J. Pseudo-Differ. Oper. Appl. (2017) 8:661–678
DOI 10.1007/s11868-017-0225-9

Inverse Weyl transform/operator

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Received: 18 May 2017 / Revised: 20 August 2017 / Accepted: 22 August 2017 /
Published online: 12 September 2017

Abstract The Weyl procedure associates a function of two ordinary variables, called the c-function or symbol, with an operator, called the Weyl operator of the symbol. One generally formulates this association by defining the operator corresponding to a given symbol. In this paper we consider the reverse problem: Given the Weyl operator, what is the matching symbol? We give a number of explicit formulas for obtaining the symbol that would generate an arbitrary Weyl operator, and we illustrate each form with an example.

Keywords Weyl transforms · Wigner distribution · Generalized phase-space distributions

Mathematics Subject Classification Primary 47G30; Secondary 81S30

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1 Introduction

Although there were antecedents in the operational calculus of Heaviside, the concept of associating ordinary functions with operators took on particular importance with the development of quantum mechanics. Since ordinary functions commute and operators generally do not, there is an infinite number of ways to associate a function of two ordinary variables with an operator. The earliest proposed rules were that of Born and Jordan [2] and of Weyl [14], and subsequently other rules have been proposed. Such rules are called rules of association, correspondence rules, ordering rules, among other names. The ordinary function is commonly called a c-function or symbol. The infinite number of rules may be characterized and generated in a simple manner [3,6]. In this paper we restrict ourselves to the Weyl rule. Historically the issue was posed as to how to define the operator for a given symbol. We consider the inverse problem, namely how to obtain the c-function given the Weyl operator. We present a number of explicit formulas and give an example for each one.

Notation Except for the operator D, all operators will be denoted by boldface type. For a symbol we shall generally use $g(x, k)$, and use $\mathbf{G}(\mathbf{x}, D)$ for the corresponding operator. The Weyl operator is generally a function of the operators $\mathbf{x}$ and D where,

$$\mathbf{x} = \begin{cases} x & \text{in the } x \text{ representation} \\ i\,\frac{d}{dk} & \text{in the Fourier representation} \end{cases} \tag{1.1}$$

$$D = \begin{cases} \frac{1}{i}\,\frac{d}{dx} & \text{in the } x \text{ representation} \\ k & \text{in the Fourier representation} \end{cases} \tag{1.2}$$

The fundamental relation between $\mathbf{x}$ and D is the commutator,

$$[\mathbf{x}, D] = \mathbf{x}D - D\mathbf{x} = i \tag{1.3}$$

Depending on the field of study these operators may be appropriately called position and spatial frequency, or position and momentum, and in time-frequency analysis, they correspond to the time and frequency operators [4,5]. To avoid confusion in some equations, instead of the pair $(\mathbf{x}, D)$ as defined above we often use the pair $(\mathbf{y}, D_y)$. All integrals range from $-\infty$ to ∞ unless otherwise indicated.

Throughout the paper all functions will be supposed to belong to suitably regular function spaces in order that all performed operations makes sense. We use the delta function and its properties freely.

2 The Weyl operator

There are two standard definitions of the Weyl procedure, the first, originally given by Weyl, and the second, by the action of the Weyl operator on a function [17].

Definition 1 For the symbol $g(x,k)$, the corresponding operator, $\mathbf{G}(\mathbf{x}, D)$, is:

$$\mathbf{G}(\mathbf{x}, D) = \iint \widehat{g}(\theta,\tau)\, e^{i\theta\mathbf{x}+i\tau D}\, d\theta\, d\tau \tag{2.1}$$

$$= \iint \widehat{g}(\theta,\tau)\; e^{i\theta\tau/2}\, e^{i\theta\mathbf{x}}\, e^{i\tau D}\, d\theta\, d\tau \tag{2.2}$$

where $\widehat{g}(\theta,\tau)$ is the Fourier transform of $g(x,k)$,

$$\widehat{g}(\theta,\tau) = \frac{1}{4\pi^2}\iint g(x,k)\, e^{-i\theta x - i\tau k}\, dx\, dk \tag{2.3}$$

$$g(x,k) = \iint \widehat{g}(\theta,\tau)\; e^{i\theta x+i\tau k}\, d\theta\, d\tau \tag{2.4}$$

In going from Eqs. (2.1) to (2.2) we used the well known identity [12,16]

$$e^{i\theta\mathbf{x}+i\tau D} = e^{i\theta\tau/2}\, e^{i\theta\mathbf{x}}\, e^{i\tau D} = e^{-i\theta\tau/2}\, e^{i\tau D}\, e^{i\theta\mathbf{x}} \tag{2.5}$$

Combining Eqs. (2.2) and (2.3), we also have

$$\mathbf{G}(\mathbf{x}, D) = \left(\frac{1}{2\pi}\right)^2 \iiiint g(x',k')\, e^{i\theta(\mathbf{x}-x')+i\tau(D-k')}\, d\theta\, d\tau dx'\, dk' \tag{2.6}$$

$$= \left(\frac{1}{2\pi}\right)^2 \iiiint g(x',k')\; e^{i\theta\tau/2} e^{i\theta(\mathbf{x}-x')} e^{i\tau(D-k')}\, d\theta\, d\tau dx'\, dk' \tag{2.7}$$

We call $\mathbf{G}(\mathbf{x}, D)$ the Weyl operator corresponding to the symbol $g(x,k)$.

Definition 2 Alternatively, one can define the Weyl procedure by how the operator transforms a function, say $\psi(x)$,

$$\mathbf{G}(\mathbf{x}, D)\; \psi(x) = \frac{1}{2\pi}\iint g\left(\tfrac{1}{2}(x+x'), k\right)\; e^{i(x-x')k}\psi(x')\; dx'\, dk \tag{2.8}$$

Hence the Weyl procedure is often called the Weyl transform because it transforms $\psi(x)$ into the right hand side of Eq. (2.8) [17].

Equivalence of the two definitions For completeness we show the equivalence of the above two definitions. Using Eq. (2.6) and operating on a function $\psi(x)$ we have

$$\mathbf{G}(\mathbf{x}, D)\psi(x) = \left[\left(\frac{1}{2\pi}\right)^2 \iiiint g(x',k')\, e^{i\theta(\mathbf{x}-x')+i\tau(D-k')}\, d\theta\, d\tau dx'\, dk'\right]\psi(x) \tag{2.9}$$

$$= \left(\frac{1}{2\pi}\right)^2 \iiiint g(x',k')\; e^{-i\theta x'-i\tau k'}\; e^{i\theta\tau/2}\, e^{i\theta\mathbf{x}}\, e^{i\tau D}\psi(x)\, d\theta\, d\tau dx'\, dk' \tag{2.10}$$

Using the fact that $e^{i\tau D}$ is the translation operator

$$e^{i\tau D}\psi(x) = e^{\tau \frac{d}{dx}}\psi(x) = \psi(x+\tau) \tag{2.11}$$

we have

$$\begin{aligned}
&\mathbf{G}(\mathbf{x}, D)\psi(x)\\
&= \left(\frac{1}{2\pi}\right)^2 \iiiint g(x', k')\ e^{-i\theta x' - i\tau k'}\, e^{i\theta\tau/2}\, e^{i\theta x}\ \psi(x+\tau)\, d\theta\, d\tau dx'\, dk' \quad (2.12)\\
&= \frac{1}{2\pi}\iiint g(x', k')\,\delta(x' - x + \tau/2)\ e^{-i\tau k'}\ \psi(x+\tau)\, d\tau dx'\, dk' \quad (2.13)\\
&= \frac{1}{2\pi}\iint g(x+\tau/2, k')\ e^{-i\tau k'}\ \psi(x+\tau)\, d\tau\, dk' \quad (2.14)
\end{aligned}$$

Making the transformation

$$x' = x + \tau \quad dx' = d\tau \tag{2.15}$$

Equation (2.8) follows straight forwardly.

In this paper, we deal mostly with the first definition, but all the formulas we derive can be transcribed to the second.

3 Preliminaries

Central to our results is the ordinary function $R(x, k)$ which is defined by

$$R(x, k) = e^{-ixk}\mathbf{G}\left(\mathbf{x}, D\right)\, e^{ixk} = e^{-ixk}\mathbf{G}\left(x, \frac{1}{i}\frac{\partial}{\partial x}\right)\, e^{ixk} \tag{3.1}$$

Using Eq. (2.1) we have

$$\begin{aligned}
R(x, k) &= e^{-ixk}\iint \widehat{g}(\theta, \tau)\, e^{i\theta\mathbf{x} + i\tau D}\, d\theta\, d\tau\, e^{ixk} \quad (3.2)\\
&= \iint e^{-ixk}\,\widehat{g}(\theta, \tau)\ e^{i\theta\tau/2}\, e^{i\theta x}\ e^{i(x+\tau)k} d\theta\, d\tau \quad (3.3)
\end{aligned}$$

giving

$$R(x, k) = \iint \widehat{g}(\theta, \tau)\ e^{i\theta\tau/2}\, e^{i\theta x + i\tau k}\, d\theta\, d\tau \tag{3.4}$$

Expressing $R(x, k)$ in terms of the symbol directly, we have

$$R(x, k) = \frac{1}{4\pi^2}\iiiint g(x', k')\ e^{i\theta\tau/2}\, e^{i\theta(x - x') + i\tau(k - k')}\, d\theta\, d\tau dx' dk' \tag{3.5}$$

Rearrangement form of an operator For an operator $\mathbf{G}(\mathbf{x}, D)$, one defines the function $R(x, k)$ by the following procedure [6]:

$$R(x,k) = \text{rearrange } \mathbf{G}(\mathbf{x}, D) \text{ so that all the } D \text{ operators are to the right of the } \mathbf{x} \text{ operators; then replace } (\mathbf{x}, D) \text{ by } (x, k). \quad (3.6)$$

To perform the rearrangement, one generally uses the commutation relation, Eq. (1.3), and variations of it. Conversely, if we have $R(x, k)$, one gets the operator, $\mathbf{G}(\mathbf{x}, D)$, by:

$$\mathbf{G}(\mathbf{x}, D) = \text{in} R(x,k), \text{ place all the } k \text{ variables to the right of the } x \text{ factors and then substitute } (\mathbf{x}, D) \text{ for } (x, k). \quad (3.7)$$

Notice that we have used the same notation $R(x, k)$ in both Eqs. (3.6) and (3.1) because indeed they are the same. To prove this, consider Eq. (2.2) which we repeat here for convenience

$$\mathbf{G}\,(\mathbf{x}, D) \;=\iint \widehat{g}(\theta,\tau)\; e^{i\theta\tau/2}\, e^{i\theta\mathbf{x}}\, e^{i\tau D}\, d\theta\, d\tau \qquad (3.8)$$

which indeed has the $\mathbf{x}$ operators to the left of the D operators. Hence

$$R(x,k) \;=\iint \widehat{g}(\theta,\tau)\; e^{i\theta\tau/2}\, e^{i\theta x}\, e^{i\tau k}\, d\theta\, d\tau \qquad (3.9)$$

which is the same as Eq. (3.4).

Example Consider the operator

$$\mathbf{G}(\mathbf{x}, D) = \mathbf{x}D^2\mathbf{x}. \qquad (3.10)$$

By making repeated use of the commutation relation, Eq. (1.3), we place all the D operators to the right of the $\mathbf{x}$ operators,

$$\mathbf{x}D^2\mathbf{x} = \mathbf{x}D(\mathbf{x}D - i) = \mathbf{x}D\mathbf{x}D - i\mathbf{x}D = \mathbf{x}(\mathbf{x}D - i)D - i\mathbf{x}D = \mathbf{x}^2D^2 - 2i\mathbf{x}D \quad (3.11)$$

and therefore

$$R(x,k) = x^2k^2 - 2ixk \qquad (3.12)$$

Alternatively, using Eq. (3.1) for $R(x, k)$ we have

$$R(x,k) = e^{-ixk}x\left(\frac{1}{i}\frac{\partial}{\partial x}\right)^2 xe^{ixk} = x^2k^2 - 2ixk \qquad (3.13)$$

which is the same as Eq. (3.12).

We note that the symbol that gives the operator in Eq. (3.10) is

$$g(x,k) = x^2k^2 + 1/2 \tag{3.14}$$

To show that, consider the Weyl correspondence for $\left(x^2k^2 + 1/2\right)$. First we calculate $\widehat{g}(\theta,\tau)$

$$\widehat{g}(\theta,\tau) = \frac{1}{4\pi^2}\iint \left(x^2k^2 + 1/2\right) e^{-i\theta x - i\tau k}\, dx\, dk \tag{3.15}$$

$$= \frac{1}{4\pi^2}\iint \left(-\frac{1}{i}\frac{\partial}{\partial\theta}\right)^2 \left(-\frac{1}{i}\frac{\partial}{\partial\tau}\right)^2 e^{-i\theta x - i\tau k}\, dx\, dk + \frac{1}{2}\delta(\theta)\delta(\tau) \tag{3.16}$$

$$= \left(\frac{\partial}{\partial\theta}\right)^2 \left(\frac{\partial}{\partial\tau}\right)^2 \delta(\theta)\delta(\tau) + \frac{1}{2}\delta(\theta)\delta(\tau) \tag{3.17}$$

Therefore, the corresponding operator, according to Eq. (2.2) is

$$\mathbf{G}\left(\mathbf{x}, D\right) = \iint \widehat{g}(\theta,\tau)\; e^{i\theta\tau/2}\, e^{i\theta\mathbf{x}}\, e^{i\tau D}\, d\theta\, d\tau \tag{3.18}$$

$$= \iint \left(\left(\frac{\partial}{\partial\theta}\right)^2 \left(\frac{\partial}{\partial\tau}\right)^2 \delta(\theta)\delta(\tau) + \frac{1}{2}\delta(\theta)\delta(\tau)\right) e^{i\theta\tau/2}\, e^{i\theta\mathbf{x}}\, e^{i\tau D}\, d\theta\, d\tau \tag{3.19}$$

which evaluates to Eq. (3.10).

4 Inversion: from Weyl operator to symbol

The issue we address is finding the symbol $g(x,k)$ for a given operator, $\mathbf{G}\left(\mathbf{x}, D\right)$, assuming that the operator was obtained by the Weyl procedure. We have found a variety of expressions. We list these results, and for each one we provide a proof, and an example. For the example, we use the one considered in Sect. 3 which we repeat here for convenience. For the symbol

$$g(x,k) = x^2k^2 + 1/2 \tag{4.1}$$

the Weyl operator is

$$\mathbf{G}(\mathbf{x}, D) = \mathbf{x}D^2\mathbf{x} \tag{4.2}$$

and the corresponding rearrangement operator is

$$R(x,k) = x^2k^2 - 2ixk. \tag{4.3}$$

4.1 Direct substitution method in G(x, D)

For an operator $\mathbf{G}(\mathbf{x}, D)$ make the substitution

$$\mathbf{x} \longrightarrow x + \frac{i}{2}\frac{\partial}{\partial k} \quad D \longrightarrow k - \frac{i}{2}\frac{\partial}{\partial x} \tag{4.4}$$

then the symbol is given by

$$g(x,k) = \mathbf{G}\left(x + \frac{i}{2}\frac{\partial}{\partial k}, k - \frac{i}{2}\frac{\partial}{\partial x}\right) 1 \tag{4.5}$$

where the right hand side is the operation on the number one as indicated.

Proof In Eq. (2.2) make the substitution indicated by Eq. (4.4),

$$\mathbf{G}\left(x + \frac{i}{2}\frac{\partial}{\partial k}, k - \frac{i}{2}\frac{\partial}{\partial x}\right) = \iint \widehat{g}(\theta,\tau) e^{i\theta\tau/2}\, e^{i\theta\left(x+\frac{i}{2}\frac{\partial}{\partial k}\right)}\, e^{i\tau\left(k-\frac{i}{2}\frac{\partial}{\partial x}\right)} d\theta\, d\tau \tag{4.6}$$

Now, operate on an arbitrary function, $f(x,k)$,

$$\begin{aligned} &\mathbf{G}\left(x + \frac{i}{2}\frac{\partial}{\partial k}, k - \frac{i}{2}\frac{\partial}{\partial x}\right) f(x,k) \\ &= \iint \widehat{g}(\theta,\tau) e^{i\theta\tau/2}\, e^{i\theta\left(x+\frac{i}{2}\frac{\partial}{\partial k}\right)}\, e^{i\tau\left(k-\frac{i}{2}\frac{\partial}{\partial x}\right)} f(x,k) d\theta\, d\tau & (4.7) \\ &= \iint \widehat{g}(\theta,\tau) e^{i\theta\tau/2}\, e^{i\theta\left(x+\frac{i}{2}\frac{\partial}{\partial k}\right)}\, e^{i\tau k} f(x+\tau/2,k) d\theta\, d\tau & (4.8) \\ &= \iint \widehat{g}(\theta,\tau)\, e^{i\theta\tau/2}\, e^{i\theta x}\, e^{i\tau(k-\theta/2)} f(x+\tau/2, k-\theta/2) d\theta\, d\tau & (4.9) \end{aligned}$$

and therefore

$$\mathbf{G}\left(x + \frac{i}{2}\frac{\partial}{\partial k}, k - \frac{i}{2}\frac{\partial}{\partial x}\right) f(x,k) = \iint \widehat{g}(\theta,\tau)\, e^{i\theta x + i\tau k} f(x+\tau/2, k-\theta/2) d\theta\, d\tau \tag{4.10}$$

If we take

$$f(x,k) = 1 \tag{4.11}$$

then Eq. (4.10) becomes identical to Eq. (4.10), and we have that

$$g(x,k) = \mathbf{G}\left(x + \frac{i}{2}\frac{\partial}{\partial k}, k - \frac{i}{2}\frac{\partial}{\partial x}\right) 1 \tag{4.12}$$

Also, we note, that the operators indicated by Eq. (4.4) have these same commutation relations as $\mathbf{x}$ and D:

$$\left[\left(x + \frac{i}{2}\frac{\partial}{\partial k}\right), \left(k - \frac{i}{2}\frac{\partial}{\partial x}\right)\right] = [\mathbf{x}, D] = i. \tag{4.13}$$

In this regard we mention that the Weyl operator may be obtained from $g(x,k)$ by way of

$$\mathbf{G}(\mathbf{x}, D) = g\left(\mathbf{x}+\alpha-\frac{i}{2}\frac{\partial}{\partial\beta}, D+\beta+\frac{i}{2}\frac{\partial}{\partial\alpha}\right)1\Big|_{\alpha,\beta=0} \tag{4.14}$$

Example Consider the operator

$$\mathbf{G}(\mathbf{x}, D) = \mathbf{x}D^2\mathbf{x} \tag{4.15}$$

Making the substitution indicated by Eq. (4.5) we have

$$g(x,k) = \mathbf{G}\left(x+\frac{i}{2}\frac{\partial}{\partial k}, k-\frac{i}{2}\frac{\partial}{\partial x}\right)1 \tag{4.16}$$

$$= \left(x+\frac{i}{2}\frac{\partial}{\partial k}\right)\left(k-\frac{i}{2}\frac{\partial}{\partial x}\right)^2\left(x+\frac{i}{2}\frac{\partial}{\partial k}\right)1 \tag{4.17}$$

$$= k^2x^2+1/2 \tag{4.18}$$

4.2 Integral transformation of $R(x,k)$

For an operator $\mathbf{G}(\mathbf{x}, D)$ with a corresponding $R(x,k)$, as defined in Sect. 3, the symbol $g(x,k)$ may obtained by way of

$$g(x,k) = \frac{1}{\pi}\iint R(x',k')\, e^{2i(x'-x)(k'-k)}\, dx'\, dk' \tag{4.19}$$

or, translating the integration variables, via

$$g(x,k) = \frac{1}{\pi}\iint R(x'+x, k'+k)\, e^{2ix'k'}\, dx'\, dk' \tag{4.20}$$

In addition,

$$g(x,k) = -\frac{1}{2\pi}\iint R(x', k-\theta/2)e^{i\theta(x-x')}dx'd\theta \tag{4.21}$$

Proof Inverting the Fourier transform in Eq. (3.4) to find $\widehat{g}(\theta,\tau)$ in terms of R, we have

$$\widehat{g}(\theta,\tau) = \frac{e^{-i\theta\tau/2}}{4\pi^2}\iint R(x',k')e^{-i\theta x'-i\tau k'}dx'dk' \tag{4.22}$$

and using Eq. (2.4) to obtain the symbol g from $\widehat{g}$,

$$g(x,k) = \frac{1}{4\pi^2}\iiiint e^{-i\theta\tau/2}R(x',k')e^{i\theta(x-x')+i\tau(k-k')}dx'dk'd\theta\, d\tau \tag{4.23}$$

which simplifies to

$$g(x,k) = \frac{1}{\pi}\iint R(x',k')\, e^{2i(x'-x)(k'-k)}\, dx'\, dk' \tag{4.24}$$

which is (4.19).

To obtain Eq. (4.21), we make the substitution

$$k' = k - \theta/2 \qquad dk' = -d\theta/2 \tag{4.25}$$

which gives

$$g(x,k) = \frac{1}{2\pi}\iint R(x', k-\theta/2)\, e^{-i\theta(x'-x)}\, dx'\, d\theta. \tag{4.26}$$

□

Example Substituting $R(x,k)$ from Eq. (4.3) into (4.20), we have

$$g(x,k) = \frac{1}{\pi}\iint \left[(x'+x)^2(k'+k)^2 - 2i(x'+x)\right] e^{2ix'k'}\, dx'\, dk', \tag{4.27}$$

which straightforwardly evaluates to

$$g(x,k) = x^2k^2 + 1/2. \tag{4.28}$$

4.3 Operator form on $R(x,k)$

Equation (4.19) can be put into an operator form giving

$$g(x,k) = \exp\left[\frac{i}{2}\frac{\partial}{\partial x}\frac{\partial}{\partial k}\right] R(x,k) \tag{4.29}$$

Proof From Eq. (4.19) we have

$$g(x,k) = \frac{1}{4\pi^2}\iiiint R(x',k')\, e^{-i\theta\tau/2} e^{-i\theta x'}\, e^{-i\tau k'} e^{i\theta x + i\tau k}\, dx' dk'\, d\theta\, d\tau \tag{4.30}$$

$$= \exp\left[\frac{i}{2}\frac{\partial}{\partial x}\frac{\partial}{\partial k}\right] \frac{1}{4\pi^2}\iiiint R(x',k') e^{-i\theta(x'-x)}\, e^{-i\tau(k'-k)} dx' dk' d\theta\, d\tau \tag{4.31}$$

which gives Eq. (4.29). □

Example Using Eq. (4.3) for $R(x,k)$

$$g(x,k) = \exp\left[\frac{i}{2}\frac{\partial}{\partial x}\frac{\partial}{\partial k}\right] (x^2k^2 - 2ixk). \tag{4.32}$$

Expanding the exponential in a power series in its argument, only the first three terms contribute to the sum

$$g(x,k) = (x^2k^2 - 2ixk) + \left[\frac{i}{2}\frac{\partial}{\partial x}\frac{\partial}{\partial k}\right](x^2k^2 - 2ixk) + \frac{1}{2}\left[\frac{i}{2}\frac{\partial}{\partial x}\frac{\partial}{\partial k}\right]^2 (x^2k^2 - 2ixk), \tag{4.33}$$

giving

$$g(x,k) = x^2k^2 + 1/2 \tag{4.34}$$

4.4 $R(x,k)$ operating on e^{-2ixk}

In $R(x,k)$ make the substitution

$$x \longrightarrow \frac{i}{2}\frac{\partial}{\partial k} \qquad k \longrightarrow \frac{i}{2}\frac{\partial}{\partial x} \tag{4.35}$$

which defines the operator $R\left(\frac{i}{2}\frac{\partial}{\partial k}, \frac{i}{2}\frac{\partial}{\partial x}\right)$. Then, the symbol $g(x,k)$ is given by

$$g(x,k) = e^{2ixk} R\left(\frac{i}{2}\frac{\partial}{\partial k}, \frac{i}{2}\frac{\partial}{\partial x}\right) e^{-2ixk} \tag{4.36}$$

Proof In Eq. (3.4) for $R(x,k)$, make the substitutions given by Eq. (4.36) to obtain that

$$e^{2ixk} R\left(\frac{i}{2}\frac{\partial}{\partial k}, \frac{i}{2}\frac{\partial}{\partial x}\right) e^{-2ixk} \tag{4.37}$$

$$= e^{2ixk} \iint \widehat{g}(\theta,\tau)\, e^{i\theta\tau/2}\, e^{i\theta\frac{i}{2}\frac{\partial}{\partial k}}\, e^{i\tau\frac{i}{2}\frac{\partial}{\partial x}} e^{-2ixk} d\theta\, d\tau \tag{4.38}$$

$$= e^{2ixk} \iint \widehat{g}(\theta,\tau)\, e^{i\theta\tau/2}\, e^{-\theta\frac{1}{2}\frac{\partial}{\partial k}}\, e^{-\tau\frac{1}{2}\frac{\partial}{\partial x}} e^{-2ixk} d\theta\, d\tau \tag{4.39}$$

$$= e^{2ixk} \iint \widehat{g}(\theta,\tau)\, e^{i\theta\tau/2}\, e^{-2i(x-\tau/2)(k-\theta/2)} d\theta\, d\tau \tag{4.40}$$

$$= \iint \widehat{g}(\theta,\tau)\, e^{i\theta x + i\tau k} d\theta\, d\tau \tag{4.41}$$

which is $g(x,k)$. □

Example For our example, where $R(x,k) = x^2k^2 - 2ixk$, the expression for the symbol g, as per Eq. (4.36), gives

$$e^{2ixk} R\left(\frac{i}{2}\frac{\partial}{\partial k}, \frac{i}{2}\frac{\partial}{\partial x}\right) e^{-2ixk}$$

$$= e^{2ixk}\left[\left(\frac{i}{2}\frac{\partial}{\partial x}\right)^2\left(\frac{i}{2}\frac{\partial}{\partial k}\right)^2 - 2i\left(\frac{i}{2}\frac{\partial}{\partial x}\right)\left(\frac{i}{2}\frac{\partial}{\partial k}\right)\right] e^{-2ixk} \tag{4.42}$$

$$= e^{2ixk}\left[\left(\frac{i}{2}\frac{\partial}{\partial x}\right)^2 x^2 + \frac{\partial}{\partial x}x\right]e^{-2ixk} \tag{4.43}$$

$$= x^2k^2 + 1/2. \tag{4.44}$$

4.5 Operating on a delta function

Since the delta function forms a complete set, one would expect that the operation of the operator on the delta function would allow one to obtain the symbol. Explicitly we show that

$$g(x,k) = \iint \delta(x - \tau/2 - y)\mathbf{G}(\mathbf{y}, D_y)\delta(x + \tau/2 - y)e^{i\tau k}d\tau dy \tag{4.45}$$

Proof Consider

$$\iint \delta(x - \tau'/2 - y)\mathbf{G}(\mathbf{y}, D_y)\delta(x + \tau'/2 - y)e^{i\tau' k}d\tau' dy$$

$$= \left(\frac{1}{2\pi}\right)^2 \iiiint \iint g(x', k')\delta(x - \tau'/2 - y)\, e^{i\theta(\mathbf{y}-x')+i\tau(D_y-k')}$$
$$\delta(x + \tau'/2 - y)e^{i\tau' k}d\tau' dy d\theta\, d\tau dx'\, dk' \tag{4.46}$$

$$= \left(\frac{1}{2\pi}\right)^2 \iiiint \iint g(x, k)\delta(x - \tau'/2 - y)\, e^{i\theta\tau/2}e^{-i\theta x' - i\tau k'}e^{i\theta\mathbf{y}}e^{i\tau D_y}$$
$$\delta(x + \tau'/2 - y)e^{i\tau' k}d\tau' dy d\theta\, d\tau dx'\, dk' \tag{4.47}$$

$$= \left(\frac{1}{2\pi}\right)^2 \iiiint \iint g(x, k)\delta(x - \tau'/2 - y)\, e^{i\theta\tau/2}e^{-i\theta x' - i\tau k'}e^{i\theta y}$$
$$\delta(x + \tau'/2 - (y+\tau))e^{i\tau' k}d\tau' dy d\theta\, d\tau dx'\, dk' \tag{4.48}$$

witch evaluates to $g(x,k)$. □

Example Using our standard example

$$g(x,k) = \iint \delta(x - \tfrac{1}{2}\tau - y)\mathbf{G}(\mathbf{y}, D_y)\delta(x + \tfrac{1}{2}\tau - y)e^{i\tau k}d\tau dy \tag{4.49}$$

$$= \iint \delta(x - \tfrac{1}{2}\tau - y)y\left(\frac{1}{i}\frac{\partial}{\partial y}\right)^2 y\,\delta(x + \tfrac{1}{2}\tau - y)e^{i\tau k}d\tau dy \tag{4.50}$$

which evaluates to $g(x,k) = x^2k^2 + 1/2$.

4.6 Operating on the exponential

Similar to Eq. (4.45), we have

$$g(x,k)=\frac{1}{2\pi}\iint e^{-i(k-\theta/2)y}\mathbf{G}(\mathbf{y},D_y)e^{i(k+\theta/2)y}e^{-i\theta x}\,d\theta dy \tag{4.51}$$

Proof Substituting Eq. (2.2) for $\mathbf{G}(\mathbf{y}, D_y)$ gives for the right hand side of Eq. (4.51),

$$\frac{1}{2\pi}\iiiint e^{-i(k-\theta/2)y}\,\widehat{g}(\theta',\tau')\;e^{i\theta'\tau'/2}\,e^{i\theta' y}\,e^{i\tau' D_y}\,d\theta'\,d\tau' e^{i(k+\theta/2)y}e^{-i\theta x}d\theta dyd\theta'\,d\tau' \tag{4.52}$$

$$=\frac{1}{2\pi}\iiiint e^{-i(k-\theta/2)y}\,\widehat{g}(\theta',\tau')\;e^{i\theta'\tau'/2}\,e^{i\theta' y}\;e^{i(k+\theta/2)(y+\tau')}e^{-i\theta x}d\theta dyd\theta'\,d\tau' \tag{4.53}$$

$$=\frac{1}{2\pi}\iiiint \widehat{g}(\theta',\tau')\,e^{i\theta y}\,e^{i\theta'\tau'/2}\,e^{i\theta' y}\;e^{i(k+\theta/2)\tau'}\,e^{-i\theta x}d\theta dyd\theta'\,d\tau' \tag{4.54}$$

which evaluates to $g(x,k)$. □

Example For our standard example

$$\mathbf{G}(\mathbf{y},D_y)=\mathbf{x}D^2\mathbf{x} \tag{4.55}$$

we have

$$g(x,k)=\frac{1}{2\pi}\iint e^{-i(k-\theta/2)y}\,y\left(\frac{1}{i}\frac{\partial}{\partial y}\right)^2 y e^{i(k+\theta/2)y}e^{-i\theta x}d\theta dy \tag{4.56}$$

$$=\frac{1}{2\pi}\iint e^{i\theta(y-x)}\left[y^2\left(k+\frac{\theta}{2}\right)^2-2iy\left(k+\frac{\theta}{2}\right)\right]d\theta dy \tag{4.57}$$

which simplifies to

$$g(x,k)=\int \delta(y-x)\left(k^2y^2+1/2\right)dy=x^2k^2+1/2 \tag{4.58}$$

4.7 Operating on the delta function directly

Another form involving the delta function is to define

$$F(x,x')=\mathbf{G}(\mathbf{x},D)\delta(x-x') \tag{4.59}$$

then the symbol is found via

$$g(x,k)=\;2\int F(x',2x-x')e^{2i(x-x')k}dx'. \tag{4.60}$$

A convenient form to evaluate Eq. (4.60) is

$$g(x,k) = 2\iint F(x',x'')\delta(x'+x''-2x)e^{2i(x''-x)k}dx'dx'' \tag{4.61}$$

Proof Using Eq. (2.8), we have that

$$F(x,x') = \mathbf{G}(\mathbf{x},D)\delta(x-x') \tag{4.62}$$

$$= \frac{1}{2\pi}\iint g\left(\tfrac{1}{2}(x+x''),k\right)e^{i(x-x'')k}\delta(x''-x')\,dx''dk \tag{4.63}$$

$$= \frac{1}{2\pi}\int g\left(\tfrac{1}{2}(x+x'),k\right)e^{i(x-x')k}dk. \tag{4.64}$$

A few manipulations and an inverse Fourier transform leads to Eq. (4.60).

Example For our usual example,

$$F(x,x') = \mathbf{G}(\mathbf{x},D)\delta(x-x') = \mathbf{x}D^2\mathbf{x}\,\delta(x-x') = x\left(\frac{1}{i}\frac{\partial}{\partial x}\right)^2 x\,\delta(x-x') \tag{4.65}$$

we have

$$g(x,k) = 2\iint x'\left[\left(\frac{1}{i}\frac{\partial}{\partial x'}\right)^2 x'\delta(x'-x'')\right]\delta(x'+x''-2x)e^{2i(x''-x)k}dx'dx'' \tag{4.66}$$

$$= -\frac{1}{\pi}\iiint x'\left[\left(\frac{\partial}{\partial x'}\right)^2 x'e^{iy(x'-x'')}\right]\delta(x'+x''-2x)e^{2i(x''-x)k}dx'dx''dy \tag{4.67}$$

$$= \frac{1}{\pi}\iiint x'\left[2iy-x'y^2\right]e^{iy(x'-x'')}\delta(x'+x''-2x)e^{2i(x''-x)k}dx'dx''dy \tag{4.68}$$

$$= \frac{1}{\pi}\iint x'\left[2iy-x'y^2\right]e^{i2y(x'-x)}e^{2i(x-x')k}dx'dy. \tag{4.69}$$

Straightforward evaluation leads to $g(x,k) = x^2k^2 + 1/2$.

4.8 Trace of the Wigner and Weyl operators

The symbol may be obtained from

$$g(x,k) = 2\pi\,\mathrm{Tr}(\mathbf{WG}) \tag{4.70}$$

where **W** and **G** are the matrix elements of the Wigner operator and the Weyl operator respectively, terms that we now define. This method is essentially the one presented by Englert [8] and in reference [1]. A similar result was obtained by Duan and Wong [7]; see discussion after Eq. (4.87).

For any complete set $u_n(y)$, we define, as is standard in quantum mechanics, the matrix elements, G_{nk} of an operator $\mathbf{G}(\mathbf{y}, D_\mathbf{y})$ by

$$G_{nm} = \int u_n^*(y)\mathbf{G}(\mathbf{y}, D_\mathbf{y})\, u_m(y) dy. \tag{4.71}$$

We define the Wigner operator by

$$\mathbf{W}_{xk}(\mathbf{y}, D_y) = \frac{1}{4\pi^2} \iint e^{i\theta(\mathbf{y}-x)+i\tau(D_y-k)}\, d\theta\, d\tau \tag{4.72}$$

$$= \frac{1}{(2\pi)^2} \iint e^{i\theta\tau/2} e^{i\theta(\mathbf{y}-x)} e^{i\tau(D_y-k)} d\theta\, d\tau \tag{4.73}$$

where the subscripts x and k are parameters. The reason this is called the Wigner operator is that its expectation value gives the Wigner distribution [1,8,15]. That is,

$$\left\langle \mathbf{W}_{xk}(\mathbf{y}, D_\mathbf{y}) \right\rangle = \int \psi^*(y)\mathbf{W}_{xk}(\mathbf{y}, D_\mathbf{y})\psi(y) dy \tag{4.74}$$

$$= \frac{1}{2\pi} \int \psi^*(x-\tau/2) e^{-i\tau k} \psi(x+\tau/2) d\tau, \tag{4.75}$$

which is the Wigner distribution in x and k. For the Wigner operator, the matrix elements are

$$W_{nm}(x,k) = \frac{1}{(2\pi)^2} \iiint u_n^*(y)\, e^{i\theta(\mathbf{y}-x)+i\tau(D_\mathbf{y}-k)}\, u_m(y)\, d\theta d\tau dy \tag{4.76}$$

$$= \frac{1}{(2\pi)^2} \iiint u_n^*(y)\, e^{i\theta\tau/2}\, e^{i\theta(y-x)-i\tau k}\, u_m(y+\tau)\, d\theta d\tau dy \tag{4.77}$$

$$= \frac{1}{(2\pi)^2} \iiint u_n^*(y-\tau/2)\, e^{i\theta(y-x)-i\tau k}\, u_m(y+\tau/2)\, d\theta d\tau dy \tag{4.78}$$

and are functions of the parameters x and k.

The nk matrix elements of the Weyl operator, $\mathbf{G}(\mathbf{y},\ D_\mathbf{y})$, given by Eq. (2.2), are

$$G_{nm} = \iiint u_n^*(y)\, \widehat{g}(\theta,\tau)\, e^{i\theta\tau/2}\, e^{i\theta\mathbf{y}}\, e^{i\tau D_\mathbf{y}}\, u_m(y)\, d\theta d\tau dy \tag{4.79}$$

$$= \iiint u_n^*(y)\, \widehat{g}(\theta,\tau)\, e^{i\theta\tau/2}\, e^{i\theta y}\, u_m(y+\tau)\, d\theta d\tau dy \tag{4.80}$$

$$= \iiint u_n^*(y-\tau/2)\, \widehat{g}(\theta,\tau)\, e^{i\theta y}\, u_m(y+\tau/2)\, d\theta d\tau dy \tag{4.81}$$

Trace The nm matrix elements of the product of two matrices is

$$(WG)_{nm} = \sum_j W_{nj} G_{jm} \tag{4.82}$$

and their trace is given by

$$\mathrm{Tr}(\mathbf{WG}) = \sum_{jn} W_{nj} G_{jn} \tag{4.83}$$

$$= \frac{1}{(2\pi)^2} \sum_{jn} \iiint \iiint u_n^*(y')\; e^{i\theta'\tau'/2} e^{i\theta'(y'-x)-i\tau' k}\; u_j(y'+\tau')$$
$$\times\; u_j^*(y)\widehat{g}(\theta,\tau)\; e^{i\theta\tau/2}\; e^{i\theta y}\;\; u_n(y+\tau)\, d\theta' d\tau' dy' d\theta d\tau dy \tag{4.84}$$

Since the u_n form a complete set, we have that in general

$$\sum_n u_n^*(y')\; u_n(y) = \delta(y-y'). \tag{4.85}$$

Applying Eq. (4.85) to Eq. (4.84) we have

$$\mathrm{Tr}(\mathbf{WG}) = \frac{1}{4\pi^2} \iiint \iiint e^{i\theta'\tau'/2} \delta(y'-y-\tau)\, \delta(y-y'-\tau') e^{i\theta'(y'-x)-i\tau' k}$$
$$\times\, \widehat{g}\,(\theta,\tau)\; e^{i\theta\tau/2}\; e^{i\theta y}\;\; d\theta' d\tau' dy' d\theta\, d\tau dy \tag{4.86}$$

which simplifies to

$$\mathrm{Tr}(\mathbf{WG}) = \frac{1}{2\pi} g(x,k) \tag{4.87}$$

and hence Eq. (4.70).

As mentioned above, a similar result was obtained by Duan and Wong [7] but with different approach and terminology.[1] It is of interest to relate their result to that of Eq. (4.70). Using our notation, they define integral representation of the Weyl transform by

$$W_g = \int\!\!\int \widehat{g}(\theta,\tau) e^{i\theta\mathbf{x}+i\tau\mathbf{k}} d\theta d\tau \tag{4.88}$$

which is what we have called the Weyl operator, $\mathbf{G}\,(\mathbf{x}, D)$. Also, they define the operator ρ^* by what we have called the Wigner operator $\mathbf{W}_{xk}(\mathbf{y}, D_y)$. Their Theorem 1.8 is that the symbol is the trace of $\rho^* W_g$ which corresponds to Eq. (4.70)

Continuous case If the complete set is continuous, $u_\alpha(y)$, the above summations become integrations and we have that

$$G_{\alpha\beta} = \iiint u_\alpha^*(y)\widehat{g}(\theta,\tau)\; e^{i\theta\tau/2}\; e^{i\theta y}\;\; u_\beta(y+\tau) d\theta\, d\tau dy \tag{4.89}$$

and

$$W_{\alpha\beta}(x,k) = \frac{1}{4\pi^2} \iiint u_\alpha^*(y)\, e^{i\theta\tau/2}\; e^{i\theta(y-x)-i\tau k}\;\; u_\beta(y+\tau) d\theta\, d\tau dy \tag{4.90}$$

[1] The authors thank the referee for making us aware of the paper by Duan and Wong.

and further

$$\mathrm{Tr}(\mathbf{WG}) = \iint W_{\beta\alpha} G_{\alpha\beta} d\alpha d\beta = \frac{1}{2\pi} g(x,k), \tag{4.91}$$

which is the continuous analog of Eq. (4.87).

Examples of complete sets Considering the continuous complete set given by

$$u_\alpha(y) = \frac{1}{\sqrt{2\pi}} e^{i\alpha y} \tag{4.92}$$

The Wigner operator matrix elements are

$$W_{\alpha\beta}(x,k) = \frac{1}{4\pi^2}\frac{1}{2\pi} \iiint e^{-i\alpha y}\ e^{i\theta(\mathbf{y}-x)+i\tau(D_y-k)}\ \ e^{i\beta y} d\theta\, d\tau dy \tag{4.93}$$

$$= \frac{1}{8\pi^3} \iiint e^{i\theta\tau/2} e^{-i\alpha y}\ e^{i\theta(y-x)-i\tau k}\ \ e^{i\beta(y+\tau)} d\theta\, d\tau dy. \tag{4.94}$$

Evaluation leads to

$$W_{\alpha\beta}(x,k) = \frac{1}{2\pi} e^{i(\beta-\alpha)x} \delta\left(k - \frac{\alpha+\beta}{2}\right). \tag{4.95}$$

For the matrix element of the Weyl operator we have

$$G_{\alpha\beta} = \frac{1}{2\pi} \iiint e^{-i\alpha y}\, \widehat{g}(\theta,\tau)\ e^{i\theta\tau/2}\, e^{i\theta y}\, e^{i\tau D_y}\ \ e^{i\beta y/} dy d\theta\, d\tau \tag{4.96}$$

$$= \frac{1}{2\pi} \iiint e^{-i\alpha y}\, \widehat{g}(\theta,\tau)\ e^{i\theta\tau/2}\, e^{i\theta y}\ \ e^{i\beta(y+\tau)} dy d\theta\, d\tau \tag{4.97}$$

which evaluates to

$$G_{\alpha\beta} = \int \widehat{g}\,(\alpha-\beta,\tau)\ \ e^{i(\alpha+\beta)\tau/2} d\tau \tag{4.98}$$

Now consider the trace of the product of $W_{\alpha,\beta}(x,k\ )$ and $G_{\alpha,\beta}$

$$\mathrm{Tr}(\mathbf{WG}) = \iint W_{\beta\alpha} G_{\alpha\beta} d\alpha d\beta \tag{4.99}$$

$$= \frac{1}{2\pi^2} \iiint e^{-i(\beta-\alpha)x} \delta\left(k - \frac{\beta+\alpha}{2}\right) \widehat{g}\,(\alpha-\beta,\tau)\ \ e^{i(\alpha+\beta)\tau/2} d\alpha d\beta d\tau \tag{4.100}$$

which evaluates to

$$\mathrm{Tr}(\mathbf{WG}) = \frac{1}{2\pi} g(x,k). \tag{4.101}$$

Complete set: delta function

We consider the complete set

$$u_\alpha(y) = \delta(y-\alpha). \tag{4.102}$$

For the Wigner matrix elements, we have

$$W_{\alpha\beta}(x,k) = \frac{1}{4\pi^2}\iiint \delta(y-\alpha)\, e^{i\theta(\mathbf{y}-x)+i\tau(D_y-k)}\ \delta(y-\beta)d\theta\, d\tau dy \tag{4.103}$$

$$= \frac{1}{4\pi^2}\iiint \delta(y-\alpha)\, e^{i\theta(\mathbf{y}-x)+i\tau(D_y-k)}\ \delta(y-\beta)d\theta\, d\tau dy \tag{4.104}$$

$$= \frac{1}{4\pi^2}\iiint e^{i\theta\tau/2}\delta(y-\alpha)\, e^{i\theta(y-x)-i\tau k}\ \delta(y-\beta+\tau)d\theta\, d\tau dy \tag{4.105}$$

which simplifies to

$$W_{\alpha\beta}(x,k) = \frac{1}{2\pi}\delta\left(x-\frac{\beta+\alpha}{2}\right)e^{-i(\beta-\alpha)k} \tag{4.106}$$

The **G** matrix elements are,

$$G_{\alpha\beta} = \int \delta(y-\alpha)\,\widehat{g}(\theta,\tau)\ e^{i\theta\tau/2}\, e^{i\theta y}\, e^{i\tau D_y}\ \delta(y-\beta)dyd\theta\, d\tau \tag{4.107}$$

$$= \int \widehat{g}(\theta,\tau)\ e^{i\theta\tau/2}\, e^{i\theta\alpha}\ \ \delta(\alpha-\beta+\tau)d\theta\, d\tau \tag{4.108}$$

which gives

$$G_{\alpha\beta} = \int \widehat{g}\,(\theta,(\beta-\alpha))\ e^{i(\beta+\alpha)\theta/2}\ d\theta. \tag{4.109}$$

Taking the trace, we have

$$\mathrm{Tr}(\mathbf{WG}) = \iint W_{\beta\alpha}G_{\alpha\beta}d\alpha d\beta \tag{4.110}$$

$$= \frac{1}{2\pi}\iiint \delta\left(x-\frac{\alpha+\beta}{2}\right)e^{-i(\alpha-\beta)k}\ \widehat{g}\,(\theta,(\beta-\alpha))\ e^{i(\beta+\alpha)\theta/2}\ d\alpha d\beta d\theta \tag{4.111}$$

which simplifies to

$$\mathrm{Tr}(\mathbf{WG}) = \iint \widehat{g}(\theta,\tau)\, e^{i\tau k}e^{ix\theta}\ d\tau d\theta = \frac{1}{2\pi}g(x,k). \tag{4.112}$$

5 Conclusion

We have presented a number of different expressions for obtaining the symbol that generates a given Weyl operator. The Weyl procedure is not the only possible one and many others have been studied, including the Born and Jordan, standard, anti-standard symmetrization rule, among others [9,11,13]. A unified approach that generates all rules has been developed [3,6,10]. For a symbol $g(x,k)$, the operator is given by

$$\mathbf{G}(\mathbf{x}, D) = \iint \widehat{g}(\theta,\tau)\Phi(\theta,\tau)\, e^{i\theta\mathbf{x}+i\tau D}\, d\theta\, d\tau \tag{5.1}$$

where $\Phi(\theta,\tau)$ is a two dimensional function, called the kernel. By choosing different kernels, different rules are obtained. In a future paper, we will deal with the inverse problem for the generalized correspondence rule indicated by Eq. (5.1).

Acknowledgements This paper originated a number of years ago with discussions that one of the authors (LC) had with Prof. Wong on the problem of the Weyl operator and its inverse. L. Cohen would like to thank Prof. Wong for the many subsequent discussions. The authors would like to thank Profs. Schleich and Scully for the lively discussions we had at the Wyoming summer school. The work of M. Kim and J. Ben-Benjamin is supported by the Office of Naval Research (Award No. N00014-16-1-3054) and the Robert A. Welch Foundation (Grant No. A-1261).

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