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Best trigonometric approximation and Dini-Lipschitz classes

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Abstract In this paper, we prove an analog of Younis's result (Younis in Int J Math Math Sci 9(2):301–312, 1986, Theorem 5.2) on the image under the discrete Fourier–Jacobi transform of a set of functions satisfying the Dini-Lipschitz functions in the space $\mathbb{L}_2^{(\alpha,\beta)}$.

Keywords Fourier–Jacobi series · Generalized translation operator · Dini-Lipschitz functions

Mathematics Subject Classification 43A30

1 Introduction

It is well known that many problems for partial differential equations are reduced to a power series expansion of the desired solution in terms of special functions or orthogonal polynomials (such as Laguerre, Hermite, Jacobi, etc., polynomials). In particular, this is associated with the separation of variables as applied to problems in mathematical physics (see, e.g., [1,2]).

In this article, we obtain an analog of Younis's Theorem [3, Theorem 5.2] on the description of the image under the discrete Fourier–Jacobi transform of a class

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of functions satisfying the Dini-Lipschitz condition in weighted function spaces on $[-1, 1]$. We now give the exact statement of this theorem.

Suppose that $f(x)$ is a function in the $L^2(\mathbb{R})$ space (all functions below are complex-valued), $\|.\|_{L^2(\mathbb{R})}$ is the norm of $L^2(\mathbb{R})$, and δ is an arbitrary number in the interval $(0, 1)$.

Theorem 1.1 [3, Theorem 5.2] *Let* $f \in L^2(\mathbb{R})$. *Then the following conditions are equivalents:*

(i) $\|f(x+h)-f(x)\|_{L^2(\mathbb{R})} = O\left(h^\delta \left(\log \frac{1}{h}\right)^{-\zeta}\right)$, *as* $h \to 0, 0 < \delta < 1, \zeta > 0$,

(ii) $\int_{|\lambda|\geq r} |\widehat{f}(\lambda)|^2 d\lambda = O(r^{-2\delta} (\log r)^{-2\zeta})$ *as* $r \to \infty$,

where $\widehat{f}$ *stands for the Fourier transform of* f.

This theorem has been generalized in the case of noncompact rank 1 Riemannian symmetric spaces [4], and was extented in [5] for the Dunkl transform in the space $L^2(\mathbb{R}^d, w_k(x)dx)$, where w_k is a weight function invariant under the action of an associated reflection group, using a generalized spherical mean operator. The Younis's Theorem 1.1 has been generalized recently for a class of functions satisfying the Dini-Lipschitz condition for the Jacobi–Dunkl transform in [6] and also for the Cherednik–Opdam transform in [7].

On the other hand, in [8, Theorem 2.17], Younis characterizes the set of functions in $L^2([-\pi, \pi])$ satisfying the Lipschitz condition by means of an asymptotic estimate growth of the norm of their discrete Fourier transform. More precisely, we have:

Theorem 1.2 [8, Theorem 2.17] *Let* $f \in L^2([-\pi, \pi])$. *Then the following conditions are equivalents:*

(i) $\|f(x+h) - f(x)\|_{L^2([-\pi,\pi])} = O(h^\delta)$, *as* $h \to 0, \delta \in (0, 1)$,
(ii) $\sum_{|n|\geq N} |\widehat{f}(n)|^2 = O(N^{-2\delta})$ *as* $N \to \infty$,

where $\widehat{f}(n)$ *stands for the nth Fourier transform coefficient of* f.

We emphasize that Younis's Theorem 1.2 has been generalized to the case of compact. Recently, it has also been extended to general compact Lie groups [9], and also for the Dunkl and Fourier–Bessel transform of a class of functions satisfying the Lipschitz condition [10,11].

In our present paper, we investigate among other things the validity of Theorem 1.2 in case of functions of the wider Dini-Lipschitz class in weighted function spaces on $[-1, 1]$. For this purpose, we use a generalized translation operator which was defined by Flensted-Jensen an Koornwinder (see [12]).

2 Preliminaries

Throughout the paper, α and β are arbitrary real numbers with $\alpha \geq \beta \geq -1/2$ and $\alpha \neq -1/2$. We put $w(x) = (1-x)^\alpha(1+x)^\beta$ and consider problems of the approximation of functions in the Hilbert spaces $L_2([-1,1], w(x)dx)$. Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi orthogonal polynomials , $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$ (see [13] or [14]). The polynomials $P_n^{(\alpha,\beta)}(x), n \in \mathbb{N}_0$, form a complete orthogonal system in the Hilbert space $L_2([-1,1], w(x)dx)$. It is known (see [14], Ch. IV) that

$$\max_{-1\leq x\leq 1} |P_n^{(\alpha,\beta)}(x)| = P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{\alpha} = \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+1)}.$$

The polynomials

$$R_n^{(\alpha,\beta)}(x) := \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)}$$

are called normalized Jacobi polynomials.

In what follows it is convenient to change the variable by the formula $x = \cos t$, $t \in I := [0, \pi]$. We use the notation

$$\rho(t) = w(\cos t)\sin t = 2^{\gamma}\left(\sin\frac{t}{2}\right)^{2\alpha+1}\left(\cos\frac{t}{2}\right)^{2\beta+1}, \quad \gamma = \alpha+\beta+1,$$

$$\varphi_n(t) = \varphi_n^{(\alpha,\beta)}(t) := R_n^{(\alpha,\beta)}(\cos t), n \in \mathbb{N}_0.$$

Let $\mathbb{L}_2^{(\alpha,\beta)}$ denote the space of square integrable functions $f(t)$ on the closed interval I with the weight function $\rho(t)$ and the norm

$$\|f\| = \sqrt{\int_0^{\pi} |f(t)|^2\rho(t)dt}.$$

The Jacobi differential operator is defined as

$$\mathcal{B} := \frac{d^2}{dt^2} + \left(\left(\alpha+\frac{1}{2}\right)\cot\frac{t}{2} - \left(\beta+\frac{1}{2}\right)\tan\frac{t}{2}\right)\frac{d}{dt}.$$

The function $\varphi_n(t)$ satisfies the differential equation

$$\mathcal{B}\varphi_n = -\lambda_n\varphi_n, \qquad \lambda_n = n(n+\gamma), n \in \mathbb{N}_0,$$

with the initial conditions $\varphi_n(0) = 1$ and $\varphi_n'(0) = 0$.

Lemma 2.1 *The following inequalities are valid for Jacobi functions* $\varphi_n(t)$

(i) For $t \in (0, \pi/2]$ *we have*

$$|\varphi_n(t)| < 1.$$

(ii) For $t \in [0, \pi/2]$ *we have*

$$1 - \varphi_n(t) \leq c_1 \lambda_n t^2.$$

(iii) For $t \in [0, 1]$ *and* $ts \leq 2$ *we have*

$$1 - \varphi_n(t) \geq c_2 \lambda_n t^2.$$

Proof See [15, Proposition 3.5. and Lemma 3.1]

Recall from [15], the Fourier–Jacobi series of a function $f \in \mathbb{L}_2^{(\alpha,\beta)}$ is defined by

$$f(t) = \sum_{n=1}^{\infty} a_n(f) \tilde{\varphi}_n(t), \tag{1}$$

where

$$\tilde{\varphi}_n = \frac{\varphi_n}{\|\varphi_n\|}, \quad a_n(f) = \langle f, \tilde{\varphi}_n \rangle = \int_0^{\pi} f(t) \tilde{\varphi}_n(t) \rho(t) dt.$$

□

The sequence $\{a_n(f), n \in \mathbb{N}_0\}$ is called the discrete Fourier–Jacobi transform of f.

Let

$$S_m f(t) = \sum_{n=1}^{m-1} a_n(f) \tilde{\varphi}_n(t),$$

be a partial sums of series (1), and let

$$_m(f) = \inf_{P_m} \|f - P_m\|,$$

denote the best approximation of $f \in \mathbb{L}_2^{(\alpha,\beta)}$ by polynomials of the form

$$P_m(t) = \sum_{n=1}^{m-1} c_n \tilde{\varphi}_n(t), \, c_n \in \mathbb{R}.$$

It is well known that

$$\|f\| = \sqrt{\sum_{n=1}^{\infty} |a_n(f)|^2},$$

$$E_m(f) = \|f - S_m f\| = \sqrt{\sum_{n=m}^{\infty} |a_n(f)|^2}.$$

The Jacobi generalized translation is defined by the formula

$$ {}_h f(t) = \int_0^{\pi} f(\theta) K(t, h, \theta) \rho(\theta) d\theta, \quad 0 < t, h < \pi, $$

where $K(t, s, \theta)$ is a certain function (see [16]).

Below are some properties (see [15]):

(i) $T_h : \mathbb{L}_2^{(\alpha,\beta)} \to \mathbb{L}_2^{(\alpha,\beta)}$ is a continuous linear operator,
(ii) $\|T_h f\| \leq \|f\|$,
(iii) $T_h(\varphi_n(t)) = \varphi_n(h)\varphi_n(t)$,
(iv) $a_n(T_h f) = \varphi_n(h) a_n(f)$,
(v) $\|T_h f - f\| \to 0, \quad h \to 0$,
(vi) $\mathcal{B}(T_h f) = T_h(\mathcal{B} f)$.

The following lemma will be needed in due course.

Lemma 2.2 [9, Lemma 4.1] *Suppose* $a \in \mathbb{R}$, $b_n \geq 0$ *and* $0 < c < d$. *Then*

$$ \sum_{n=1}^{N} n^d b_n = O(N^c (\log N)^a) \ \text{ iff } \ \sum_{n=N}^{\infty} b_n = O(N^{c-d} (\log N)^a). $$

3 Main results

For every function $f \in \mathbb{L}_2^{(\alpha,\beta)}$ we define the differences $\Delta_h^k f$ of order, $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$, with step h, $0 < h < \pi$, by the formulae

$$ \Delta_h^1 f(t) = \Delta_h f(t) = (T_h - I) f(t), $$

where I is the identity operator in $\mathbb{L}_2^{(\alpha,\beta)}$.

$$ \Delta_h^k f(t) = \Delta_h(\Delta_h^{k-1} f(t)) = (T_h - I)^k f(t) = \sum_{i=0}^{k} (-1)^{k-1} \binom{k}{i} T_h^i f(t), \quad k > 1, $$

where

$$ T_h^0 f(t) = f(t), \quad T_h^i f(t) = T_h(T_h^{i-1} f(t)), \quad i = 1, 2, \ldots, k. $$

Let W_2^k be the Sobolev space constructed by the Jacobi operator $\mathcal{B}$. that is:

$$ W_2^k := \{ f \in \mathbb{L}_2^{(\alpha,\beta)} : \mathcal{B}^j f \in \mathbb{L}_2^{(\alpha,\beta)}, j = 1, 2, \ldots, k \}, $$

where $\mathcal{B}^0 f = f, \mathcal{B}^j f = \mathcal{B}(\mathcal{B}^{j-1} f)$, $j = 1, 2, \ldots, k$.

Lemma 3.1 *If* $f \in W_2^k$, *then*

$$a_n(f) = (-1)^r \frac{1}{\lambda_n^r} a_n(\mathcal{B}^r f), r \in \mathbb{N}_0,$$

where $r = 0, 1, \ldots, k$.

Proof Since $\mathcal{B}$ is self-adjoint (see [15]), we have

$$\begin{aligned} a_n(f) &= \langle f, \tilde{\varphi}_n \rangle = -\frac{1}{\lambda_n} \langle f, \mathcal{B}\tilde{\varphi}_n \rangle \\ &= -\frac{1}{\lambda_n} \langle \mathcal{B}f, \tilde{\varphi}_n \rangle = -\frac{1}{\lambda_n} a_n(\mathcal{B}f). \end{aligned}$$

This completes the proof of lemma □

Lemma 3.2 *If*

$$f(t) = \sum_{n=1}^{\infty} a_n(f)\tilde{\varphi}_n(t),$$

then

$$T_h f(t) = \sum_{n=1}^{\infty} \varphi_n(h) a_n(f)\tilde{\varphi}_n(t).$$

Here, the convergence of the series on the right-hand side is understood in the sense of $\mathbb{L}_2^{(\alpha,\beta)}$.

Proof By the definition of the operator T_h,

$$T_h(\tilde{\varphi}_n(t)) = \varphi_n(h)\tilde{\varphi}_n(t).$$

Therefore, for any polynomial

$$Q_N(t) = \sum_{n=1}^{N} a_n(f)\tilde{\varphi}_n(t).$$

Since T_h is linear, we have

$$T_h Q_N(t) = \sum_{n=1}^{N} \varphi_n(h) a_n(f)\tilde{\varphi}_n(t). \tag{2}$$

Since T_h is a linear bounded operator in $\mathbb{L}_2^{(\alpha,\beta)}$ and the set of all polynomials $Q_N(t)$ is everywhere dense in $\mathbb{L}_2^{(\alpha,\beta)}$, passage to the limit in (2) give the required equality. □

Remark Since

$$T_h f(t) - f(t) = \sum_{n=1}^{\infty} (\varphi_n(h) - 1) a_n(f) \tilde{\varphi}_n(t),$$

the Parseval's identity gives

$$\|T_h f - f\|^2 = \sum_{n=1}^{\infty} (1 - \varphi_n(h))^2 \, |a_n(f)|^2 \, .$$

If $f \in W_2^k$, from Lemma 3.1, we have

$$\|\Delta_h^k(\mathcal{B}^r f)\|^2 = \sum_{n=1}^{\infty} (1 - \varphi_n(h))^{2k} \lambda_n^{2r} \, |a_n(f)|^2 \, . \tag{3}$$

where $r = 0, 1, \ldots, k$.

Definition 3.1 Let $\zeta \in \mathbb{R}$ and $\delta \in (0, 1)$. A function $f \in W_2^k$ is said to be in the $(\delta, \zeta, 2)$-Jacobi Dini-Lipschitz class, denoted by $Lip(\delta, \zeta, 2)$, if

$$\|\Delta_h^k(\mathcal{B}^r f)\| = O\left(h^\delta \left(\log \frac{1}{h}\right)^\zeta\right) \quad \text{as} \quad h \to 0,$$

where $r = 0, 1, \ldots, k$.

Theorem 3.1 *Let $f \in W_2^k$. The following two conditions are equivalent:*

(a) $f \in Lip(\delta, \zeta, 2)$,
(b) $\sum_{n \geq N} \lambda_n^{4r} |a_n(f)|^2 = O(N^{-2\delta} (\log N)^{2\zeta}), \quad as \quad N \to \infty.$

Proof $(a) \Rightarrow (b)$ Let $f \in Lip(\delta, \zeta, 2)$. Then we have

$$\|\Delta_h^k(\mathcal{B}^r f)\| = O\left(h^\delta \left(\log \frac{1}{h}\right)^\zeta\right) \quad \text{as} \quad h \to 0,$$

It follows from (3) that

$$\sum_{n=1}^{\infty} (1 - \varphi_n(h))^{2k} \lambda_n^{2r} |a_n(f)|^2 = O\left(h^{2\delta} \left(\log \frac{1}{h}\right)^{2\zeta}\right).$$

If $0 \leq n \leq \frac{1}{h}$, then $nh \leq 2$, and from the third inequality of lemma 2.1, we obtain

$$1 - \varphi_n(h) \geq c_2 \lambda_n h^2.$$

Therefore,

$$\sum_{n=1}^{[\frac{1}{h}]} \lambda_n^{2k} h^{4k} \lambda_n^{2r} |a_n(f)|^2 = O\left(h^{2\delta}\left(\log\frac{1}{h}\right)^{2\zeta}\right),$$

and, by $\lambda_n \geq n^2$,

$$\sum_{n=1}^{[\frac{1}{h}]} n^{4k} \lambda_n^{2r} |a_n(f)|^2 = O\left(h^{2\delta-4k}\left(\log\frac{1}{h}\right)^{2\zeta}\right).$$

Putting $N = \frac{1}{h}$, we may write this inequality in the following form:

$$\sum_{n=1}^{N} n^{4k} \lambda_n^{2r} |a_n(f)|^2 = O\left(N^{4k-2\delta}(\log N)^{2\zeta}\right).$$

From Lemma 2.2, we have

$$\sum_{N}^{\infty} \lambda_n^{2r} |a_n(f)|^2 = O\left(N^{4k-2\delta-4k}(\log N)^{2\zeta}\right) = O\left(N^{-2\delta}(\log N)^{2\zeta}\right).$$

Thus, the first implication is proved.

$(b) \Rightarrow (a)$. Suppose now that

$$\sum_{n \geq N} \lambda_n^{2r} |a_n(f)|^2 = O\left(N^{-2\delta}(\log N)^{2\zeta}\right), \quad as \quad N \to \infty.$$

It follows from Lemma 2.2 that

$$\sum_{n=1}^{N} n^{4k} \lambda_n^{2r} |a_n(f)|^2 = O\left(N^{4k-2\delta}(\log N)^{2\zeta}\right).$$

According to (3), we write

$$\|\Delta_h^k(\mathcal{B}^r f)\|^2 \leq \sum_{n=1}^{N}(1-\varphi_n(h))^{2k}\lambda_n^{2r}|a_n(f)|^2 + \sum_{n \geq N}(1-\varphi_n(h))^{2k}\lambda_n^{2r}|a_n(f)|^2.$$

Note that

$$\lambda_n \leq n^2\left(1+\frac{\gamma}{n}\right) \leq n^2(1+\gamma), \quad n = 1, 2, \ldots. \tag{4}$$

It follows from (4) and the second inequality in Lemma 2.1 that

$$\begin{aligned}\sum_{n=1}^{N}(1-\varphi_n(h))^{2k}\lambda_n^{2r}|a_n(f)|^2 &\leq c_1 h^{4k}\sum_{n=1}^{N}\lambda_n^{2k}\lambda_n^{2r}|a_n(f)|^2\\ &\leq c_1(1+\gamma)^{2k}h^{4k}\sum_{n=1}^{N}n^{4k}\lambda_n^{2r}|a_n(f)|^2\\ &= O\left(N^{4k-2\delta-4k}(\log N)^{2\zeta}\right)\\ &= O\left(N^{-2\delta}(\log N)^{2\zeta}\right).\end{aligned}$$

On the other hand, it follows from the first inequality of lemma 2.1 that

$$\begin{aligned}\sum_{n\geq N}(1-\varphi_n(h))^{2k}\lambda_n^{2r}|a_n(f)|^2 &\leq 2^{2k}\sum_{n\geq N}\lambda_n^{2r}|a_n(f)|^2\\ &= O\left(N^{-2\delta}(\log N)^{2\zeta}\right).\end{aligned}$$

Consequently,

$$\|\Delta_h^k(\mathcal{B}^r f)\| = O\left(h^\delta\left(\log\frac{1}{h}\right)^\zeta\right),$$

and this ends the proof of the Theorem. □

We conclude this work by the following immediate consequence.

Corollary 3.1 *Let $f\in W_2^k$, and let*

$$f\in Lip(\delta,\zeta,2).$$

Then

$$E_N(f) = O\left(N^{-\delta-2r}(\log N)^\zeta\right),\quad as \quad N\to\infty.$$

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