space $\mathcal{H}_{\mu,t}(\mathbb{C}^n)$ is Hilbert space with respect to the following inner product:

$$\langle F, G\rangle_{\mathcal{H}_{\mu,t}(\mathbb{C}^n)} := \langle \mathcal{G}F, \mathcal{G}G\rangle_{\mathcal{C}_{\mu,\frac{1}{2}}(\mathbb{C}^n)}, \text{ for } F, G \in \mathcal{H}_{\mu,t}(\mathbb{C}^n).$$

Theorem 3.1 *The operator $e^{it\Delta_\mu} : L^2(\mathbb{R}^n, h_\mu(u)e^{u^2}du) \to \mathcal{H}_{\mu,t}(\mathbb{C}^n)$ is unitary. Moreover, $\mathcal{H}_{\mu,t}(\mathbb{C}^n)$ is a reproducing kernel Hilbert space.*

Proof Let $f \in L^2(\mathbb{R}^n, e^{u^2}h_\mu(u)du)$ and set $g(u) = f(u)e^{\frac{u^2}{2}}e^{\frac{i}{4t}u^2}$. Then it is easy to see that $g \in L^2(\mathbb{R}^n, h_\mu(u)du)$. Since $\left(\widehat{F_\mu}(\frac{1}{2}, .)\right)(y) = \frac{1}{c_\mu}e^{-\frac{1}{2}y^2}$, $y \in \mathbb{R}^n$, we have

$$\begin{aligned}\left(\widehat{g} *_\mu F_\mu(\frac{1}{2}, .)\right)(x) &= \int_{\mathbb{R}^n}\widehat{\widehat{g}}(y)\left(\widehat{F_\mu}(\frac{1}{2}, .)\right)(y)E_\mu(ix, y)h_\mu(y)dy \\ &= \frac{1}{c_\mu}\int_{\mathbb{R}^n}g(u)e^{-\frac{1}{2}u^2}E_\mu(-ix, u)h_\mu(u)du.\end{aligned}$$

By using the above equation, $e^{it\Delta_\mu}f$ can be written as:

$$\begin{aligned} e^{it\Delta_\mu}f(x) &= \frac{M_\mu}{(it)^{\nu_\mu+\frac{n}{2}}} e^{\frac{i}{4t}x^2} \int_{\mathbb{R}^n} f(u) e^{\frac{i}{4t}u^2} E_\mu\left(-\frac{i}{2t}x, u\right) h_\mu(u)du \\ &= \frac{1}{(2it)^{\nu_\mu+\frac{n}{2}} c_\mu} e^{\frac{i}{4t}x^2} \int_{\mathbb{R}^n} g(u) e^{-\frac{1}{2}u^2} E_\mu\left(-\frac{i}{2t}x, u\right) h_\mu(u)du \\ &= \frac{1}{(2it)^{\nu_\mu+\frac{n}{2}}} e^{\frac{i}{4t}x^2} \widehat{g} *_\mu F_\mu\left(\frac{1}{2}, .\right)\left(\frac{x}{2t}\right). \end{aligned}$$

Since the right hand side function can be extended as an analytic function on $\mathbb{C}^n$, we write

$$e^{it\Delta_\mu}f(z) = \frac{1}{(2it)^{\nu_\mu+\frac{n}{2}}} e^{\frac{i}{4t}z^2} \left(\widehat{g} *_\mu F_\mu\left(\frac{1}{2}, .\right)\right)\left(\frac{z}{2t}\right), \quad \forall z \in \mathbb{C}^n. \tag{3.1}$$

Then by using change of variable we have,

$$(2it)^{\nu_\mu+\frac{n}{2}} e^{-\frac{i}{4t}(2tz)^2} e^{it\Delta_\mu} f(2tz) = \left(\widehat{g} *_\mu F_\mu\left(\frac{1}{2}, .\right)\right)(z), \quad \forall z \in \mathbb{C}^n.$$

Since $g \in L^2(\mathbb{R}^n, h_\mu(u)du)$, using Theorem 2.5 and Plancherel theorem for Dunkl transform [9] we have,

$$\begin{aligned} \left\|\widehat{g} *_\mu F_\mu(\frac{1}{2}, .)\right\|^2_{\mathcal{C}_{\mu,\frac{1}{2}}(\mathbb{C}^n)} &= \|\widehat{g}\|^2_{L^2(\mathbb{R}^n, h_\mu(u)du)} \\ &= \|g\|^2_{L^2(\mathbb{R}^n, h_\mu(u)du)} = \|f\|^2_{L^2_\mu(\mathbb{R}^n)}. \end{aligned}$$

From above equations we can see that, $\mathcal{G}(e^{it\Delta_\mu}f) \in \mathcal{C}_{\mu,\frac{1}{2}}(\mathbb{C}^n)$ and this implies that $e^{it\Delta_\mu}f \in \mathcal{H}_{\mu,t}(\mathbb{C}^n)$ for all $f \in L^2(\mathbb{R}^n, e^{u^2}h_\mu(u)du)$. Hence $e^{it\Delta_\mu} : L^2(\mathbb{R}^n, e^{u^2}h_\mu(u)du) \to \mathcal{H}_{\mu,t}(\mathbb{C}^n)$ is an isometry.

Now we will show that $e^{it\Delta_\mu}$ is onto. Let $F \in \mathcal{H}_{\mu,t}(\mathbb{C}^n)$, then $\mathcal{G}F \in \mathcal{C}_{\mu,\frac{1}{2}}(\mathbb{C}^n)$. Since the operator $e^{\frac{1}{2}\Delta_\mu} : L^2(\mathbb{R}^n, e^{u^2}h_\mu(u)du) \to C_{\mu,\frac{1}{2}}(\mathbb{C}^n)$ is unitary, then there exists $\phi \in L^2(\mathbb{R}^n, h_\mu(u)du)$ such that

$$\mathcal{G}F(z) = e^{\frac{1}{2}\Delta_\mu}\widehat{\phi}(z) = \widehat{\phi} *_\mu F_\mu\left(\frac{1}{2}, .\right)(z).$$

Set $f(u) = \phi(u)e^{-\frac{1}{2}u^2}e^{-\frac{i}{4t}u^2}$, then it is easy to see that

$$f \in L^2(\mathbb{R}^n, e^{u^2}h_\mu(u)du) \quad \text{and} \quad \left(\mathcal{G}(e^{it\Delta_\mu}f)\right)(z) = \widehat{\phi} *_\mu F_\mu\left(\frac{1}{2}, .\right)(z).$$

This implies that $e^{it\Delta_\mu} f = F$. That is, $e^{it\Delta_\mu} : L^2(\mathbb{R}^n, h_\mu(u)e^{u^2}du) \to \mathcal{H}_{\mu,t}(\mathbb{C}^n)$ is an onto map.

Now we show that $\mathcal{H}_{\mu,t}(\mathbb{C}^n)$ is a reproducing kernel Hilbert space. This can be seen with the help of the function $\mathcal{G}$. Since $\mathcal{G}e^{it\Delta_\mu} f \in \mathcal{C}_{\mu,\frac{1}{2}}(\mathbb{C}^n)$ for every $f \in L^2(\mathbb{R}^n, e^{u^2}h_\mu(u)du)$ and $\mathbb{K}_{\mu,\frac{1}{2}}(z,w)$ is a reproducing kernel for $\mathcal{C}_{\mu,t}(\mathbb{C}^n)$, we have

$$\mathcal{G}e^{it\Delta_\mu} f(z) = \left\langle \mathcal{G}e^{it\Delta_\mu} f, \mathbb{K}_{\mu,\frac{1}{2}}(\,.\,,z)\right\rangle_{\mathcal{C}_{\mu,\frac{1}{2}}(\mathbb{C}^n)}.$$

Applying change of variable from z to $\frac{z}{2t}$, we have

$$e^{it\Delta_\mu} f(z) = \left\langle \mathcal{G}e^{it\Delta_\mu} f\,,\ \overline{(2it)^{-(\nu_\mu+\frac{n}{2})}}e^{-\frac{it}{4t}\bar{z}^2}\mathbb{K}_{\mu,\frac{1}{2}}\left(\,.\,,\frac{z}{2t}\right)\right\rangle_{\mathcal{C}_{\mu,\frac{1}{2}}(\mathbb{C}^n)}.$$

Let us define a function $\overline{\mathbb{K}}_{\mu,t}(w,z)$ on $\mathbb{C}^n \times \mathbb{C}^n$ by,

$$\overline{\mathbb{K}}_{\mu,t}(w,z) = |(2ti)^{-(\nu_\mu+\frac{n}{2})}|^2 e^{\frac{i}{4t}(-\bar{z}^2+w^2)}\mathbb{K}_{\mu,\frac{1}{2}}\left(\frac{w}{2t},\frac{z}{2t}\right).$$

Then it is easy to see that the above function satisfies the following relation

$$\left(\mathcal{G}\overline{\mathbb{K}}_{\mu,t}(\,.\,,z)\right)(w) = \overline{(2it)^{-(\nu_\mu+\frac{n}{2})}}e^{-\frac{it}{4t}\bar{z}^2}\mathbb{K}_{\mu,\frac{1}{2}}\left(w,\frac{z}{2t}\right).$$

So, for every $f \in L^2(\mathbb{R}^n, e^{u^2}h_\mu(u)du)$, we have

$$e^{it\Delta_\mu} f(z) = \left\langle \mathcal{G}e^{it\Delta_\mu} f\,,\ \mathcal{G}\overline{\mathbb{K}}_{\mu,t}(\,.\,,z)\right\rangle_{\mathcal{C}_{\mu,\frac{1}{2}}} = \left\langle e^{it\Delta_\mu} f\,,\ \overline{\mathbb{K}}_{\mu,t}(\,.\,,z)\right\rangle_{\mathcal{H}_{\mu,t}(\mathbb{C}^n)}. \tag{3.2}$$

This implies that the space $\mathcal{H}_{\mu,t}(\mathbb{C}^n)$ is a reproducing kernel Hilbert space with reproducing kernels $\left\{\overline{\mathbb{K}}_{\mu,t}(z,w) : z, w \in \mathbb{C}^n\right\}$ and it is unitarily equivalent to the Hilbert space $e^{it\Delta_\mu}(L^2_\mu(\mathbb{R}^n))$. □

For $n = 1$, the above theorem can be written more precisely as follows: let $z \in \mathbb{C}$ and consider the weight functions, $u^\mu_{o,t}(z) = (2t)^{2\mu-1}e^{\frac{xy}{t}}\nu_{o,\mu,\frac{1}{2}}(\frac{z}{2t})$ and $u^\mu_{e,t}(z) = (2t)^{2\mu-1}e^{\frac{xy}{t}}\nu_{e,\mu,\frac{1}{2}}(\frac{z}{2t})$. Define

$$\mathcal{H}_{\mu,t}(\mathbb{C}) := \left\{F \in \mathcal{O}(\mathbb{C}) : F_o \in L^2(\mathbb{C}, u^\mu_{o,t}(z)dz) \text{ and } F_e \in L^2(\mathbb{C}, u^\mu_{e,t}(z)dz)\right\}.$$

Then the space $\mathcal{H}_{\mu,t}(\mathbb{C})$ is a Hilbert space with respect to the inner product

$$\langle F, G\rangle_{\mathcal{H}_{\mu,t}(\mathbb{C})} := \langle F_o, G_o\rangle_{L^2(\mathbb{C},u^\mu_{o,t}(z)dz)} + \langle F_e, G_e\rangle_{L^2(\mathbb{C},u^\mu_{e,t}(z)dz)},$$

where $F, G \in \mathcal{H}_{\mu,t}(\mathbb{C})$.

Theorem 3.2 *The operator* $e^{it\Delta_\mu} : L^2(\mathbb{R}, |u|^{2\mu} e^{u^2} du) \to \mathcal{H}_{\mu,t}(\mathbb{C})$ *is unitary.*

Proof Proof of this theorem easily follows from the Eq. (3.1) and the Theorem 2.4. □

Consider the following Hilbert spaces,

$$\mathcal{H}^o_{\mu,t}(\mathbb{C}) = \left\{ F \in \mathcal{O}(\mathbb{C}) : F \text{ is odd and } F \in L^2(\mathbb{C}, u^\mu_{o,t}(z)dz) \right\}$$

and

$$\mathcal{H}^e_{\mu,t}(\mathbb{C}) = \left\{ F \in \mathcal{O}(\mathbb{C}) : F \text{ is even and } F \in L^2(\mathbb{C}, u^\mu_{e,t}(z)dz) \right\}$$

with the inner product on $\mathcal{H}_{\mu,t}(\mathbb{C})^o$ and $\mathcal{H}_{\mu,t}(\mathbb{C})^e$ is given by $\langle F, G\rangle_{\mathcal{H}^o_{\mu,t}(\mathbb{C})} := \langle F, G\rangle_{L^2(\mathbb{C}, u^\mu_{o,t}(z)dz)}$ and $\langle F, G\rangle_{\mathcal{H}^e_{\mu,t}(\mathbb{C})} := \langle F, G\rangle_{L^2(\mathbb{C}, u^\mu_{e,t}(z)dz)}$, respectively.

Notice that the above spaces are subspaces of $\mathcal{H}_{\mu,t}(\mathbb{C})$ and $\mathcal{H}_{\mu,t}(\mathbb{C})$ is the direct sum of $\mathcal{H}^o_{\mu,t}(\mathbb{C})$ and $\mathcal{H}_{\mu,t}(\mathbb{C})^e$. That is $\mathcal{H}_{\mu,t}(\mathbb{C}) = \mathcal{H}^o_{\mu,t}(\mathbb{C}) \bigoplus \mathcal{H}^e_{\mu,t}(\mathbb{C})$. The direct sum of these kind of Hilbert spaces are called Fock type spaces.

In Theorem 3.1 we have identified the image of $L^2_\mu(\mathbb{R}^n)$ under $e^{it\Delta_\mu}$ (for any finite reflection group G) as a reproducing kernel Hilbert space $\mathcal{H}_{\mu,t}(\mathbb{C}^n)$. In the special case $G = \mathbb{Z}_2^n$, we explicitly construct an orthonormal basis which will help us to characterize the image of Sobolev space under $e^{it\Delta_\mu}$.

Now consider the Dunkl–Laplacian associated to the group $\mathbb{Z}_2^n$ and the root system $\mathcal{R} = \{\pm e_i : i \in \{1, 2, \ldots, n\}\}$. For any non negative multiplicative function μ on $\mathcal{R}$ ($\mu(e_j) = \mu_j$ for every $j = 1, \ldots, n$), the corresponding weight function is given by $h_\mu(u) = \prod_{j=1}^n |u_j|^{2\mu_j}$, where $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$. In this case, the Dunkl kernel is given by

$$E_\mu(x, y) = \prod_{k=1}^n e_{\mu_k}(x_k y_k) \quad \text{for } x, y \in \mathbb{R}^n.$$

Now consider the multiplication operator

$$\mathcal{S} f(u) = f(u) e^{-\frac{i}{4t}u^2}, \quad \text{for } f \in L^2(\mathbb{R}^n, h_\mu(u) e^{u^2} du).$$

Since $|e^{-\frac{i}{4t}u^2}| = 1$ for $u \in \mathbb{R}^n$, $\mathcal{S} : L^2(\mathbb{R}^n, h_\mu(u)e^{u^2}du) \to L^2(\mathbb{R}^n, h_\mu(u)e^{u^2}du)$ is a unitary. For each $k \in \mathbb{N}$ and $\tilde{\mu} > 0$, we define

$$\psi^{\tilde{\mu}}_k(u) = \left(\frac{\gamma_{\tilde{\mu}}(k)}{\Gamma(\tilde{\mu} + \frac{1}{2})} \right)^{\frac{1}{2}} \frac{1}{2^{\frac{k}{2}} k!} H^{\tilde{\mu}}_k(u) e^{-u^2}, \quad u \in \mathbb{R}. \tag{3.3}$$

From the equation 3.5.1 in [16], we can conclude that $\left\{ \psi^{\tilde{\mu}}_k : k \in \mathbb{N} \right\}$ forms an orthonormal basis for $L^2(\mathbb{R}, e^{u^2} |u|^{2\tilde{\mu}} du)$. For $\alpha \in \mathbb{N}^n$ and $u \in \mathbb{R}^n$, let

$$\Psi_\alpha^\mu(u) = \prod_{k=1}^{n} \psi_{\alpha_k}^{\mu_k}(u_k).$$

Then it is easy to show that $\left\{\Psi_\alpha^\mu : \alpha \in \mathbb{N}^n\right\}$ forms an orthonormal basis for $L^2(\mathbb{R}^n, e^{u^2} h_\mu(u)du)$.

Proposition 3.3 *For $\alpha \in \mathbb{N}^n$ and $z \in \mathbb{C}^n$ we have*

$$e^{it\Delta_\mu} \mathcal{S}\Psi_\alpha^\mu(z) = \frac{M_\mu}{(it)^{\nu_\mu + \frac{n}{2}}} \left(-\frac{i}{2t}\right)^{|\alpha|} \left(\frac{\prod_{k=1}^n \Gamma(\mu_k + \frac{1}{2})}{\gamma_\mu(\alpha) 2^{|\alpha|}}\right)^{\frac{1}{2}} z^\alpha e^{\left(\frac{i}{4t} - \frac{1}{16t^2}\right) z^2}.$$

Proof First we will prove this result for $n = 1$. For $k \in \mathbb{N}$, $\mu > 0$ and $x \in \mathbb{R}$, from the Eqs. 2.11 and 2.14 we have the following,

$$\begin{aligned}
&\int_{\mathbb{R}} H_k^\mu(u) e_\mu\left(-\frac{i}{2t} xu\right) e^{-u^2} |u|^{2\mu} du \\
&= k! \sum_{p=0}^{[\frac{k}{2}]} \frac{(-1)^p 2^{k-2p}}{p! \gamma_\mu(k-2p)} \int_{\mathbb{R}} u^{k-2p} e_\mu\left(-\frac{i}{2t} xu\right) e^{-u^2} |u|^{2\mu} du \\
&= k!(-i)^k \Gamma\left(\mu + \frac{1}{2}\right) \sum_{p=0}^{[\frac{k}{2}]} \frac{H_{k-2p}^\mu\left(\frac{x}{4t}\right)}{p!(k-2p)!} e^{-\frac{x^2}{16t^2}} \\
&= \frac{k!(-i)^k \Gamma(\mu + \frac{1}{2})}{\gamma_\mu(k)(2t)^k} x^k e^{-\frac{x^2}{16t^2}}.
\end{aligned}$$

Using the above, we can calculate, for $x \in \mathbb{R}$,

$$\begin{aligned}
e^{it\Delta_\mu} \mathcal{S}\psi_k^\mu(x) &= \frac{M_\mu}{(it)^{\mu + \frac{n}{2}}} e^{\frac{i}{4t} x^2} \int_{\mathbb{R}} \mathcal{S}\psi_k^\mu(u) e^{\frac{i}{4t} u^2} E_\mu\left(-\frac{i}{2t} x, u\right) h_\mu(u) du \\
&= \frac{M_\mu}{(it)^{\mu + \frac{1}{2}}} e^{\frac{i}{4t} x^2} \int_{\mathbb{R}} \psi_k^\mu(u) e_\mu\left(-\frac{i}{2t} xu\right) h_\mu(u) du \\
&= \frac{M_\mu}{(it)^{\mu + \frac{n}{2}}} e^{\frac{i}{4t} x^2} \left(\frac{\gamma_\mu(k)}{\Gamma\left(\mu + \frac{1}{2}\right)}\right)^{\frac{1}{2}} \frac{1}{2^{\frac{k}{2}} k!} \\
&\quad \times \int_{\mathbb{R}} H_k^\mu(u) e_\mu\left(-\frac{i}{2t} xu\right) e^{-u^2} |u|^{2\mu} du \\
&= \frac{M_\mu}{(it)^{\mu + \frac{n}{2}}} e^{\frac{i}{4t} x^2} \left(-\frac{i}{2t}\right)^k \left(\frac{\Gamma(\mu + \frac{1}{2})}{\gamma_\mu(k) 2^k}\right)^{\frac{1}{2}} x^k e^{-\frac{1}{16t^2} x^2}.
\end{aligned}$$

Now $e^{it\Delta_\mu}\mathcal{S}\psi_k^\mu$ can be extended analytically to the complex plane as

$$e^{it\Delta_\mu}\mathcal{S}\psi_k^\mu(z) = \frac{M_\mu}{(it)^{\mu+\frac{1}{2}}}\left(-\frac{i}{2t}\right)^k \left(\frac{\Gamma(\mu+\frac{1}{2})}{\gamma_\mu(k)2^k}\right)^{\frac{1}{2}} z^k e^{\left(\frac{i}{4t}-\frac{1}{16t^2}\right)z^2}, \quad z \in \mathbb{C}.$$

This establishes the proposition for $n = 1$.

The general case follows by observing that $e^{it\Delta_\mu}\mathcal{S}\Psi_\alpha^\mu$ can be expressed as a product of one dimensional expressions. For $\alpha \in \mathbb{N}^n$ and $\mu = (\mu_k)_{k=1}^n$, we have

$$\begin{aligned} e^{it\Delta_\mu}\mathcal{S}\Psi_\alpha^\mu(z) &= \frac{M_\mu}{(it)^{\nu_\mu+\frac{n}{2}}} e^{\frac{i}{4t}z^2} \int_{\mathbb{R}^n} \mathcal{S}\Psi_\alpha^\mu(u) e^{\frac{i}{4t}u^2} E_\mu\left(-\frac{i}{2t}z, u\right) h_\mu(u)du \\ &= \frac{M_\mu}{(it)^{\nu_\mu+\frac{n}{2}}} e^{\frac{i}{4t}z^2} \prod_{k=1}^n \int_{\mathbb{R}} \psi_{\alpha_k}^{\mu_k}(u_k) e_{\mu_k}\left(-\frac{i}{2t}z_k u_k\right) h_{\mu_k}(u_k)du_k. \\ &= \prod_{k=1}^n e^{it\Delta_{\mu_k}}\mathcal{S}\psi_{\alpha_k}^{\mu_k}(z_k). \end{aligned}$$

□

Since the set $\left\{\Psi_\alpha^\mu : \alpha \in \mathbb{N}^n\right\}$ is a complete orthonormal basis for the space $L^2(\mathbb{R}^n, h_\mu(u)e^{u^2}du)$ and the operator $e^{it\Delta_\mu}\mathcal{S} : L^2(\mathbb{R}^n, h_\mu(u)e^{u^2}du) \to \mathcal{H}_{\mu,t}(\mathbb{C}^n)$ is unitary, the set $\{\Upsilon_\alpha^{\mu,t} := e^{it\Delta_\mu}\mathcal{S}\Psi_\alpha^\mu : \alpha \in \mathbb{N}^n\}$ forms a complete orthonormal basis for $\mathcal{H}_{\mu,t}(\mathbb{C}^n)$. Also from the above theorem we can see that, the n-dimensional vectors $\Upsilon_\alpha^{\mu,t}$ is a pointwise product of one dimensional vectors $\Upsilon_{\alpha_k}^{\mu_k,t}$, $k = 1, 2, \ldots, n$. That is, for $\alpha \in \mathbb{N}^n$ and $z \in \mathbb{C}^n$, $\Upsilon_\alpha^{\mu,t}(z) = \prod_{k=1}^n \Upsilon_{\alpha_k}^{\mu_k,t}(z_k)$, where $\left\{\Upsilon_{\alpha_k}^{\mu_k,t} : \alpha_k \in \mathbb{N}\right\}$ is a complete orthonormal basis for $\mathcal{H}_{\mu_k,t}(\mathbb{C})$. This fact forces to consider the tensor product of Hilbert spaces $\bigotimes_{k=1}^n \mathcal{H}_{\mu_k,t}(\mathbb{C})$.

Theorem 3.4 *The Hilbert space $\mathcal{H}_{\mu,t}(\mathbb{C}^n)$ is unitarily equivalent to tensor product of Hilbert spaces $\bigotimes_{k=1}^n \mathcal{H}_{\mu_k,t}(\mathbb{C})$.*

The above theorem says that the image of $L_\mu^2(\mathbb{R}^n)$ is also identified with tensor product of Fock type spaces.

4 Image of Dunkl–Sobolev space under Schrödinger semigroup

In this section we define Dunkl–Sobolev spaces using the Dunkl operator associated to the reflection group $\mathbb{Z}_2^n$. The reason to consider this particular group is that, the Dunkl kernel and the Hermite polynomials are explicitly known in this setting. By using these explicit functions, we can choose a basis for Dunkl–Sobolev space such that the Schrödinger semigroup composed with a "multiplication operator" takes this basis to "nice class of functions". On observing those "nice class of functions" we can easily guess the reproducing kernel Hilbert space, which is same as the image of the Dunkl–Sobolev space under $e^{it\Delta_\mu}$, upto equivalence of norms.

For the group $\mathbb{Z}_2^n$ the associated Dunkl derivatives corresponding to the vector e_k is given by,

$$D_{k,\mu} f(x) = \partial_k f(x) + \mu_k \frac{f(x) - f(\sigma_k(x))}{x_k} \quad \text{for } k = 1, 2, \ldots, n. \tag{4.1}$$

For $\alpha \in \mathbb{N}^n$, let $D_\mu^\alpha := D_{1,\mu}^{\alpha_1} \cdots D_{n,\mu}^{\alpha_n}$. Now for $m \in \mathbb{N}$, we define the Dunkl–Sobolev space $W_\mu^{m,2}(\mathbb{R}^n)$ in $L_\mu^2(\mathbb{R}^n) := L^2(\mathbb{R}^n, h_\mu(u)e^{u^2}du)$ by

$$W_\mu^{m,2}(\mathbb{R}^n) := \left\{ f \in L_\mu^2(\mathbb{R}^n) : D_\mu^\alpha f \in L_\mu^2(\mathbb{R}^n), \forall \alpha \in \mathbb{N}^n \text{ and } |\alpha| \le m \right\}. \tag{4.2}$$

Then the space $W_\mu^{m,2}(\mathbb{R}^n)$ is a Hilbert space with respect to the inner product

$$\langle f, g \rangle_{W_\mu^{m,2}(\mathbb{R}^n)} := \sum_{|\alpha| \le m} \langle D_\mu^\alpha f, D_\mu^\alpha g \rangle_{L_\mu^2(\mathbb{R}^n)}, \quad f, g \in W_\mu^{m,2}(\mathbb{R}^n).$$

Since $W_\mu^{m,2}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n, h_\mu(u)e^{u^2}du)$, we can consider

$$e^{it\Delta_\mu}\left(W_\mu^{m,2}(\mathbb{R}^n)\right) := \left\{ e^{it\Delta_\mu} f \in \mathcal{O}(\mathbb{C}^n) : f \in W_\mu^{m,2}(\mathbb{R}^n) \right\}$$

as a subspace of $\mathcal{O}(\mathbb{C}^n)$. It is clear that $e^{it\Delta_\mu} : W_\mu^{m,2}(\mathbb{R}^n) \to e^{it\Delta_\mu}\left(W_\mu^{m,2}(\mathbb{R}^n)\right)$ is a bijective linear map. Since it is injective, it's image $e^{it\Delta_\mu}\left(W_\mu^{m,2}(\mathbb{R}^n)\right)$ is made into a Hilbert space simply by transferring the Hilbert space structure of $W_\mu^{m,2}(\mathbb{R}^n)$ to $e^{it\Delta_\mu}(W_\mu^{m,2}(\mathbb{R}^n))$ so that the Schrödinger semigroup $e^{it\Delta_\mu}$ is an isometric isomorphism from $W_\mu^{m,2}(\mathbb{R}^n)$ onto $e^{it\Delta_\mu}\left(W_\mu^{m,2}(\mathbb{R}^n)\right)$. This means that

$$\left\langle e^{it\Delta_\mu} f, e^{it\Delta_\mu} g \right\rangle_{e^{it\Delta_\mu}(W_\mu^{m,2}(\mathbb{R}^n))} := \langle f, g \rangle_{W_\mu^{m,2}(\mathbb{R}^n)} \quad \text{for } f, g \in W_\mu^{m,2}(\mathbb{R}^n),$$

where $f, g \in W_\mu^{m,2}(\mathbb{R}^n)$. Our aim is to identify $e^{it\Delta_\mu}\left(W_\mu^{m,2}(\mathbb{R}^n)\right)$ as a reproducing kernel Hilbert space up to an equivalence of norms. For the notational convenience, we will do this image identification for $n = 1$ and the general n follows in the similar way.

Note 4.1 *For $\mu = 0, m \ge 1$ and $f \in W_0^{m,2}(\mathbb{R})$, we have*

$$\lim_{|x| \to \infty} \left| D^k f(x) p(x) \right| = 0 \tag{4.3}$$

for any polynomial p on $\mathbb{R}$ and $k \in \mathbb{N}$ with $k \le m - 1$.

The above note follows from the particular case $m = 1$. This particular case is a consequence of the following: using classical Sobolev embedding lemma we have,

$$\left(f(t)e^{\frac{t^2}{2}}\right)' = -tf(t)e^{\frac{t^2}{2}} + f'(t)e^{\frac{t^2}{2}}.$$

Integrating both sides from a to b and using the Schwarz inequality to estimate the integral on the right, we have

$$\left(\left|f(t)e^{\frac{t^2}{2}}\right|\right)_a^b \leq \left(\frac{b^3 - a^3}{3}\int_a^b |f(t)|^2 e^{t^2} dt\right)^{\frac{1}{2}} + \left((b-a)\int_a^b \left|f^{'}(t)\right|^2 e^{t^2} dt\right)^{\frac{1}{2}}.$$

From the above inequality we can find $\tilde{B} > 0$ such that

$$|f(u)|\, e^{u^2} \leq \tilde{B}(1 + |u|)^{\frac{3}{2}}, \quad \text{all } u \in \mathbb{R}.$$

Definition 4.2 We say that a function $f \in L^2(\mathbb{R}, |u|^{2\mu} e^{u^2} du)$ has a Dunkl primitive in $L^2(\mathbb{R}, |u|^{2\mu} e^{u^2} du)$ if there exists $g \in L^2(\mathbb{R}, |u|^{2\mu} e^{u^2} du)$ such that $D_\mu g = f$.

Lemma 4.3 *The function* $\psi_0(u) = e^{-u^2}$ *does not have Dunkl primitive in* $L^2(\mathbb{R}, |u|^{2\mu} e^{u^2} du)$.

Proof Suppose there exist a $g \in L^2(\mathbb{R}, |u|^{2\mu} e^{u^2} du)$ such that $D_\mu g(u) = e^{-u^2}$. Then by Sobolev embedding lemma associated to Dunkl operators [9], g is infinitely differentiable function. In particular g is continuous. This gives us

$$\int_{|u|\leq 1} |g(u)|^2 e^{u^2} du < \infty$$

and

$$\int_{|u|\geq 1} |g(u)|^2 e^{u^2} du \leq \int_{|u|\geq 1} |g(u)|^2 e^{u^2} |u|^{2\mu} du < \infty.$$

Hence $g \in L^2(\mathbb{R},\ e^{u^2} du)$. Write $g = g_o + g_e$, where g_o, g_e are odd and even parts of g, respectively. Now

$$e^{-u^2} = D_\mu g(u) = \frac{d}{du} g_o(u) + \frac{d}{du} g_e(u) + \frac{\mu}{u}(g_o(u) - g_o(-u)).$$

The above equation can be written as

$$\frac{d}{du} g_e(u) = -\frac{d}{du} g_o(u) - \frac{\mu}{u}(g_o(u) - g_o(-u)) + e^{-u^2}.$$

Since the right hand side of the above equation is even, we can conclude that $\frac{d}{du}g_e(u)$ is also even. At the same time derivative of an even function is odd. These together forces that g_e has to be a constant. We know that $g_e \in L^2(\mathbb{R}, |u|^{2\mu}e^{u^2}du)$ this implies $g_e = 0$. As the even part g_e vanished, now g is an odd function. This implies $\frac{d}{du}g(u) + \frac{2\mu}{u}g(u) = e^{-u^2}$. Since g and e^{-u^2} are infinitely differentiable, $\frac{g(u)}{u}$ is a continuous function on $\mathbb{R}$. Using this it is easy to see that $\frac{2g(u)}{u} \in L^2(\mathbb{R}, e^{u^2}du)$. Consequently we have $\frac{d}{du}g \in L^2(\mathbb{R}, e^{u^2}du)$. That implies $g \in W_0^{1,2}(\mathbb{R})$. From the note 4.1 we conclude that, for all complex polynomial $p(u)$

$$\lim_{|u|\to\infty} |g(u)p(u)| = 0\cdot \tag{4.4}$$

For $a > 0$,

$$\int_{\epsilon}^{a} \frac{d}{du}g(u)|u|^{2\mu}du = g(a)|a|^{2\mu} - g(\epsilon)|\epsilon|^{2\mu} - 2\mu\int_{\epsilon}^{a}\frac{g(u)}{u}|u|^{2\mu}du. \tag{4.5}$$

Similarly,

$$\int_{-a}^{-\epsilon} \frac{d}{du}g(u)|u|^{2\mu}du = g(-\epsilon)|\epsilon|^{2\mu} - g(-a)|a|^{2\mu} - 2\mu\int_{-a}^{-\epsilon}\frac{g(u)}{u}|u|^{2\mu}du. \tag{4.6}$$

Letting $\epsilon \to 0$ in the above equations, we have

$$\lim_{\epsilon\to 0}\int_{\epsilon}^{a} \frac{d}{du}g(u)|u|^{2\mu}du = g(a)a^{2\mu} - 2\mu\int_{0}^{a}\frac{g(u)}{u}|u|^{2\mu}du$$
$$\lim_{\epsilon\to 0}\int_{-a}^{-\epsilon} \frac{d}{du}g(u)|u|^{2\mu}du = g(a)a^{2\mu} - 2\mu\int_{-a}^{0}\frac{g(u)}{u}|u|^{2\mu}du.$$

So, for every $a > 0$,

$$\int_{-a}^{a} D_\mu(g)(u)|u|^{2\mu}du = 2g(a)a^{2\mu}. \tag{4.7}$$

Letting a tends to ∞ in the above equation and using (4.4) we have

$$\int_{\mathbb{R}} e^{-u^2}|u|^{2\mu}du = \lim_{a\to\infty}\int_{-a}^{a} D_\mu(g)(u)|u|^{2\mu}du = 0,$$

which is a contradiction to $\int_{\mathbb{R}} e^{-u^2}|u|^{2\mu}du \neq 0$. So ψ_0 does not have Dunkl primitive in $L^2(\mathbb{R}, |u|^{2\mu}e^{u^2}du)$. □

Lemma 4.4 *Dunkl primitive for a function $f \in L^2(\mathbb{R}, e^{u^2}|u|^{2\mu}du)$ is unique.*

Proof It is enough to prove that the zero function is the only solution to the equation $D_\mu g = 0$, for $g \in L^2(\mathbb{R}, e^{u^2}|u|^{2\mu}du)$. To prove this, we argue as in the Lemma 4.3 up to the Eq. (4.7), so that we have g as an odd function with $g(a) = 0$, forall $a > 0$. That implies $g = 0$. □

For the functions $f \in W_\mu^{m,2}(\mathbb{R})$, using the Lemmas 4.3 and 4.4 we will get a $L^2(\mathbb{R}, e^{u^2}|u|^{2\mu}du)$ representation of Dunkl derivatives of f in terms of the Fourier coefficients of f.

Lemma 4.5 *Let $f \in W_\mu^{m,2}(\mathbb{R})$. Then $D_\mu^m f$ has the following representation in $L^2(\mathbb{R}, e^{u^2}h_\mu(u)du)$,*

$$D_\mu^m f = \sum_{k=0}^{\infty}\left(\prod_{j=1}^{m}(-1)^j\left(2(k+j+2\mu\theta_{k+j})\right)^{\frac{1}{2}}\right)\langle f, \psi_k^\mu\rangle_{L_\mu^2(\mathbb{R})}\psi_{k+j}^\mu.$$

Proof First we will prove this for $m = 1$ and the general case follows inductively. Using the Eq. 2.12 it is easy to see the following

$$D_\mu\psi_k^\mu(u) = -\,(2(k+1+2\mu\theta_{k+1}))^{\frac{1}{2}}\,\psi_{k+1}^\mu(u). \tag{4.8}$$

If $f \in W_\mu^{1,2}(\mathbb{R})$ then we have the following representations,

$$f = \sum_{k=0}^{\infty}\langle f, \psi_k\rangle_{L_\mu^2(\mathbb{R})}\psi_k^\mu \qquad \text{and}$$

$$D_\mu f = \langle D_\mu f, \psi_0\rangle_{L_\mu^2(\mathbb{R})}\psi_0 + \sum_{k=1}^{\infty}\langle D_\mu f, \psi_k\rangle_{L_\mu^2(\mathbb{R})}\psi_k^\mu.$$

Define $g_1 := \sum_{k=1}^{\infty}\langle D_\mu f, \psi_k\rangle_{L_\mu^2(\mathbb{R})}\psi_k^\mu$ and $f_1 := \sum_{k=1}^{\infty}\frac{\langle D_\mu f,\psi_k^\mu\rangle_{L_\mu^2(\mathbb{R})}}{-(2(k+2\mu\theta_k))^{\frac{1}{2}}}\psi_{k-1}^\mu$. Since $f \in L^2(\mathbb{R}, |u|^{2\mu}e^{u^2}du)$ we can see that $f_1, g_1 \in L^2(\mathbb{R}, |u|^{2\mu}e^{u^2}du)$. Also it can be observed that the Dunkl primitives of g_1 is f_1. That is $D_\mu f_1 = g_1$. This implies that $\langle D_\mu f, \psi_0\rangle_{L_\mu^2(\mathbb{R})}\psi_0 = D_\mu f - g_1$ has primitive in $L^2(\mathbb{R}, |u|^{2\mu}e^{u^2}du)$. From the Lemma 4.3 we know that ψ_0 does not have a primitive. This implies that $\langle D_\mu f, \psi_0\rangle_{L_\mu^2(\mathbb{R})} = 0$. Consequently we have

$$D_\mu f = \sum_{k=1}^{\infty}\langle D_\mu f, \psi_k\rangle_{L_\mu^2(\mathbb{R})}\psi_k^\mu \quad \text{for every } f \in W_\mu^{1,2}(\mathbb{R}).$$

From above we can observe that f_1 and f are Dunkl primitives of $D_\mu f$ in $L^2(\mathbb{R}, |u|^{2\mu}e^{u^2}du)$. But Dunkl primitive is unique in $L^2(\mathbb{R}, e^{u^2}|u|^{2\mu}du)$. This leads to

$$f = \sum_{k=0}^{\infty}\langle f, \psi_k^\mu\rangle_{L_\mu^2(\mathbb{R})}\psi_k^\mu = \sum_{k=1}^{\infty}\frac{\langle D_\mu f, \psi_k^\mu\rangle_{L_\mu^2(\mathbb{R})}}{-(2k+2\mu\theta_k)^{\frac{1}{2}}}\psi_{k-1}^\mu.$$

Comparing the coefficients of ψ_k from the above equation we have

$$\langle D_\mu f, \psi_k^\mu\rangle_{L_\mu^2(\mathbb{R})} = -\,(2(k+2\mu\theta_k))^{\frac{1}{2}}\,\langle f, \psi_{k-1}^\mu\rangle_{L_\mu^2(\mathbb{R})}.$$

This implies that

$$D_\mu f = \sum_{k=0}^{\infty}(-1)(2(k+1+2\mu\theta_{k+1}))^{\frac{1}{2}}\langle f, \psi_k^\mu\rangle_{L^2_\mu(\mathbb{R})}\psi_{k+1}^\mu$$
$$= \sum_{k=0}^{\infty}\langle f, \psi_k^\mu\rangle_{L^2_\mu(\mathbb{R})} D_\mu\psi_k^\mu .$$

□

Next we are going to prove that the collection $\{\psi_k^\mu : k \in \mathbb{N}\}$ is a complete orthogonal set in $W_\mu^{m,2}(\mathbb{R})$.

Theorem 4.6 *The system of vectors* $\left\{\frac{\psi_k^\mu}{\|\psi_k^\mu\|_{W_\mu^{m,2}(\mathbb{R})}} : k \in \mathbb{N}\right\}$ *forms a complete orthonormal basis for* $W_\mu^{m,2}(\mathbb{R})$.

Proof For $l \neq n$,

$$\langle\psi_l^\mu, \psi_n^\mu\rangle_{W_\mu^{m,2}(\mathbb{R})} = \sum_{k=0}^{m}\langle D_\mu^k\psi_l^\mu, D_\mu^k\psi_n^\mu\rangle_{L^2_\mu(\mathbb{R})}$$
$$= \sum_{k=0}^{m} r_k^\mu(l)r_k^\mu(n)\langle\psi_{l+k}^\mu, \psi_{n+k}^\mu\rangle_{L^2_\mu(\mathbb{R})} = 0,$$

where $r_k^\mu(l) = (-1)^k\prod_{j=1}^{k}(2(l+j+2\mu\theta_{l+j}))^{\frac{1}{2}}$ and $r_0^\mu(l) = 1$ for $l \in \mathbb{N}, k \in \mathbb{N}\backslash\{0\}$.

Suppose $\langle f, \psi_k^\mu\rangle_{W_\mu^{m,2}(\mathbb{R})} = 0$ for $f \in W_\mu^{m,2}(\mathbb{R})$ and $k \in \mathbb{N}$. From Lemma 4.5 we can conclude that $\langle f, \psi_k^\mu\rangle_{L^2_\mu(\mathbb{R})} = 0$ for every $k \in \mathbb{N}$. So the system of vectors $\left\{\frac{\psi_k^\mu}{\|\psi_k^\mu\|_{W_\mu^{m,2}(\mathbb{R})}} : k \in \mathbb{N}\right\}$ is a complete orthonormal basis for $W_\mu^{m,2}(\mathbb{R})$. □

Note 4.7 *For* $k \in \mathbb{N}$,

$$\|\psi_k^\mu\|^2_{W_m^\mu(\mathbb{R})} = \sum_{j=0}^{m}\prod_{s=1}^{j}2(k+s+2\mu\theta_{k+s}).$$

Let $\mathcal{B}_\mu = span\{\psi_k^\mu : k \in \mathbb{N}\}$. For $f \in \mathcal{B}_\mu$ we have

$$f = \sum_{k=0}^{l} a_k\psi_k^\mu \quad \text{for some } a_k \in \mathbb{C} \text{ and } l \in \mathbb{N}.$$

For $f \in \mathcal{B}_\mu$ recall that $\mathcal{S}f(u) = f(u)e^{-\frac{i}{4t}u^2}$. From the Eq. 2.13 it is easy to see that

$$u\psi_k^\mu(u) = \left(\frac{k+2\mu\theta_k}{2}\right)^{\frac{1}{2}} \psi_{k-1}^\mu(u) + \left(\frac{k+1+2\mu\theta_{k+1}}{2}\right)^{\frac{1}{2}} \psi_{k+1}^\mu(u) \quad \text{for } k \in \mathbb{N}. \tag{4.9}$$

Using this equation we can conclude that the map $\mathcal{S} : \mathcal{B}_\mu \to \mathcal{B}_\mu$ is a linear map.

Theorem 4.8 *The map* $\mathcal{S} : W_\mu^{m,2}(\mathbb{R}) \to W_\mu^{m,2}(\mathbb{R})$ *is a bounded and invertible operator.*

Proof First we will prove this theorem for Dunkl–Sobolev space of order one and the rest follows from the induction on m. Let $f \in \mathcal{B}_\mu$, then f can be written as

$$f = \sum_{k=0}^{l} a_k \psi_k^\mu \quad \text{for some } a_k \in \mathbb{C} \text{ and } l \in \mathbb{N}.$$

From the Eq. (4.9) and using triangle inequality we have,

$$\begin{aligned}
&\|uf\|_{L^2_\mu(\mathbb{R})} \\
&= \left\| \sum_{k=1}^{l} \left(\frac{k+2\mu\theta_k}{2}\right)^{\frac{1}{2}} a_k\psi_{k-1}^\mu + \sum_{k=0}^{l} \left(\frac{k+1+2\mu\theta_{k+1}}{2}\right)^{\frac{1}{2}} a_k\psi_{k+1}^\mu \right\|_{L^2_\mu(\mathbb{R})} \\
&\leq \left\| \sum_{k=1}^{l} \left(\frac{k+2\mu\theta_k}{2}\right)^{\frac{1}{2}} a_k\psi_{k-1}^\mu \right\|_{L^2_\mu(\mathbb{R})} + \left\| \sum_{k=0}^{l} \left(\frac{k+1+2\mu\theta_{k+1}}{2}\right)^{\frac{1}{2}} a_k\psi_{k+1}^\mu \right\|_{L^2_\mu(\mathbb{R})} \\
&= \left(\sum_{k=1}^{l} \left(\frac{k+2\mu\theta_k}{2}\right) |a_k|^2 \right)^{\frac{1}{2}} + \left(\sum_{k=0}^{l} \left(\frac{k+1+2\mu\theta_{k+1}}{2}\right) |a_k|^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Observe that there exists $\tilde{s}_\mu,\ s_\mu > 0$ (independent of l) such that

$$d_\mu \left(\frac{k+1+2\mu\theta_{k+1}}{2}\right) \leq \left(\frac{k+2\mu\theta_k}{2}\right) \leq s_\mu \left(\frac{k+1+2\mu\theta_{k+1}}{2}\right) \quad \text{for all } k \in \mathbb{N}.$$

Then we have,

$$\|uf\|_{L^2_\mu(\mathbb{R})} \leq \tilde{s}_\mu \left(\sum_{k=0}^{l} 2\,(k+1+2\mu\theta_{k+1})\,|a_k|^2 \right)^{\frac{1}{2}} = \tilde{s}_\mu \left\| D_\mu f \right\|_{L^2_\mu(\mathbb{R})}, \tag{4.10}$$

where $\tilde{s}_\mu = \max\{1, \sqrt{s_\mu}\}$. Also by comparing the $L^2_\mu(\mathbb{R})$ norms of $D_\mu f$ and uf, we have

$$\left\| D_\mu f \right\|_{L^2_\mu(\mathbb{R})} \leq \tilde{d}_\mu \,\|uf\|_{L^2_\mu(\mathbb{R})}, \quad \text{where } \tilde{d}_\mu = \sqrt{\max\left\{\frac{1}{2}, \frac{1}{2d_\mu}\right\}}. \tag{4.11}$$

Using the Eq. 4.10 we get,

$$\begin{aligned}\|S(f)\|^2_{W^{1,2}_\mu(\mathbb{R})} &= \|f\|^2_{L^2_\mu(\mathbb{R})} + \left\| D_\mu \left(f(u) e^{-\frac{i}{4t}u^2} \right) \right\|^2_{L^2_\mu(\mathbb{R})} \\ &= \|f\|^2_{L^2_\mu(\mathbb{R})} + \left\| D_\mu f(u) e^{-\frac{i}{4t}u^2} - \frac{i}{2t} u f(u) \right\|^2_{L^2_\mu(\mathbb{R})} \\ &\leq \|f\|^2_{L^2_\mu(\mathbb{R})} + \left(\|D_\mu f\|_{L^2_\mu(\mathbb{R})} + \frac{1}{2t} \|uf\|_{L^2_\mu(\mathbb{R})} \right)^2 \\ &\leq \|f\|^2_{L^2_\mu(\mathbb{R})} + \left(\max\{1, \frac{\tilde{s_\mu}}{2t}\} \right)^2 \|D_\mu f\|^2_{L^2_\mu(\mathbb{R})}.\end{aligned}$$

Hence we have,

$$\|\mathcal{S}(f)\|^2_{W^{1,2}_\mu(\mathbb{R})} \leq B_{t,1,\mu} \|f\|^2_{W^{1,2}_\mu(\mathbb{R})},$$

where $B_{t,1,\mu} = \max\{1, \left(\max\{1, \frac{\tilde{s}_\mu}{2t}\}\right)^2\}$. As $\mathcal{B}_\mu$ is dense in $W^{1,2}_\mu(\mathbb{R})$ and the above inequality is true for all $f \in \mathcal{B}_\mu$, the operator $\mathcal{S} : W^{1,2}_\mu(\mathbb{R}) \to W^{1,2}_\mu(\mathbb{R})$ is bounded. Already we know that $\mathcal{S}$ is a unitary operator on $L^2(\mathbb{R}, e^{u^2} h_\mu(u) du)$. That says $\mathcal{S} : W^{1,2}_\mu(\mathbb{R}) \to W^{1,2}_\mu(\mathbb{R})$ is a one to one map as well. From the Eq. (4.10) we can conclude that for any $f \in W^{1,2}_\mu(\mathbb{R})$ we have $uf \in L^2_\mu(\mathbb{R})$. This implies $D_\mu \mathcal{S}^{-1} f(u) = D_\mu(f)(u) e^{\frac{i}{4t}u^2} + \frac{i}{2t} u f(u) e^{\frac{i}{4t}u^2} \in L^2_\mu(\mathbb{R})$. Consequently we have $\mathcal{S}^{-1} f \in W^{1,2}_\mu(\mathbb{R})$ and $\mathcal{S} : W^{1,2}_\mu(\mathbb{R}) \to W^{1,2}_\mu(\mathbb{R})$ is onto. Since $\mathcal{S} : W^{1,2}_\mu(\mathbb{R}) \to W^{1,2}_\mu(\mathbb{R})$ is a one to one, onto and bounded operator, by bounded inverse theorem it is a bounded and invertible operator. This proves the stated theorem for $m = 1$.

For general $m \in \mathbb{N}$ and $f \in \mathcal{B}_\mu$,

$$\begin{aligned}&\|\mathcal{S} f\|^2_{W^{m,2}_\mu(\mathbb{R})} \\ &\quad = \|\mathcal{S}(f)\|^2_{W^{m-1,2}_\mu(\mathbb{R})} + \left\| D^m_\mu (\mathcal{S} f) \right\|^2_{L^2_\mu(\mathbb{R})} \\ &\quad = \|\mathcal{S}(f)\|^2_{W^{m-1,2}_\mu(\mathbb{R})} + \left\| D^{m-1}_\mu \left(\mathcal{S}(D_\mu f) - \frac{i}{2t} u \mathcal{S} f \right) \right\|^2_{L^2_\mu(\mathbb{R})} \\ &\quad \leq \|\mathcal{S}(f)\|^2_{W^{m-1,2}_\mu(\mathbb{R})} + \left(\left\| D^{m-1}_\mu \mathcal{S}(D_\mu f) \right\|_{L^2_\mu(\mathbb{R})} + \frac{1}{2t} \|D^{m-1}_\mu (u \mathcal{S} f)\|_{L^2_\mu(\mathbb{R})} \right)^2.\end{aligned}$$

Using induction on m and from the Eqs. (4.10) and (4.11) there exists $B_{t,m,\mu} > 0$ such that,

$$\|\mathcal{S} f\|^2_{W^{m,2}_\mu(\mathbb{R})} \leq B_{t,m,\mu} \|f\|^2_{W^{m,2}_\mu(\mathbb{R})}.$$

Since $\mathcal{B}_\mu$ is dense in $W^{m,2}_\mu(\mathbb{R})$, from bounded inverse theorem we conclude that $\mathcal{S}$ is a bounded and invertible operator. ◻

Note 4.9 *For $k = 0, \ldots, m$, from the Eq. (4.10) we can conclude the following,*

1. *If $f \in W_\mu^{m,2}(\mathbb{R})$ then $u^k f \in W_\mu^{m-k,2}(\mathbb{R})$.*
2. *If $f \in W_\mu^{m,2}(\mathbb{R})$ then there exist $V_{\mu,t,k} > 0$ such that, $\left\|u^k f\right\|_{L^2_\mu(\mathbb{R})} \leq V_{\mu,t,k} \left\|D_\mu^k f\right\|_{L^2_\mu(\mathbb{R})}$.*

Now we are in the position to characterize the image of sobolev space under Schrödinger semigroup. Since $e^{it\Delta_\mu}\mathcal{S}f(x) = \frac{M_\mu}{(it)^{\mu+\frac{1}{2}}} e^{\frac{i}{4t}x^2}\widehat{f}(\frac{x}{2t})$ for $f \in W_\mu^{m,2}(\mathbb{R})$ and $x \in \mathbb{R}$ and it is known that Dunkl transform translate Dunkl derivatives into multiplication with polynomials, then we have the following:

$$e^{it\Delta_\mu}\mathcal{S}D_\mu^m f(x) = \frac{M_\mu}{(it)^{\mu+\frac{1}{2}}} e^{\frac{i}{4t}x^2} \widehat{D_\mu^m f}\left(\frac{x}{2t}\right) = \left(i\frac{x}{2t}\right)^m e^{it\Delta_\mu}\mathcal{S}f(x).$$

Consequently we have $z^k e^{it\Delta_\mu}\mathcal{S}f(z)$ in $\mathcal{H}_{\mu,t}(\mathbb{C})$ for $k \in \{0, \ldots, m\}$ if $f \in W_\mu^{m,2}(\mathbb{R})$. This fact forces us to consider the Hilbert space

$$\mathcal{H}_{\mu,t}^m(\mathbb{C}) = \left\{F \in \mathcal{O}(\mathbb{C}) : \sum_{k=0}^m \left(\frac{1}{2t}\right)^{2k} \left\|z^k F\right\|^2_{\mathcal{H}_{\mu,t}(\mathbb{C})} < \infty\right\}$$

with the inner product on $\mathcal{H}_{\mu,t}^m(\mathbb{C})$ by

$$\langle F, G\rangle_{\mathcal{H}_{\mu,t}^m(\mathbb{C})} := \sum_{k=0}^m \left(\frac{1}{2t}\right)^{2k} \left\langle z^k F, z^k G\right\rangle_{\mathcal{H}_{\mu,t}(\mathbb{C})}.$$

Theorem 4.10 *The set $\left\{\Upsilon_{k,m}^{\mu,t} := \frac{\Upsilon_k^{\mu,t}}{\|\Upsilon_k^{\mu,t}\|_{\mathcal{H}_{\mu,t}^m(\mathbb{C})}} : k \in \mathbb{N}\right\}$ forms an orthonormal basis for $\mathcal{H}_{\mu,t}^m(\mathbb{C})$.*

Proof We know that $\Omega := \left\{\Upsilon_k^{\mu,t} : k \in \mathbb{N}\right\}$ forms a complete orthonormal basis for $\mathcal{H}_{\mu,t}(\mathbb{C})$. By using this we prove Ω is an orthogonal set in $\mathcal{H}_{\mu,t}^m(\mathbb{C})$. For that $j \neq l \in \mathbb{N}^n$,

$$\begin{aligned}
\left\langle \Upsilon_j^{\mu,t}, \Upsilon_l^{\mu,t}\right\rangle_{\mathcal{H}_{\mu,t}^m(\mathbb{C})} &= \sum_{k=0}^m \left(\frac{1}{2t}\right)^{2k} \left\langle z^k \Upsilon_j^{\mu,t}, z^k \Upsilon_l^{\mu,t}\right\rangle_{\mathcal{H}_{\mu,t}(\mathbb{C})} \\
&= \sum_k^m |r_k^\mu(j)||r_k^\mu(l)| \left\langle \Upsilon_{j+s}^{\mu,t}, \Upsilon_{l+s}^{\mu,t}\right\rangle_{\mathcal{H}_{\mu,t}(\mathbb{C})} \\
&= 0.
\end{aligned}$$

Here $r_k^\mu(l) = (-1)^k \prod_{j=0}^k (2(l + j + 2\mu\theta_{l+j}))^{\frac{1}{2}}$ for $l, k \in \mathbb{N}$. This proves the orthogonality.

For $F \in \mathcal{H}^m_{\mu,t}(\mathbb{C})$, we have the following representation,

$$F = \sum_{k=0}^{\infty} \left\langle F, \Upsilon_k^{\mu,t} \right\rangle_{\mathcal{H}_{\mu,t}(\mathbb{C})} \Upsilon_k^{\mu,t}.$$

So $zF(z) = \sum_{k=0}^{\infty} \left\langle F, \Upsilon_k^{\mu,t} \right\rangle_{\mathcal{H}_{\mu,t}(\mathbb{C})} z\Upsilon_k^{\mu,t}(z)$. Similarly for $k \in \{0, 1, \ldots, m\}$, we have

$$\begin{aligned} z^k F(z) &= \sum_{j=0}^{\infty} \left\langle F, \Upsilon_j^{\mu,t} \right\rangle_{\mathcal{H}_{\mu,t}(\mathbb{C})} z^k \Upsilon_j^{\mu,t}(z), \\ &= \left(\frac{-i}{2t}\right)^k \sum_{j=0}^{\infty} \left\langle F, \Upsilon_j^{\mu,t} \right\rangle_{\mathcal{H}_{\mu,t}(\mathbb{C})} |r_k^{\mu}(j)| \Upsilon_{j+k}^{\mu,t}(z). \end{aligned}$$

Suppose $\left\langle F, \Upsilon_l^{\mu,t} \right\rangle_{\mathcal{H}^m_{\mu,t}(\mathbb{C})} = 0$ for all $l \in \mathbb{N}$. This implies that,

$$\begin{aligned} 0 &= \sum_{k=0}^{m} \left(\frac{1}{2t}\right)^{2k} \left\langle z^k F, z^k \Upsilon_l \right\rangle_{\mathcal{H}_{\mu,t}(\mathbb{C})} \\ &= \sum_{k=0}^{m} \left\langle \sum_{j=0}^{\infty} |r_k^{\mu}(j)| \left\langle F, \Upsilon_j^{\mu,t} \right\rangle_{\mathcal{H}_{\mu,t}(\mathbb{C})} \Upsilon_{j+k}^{\mu,t}\, ,\ |r_k(l)| \Upsilon_{l+k} \right\rangle_{\mathcal{H}_{\mu,t}(\mathbb{C})} \\ &= \langle F, \Upsilon_l^{\mu,t} \rangle_{\mathcal{H}_{\mu,t}(\mathbb{C})} \left\{ \sum_{k=0}^{m} \prod_{s=1}^{k} (2(l+s+2\mu\theta_{l+s})) \right\}. \end{aligned}$$

Since $\prod_{s=1}^{k} (2(l+s+2\mu\theta_{l+s})) > 0$ for every $k \in \{0, 1, \ldots, m\}$, we have $\langle F, \Upsilon_l^{\mu,t} \rangle_{\mathcal{H}_{\mu,t}(\mathbb{C})} = 0$. This implies that $F = 0$. This proves the completeness of Ω in $\mathcal{H}^m_{\mu,t}(\mathbb{C})$. □

Now we have a complete orthonormal basis for $\mathcal{H}^m_{\mu,t}(\mathbb{C})$ as well as for $W^{m,2}_{\mu}(\mathbb{R})$. From this we can establish the characterization of the image of Sobolev space under Schrödinger semigroup.

Theorem 4.11 *The map $e^{it\Delta_\mu} : W^{m,2}_{\mu}(\mathbb{R}) \to \mathcal{H}^m_{\mu,t}(\mathbb{C})$ is a bounded and invertible operator.*

Proof We know that $\left\{ \frac{\psi_k^{\mu}}{\| \psi_k^{\mu} \|_{W^{m,2}_{\mu}(\mathbb{R})}} : k \in \mathbb{N} \right\}$ is a complete orthonormal basis for $W^{m,2}_{\mu}(\mathbb{R})$ and $\left\{ \Upsilon_{k,m}^{\mu,t} : k \in \mathbb{N} \right\}$ forms an complete orthonormal basis for $\mathcal{H}^m_{\mu,t}(\mathbb{C})$. For $k \in \mathbb{N}$,

$$\left\| \Upsilon_k^{\mu,t} \right\|^2_{\mathcal{H}_{\mu,t}(\mathbb{C})} = \| \psi_k^{\mu} \|^2_{W^{m,2}_{\mu}(\mathbb{R})} = \sum_{j=0}^{m} \prod_{s=1}^{j} 2(k+s+2\mu\theta_{k+j}).$$

Moreover, for $k \in \mathbb{N}$ and $z \in \mathbb{C}$ from proposition 3.3, we have

$$\left(e^{it\Delta_\mu} \mathcal{S} \frac{\psi_k^{\mu,t}}{\left\| \psi_k^{\mu,t} \right\|_{W_\mu^{m,2}(\mathbb{R})}} \right)(z) = \Upsilon_{k,m}^{\mu,t}(z).$$

So, $e^{it\Delta_\mu}\mathcal{S}$ takes an orthonormal basis in $W_\mu^{m,2}(\mathbb{R})$ to an orthonormal basis in $\mathcal{H}_{\mu,t}^m(\mathbb{C})$. This implies that $e^{it\Delta_\mu}\mathcal{S} : W_\mu^{m,2}(\mathbb{R}) \to \mathcal{H}_{\mu,t}^m(\mathbb{C})$ is a unitary operator. In particular, This implies that,

$$\left\| e^{it\Delta_\mu} \mathcal{S} f \right\|_{\mathcal{H}_{\mu,t}^m(\mathbb{C})} = \| f \|_{W_\mu^{m,2}(\mathbb{R})} \quad \text{for all } f \in W_\mu^{m,2}(\mathbb{R}).$$

Since $\mathcal{S} : W_\mu^{m,2}(\mathbb{R}) \to W_\mu^{m,2}(\mathbb{R})$ is bounded and invertible operator, there exists $M_1, M_2 > 0$ such that

$$M_1 \| f \|_{W_\mu^{m,2}(\mathbb{R})} \leq \left\| e^{it\Delta_\mu} f \right\|_{\mathcal{H}_{\mu,t}^m(\mathbb{C})} \leq M_2 \| f \|_{W_\mu^{m,2}(\mathbb{R})} \quad \text{for all } f \in W_\mu^{m,2}(\mathbb{R}).$$

□

The $n-$dimensional characterization of the image of Sobolev space under Schrödinger semigroup is analogous to the one dimensional case. The set $\left\{ \frac{\Psi_\alpha^\mu}{\left\| \Psi_\alpha^\mu \right\|_{W_\mu^{m,2}(\mathbb{R}^n)}} : \alpha \in \mathbb{N}^n \right\}$ forms a complete orthonormal basis for $W_\mu^{m,2}(\mathbb{R}^n)$ and the transform $\mathcal{S} : W_\mu^{m,2}(\mathbb{R}^n) \to W_\mu^{m,2}(\mathbb{R}^n)$ is a bounded and invertible operator. For $\alpha \in \mathbb{N}^n$, from the Eq. (4.8) we have

$$D_{k,\mu} \Psi_\alpha^\mu = -\left(2(2\alpha_k + 1 + 2\mu_k \theta_{\alpha_k+1})\right)^{\frac{1}{2}} \Psi_{\alpha+e_k}^\mu, \quad \alpha \in \mathbb{N}^n. \tag{4.12}$$

Similarly for $\beta \in \mathbb{N}^n$,

$$D_\mu^\beta \Psi_\alpha^\mu = (-1)^{|\beta|} \prod_{k=1}^{n} \prod_{j=1}^{\beta_k} \left(2(\alpha_k + j + 2\mu_k \theta_{\alpha_k+j})\right)^{\frac{1}{2}} \Psi_{\alpha+\beta}. \tag{4.13}$$

Consider the Hilbert space

$$\mathcal{H}_{\mu,t}^m(\mathbb{C}^n) := \left\{ F \in \mathcal{O}(\mathbb{C}^n) : z^\beta F \in \mathcal{H}_{\mu,t}(\mathbb{C}^n), 0 \leq |\beta| \leq m \right\}$$

with the inner product

$$\langle F, G \rangle_{\mathcal{H}_{\mu,t}^m(\mathbb{C}^n)} := \sum_{|\beta| \leq m} \left(\frac{1}{2t} \right)^{2|\beta|} \left\langle z^\beta F, z^\beta G \right\rangle_{\mathcal{H}_{\mu,t}(\mathbb{C}^n)}, \quad F, G \in \mathcal{H}_{\mu,t}^m(\mathbb{C}^n).$$

The system of vectors $\left\{\Upsilon^{\mu,t}_{\alpha,m} = \frac{\Upsilon^{\mu,t}_{\alpha}}{\|\Upsilon^{\mu,t}_{\alpha}\|_{\mathcal{H}^m_{\mu,t}(\mathbb{C}^n)}} : \alpha \in \mathbb{N}^n\right\}$ forms a complete orthonormal basis for $\mathcal{H}^m_{\mu,t}(\mathbb{C}^n)$.

Theorem 4.12 *The operator* $e^{it\Delta_\mu} : W^{m,2}_\mu(\mathbb{R}^n) \to \mathcal{H}^m_{\mu,t}(\mathbb{C}^n)$ *is bounded and invertible. Moreover, the space* $\mathcal{H}^m_{\mu,t}(\mathbb{C}^n)$ *is a reproducing kernel Hilbert space.*

Proof Using the above details and following similar technique as in the Theorem 4.11, we can prove that $e^{it\Delta_\mu} : W^{m,2}_\mu(\mathbb{R}^n) \to \mathcal{H}^m_{\mu,t}(\mathbb{C}^n)$ is a bounded and invertible operator. For every $F \in \mathcal{H}^m_{\mu,t}(\mathbb{C}^n)$ and fixed $z \in \mathbb{C}^n$ we have the following representation,

$$F(z) = \sum_{\alpha\in\mathbb{N}^n} \langle F, \Upsilon^{\mu,t}_{\alpha,m}\rangle_{\mathcal{H}^m_{\mu,t}(\mathbb{C}^n)} \Upsilon^{\mu,t}_{\alpha,m}(z).$$

Since $\sum_{\alpha\in\mathbb{N}^n} |\left\langle F, \Upsilon^{\mu,t}_{\alpha,m}\right\rangle|^2_{\mathcal{H}^m_{\mu,t}(\mathbb{C}^n)} < \infty$ and

$$\begin{aligned}\sum_{\alpha\in\mathbb{N}^n} |\Upsilon^{\mu,t}_{\alpha,m}(z)|^2 &\le \sum_{\alpha\in\mathbb{N}^n} \frac{|z^\alpha|^2}{\|\Upsilon_\alpha\|^2_{\mathcal{H}^m_{\mu,t}(\mathbb{C}^n)} 2^{2|\alpha|}\gamma_\mu(\alpha)} \\ &= E_\mu\left(\left(\frac{|z_1|^2}{2}, \dots, \frac{|z_1|^2}{2}\right), (1, \dots, 1)\right) < \infty,\end{aligned}$$

we have

$$|F(z)|^2 \le \left|\frac{M_\mu}{(it)^{\nu_\mu + \frac{n}{2}}}\right|^2 \|F\|^2_{\mathcal{H}^m_{\mu,t}(\mathbb{C}^n)} \sum_{\alpha\in\mathbb{N}^n} \frac{|z^\alpha|^2}{2^{2|\alpha|}\gamma_\mu(\alpha) \left\|\Upsilon^{\mu,t}_\alpha\right\|^2_{\mathcal{H}^m_{\mu,t}(\mathbb{C}^n)}}.$$

From the above equation we can conclude that pointwise evalutaions are continuous on $\mathcal{H}^m_{\mu,t}(\mathbb{C}^n)$. This proves that $\mathcal{H}^m_{\mu,t}(\mathbb{C}^n)$ is a reproducing kernel Hilbert space. Since $\left\{\Upsilon^{\mu,t}_{\alpha,m} : \alpha \in \mathbb{N}^n\right\}$ forms a complete orthonormal basis for $\mathcal{H}^m_{\mu,t}(\mathbb{C}^n)$, one can verify that the reproducing kernels are given by

$$\mathbb{K}^m_{\mu,t}(z, w) = \sum_{\alpha\in\mathbb{N}^n} \frac{z^\alpha \overline{w^\alpha}}{2^{2|\alpha|}\gamma_\mu(\alpha) \left\|\Upsilon^{\mu,t}_\alpha\right\|^2_{\mathcal{H}^m_{\mu,t}(\mathbb{C}^n)}} e^{-\frac{1}{16t^2}(z^2+\overline{w}^2)} e^{\frac{i}{4t}(z^2-\overline{w}^2)},$$

where $\|\Upsilon^{\mu,t}_\alpha\|^2_{\mathcal{H}^m_{\mu,t}(\mathbb{C}^n)} = \sum_{|\beta|\le m} \prod_{k=1}^n \prod_{j=1}^{\beta_k} \left(2(\alpha_k + j + 2\mu_k\theta_{\alpha_k+j})\right).$ □

Remark 4.13 As in Theorem 3.4 we can see that $\mathcal{H}^m_{\mu,t}(\mathbb{C}^n)$ is unitarily equivalent to $\bigotimes_{k=1}^n \mathcal{H}^m_{\mu_k,t}(\mathbb{C})$.

5 Image of Dunkl–Hermite–Sobolev space under Schrödinger semigroup associated to the Dunkl–Hermite operator

Let G be a finite reflection group on $\mathbb{R}^n$ and μ be a non-negative multiplicity function, on some fixed root system $\mathcal{R}$, that is invariant under the group action G on $\mathbb{R}^n$. The Dunkl–Hermite operator on $\mathbb{R}^n$ is given by

$$H_\mu = -\Delta_\mu + x^2.$$

In this section we define Dunkl–Hermite–Sobolev space and wish to discuss the image of it under Schrödinger semigroup associated to the operator H_μ.

It is known that from [17], the generalized Hermite functions $\Phi_{\alpha,\mu}$ for $\alpha \in \mathbb{N}^n$ are eigen vectors of H_μ with eigen values $(2|\alpha| + 2\nu_\mu + n)$. They form a complete orthonormal basis for $L^2(\mathbb{R}^n, h_\mu(u)du)$. So, the spectral decomposition of H_μ is given by

$$H_\mu = \sum_{k=0}^{\infty} P_k f \tag{5.1}$$

where $P_k f = \sum_{|\alpha,|=k} \langle f, \Phi_{\alpha,\mu}\rangle \Phi_{\alpha,\mu}$. Using generalized Mehler's formula [17], we can see that the heat semigroup associated to the Dunkl–Hermite operator e^{-tH_μ} on $L^2(\mathbb{R}^n, h_\mu(u)du)$ given by,

$$e^{-tH_\mu} f(x) = \int_{\mathbb{R}^n} f(u) K_{H_\mu,t}(x,u) h_\mu(u)du, \tag{5.2}$$

where the kernel is

$$K_{H_\mu,t}(x,u) = (2\sin ht)^{-(\nu_\mu+\frac{n}{2})}\, e^{-\frac{1}{2}\coth 2t(x^2+u^2)} E_\mu\left(\csc h2tx, u\right),$$

for all $x, u \in \mathbb{R}^n$. For $n = 1$, the heat semigroup e^{-tH_μ} is considered and studied in [18,19] by Ben Salem and Nefzi. For $t \neq k\pi$ where $k \in \mathbb{N}$, the Schrödinger semigroup is an integral operator on $L^2(\mathbb{R}^n, h_\mu(u)du)$ given by

$$e^{-itH_\mu} f(x) = \int_{\mathbb{R}^n} f(u) K_{H_\mu,it}(u,x) e^{-u^2} h_\mu(u)du, \tag{5.3}$$

where

$$K_{H_\mu,it}(u,x) = (2i\sin 2t)^{-(\nu_\mu+\frac{n}{2})} \exp\left(\frac{i}{2}\cot 2t(u^2+x^2)\right) E_\mu(-i\csc 2tu, x).$$

If $f \in L^2(\mathbb{R}^n, e^{u^2} h_\mu(u)du)$ then $e^{-itH_\mu} f$ can be extended as an analytic function on $\mathbb{C}^n$, by the discussion given in Sect. 3. Also it can be seen that

$$e^{-itH_\mu} f(z) = (2i\sin 2t)^{-(\mu+\frac{n}{2})} e^{\frac{i}{2}\cot 2tz^2} \left(g * F_\mu(s,.)\right)(z).$$

Let $\mathcal{HL}_{\overline{\mathbb{K}}_{it,H_\mu}}(\mathbb{C}^n)$ be a reproducing kernel Hibert space of holomorphic fuctions whose reproducing kernel function is

$$\overline{\mathbb{K}}_{it,H_\mu}(z,w) = e^{\frac{i}{2}\cot 2t(z^2-\overline{w}^2)} e^{-\frac{1}{4}\csc^2 2t(z^2+\overline{w}^2)} E_\mu\left(\frac{\csc 2tz}{\sqrt{2}}\ ,\ \frac{\csc 2t\overline{w}}{\sqrt{2}}\right).$$

The proof of the following theorem is similar to the proof of the Theorem 3.1.

Theorem 5.1 *The operator* $e^{-itH_\mu} : L^2(\mathbb{R}^n, e^{u^2}h_\mu(u)du) \to \mathcal{HL}_{\overline{\mathbb{K}}_{it,H_\mu}}(\mathbb{C}^n)$ *is a unitary operator.*

For rest of the section we fix $G = \mathbb{Z}_2^n$, the root system $\mathcal{R} = \{\pm e_i : i = 1, 2, \ldots, n\}$ and μ be any nonnegative function on $\mathcal{R}$ which is invariant under G. For $k \in \mathbb{N}$, let us consider the operator $A_{k,\mu} = -D_{k,\mu} + x_k$ on suitable dense class of $L^2(\mathbb{R}^n, h_\mu(u)du)$. Its adjoint is given by $A^*_{k,\mu} = D_{k,\mu} + x_k$. Then the Dunkl–Hermite operator can be written as

$$H_\mu = \frac{1}{2}\sum_{k=1}^{n}(A_{k,\mu}A^*_{k,\mu} + A^*_{k,\mu}A_{k,\mu}).$$

Definition 5.2 Let $m \in \mathbb{N}$, the Dunkl–Hermite–Sobolev spaces is defined by,

$$W^{m,2}_{H_\mu}(\mathbb{R}^n) := \left\{ f \in L^2_\mu(\mathbb{R}^n) : A^\alpha_\mu \overline{A_\mu}^\beta f \in L^2_\mu(\mathbb{R}^n), |\alpha| + |\beta| \le m \right\}, \tag{5.4}$$

where $A^\alpha_\mu = A^{\alpha_1}_{1,\mu} \cdots A^{\alpha_n}_{n,\mu}$ and $\overline{A}^\beta_\mu = (A^*_{1,\mu})^{\beta_1} \cdots (A^*_{n,\mu})^{\beta_n}$.

The Sobolev space $W^{m,2}_{H_\mu}(\mathbb{R}^n)$ is a Hilbert space under the inner product

$$\langle f\ ,\ g\rangle_{W^{m,2}_{H_\mu}(\mathbb{R}^n)} := \sum_{|\alpha|+|\beta|\le m} \langle A^\alpha_\mu A^\beta_\mu f\ ,\ A^\alpha_\mu A^\beta_\mu g\rangle_{L^2_\mu(\mathbb{R}^n)}.$$

Consider the map $e^{itH_\mu} : W^{m,2}_{H_\mu}(\mathbb{R}^n) \to e^{-itH_\mu}\left(W^{m,2}_{H_\mu}(\mathbb{R}^n)\right)$, where

$$e^{-itH_\mu}\left(W^{m,2}_{H_\mu}(\mathbb{R}^n)\right) := \left\{ e^{itH_\mu} f \in \mathcal{O}(\mathbb{C}^n) : f \in W^{m,2}_{H_\mu}(\mathbb{R}^n)\right\}.$$

Clealry it is linear and bijective. The space $e^{-itH_\mu}\left(W^{m,2}_{H_\mu}(\mathbb{R}^n)\right)$ is made into a Hilbert space simply by transferring the Hilbert space structure of $W^{m,2}_{H_\mu}(\mathbb{R}^n)$ to $e^{itH_\mu}(W^{m,2}_{H_\mu}(\mathbb{R}^n))$ so that the Schrödinger semigroup e^{-itH_μ} is an isometric isomorphism from $W^{m,2}_{H_\mu}(\mathbb{R}^n)$ onto $e^{itH_\mu}(W^{m,2}_{H_\mu}(\mathbb{R}^n))$. This means that

$$\left\langle e^{-itH_\mu} f\ ,\ e^{-itH_\mu} g\right\rangle := \langle f\ ,\ g\rangle_{W^{m,2}_{H_\mu}(\mathbb{R}^n)},$$

where $f, g \in W^{m,2}_{H_\mu}(\mathbb{R}^n)$).

Note that $A_{k,\mu} + A^*_{k,\mu} = 2x_k$ and $A_{k,\mu} - A^*_{k,\mu} = 2D_{k,\mu}$ for $k = 1, 2, \ldots, n$. Using these relations it is easy to see that $W^{m,2}_{H_\mu}(\mathbb{R}^n)$ and $W^{m,2}_{\mu}(\mathbb{R}^n)$ represent the same vector space. From the note 4.9, it is easy to see that there exist $B > 0$ (depends only on m) such that

$$\|f\|_{W^{m,2}_{H_\mu}(\mathbb{R}^n)} \leq B\,\|f\|_{W^{m,2}_{\mu}(\mathbb{R}^n)}$$

for any $f \in W^{m,2}_{\mu}(\mathbb{R}^n)$. That is, the identity map from $W^{m,2}_{\mu}(\mathbb{R}^n)$ to $W^{m,2}_{H_\mu}(\mathbb{R}^n)$ is bounded bijective linear map. From bounded inverse theorem it follows that the norms $\|\cdot\|_{W^{m,2}_{H_\mu}(\mathbb{R}^n)}, \|\cdot\|_{W^{m,2}_{\mu}(\mathbb{R}^n)}$ are equivalent. Hence characterizing the image of $W^{m,2}_{H_\mu}(\mathbb{R}^n)$ under e^{-itH_μ} is equivalent to characterizing the image of $W^{m,2}_{\mu}(\mathbb{R}^n)$ under e^{itH_μ}.

Let $\widetilde{S}_t f(u) = e^{-\frac{i}{2}\cot 2tu^2} f(u)$, for $f \in W^{m,2}_{\mu}(\mathbb{R}^n)$. It is easy to prove that the operator $\widetilde{\mathcal{S}}_t : W^{m,2}_{\mu}(\mathbb{R}^n) \to W^{m,2}_{\mu}(\mathbb{R}^n)$ is bounded and invertible as in the Theorem 4.8 . For $\alpha \in \mathbb{N}^n$ we have,

$$\Upsilon^{H_\mu,t}_{\alpha}(z) := \left(e^{-itH_\mu}\widetilde{\mathcal{S}}\Psi_\alpha\right)(z) = \tilde{a}_{\alpha,\mu,t} z^\alpha e^{(\frac{i}{2}\cot 2t - \frac{1}{4}\csc^2 2t)z^2}, \tag{5.5}$$

where $\tilde{a}_{\alpha,\mu,t} = (2i\sin 2t)^{-\frac{n}{2}} \left(\frac{\prod_{k=1}^n \Gamma(\mu_k + \frac{1}{2})}{2^{|\alpha|}\gamma_\mu(\alpha)}\right)^{\frac{1}{2}} (-i\csc 2t)^{|\alpha|}$.

Consider the Hilbert space

$$\mathcal{HL}_{\overline{\mathbb{K}}^m_{it,H_\mu}}(\mathbb{C}^n) := \left\{ F \in \mathcal{O}(\mathbb{C}^n) : \sum_{|\beta| \leq m} (\csc 2t)^{2|\beta|} \left\|z^\beta F\right\|^2_{\mathcal{HL}_{\overline{\mathbb{K}}_{it,H_\mu}}(\mathbb{C}^n)} < \infty \right\}$$

with the inner product

$$\langle F\ ,\ G\rangle_{\mathcal{HL}^m_{\overline{\mathbb{K}}_{it,H_\mu}}(\mathbb{C}^n)} = \sum_{|\beta| \leq m} (\csc 2t)^{2|\beta|} \left\langle z^\beta F, z^\beta G\right\rangle_{\mathcal{HL}_{\overline{\mathbb{K}}_{it,H_\mu}}(\mathbb{C}^n)},$$

where $F, G \in \mathcal{HL}_{\overline{\mathbb{K}}^m_{it,H_\mu}}(\mathbb{C}^n)$. Moreover, it is a reproducing kernel Hilbert space with the reproducing kernel function is given by

$$\overline{\mathbb{K}}^m_{H_\mu,t}(z,w) = \sum_{\alpha \in \mathbb{N}^n} \frac{z^\alpha \overline{w^\alpha}}{2^{2|\alpha|}\gamma_\mu(\alpha) \left\|\Upsilon^{H_\mu,t}_\alpha\right\|^2_{\mathcal{HL}_{\overline{\mathbb{K}}^m_{it,H_\mu}}(\mathbb{C}^n)}} e^{\frac{1}{4}\csc^2 2t(z^2 + \overline{w}^2)} e^{\frac{i}{2}cot2t(z^2 - \overline{w}^2)}.$$

Theorem 5.3 *The map* $e^{-itH_\mu} : W^{m,2}_{H_\mu}(\mathbb{R}^n) \to \mathcal{HL}^m_{\overline{\mathbb{K}}_{it,H_\mu}}(\mathbb{C}^n)$ *is a bounded and invertible operator.*

From the above theorem we can see that, the image of Dunkl–Hermite–Sobolev space $W^{m,2}_{H_\mu}(\mathbb{R}^n)$ under e^{itH_μ} is identified with the reproducing kernel Hilbert space $\mathcal{HL}_{\overline{\mathbb{K}}^m_{it,H_\mu}}(\mathbb{C}^n)$ upto equivalance of norms.

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