

Inhomogeneous microlocal propagation of singularities in Fourier Lebesgue spaces

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Abstract In the paper some results of microlocal continuity for pseudodifferential operators whose symbols belong to weighted Fourier Lebesgue spaces are given. Inhomogeneous local and microlocal propagation of singularities of Fourier Lebesgue type are then studied, with applications to some classes of semilinear equations.

Keywords Microlocal analysis · Pseudodifferential operators · Fourier Lebesgue spaces

1 Introduction

Consider the general nonlinear partial differential equation

$$F(x, \{\partial^\alpha u\}_{|\alpha| \leq m}) = 0, \quad (1)$$

where $F(x, \zeta) \in C^\infty(\mathbb{R}^n \times \mathbb{C}^N)$ for suitable positive integer N .

In order to investigate the regularity of the solutions, we can reduce the study to the linearized equation, obtained by differentiation with respect to x_j

$$\sum_{|\alpha| \leq m} \frac{\partial F}{\partial \zeta^\alpha}(x, \{\partial^\beta u\}_{|\beta| \leq m}) \partial^\alpha \partial_{x_j} u = - \frac{\partial F}{\partial x_j}(x, \{\partial^\beta u\}_{|\beta| \leq m}). \quad (2)$$

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Notice that the regularity of the coefficients $a_\alpha(x) = \frac{\partial F}{\partial \zeta^\alpha}(x, \{\partial^\beta u\}_{|\beta| \leq m})$, depends on the solution u and the function $F(x, \zeta)$. We need then to study as first step the algebra properties of the function spaces wherein we are intended to operate, as well as the behavior of the pseudodifferential operators with symbols in such spaces.

When working in Hölder spaces and Sobolev spaces $H^{s,2}$, we can refer to the paradifferential calculus, developed by Bony and Meyer [2,25,31]. Generalizations of these arguments to symbols in weighted spaces can be found in [7,8,11,36,37].

In this paper we fix the attention on pseudodifferential operators with symbols in weighted Fourier Lebesgue spaces $\mathcal{FL}_\omega^p(\mathbb{R}^n)$, following an idea of Pilipović–Teofanov–Toft [26,27].

Passing now to consider the microlocal regularity properties, let us notice that the Hörmander wave front set, introduced in [20] for smooth singularities and extended to the Sobolev spaces $H^{s,2}$ in [22], uses as basic tool the conic neighborhoods in $\mathbb{R}^n \setminus \{0\}$. Thus the homogeneity properties of the symbol $p(x, \xi)$ and the characteristic set $\text{Char } P$ of the (pseudo) differential operator $P = p(x, D)$, play a key role. In order to better adapt the study to a wider class of equations, starting from the fundamental papers of Beals [1], Hörmander [21], an extensive literature about weighted pseudodifferential operators has been developed, see e.g. [3,4,24,36].

We are particularly interested here in the generalizations of the wave front set not involving the use of conic neighborhoods. In some cases, for example in the study of propagation of singularities of the Schroedinger operator, $i\partial_t - \Delta$, we can use the quasi-homogeneous wave front set, introduced in [23], see further [30,37]. More generally, failing of any homogeneity properties, the propagation of the microlocal singularities are described in terms of filter of neighborhoods, introduced in [28] and further developed in [6–8,10,11,16–18].

In some previous works of the authors, continuity and microlocal properties are considered in Sobolev spaces in L^p setting, see [12,13,15–18].

In the present paper we prove a result of propagation of singularities of Fourier Lebesgue type for partial (pseudo) differential equations, whose symbol satisfies generalized elliptic properties. Namely we obtain an extension of the well known propagation of singularities given by Hörmander [22] for the Sobolev wave front set $WF_{H^{s,2}}$ and operators of order m :

$$WF_{H^{s-m,2}}(Pf) \subset WF_{H^{s,2}}(f) \subset WF_{H^{s-m,2}}(Pf) \cup \text{Char}(P),$$

given in terms of filter of microlocal singularities and quasi-homogeneous wave front set.

Applications to semilinear partial differential equations are given at the end. The plan of the paper is the following: in Sect. 2 the weight functions ω and the Fourier Lebesgue spaces $\mathcal{FL}_\omega^p(\mathbb{R}^n)$ are introduced and their properties studied. In Sects. 3, 4, under suitable additional conditions on the weight function, algebra properties in $\mathcal{FL}_\omega^p(\mathbb{R}^n)$ and continuity of pseudodifferential operators with symbols in Fourier Lebesgue spaces are studied. The microlocal regularity, in terms of inhomogeneous neighborhoods, is introduced and studied in Sect. 5, while the propagation of microlocal singularities is given in Sect. 6, namely in Proposition 15. In Sect. 6 applications

to semilinear equations are studied, with specific examples in the field of quasi-homogeneous partial differential equations.

2 Preliminaries

2.1 Weight functions

Throughout the paper, we call *weight function* any positive measurable map $\omega : \mathbb{R}^n \rightarrow]0, +\infty[$ satisfying the following *temperance* condition

$$(T) \quad \omega(\xi) \leq C(1 + |\xi - \eta|)^N \omega(\eta), \quad \forall \xi, \eta \in \mathbb{R}^n,$$

for suitable positive constants C and N .

In the current literature, a positive function ω obeying condition (T) is said to be either *temperate* (see [11], [20]) or, in the field of Modulation Spaces, *polynomially moderated* (cf. [5], [19], [26], [27]).

For ω, ω_1 weight functions; we write $\omega \leq \omega_1$ to mean that, for some $C > 0$

$$\omega(x) \leq C\omega_1(x), \quad \forall x \in \mathbb{R}^n;$$

moreover we say that ω, ω_1 are *equivalent*, writing $\omega \asymp \omega_1$ in this case, if

$$\omega \leq \omega_1 \quad \text{and} \quad \omega_1 \leq \omega. \tag{3}$$

Applying (T) it yields at once that $\omega(\xi) \leq C(1 + |\xi|)^N \omega(0)$ and $\omega(0) \leq C(1 + |-\xi|)^N \omega(\xi) = C(1 + |\xi|)^N \omega(\xi)$, for any $\xi \in \mathbb{R}^n$.

Thus, for every weight function ω there exist constants $C \geq 1$ and $N > 0$ such that

$$\frac{1}{C}(1 + |\xi|)^{-N} \leq \omega(\xi) \leq C(1 + |\xi|)^N, \quad \forall \xi \in \mathbb{R}^n. \tag{4}$$

Proposition 1 *Let ω, ω_1 be two weight functions and $s \in \mathbb{R}$. Then $\omega\omega_1, 1/\omega$ and ω^s are again weight functions.*

Proof Assume that for suitable constants C, C_1, N, N_1

$$\begin{aligned} \omega(\xi) &\leq C(1 + |\xi - \eta|)^N \omega(\eta), \\ \omega_1(\xi) &\leq C_1(1 + |\xi - \eta|)^{N_1} \omega_1(\eta) \quad \forall \xi, \eta \in \mathbb{R}^n. \end{aligned} \tag{5}$$

Then we deduce that

$$\begin{aligned} \omega(\xi)\omega_1(\xi) &\leq CC_1(1 + |\xi - \eta|)^{N+N_1} \omega(\eta)\omega_1(\eta), \\ 1/\omega(\eta) &\leq C(1 + |\eta - \xi|)^N 1/\omega(\xi), \quad \forall \xi, \eta \in \mathbb{R}^n, \end{aligned}$$

which show that $\omega\omega_1$ and $1/\omega$ are temperate.

If $s \geq 0$ then condition (T) for ω^s follows at once from (5). If $s < 0$ it suffices to observe that $\omega^s = 1/\omega^{-s}$ and then combining the preceding results.

We introduce now some further conditions on the weight function ω which will be repeatedly used in the following.

(SV) *Slowly varying condition* there exist positive constants $C \geq 1$, N such that

$$\frac{1}{C} \leq \frac{\omega(\eta)}{\omega(\xi)} \leq C, \quad \text{when } |\eta - \xi| \leq \frac{1}{C} \omega(\xi)^{1/N}; \quad (6)$$

(SA) *Sub additive condition* for some positive constant C

$$\omega(\xi) \leq C \{ \omega(\xi - \eta) + \omega(\eta) \}, \quad \forall \xi, \eta \in \mathbb{R}^n; \quad (7)$$

(SM) *Sub multiplicative condition* for some positive constant C

$$\omega(\xi) \leq C \omega(\xi - \eta) \omega(\eta), \quad \forall \xi, \eta \in \mathbb{R}^n; \quad (8)$$

(G) δ *condition* for some positive constants C and $0 < \delta < 1$

$$\omega(\xi) \leq C \{ \omega(\eta) \omega(\xi - \eta)^\delta + \omega(\eta)^\delta \omega(\xi - \eta) \}, \quad \forall \xi, \eta \in \mathbb{R}^n; \quad (9)$$

(B) *Beurling's condition* for some positive constant C

$$\sup_{\xi \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\omega(\xi)}{\omega(\xi - \eta) \omega(\eta)} d\eta \leq C. \quad (10)$$

For a thorough account on the relations between the properties introduced above, we refer to [11]. For reader's convenience, here we quote and prove only the following result.

Proposition 2 *For the previous conditions the following relationships are true.*

(i) Assume that ω is uniformly bounded from below in \mathbb{R}^n , that is

$$\inf_{\xi \in \mathbb{R}^n} \omega(\xi) = c > 0. \quad (11)$$

Then (SV) \Rightarrow (T) and (G) \Rightarrow (SM).

(ii) Assume that

$$\frac{1}{\omega} \in L^1(\mathbb{R}^n). \quad (12)$$

Then (SA) \Rightarrow (B) and (G) \Rightarrow $\omega^{\frac{1}{1-\delta}}$ satisfies (B).

Proof Statement i Let the constants C, N be fixed as in (6). For $\xi, \eta \in \mathbb{R}^n$ such that $|\xi - \eta| \leq \frac{1}{C}\omega(\xi)^{1/N}$, it follows directly from (6) that $\omega(\xi) \leq C\omega(\eta) \leq C\omega(\eta)(1 + |\xi - \eta|)^N$. On the other hand, when $|\xi - \eta| > \frac{1}{C}\omega(\xi)^{1/N}$ from (11) we deduce at once $\omega(\xi) \leq C^N |\xi - \eta|^N \leq \frac{C^N}{c}\omega(\eta)(1 + |\xi - \eta|)^N$.

This shows the validity of the first implication. As for the second one, it is sufficient to observe that $\omega(\xi) \geq c > 0$ and $0 < \delta < 1$ yield at once

$$\omega(\xi)^\delta \leq c^{\delta-1}\omega(\xi), \quad \forall \xi \in \mathbb{R}^n. \tag{13}$$

Then the result follows from estimating by (13) the function ω^δ in the right-hand side of (9).

Statement ii For every $\xi \in \mathbb{R}^n$, using (7) we get

$$\frac{\omega(\xi)}{\omega(\xi - \eta)\omega(\eta)} \leq C \left\{ \frac{1}{\omega(\eta)} + \frac{1}{\omega(\xi - \eta)} \right\};$$

hence the first implication follows observing that, by a suitable change of variables, the right-hand side is an integrable function on \mathbb{R}^n , whose integral is independent of ξ .

Concerning the second implication, for every $\xi \in \mathbb{R}^n$, from (9) we get

$$\begin{aligned} \left(\frac{\omega(\xi)}{\omega(\xi - \eta)\omega(\eta)} \right)^{\frac{1}{1-\delta}} &\leq C_\delta \left(\omega(\xi - \eta)^{\delta-1} + \omega(\eta)^{\delta-1} \right)^{\frac{1}{1-\delta}} \\ &\leq C_\delta \left\{ \left(\omega(\xi - \eta)^{\delta-1} \right)^{\frac{1}{1-\delta}} + \left(\omega(\eta)^{\delta-1} \right)^{\frac{1}{1-\delta}} \right\} \\ &= C_\delta \left\{ \frac{1}{\omega(\xi - \eta)} + \frac{1}{\omega(\eta)} \right\}, \end{aligned}$$

for a suitable constant $C_\delta > 0$ depending on δ . Now we conclude as in the proof of the first implication. □

Example 1. The standard homogeneous weight

$$\langle \xi \rangle^m := \left(1 + |\xi|^2 \right)^{m/2}, \quad \xi \in \mathbb{R}^n, \quad m \in \mathbb{R}, \tag{14}$$

is a weight function according to the definition given at the beginning of this section.

The well-known Peetre inequality

$$\langle \xi \rangle^m \leq 2^{|m|} \langle \xi - \eta \rangle^{|m|} \langle \eta \rangle^m, \quad \forall \xi, \eta \in \mathbb{R}^n, \tag{15}$$

shows that $\langle \cdot \rangle^m$ satisfies the condition (T) for every $m \in \mathbb{R}$ (with $N = |m|$) as well as the condition (SM) for $m \geq 0$. For every $m \geq 0$, the function $\langle \cdot \rangle^m$ also fulfils (SV) (where $N = m$) as a consequence of a Taylor expansion, and (SA). Finally $1/\langle \cdot \rangle^m$ satisfies the integrability condition (12) as long as $m > n$; hence $\langle \cdot \rangle^m$ satisfies condition (B) for $m > n$, in view of the statement ii of Proposition 2.

2. For $M = (m_1, \dots, m_n) \in \mathbb{N}^n$, the *quasi-homogeneous weight* is defined as

$$\langle \xi \rangle_M := \left(1 + \sum_{j=1}^n \xi_j^{2m_j} \right)^{1/2}, \quad \forall \xi \in \mathbb{R}^n. \quad (16)$$

The quasi-homogeneous weight obeys the polynomial growth condition

$$\frac{1}{C} \langle \xi \rangle^{m_*} \leq \langle \xi \rangle_M \leq C \langle \xi \rangle^{m_*}, \quad \forall \xi \in \mathbb{R}^n, \quad (17)$$

for some positive constant C and $m_* := \min_{1 \leq j \leq n} m_j$, $m^* := \max_{1 \leq j \leq n} m_j$. Moreover, for all $s \in \mathbb{R}$, the derivatives of $\langle \cdot \rangle_M^s$ decay according to the estimates below

$$\left| \partial_\xi^\alpha \langle \xi \rangle_M^s \right| \leq C_\alpha \langle \xi \rangle_M^{s - \langle \alpha, \frac{1}{M} \rangle}, \quad \forall \xi \in \mathbb{R}^n, \quad \forall \alpha \in \mathbb{Z}_+^n, \quad (18)$$

where $\langle \alpha, \frac{1}{M} \rangle := \sum_{j=1}^n \frac{\alpha_j}{m_j}$ and $C_\alpha > 0$ is a suitable constant. Using (18) with $s = \frac{1}{m^*}$ we may prove that $\langle \cdot \rangle_M$ fulfils condition (SV) with $N = m^*$; indeed from the trivial identities

$$\langle \xi \rangle_M^{1/m^*} - \langle \eta \rangle_M^{1/m^*} = \sum_{j=1}^n (\xi_j - \eta_j) \int_0^1 \partial_j \left(\langle \cdot \rangle_M^{1/m^*} \right) (\eta + t(\xi - \eta)) dt \quad (19)$$

and (18), we deduce

$$\begin{aligned} \left| \langle \xi \rangle_M^{1/m^*} - \langle \eta \rangle_M^{1/m^*} \right| &\leq \sum_{j=1}^n C_j |\xi_j - \eta_j| \int_0^1 \langle \eta + t(\xi - \eta) \rangle_M^{1/m^* - 1/m_j} dt \\ &\leq \sum_{j=1}^n C_j |\xi_j - \eta_j|, \end{aligned}$$

since $m^* \geq m_j$ for every j . Now for ξ, η satisfying $|\xi - \eta| \leq \varepsilon \langle \xi \rangle_M^{1/m^*}$, from the previous inequality we deduce $\left| \langle \xi \rangle_M^{1/m^*} - \langle \eta \rangle_M^{1/m^*} \right| \leq \widehat{C} \varepsilon \langle \xi \rangle_M^{1/m^*}$, with $\widehat{C} := \sum_{j=1}^n C_j$, that is $(1 - \widehat{C} \varepsilon) \langle \xi \rangle_M^{1/m^*} \leq \langle \eta \rangle_M^{1/m^*} \leq (1 + \widehat{C} \varepsilon) \langle \xi \rangle_M^{1/m^*}$, from which we get the conclusion, if we assume for instance $0 < \varepsilon \leq \frac{1}{2\widehat{C}}$.

From (SV) and the trivial inequality $\langle \xi \rangle_M \geq 1$, using the statement i of Proposition 2 we obtain that (T) is also satisfied with $N = m^*$.

Also, the weight $\langle \cdot \rangle_M$ satisfies condition (SA) and, because of the left inequality in (17), $1/\langle \cdot \rangle_M^s$ satisfies condition (12) provided that $s > \frac{n}{m^*}$. Then from the statement ii of Proposition 2, $\langle \cdot \rangle_M^s$ satisfies condition (B) for $s > \frac{n}{m^*}$.

At the end, let us observe that for $M = (m, \dots, m)$, with a given $m \in \mathbb{N}$, $\langle \xi \rangle_M \asymp \langle \xi \rangle^m$.

3. Let \mathcal{P} be a *complete polyhedron* of \mathbb{R}^n in the sense of Volevich-Gindikin, [35]. The *multi-quasi-elliptic weight function* is defined by

$$\lambda_{\mathcal{P}}(\xi) := \left(\sum_{\alpha \in V(\mathcal{P})} \xi^{2\alpha} \right)^{1/2}, \quad \xi \in \mathbb{R}^n, \tag{20}$$

where $V(\mathcal{P})$ denotes the set of *vertices* of \mathcal{P} .

We recall that a *convex polyhedron* $\mathcal{P} \subset \mathbb{R}^n$ is the convex hull of a finite set $V(\mathcal{P}) \subset \mathbb{R}^n$ of convex-linearly independent points, called *vertices* of \mathcal{P} , and univocally determined by \mathcal{P} itself. Moreover, if \mathcal{P} has non empty interior, it is completely described by

$$\mathcal{P} = \{ \xi \in \mathbb{R}^n; \nu \cdot \xi \geq 0, \forall \nu \in \mathcal{N}_0(\mathcal{P}) \} \cap \{ \xi \in \mathbb{R}^n; \nu \cdot \xi \leq 1, \forall \nu \in \mathcal{N}_1(\mathcal{P}) \};$$

where $\mathcal{N}_0(\mathcal{P}) \subset \{ \nu \in \mathbb{R}^n; |\nu| = 1 \}$, $\mathcal{N}_1(\mathcal{P}) \subset \mathbb{R}^n$ are finite sets univocally determined by \mathcal{P} and, as usual, $\nu \cdot \xi = \sum_{j=1}^n \nu_j \xi_j$. The boundary of \mathcal{P} , $\mathcal{F}(\mathcal{P})$, is made of faces $\mathcal{F}_\nu(\mathcal{P})$ which are the convex hulls of the vertices of \mathcal{P} lying on the hyper-planes H_ν orthogonal to $\nu \in \mathcal{N}_0(\mathcal{P}) \cup \mathcal{N}_1(\mathcal{P})$, of equation:

$$\nu \cdot \xi = 0 \quad \text{if } \nu \in \mathcal{N}_0(\mathcal{P}), \quad \nu \cdot \xi = 1 \quad \text{if } \nu \in \mathcal{N}_1(\mathcal{P}).$$

A *complete polyhedron* is a convex polyhedron $\mathcal{P} \subset \mathbb{R}_+^n =: \{ \xi \in \mathbb{R}^n : \xi_j \geq 0, j = 1, \dots, n \}$ such that:

- (i) $V(\mathcal{P}) \subset \mathbb{N}^n$;
- (ii) $(0, \dots, 0) \in V(\mathcal{P})$, and $V(\mathcal{P}) \neq \{(0, \dots, 0)\}$;
- (iii) $\mathcal{N}_0(\mathcal{P}) = \{e_1, \dots, e_n\}$ with $e_j = (0, \dots, 1_{j\text{-entry}}, \dots, 0) \in \mathbb{R}_+^n$;
- (iv) every $\nu \in \mathcal{N}_1(\mathcal{P})$ has strictly positive components $\nu_j, j = 1, \dots, n$.

On can prove that the multi-quasi-elliptic weight growths at infinity according to the following estimates

$$\frac{1}{C} \langle \xi \rangle^{\mu_0} \leq \lambda_{\mathcal{P}}(\xi) \leq C \langle \xi \rangle^{\mu_1}, \quad \forall \xi \in \mathbb{R}^n, \tag{21}$$

for a suitable positive constant C and where

$$\mu_0 := \min_{\gamma \in V(\mathcal{P}) \setminus \{0\}} |\gamma| \quad \text{and} \quad \mu_1 := \max_{\gamma \in V(\mathcal{P})} |\gamma| \tag{22}$$

are called *minimum* and *maximum order* of \mathcal{P} respectively. Moreover it can be proved that for all $s \in \mathbb{R}$ the derivatives of $\lambda_{\mathcal{P}}^s$ decay according to the estimates below

$$\left| \partial_{\xi}^{\alpha} \lambda_{\mathcal{P}}^s(\xi) \right| \leq C_{\alpha} \lambda_{\mathcal{P}}(\xi)^{s - \frac{1}{\mu} |\alpha|}, \quad \forall \xi \in \mathbb{R}^n, \tag{23}$$

where

$$\mu := \max \{1/v_j, \quad j = 1, \dots, n, \quad v \in \mathcal{N}_1(\mathcal{P})\}, \quad (24)$$

satisfying $\mu \geq \mu_1$, is the so-called *formal order* of \mathcal{P} , see e.g. [3] (see also [12] where more general decaying estimates for $\lambda_{\mathcal{P}}^s$ are established).

Representing $\lambda_{\mathcal{P}}(\xi)^{1/\mu} - \lambda_{\mathcal{P}}(\eta)^{1/\mu}$ as in (19), for arbitrary $\xi, \eta \in \mathbb{R}^n$, and using (23) (with $s = 1/\mu$), we deduce that $\lambda_{\mathcal{P}}$ satisfies (SV) with $N = \mu$. Using also that $\lambda_{\mathcal{P}}(\xi) \geq 1$ (recall that $0 \in V(\mathcal{P})$), in view of Proposition 2 it follows that $\lambda_{\mathcal{P}}$ also satisfies (T) with $N = \mu$, hence it is a weight function agreeing to the definition given at the beginning of this section.

The weight function $\lambda_{\mathcal{P}}$ does not satisfy condition (SA); on the other hand it can be shown (see [8]) that condition (G) is verified taking

$$\delta = \max_{\beta \in \mathcal{P} \setminus \mathcal{F}(\mathcal{P})} \max_{v \in \mathcal{N}_1(\mathcal{P})} \{v \cdot \beta\}. \quad (25)$$

Since, from the left inequality in (21) we also derive that $\lambda_{\mathcal{P}}^{-s}$ satisfies (12) for $s > \frac{n}{\mu_0}$, we conclude from the statement ii of Proposition 2 that $\lambda_{\mathcal{P}}^r$ satisfies condition (B) if $r > \frac{n}{(1-\delta)\mu_0}$ for δ defined above.

In the end, we notice that $\lambda_{\mathcal{P}}$ verifies (SM) as a consequence of (G), since $\lambda_{\mathcal{P}}(\xi) \geq 1$ and $0 < \delta < 1$, cf. Proposition 2, statement i.

Remark 1 We notice that the quasi-homogeneous weight $\langle \cdot \rangle_M$, with $M = (m_1, \dots, m_n) \in \mathbb{N}^n$, considered in the example 2 is just the multi-quasi-elliptic weight $\lambda_{\mathcal{P}}$ introduced in the example 3 corresponding to the complete polyhedron \mathcal{P} defined by the convex hull of the finite set $V(\mathcal{P}) = \{0, m_j e_j, j = 1, \dots, n\}$; in particular, the growth estimates (17) are the particular case of (21) corresponding to the previous polyhedron \mathcal{P} (in which case $\mu_0 = m_*$ and $\mu_1 = m^*$). Notice however that the decaying estimates (18) satisfied by the quasi-homogeneous weight $\langle \cdot \rangle_M$ do not admit a counterpart in the case of the general multi-quasi-elliptic weight $\lambda_{\mathcal{P}}$. Estimates (18) give a precise decay in each coordinate direction: the decrease of $\langle \xi \rangle_M$ corresponding to one derivative with respect to ξ_j is measured by $\langle \xi \rangle_M^{-1/m_j}$.¹ The lack of homogeneity in the weight associated to a general complete polyhedron \mathcal{P} in (20), prevents from extending to derivatives of $\lambda_{\mathcal{P}}(\xi)$ the decay properties in (18): estimates (23) do not take account of the decay corresponding *separately to each coordinate direction*.

2.2 Weighted Lebesgue and Fourier Lebesgue spaces

Let $\omega : \mathbb{R}^n \rightarrow]0, +\infty[$ be a weight function.

Definition 1 For every $p \in [1, +\infty[$, the weighted Lebesgue space $L_{\omega}^p(\mathbb{R}^n)$ is defined as the set of the (equivalence classes of) measurable functions $f : \mathbb{R}^n \rightarrow \mathbb{C}$ such that

¹ In other words the decay of the derivatives is measured here by the *vector weight* $(\langle \xi \rangle_M^{1/m_1}, \dots, \langle \xi \rangle_M^{1/m_n})$, in the sense of *vector weighted* symbol classes, see [7, 18, 28].

$$\int_{\mathbb{R}^n} \omega(x)^p |f(x)|^p dx < +\infty, \quad \text{if } p < +\infty, \tag{26}$$

ωf is essentially bounded in \mathbb{R}^n , if $p = +\infty$.

For every $p \in [1, +\infty]$, $L^p_\omega(\mathbb{R}^n)$ is a Banach space with respect to the natural norm

$$\|f\|_{L^p_\omega} := \begin{cases} \left(\int_{\mathbb{R}^n} \omega(x)^p |f(x)|^p dx\right)^{1/p}, & \text{if } p < +\infty, \\ \text{ess sup}_{x \in \mathbb{R}^n} \omega(x) |f(x)|, & \text{if } p = +\infty. \end{cases} \tag{27}$$

Remark 2 It is easy to see that for all $p \in [1, +\infty]$

$$L^p_{\omega_2}(\mathbb{R}^n) \hookrightarrow L^p_{\omega_1}(\mathbb{R}^n), \quad \text{if } \omega_1 \leq \omega_2. \tag{28}$$

If in particular $\omega_1 \asymp \omega_2$ then $L^p_{\omega_1}(\mathbb{R}^n) \equiv L^p_{\omega_2}(\mathbb{R}^n)$, and the norms defined in (27) corresponding to ω_1 and ω_2 are equivalent. When the weight function ω is constant the related weighted space $L^p_\omega(\mathbb{R}^n)$ reduces to the standard Lebesgue space of order p , denoted as usual by $L^p(\mathbb{R}^n)$.

Remark 3 For an arbitrary $p \in [1, +\infty]$, $f \in L^p_\omega(\mathbb{R}^n)$ and every $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we obtain

$$\int_{\mathbb{R}^n} f(x)\varphi(x) dx \leq \left\| \frac{\varphi}{\omega} \right\|_{L^q} \| \omega f \|_{L^p}, \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{29}$$

From (29) and the estimates (4), we deduce at once that

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow L^p_\omega(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

Moreover, $C^\infty_0(\mathbb{R}^n)$ is a dense subspace of $L^p_\omega(\mathbb{R}^n)$ when $p < +\infty$, see [9].

Remark 4 For Ω open subset of \mathbb{R}^n , $L^p_\omega(\Omega)$, for any $p \in [1, +\infty]$, is the set of (equivalence classes of) measurable functions on Ω such that

$$\|f\|_{L^p_\omega(\Omega)}^p := \int_{\Omega} \omega(x)^p |f(x)|^p dx < +\infty \tag{30}$$

(obvious modification for $p = +\infty$). $L^p_\omega(\Omega)$ is endowed with a structure of Banach space with respect to the natural norm defined by (30).

Definition 2 For every $p \in [1, +\infty]$ and ω weight function, the weighted Fourier Lebesgue space $\mathcal{FL}^p_\omega(\mathbb{R}^n)$ is the vector space of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\widehat{f} \in L^p_\omega(\mathbb{R}^n), \tag{31}$$

equipped with the natural norm

$$\|f\|_{\mathcal{FL}_\omega^p} := \|\widehat{f}\|_{L_\omega^p}. \quad (32)$$

Here \widehat{f} is the Fourier transform $\widehat{f}(\xi) = \int e^{-i\xi \cdot x} f(x) dx$, defined in $\mathcal{S}(\mathbb{R}^n)$ and extended to $\mathcal{S}'(\mathbb{R}^n)$.

The spaces $\mathcal{FL}_\omega^p(\mathbb{R}^n)$ were introduced in Hörmander [20], with the notation $\mathcal{B}_{p,k}$, $k(\xi)$ weight function, for the study of the regularity of solutions to hypoelliptic partial differential equations with constant coefficients, see also [9, 10].

From the mapping properties of the Fourier transform on $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ and the above stated properties of weighted Lebesgue spaces we can conclude, see again [9], that for all $p \in [1, +\infty]$ and ω weight function

- (a) $\mathcal{FL}_\omega^p(\mathbb{R}^n)$ is a Banach space with respect to the norm (32);
- (b) $\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{FL}_\omega^p(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$;
- (c) $C_0^\infty(\mathbb{R}^n)$ is a dense subspace of $\mathcal{FL}_\omega^p(\mathbb{R}^n)$ when $p < +\infty$;
- (d) $\mathcal{FL}_{\omega_2}^p(\mathbb{R}^n) \hookrightarrow \mathcal{FL}_{\omega_1}^p(\mathbb{R}^n)$ if $\omega_1 \leq \omega_2$; in particular, we have $\mathcal{FL}_{\omega_2}^p(\mathbb{R}^n) \equiv \mathcal{FL}_{\omega_1}^p(\mathbb{R}^n)$ as long as $\omega_1 \asymp \omega_2$ and the norms corresponding to ω_1 and ω_2 by (32) are equivalent in this case.

When ω is a positive constant the weighted space $\mathcal{FL}_\omega^p(\mathbb{R}^n)$ is simply denoted by $\mathcal{FL}^p(\mathbb{R}^n)$. Moreover we will adopt the shortcut notations $L_s^p(\mathbb{R}^n) := L_{\langle \cdot \rangle^s}^p(\mathbb{R}^n)$, $\mathcal{FL}_s^p(\mathbb{R}^n) := \mathcal{FL}_{\langle \cdot \rangle^s}^p(\mathbb{R}^n)$ for the corresponding Lebesgue and Fourier Lebesgue spaces.

Analogously, when $\omega(\xi) = \langle \xi \rangle_M^s$ or $\omega(\xi) = \lambda_{\mathcal{P}}(\xi)^s$, for $s \in \mathbb{R}$, the corresponding Lebesgue and Fourier Lebesgue spaces will be denoted $L_{s,M}^p(\mathbb{R}^n)$, $\mathcal{FL}_{s,M}^p(\mathbb{R}^n)$ and $L_{s,\mathcal{P}}^p(\mathbb{R}^n)$, $\mathcal{FL}_{s,\mathcal{P}}^p(\mathbb{R}^n)$ respectively.

A local counterpart of Fourier Lebesgue spaces can be introduced in the following natural way (see [26]).

Definition 3 For ω weight function, Ω open subset of \mathbb{R}^n and any $p \in [1, +\infty]$, $\mathcal{FL}_{\omega, \text{loc}}^p(\Omega)$ is the class of all distributions $f \in \mathcal{D}'(\Omega)$ such that $\varphi f \in \mathcal{FL}_\omega^p(\mathbb{R}^n)$ for every $\varphi \in C_0^\infty(\Omega)$.

For $x_0 \in \Omega$, $f \in \mathcal{FL}_{\omega, \text{loc}}^p(x_0)$ if there exists $\phi \in C_0^\infty(\Omega)$, with $\phi(x_0) \neq 0$, such that $\phi f \in \mathcal{FL}_\omega^p(\mathbb{R}^n)$.

The family of semi-norms

$$f \mapsto \|\varphi f\|_{\mathcal{FL}_\omega^p}, \quad \varphi \in C_0^\infty(\Omega), \quad (33)$$

provides $\mathcal{FL}_{\omega, \text{loc}}^p(\Omega)$ with a natural Fréchet space topology. Moreover the following inclusions hold true with continuous embedding

$$C^\infty(\Omega) \hookrightarrow \mathcal{FL}_{\omega, \text{loc}}^p(\Omega) \hookrightarrow \mathcal{D}'(\Omega) \quad (34)$$

and for $\Omega_1 \subset \Omega_2$ open sets

$$\mathcal{FL}_\omega^p(\mathbb{R}^n) \hookrightarrow \mathcal{FL}_{\omega, \text{loc}}^p(\Omega_2) \hookrightarrow \mathcal{FL}_{\omega, \text{loc}}^p(\Omega_1). \quad (35)$$

Remark 5 It is worth noticing that, as it was proved in [29], locally the weighted Fourier Lebesgue spaces $\mathcal{FL}_\omega^q(\mathbb{R}^n)$ are the same as the weighted modulation spaces $M_\omega^{p,q}(\mathbb{R}^n)$ and the Wiener amalgam spaces $W_\omega^{p,q}(\mathbb{R}^n)$, in the sense that for $p \in [1, +\infty]$

$$\mathcal{FL}_\omega^q(\mathbb{R}^n) \cap \mathcal{E}'(\mathbb{R}^n) = M_\omega^{p,q}(\mathbb{R}^n) \cap \mathcal{E}'(\mathbb{R}^n) = W_\omega^{p,q}(\mathbb{R}^n) \cap \mathcal{E}'(\mathbb{R}^n).$$

We refer to Feichtinger [5] and Gröchenig [19] for the definition and basic properties of modulation and amalgam spaces.

Agreeing with the previous notations, when the weight function ω reduces to those considered in the examples 1, 2, 3 above, the corresponding local Fourier Lebesgue spaces will be denoted respectively by $\mathcal{FL}_{s,\text{loc}}^p(\Omega)$, $\mathcal{FL}_{s,M,\text{loc}}^p(\Omega)$, $\mathcal{FL}_{s,\mathcal{P},\text{loc}}^p(\Omega)$.

Notice at the end that, from Plancherel Theorem, when $p = 2$ the global and local weighted Fourier Lebesgue spaces $\mathcal{FL}_\omega^2(\mathbb{R}^n)$, $\mathcal{FL}_{\omega,\text{loc}}^2(\Omega)$ coincide with weighted spaces of Sobolev type, see Garello [11] for an extensive study of such spaces.

3 Algebra conditions in spaces $\mathcal{FL}_\omega^p(\mathbb{R}^n)$

In order to seek conditions on the weight function ω which allow the Fourier Lebesgue space $\mathcal{FL}_\omega^p(\mathbb{R}^n)$ to be an algebra with respect to the point-wise product, let us first state a general continuity result in the framework of suitable mixed-norm spaces of Lebesgue type.

Following [5, 19] and in particular [26], for $p, q \in [1, +\infty]$ we denote by $\mathcal{L}_1^{p,q}(\mathbb{R}^{2n})$ the space of all (equivalence classes of) measurable functions $F = F(\zeta, \eta)$ in $\mathbb{R}^n \times \mathbb{R}^n$ such that the mixed norm

$$\|F\|_{\mathcal{L}_1^{p,q}} := \left(\int \left(\int |F(\zeta, \eta)|^p d\zeta \right)^{q/p} d\eta \right)^{1/q} \tag{36}$$

is finite (with obvious modifications if p or q equal $+\infty$).

We also define $\mathcal{L}_2^{p,q}(\mathbb{R}^{2n})$ to be the space of measurable functions $F = F(\xi, \eta)$ in $\mathbb{R}^n \times \mathbb{R}^n$ such that the norm

$$\|F\|_{\mathcal{L}_2^{p,q}} := \left(\int \left(\int |F(\zeta, \eta)|^q d\eta \right)^{p/q} d\zeta \right)^{1/p} \tag{37}$$

is finite.

Lemma 1 For $p, q \in [1, +\infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$, let $f = f(\zeta, \eta) \in \mathcal{L}_1^{p,\infty}(\mathbb{R}^{2n})$ and $F = F(\zeta, \eta) \in \mathcal{L}_2^{\infty,q}(\mathbb{R}^{2n})$. Then the linear map

$$\begin{aligned} T : C_0^\infty(\mathbb{R}^n) &\rightarrow \mathcal{S}'(\mathbb{R}^n) \\ g &\mapsto Tg := \int F(\xi, \eta) f(\xi - \eta, \eta) g(\eta) d\eta. \end{aligned} \tag{38}$$

extends uniquely to a continuous map from $L^p(\mathbb{R}^n)$ into itself, still denoted by T ; moreover its operator norm is bounded as follows

$$\|T\|_{\mathcal{L}(L^p)} \leq \|f\|_{\mathcal{L}_1^{p,\infty}} \|F\|_{\mathcal{L}_2^{\infty,q}}. \quad (39)$$

The proof is given in [9, Lemma 2.1], where the statement reads in quite different formulation. The reader can find a restricted version, independently proved, in [26, Proposition 3.2].

Proposition 3 *Assume that ω , ω_1 , ω_2 are weight functions such that*

$$C_q := \sup_{\xi \in \mathbb{R}^n} \left\| \frac{\omega(\xi)}{\omega_1(\xi - \cdot)\omega_2(\cdot)} \right\|_{L^q} < +\infty, \quad (40)$$

for some $q \in [1, +\infty]$, and let $p \in [1, +\infty]$ be the conjugate exponent of q . Then

1. the point-wise product map $(f_1, f_2) \mapsto f_1 f_2$ from $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ extends uniquely to a continuous bilinear map from $\mathcal{FL}_{\omega_1}^p(\mathbb{R}^n) \times \mathcal{FL}_{\omega_2}^p(\mathbb{R}^n)$ to $\mathcal{FL}_{\omega}^p(\mathbb{R}^n)$. Moreover for all $f_i \in \mathcal{FL}_{\omega_i}^p(\mathbb{R}^n)$, $i = 1, 2$, the following holds:

$$\|f_1 f_2\|_{\mathcal{FL}_{\omega}^p} \leq C_q \|f_1\|_{\mathcal{FL}_{\omega_1}^p} \|f_2\|_{\mathcal{FL}_{\omega_2}^p}. \quad (41)$$

2. for every open set $\Omega \subseteq \mathbb{R}^n$ the point-wise product map $(f_1, f_2) \mapsto f_1 f_2$ from $C_0^\infty(\Omega) \times C_0^\infty(\Omega)$ to $C_0^\infty(\Omega)$ extends uniquely to a continuous bilinear map from $\mathcal{FL}_{\omega_1, \text{loc}}^p(\Omega) \times \mathcal{FL}_{\omega_2, \text{loc}}^p(\Omega)$ to $\mathcal{FL}_{\omega, \text{loc}}^p(\Omega)$.

Proof The proof of statement (2) follows at once from that of statement (1).

As for the proof of statement (1), for given $f_1, f_2 \in \mathcal{S}(\mathbb{R}^n)$ one easily computes:

$$\begin{aligned} \omega(\xi) \widehat{f_1 f_2}(\xi) &= (2\pi)^{-n} \int \omega(\xi) \widehat{f_1}(\xi - \eta) \widehat{f_2}(\eta) d\eta \\ &= \int F(\xi, \eta) f(\xi - \eta) g(\eta) d\eta, \end{aligned} \quad (42)$$

where

$$F(\zeta, \eta) = \frac{\omega(\zeta)}{\omega_1(\zeta - \eta)\omega_2(\eta)}, \quad f(\zeta) = \omega_1(\zeta) \widehat{f_1}(\zeta), \quad g(\zeta) = \omega_2(\zeta) \widehat{f_2}(\zeta).$$

The right-hand side of (42) provides a representation of $\widehat{\omega f_1 f_2}$ as an integral operator of the form (38). Condition (40) just means that the function $F(\zeta, \eta) \in \mathcal{L}_2^{\infty,q}(\mathbb{R}^{2n})$ (cf. (37)) and of course the η -independent function $f = f(\zeta) \in \mathcal{S}(\mathbb{R}^n)$ also belongs to $\mathcal{L}_1^{p,\infty}(\mathbb{R}^{2n})$. Then applying to (42) the result of Lemma 1, together with the definition of the norm in Fourier Lebesgue spaces, we obtain that the point-wise product $f_1 f_2$ satisfies the estimates in (41) and the proof is concluded.

When the weight functions ω , ω_1 and ω_2 in the statement of Proposition 3 coincide, condition (40) provides a sufficient condition for $\mathcal{FL}_{\omega}^p(\mathbb{R}^n)$ (or its localized counterpart $\mathcal{FL}_{\omega, \text{loc}}^p(\Omega)$) is an algebra for the point-wise product. Then we have the following

Corollary 1 *Let ω be a weight function such that*

$$C_q := \sup_{\xi \in \mathbb{R}^n} \left\| \frac{\omega(\xi)}{\omega(\xi - \cdot)\omega(\cdot)} \right\|_{L^q} < +\infty, \tag{43}$$

for $q \in [1, +\infty]$, and $p \in [1, +\infty]$ the conjugate exponent of q . Then

1. $(\mathcal{FL}_\omega^p(\mathbb{R}^n), \cdot)$ is an algebra and for $f_1, f_2 \in \mathcal{FL}_\omega^p(\mathbb{R}^n)$

$$\|f_1 f_2\|_{\mathcal{FL}_\omega^p} \leq C_q \|f_1\|_{\mathcal{FL}_\omega^p} \|f_2\|_{\mathcal{FL}_\omega^p}. \tag{44}$$

2. for every open set $\Omega \subseteq \mathbb{R}^n$, $(\mathcal{FL}_{\omega, \text{loc}}^p(\Omega), \cdot)$ is an algebra.

The algebra properties of Corollary 1 allow us to handle the composition of a Fourier Lebesgue distribution with an entire analytic function; namely we have the following result, see [9, Corollary 2.1].

Corollary 2 *Under the same assumptions of Corollary 1 on ω and p , let $F : \mathbb{C} \rightarrow \mathbb{C}$ be an entire analytic function such that $F(0) = 0$. Then $F(u) \in \mathcal{FL}_\omega^p(\mathbb{R}^n)$ for every $u \in \mathcal{FL}_\omega^p(\mathbb{R}^n)$, and*

$$\|F(u)\|_{\mathcal{FL}_\omega^p} \leq C \|u\|_{\mathcal{FL}_\omega^p}, \tag{45}$$

with $C = C(p, F, \|u\|_{\mathcal{FL}_\omega^p})$.

Remark 6 A counterpart of Corollary 2 for the local space $\mathcal{FL}_{\omega, \text{loc}}^p(\Omega)$ can be obtained by replacing $F = F(z)$ above with a function $F = F(x, \zeta)$ mapping $\Omega \times \mathbb{C}^M$ into \mathbb{C} , which is locally smooth with respect to the real variable $x \in \Omega$ and entire analytic in the complex variable $\zeta \in \mathbb{C}^M$ uniformly on compact subsets of Ω ; namely:

$$F(x, \zeta) = \sum_{\beta \in \mathbb{Z}_+^M} c_\beta(x) \zeta^\beta, \quad c_\beta \in C^\infty(\Omega), \zeta \in \mathbb{C}^M,$$

where, for any compact set $K \subset \Omega$, $\alpha \in \mathbb{Z}_+^n$, $\beta \in \mathbb{Z}_+^M$, $\sup_{x \in K} |\partial_x^\alpha c_\beta(x)| \leq c_{\alpha, K} \lambda_\beta$ and $F_1(\zeta) := \sum_{\beta \in \mathbb{Z}_+^M} \lambda_\beta \zeta^\beta$ is entire analytic.

Under the assumptions of Corollaries 1, 2, we get $F(x, u) \in \mathcal{FL}_{\omega, \text{loc}}^p(\Omega)$ as long as the components of the vector $u = (u_1, \dots, u_M)$ belong to $\mathcal{FL}_{\omega, \text{loc}}^p(\Omega)$.

Remark 7 Let us notice that for $1 \leq q < +\infty$, condition (43) on ω is nothing but condition (B) for the weight function ω^q , while for $q = +\infty$ (43) reduces to condition (SM) for ω . The latter case means that $(\mathcal{FL}_\omega^1(\mathbb{R}^n), \cdot)$ is an algebra provided that the weight function ω is sub-multiplicative, which is in agreement with the more general result of [26, Lemma 1.6].

The next result shows that the *sub-multiplicative* condition (SM) on a weight function is a necessary condition for the corresponding scale of weighted Fourier Lebesgue spaces to possess the algebra property.

Proposition 4 *Let $\omega_1, \omega_2, \omega$ be weight functions and $p_1, p_2, p \in [1, +\infty]$. If we assume that the map $(f_1, f_2) \mapsto f_1 f_2$ from $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ extends uniquely to a continuous bilinear map from $\mathcal{FL}_{\omega_1}^{p_1}(\mathbb{R}^n) \times \mathcal{FL}_{\omega_2}^{p_2}(\mathbb{R}^n)$ to $\mathcal{FL}_{\omega}^p(\mathbb{R}^n)$, then a positive constant C exists such that*

$$\omega(\eta + \theta) \leq C \omega_1(\eta) \omega_2(\theta), \quad \forall \eta, \theta \in \mathbb{R}^n. \quad (46)$$

Proof By assumption, there exists a constant $C' > 0$ such that for all $f \in \mathcal{FL}_{\omega_1}^{p_1}(\mathbb{R}^n)$, $g \in \mathcal{FL}_{\omega_2}^{p_2}(\mathbb{R}^n)$

$$\|fg\|_{\mathcal{FL}_{\omega}^p} \leq C' \|f\|_{\mathcal{FL}_{\omega_1}^{p_1}} \|g\|_{\mathcal{FL}_{\omega_2}^{p_2}}. \quad (47)$$

From condition (T) (cf. also (4)) we may find some constants $\varepsilon > 0, C > 0$ such that

$$\varepsilon \leq \frac{\omega_i(\eta)}{\omega_i(\xi)} \leq \varepsilon^{-1} \quad (i = 1, 2) \quad \text{and} \quad \varepsilon \leq \frac{\omega(\eta)}{\omega(\xi)} \leq \varepsilon^{-1}, \quad \text{when } |\xi - \eta| \leq \frac{\varepsilon}{C} \quad (48)$$

We follow here the same arguments of the proof of [11, Theorem 3.8]. Let us take a function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\widehat{\varphi}(\xi) \geq 0 \quad \text{and} \quad \text{supp } \widehat{\varphi} \subseteq \left\{ \xi \in \mathbb{R}^n : |\xi| \leq \frac{\varepsilon}{2C} \right\}.$$

For arbitrary points $\eta, \theta \in \mathbb{R}^n$, let us define

$$f(x) = e^{i\eta \cdot x} \varphi(x), \quad g(x) = e^{i\theta \cdot x} \varphi(x), \quad (49)$$

hence

$$f(x)g(x) = e^{i(\eta+\theta) \cdot x} \varphi^2(x).$$

In view of the assumption on the support of $\widehat{\varphi}$ we compute:

$$\begin{aligned} \|f\|_{\mathcal{FL}_{\omega_1}^{p_1}}^{p_1} &= \int \omega_1(\xi)^{p_1} \widehat{\varphi}(\xi - \eta)^{p_1} d\xi = \int_{|\xi - \eta| \leq \frac{\varepsilon}{2C}} \omega_1(\xi)^{p_1} \widehat{\varphi}(\xi - \eta)^{p_1} d\xi, \\ \|g\|_{\mathcal{FL}_{\omega_2}^{p_2}}^{p_2} &= \int \omega_2(\xi)^{p_2} \widehat{\varphi}(\xi - \theta)^{p_2} d\xi = \int_{|\xi - \theta| \leq \frac{\varepsilon}{2C}} \omega_2(\xi)^{p_2} \widehat{\varphi}(\xi - \theta)^{p_2} d\xi, \end{aligned} \quad (50)$$

In the domain of the integrals above (48) holds, then we get

$$\|f\|_{\mathcal{FL}_{\omega_1}^{p_1}}^{p_1} \leq \varepsilon^{-p_1} \omega_1(\eta)^{p_1} \int_{|\xi - \eta| \leq \frac{\varepsilon}{2C}} \widehat{\varphi}(\xi - \eta)^{p_1} d\xi = c_1^{p_1} \varepsilon^{-p_1} \omega_1(\eta)^{p_1},$$

hence

$$\|f\|_{\mathcal{FL}_{\omega_1}^{p_1}} \leq c_1 \varepsilon^{-1} \omega_1(\eta), \quad (51)$$

where $c_1 := \|\widehat{\varphi}\|_{L^{p_1}}$. The same holds true for the norm of g in $\mathcal{FL}_{\omega_2}^{p_2}(\mathbb{R}^n)$, by replacing η with θ , that is

$$\|g\|_{\mathcal{FL}_{\omega_2}^{p_2}} \leq c_2 \varepsilon^{-1} \omega_2(\theta), \quad \text{for } c_2 = \|\widehat{\varphi}\|_{L^{p_2}}. \tag{52}$$

The preceding calculations are performed under the assumption that both p_1 and p_2 are finite; however the same estimates (51), (52) can be easily extended to the case when p_1 or p_2 equals $+\infty$.

As for the norm in $\mathcal{FL}_{\omega}^p(\mathbb{R}^n)$ of fg we compute

$$\|fg\|_{\mathcal{FL}_{\omega}^p}^p = \int_{|\xi-\eta-\theta| \leq \frac{\varepsilon}{c}} \omega(\xi)^p \widehat{\varphi}^2(\xi - \eta - \theta)^p d\xi, \tag{53}$$

where we used that $\text{supp } \widehat{\varphi}^2 = \text{supp } (\widehat{\varphi} * \widehat{\varphi}) \subseteq \{|\xi| \leq \frac{\varepsilon}{c}\}$ and it is assumed $p < +\infty$ (to fix the ideas). Then recalling again that (48) holds true for ω on $\text{supp } \widehat{\varphi}^2$ we obtain

$$\|fg\|_{\mathcal{FL}_{\omega}^p} \geq c\varepsilon\omega(\eta + \theta), \quad \text{with } c := \|\widehat{\varphi}^2\|_{L^p}. \tag{54}$$

The same estimate (54) can be easily recovered in the case $p = +\infty$.

We use now (51), (52) and (54) to estimate the right- and left-hand sides of (47) written for f and g defined in (49) to get

$$c\varepsilon\omega(\eta + \theta) \leq C'c_1c_2\varepsilon^{-2}\omega_1(\eta)\omega_2(\theta).$$

In view of the arbitrariness of η, θ and since the constants c_1, c_2 and ε are independent of η and θ the preceding inequality gives (46) with $C = C'c_1c_2c^{-1}\varepsilon^{-3}$.

Remark 8 It is worth observing that any specific relation is assumed on the exponents $p_1, p_2, p \in [1, +\infty]$ in the statement of Proposition 4. Notice also that condition (46) is just (40) for $q = +\infty$. When in particular $\omega_1 = \omega_2 = \omega$, it reduces to (SM).

Notice also that from the results given by Corollary 1 (see also Remark 9) and Proposition 4, we derive that condition (SM) is *necessary and sufficient* to make the Fourier Lebesgue space $\mathcal{FL}_{\omega}^1(\mathbb{R}^n)$ an algebra for the point-wise product.²

Combining the results of Corollary 1 with the remarks made about the weight functions quoted in the examples 1–3 at the end of Sect. 2.1 we can easily prove the following result.

Corollary 3 *Let $r \in \mathbb{R}$, $M = (m_1, \dots, m_n) \in \mathbb{N}^n$ and \mathcal{P} a complete polyhedron of \mathbb{R}^n be given and assume that $p, q \in [1, +\infty]$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then*

- (i) $(\mathcal{FL}_r^p(\mathbb{R}^n), \cdot)$ is an algebra if $r > \frac{n}{q}$;
- (ii) $(\mathcal{FL}_{r,M}^p(\mathbb{R}^n), \cdot)$ is an algebra if $r > \frac{n}{m_*q}$, where $m_* = \min_{1 \leq j \leq n} m_j$;

² However condition (SM) is far from being sufficient for $\mathcal{FL}_{\omega}^p(\mathbb{R}^n)$ to be an algebra, as long as $p > 1$.

(iii) $(\mathcal{F}L_{r,\mathcal{P}}^p(\mathbb{R}^n), \cdot)$ is an algebra if $r > \frac{n}{(1-\delta)\mu_0q}$, where μ_0 and δ are defined in (22) and (25).

Analogous statements hold true for the localized version of the previous spaces on an open subset Ω of \mathbb{R}^n , defined according to Definition 3.

Proof Let us prove the statement iii of the Theorem; the proof of the other statements is completely analogous. Assume that $p > 1$, thus $q < +\infty$. From (21) we have that $s > \frac{n}{\mu_0q}$ implies $\lambda_{\mathcal{P}}^{-sq} \in L^1(\mathbb{R}^n)$; on the other hand, $\lambda_{\mathcal{P}}^{sq}$ satisfies (\mathcal{G}) with δ defined as in (25), see Example 3 in Sect. 2. From Proposition 2 applied to $\lambda_{\mathcal{P}}^{sq}$ we derive that $\lambda_{\mathcal{P}}^{r,q}$ fulfils condition (\mathcal{B}) , which amounts to say that $\lambda_{\mathcal{P}}^r$ satisfies (43), where $r = \frac{s}{1-\delta}$. Then the result of Corollary 1 applies to $\mathcal{F}L_{\lambda_{\mathcal{P}}^r}^p(\mathbb{R}^n) = \mathcal{F}L_{r,\mathcal{P}}^p(\mathbb{R}^n)$ and gives the statement iii. Notice that condition $r > \frac{n}{(1-\delta)\mu_0q}$ reduces to $r > 0$ when $q = +\infty$ (corresponding to $p = 1$). That $\mathcal{F}L_{r,\mathcal{P}}^1(\mathbb{R}^n)$ with $r > 0$ is an algebra for the point-wise product follows again from Corollary 1 by observing that $\lambda_{\mathcal{P}}^r$ satisfies (\mathcal{SM}) (that is (43) for $q = +\infty$).

Remark 9 In agreement with the observation made at the end of Section 2.2, for $p = 2$ the lower bounds of r given in i-iii of Corollary 3 are exactly the same required to ensure the algebra property for the corresponding weighted Sobolev spaces (see [11] and [13] for the case of a general $1 < p < +\infty$).

To the end of this Section, let us observe that, as a byproduct of Propositions 3 and 4, the following result can be proved.

Proposition 5 *Assume that $\omega, \omega_1, \omega_2$ are weight functions satisfying condition (40) for some $1 \leq q < +\infty$. Then $\omega, \omega_1, \omega_2$ also satisfy condition (46). In particular, if ω is a weight function satisfying condition (43) for some $1 \leq q < +\infty$ then it also satisfies condition (\mathcal{SM}) .*

Remark 10 The second part of Proposition 5 slightly improves the result of [11, Proposition 2.4], where the sub-multiplicative condition (\mathcal{SM}) was deduced from Beurling condition (\mathcal{B}) (corresponding to (43) with $q = 1$) and conditions (\mathcal{SV}) and (11); here $\omega, \omega_1, \omega_2$ are only required to satisfy condition (\mathcal{T}) (included in our definition of weight function), which is implied by (\mathcal{SV}) and (11) in view of Proposition 2.i.

4 Pseudodifferential operators with symbols in weighted Fourier Lebesgue spaces

This Section is devoted to the study of a class of pseudodifferential operators whose symbols $a(x, \xi)$ have a finite regularity of weighted Fourier Lebesgue type with respect to x .

Let us first recall that, under the only assumption $a(x, \xi) \in \mathcal{S}'(\mathbb{R}^{2n})$, the pseudodifferential operator defined by

$$a(x, D)f = (2\pi)^{-n} \int e^{ix \cdot \xi} a(x, \xi) \widehat{f}(\xi) d\xi, \quad f \in \mathcal{S}(\mathbb{R}^n), \quad (55)$$

maps continuously $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$.³ Similarly, if Ω is an open subset of \mathbb{R}^n and $a(x, \xi) \in \mathcal{D}'(\Omega \times \mathbb{R}^n)$ is such that $\varphi(x)a(x, \xi) \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ for every $\varphi \in C_0^\infty(\Omega)$, then (55) defines a linear continuous operator from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{D}'(\Omega)$.

Let us also recall that, as a linear continuous operator from $C_0^\infty(\Omega)$ to $\mathcal{D}'(\Omega)$, every pseudodifferential operator with symbol $a(x, \xi) \in \mathcal{D}'(\Omega \times \mathbb{R}^n)$ admits a (uniquely defined) Schwartz kernel $\mathcal{K}_a(x, y) \in \mathcal{D}'(\Omega \times \Omega)$ such that

$$\langle a(x, D)\psi, \varphi \rangle = \langle \mathcal{K}_a, \varphi \otimes \psi \rangle, \quad \forall \varphi, \psi \in C_0^\infty(\Omega).$$

The operator $a(x, D)$ is said to be *properly supported* on Ω when the support of \mathcal{K}_a is a *proper* subset of $\Omega \times \Omega$, that is $\text{supp } \mathcal{K}_a \cap (\Omega \times K)$ and $\text{supp } \mathcal{K}_a \cap (K \times \Omega)$ are compact subsets of $\Omega \times \Omega$, for every compact set $K \subset \Omega$. It is well known that every properly supported pseudodifferential operator continuously maps $C_0^\infty(\Omega)$ into the space $\mathcal{E}'(\Omega)$ of compactly supported distributions and it extends as a linear continuous operator from $C^\infty(\Omega)$ into $\mathcal{D}'(\Omega)$. In particular for every function $\phi \in C_0^\infty(\Omega)$ another function $\tilde{\phi} \in C_0^\infty(\Omega)$ can be found in such a way that

$$\phi(x)a(x, D)u = \phi(x)a(x, D)(\tilde{\phi}u), \quad \forall u \in C^\infty(\Omega). \tag{56}$$

Following [9], we introduce some local and global classes of symbols with finite Fourier Lebesgue regularity.

Definition 4 Let $\omega = \omega(\xi)$, $\gamma = \gamma(\xi)$ be arbitrary weight functions.

1. A distribution $a(x, \xi) \in \mathcal{S}'(\mathbb{R}^{2n})$ is said to belong to the class $\mathcal{FL}_\omega^p S_\gamma$ if $\xi \mapsto a(\cdot, \xi)$ is a measurable $\mathcal{FL}_\omega^p(\mathbb{R}^n)$ -valued function on \mathbb{R}_ξ^n such that

$$\left\| \frac{a(\cdot, \xi)}{\gamma(\xi)} \right\|_{\mathcal{FL}_\omega^p} \leq C, \quad \forall \xi \in \mathbb{R}^n, \tag{57}$$

for some constant $C > 0$. More explicitly, the above means that

$$\frac{\omega(\eta)\widehat{a(\cdot, \xi)}(\eta)}{\gamma(\xi)} \in L^p(\mathbb{R}_\eta^n),$$

with norm uniformly bounded with respect to ξ .

2. We say that a distribution $a(x, \xi) \in \mathcal{D}'(\Omega \times \mathbb{R}^n)$, where Ω is an open subset of \mathbb{R}^n , belongs to $\mathcal{FL}_\omega^p S_\gamma(\Omega)$ if $\phi(x)a(x, \xi) \in \mathcal{FL}_\omega^p S_\gamma$ for every $\phi \in C_0^\infty(\Omega)$ (which amounts to have that $a(\cdot, \xi)/\gamma(\xi) \in \mathcal{FL}_{\omega, \text{loc}}^p(\Omega)$ uniformly in ξ).

Remark 11 For ω, γ as in Definition 4, $\mathcal{FL}_\omega^p S_\gamma$ is a Banach space with respect to the norm

$$\|a\|_{\mathcal{FL}_\omega^p S_\gamma} := \sup_{\xi \in \mathbb{R}^n} \left\| \frac{a(\cdot, \xi)}{\gamma(\xi)} \right\|_{\mathcal{FL}_\omega^p}, \tag{58}$$

³ The integral in the right-hand side of (55) must be understood here in a weak (distributional) sense.

while $\mathcal{FL}_\omega^p S_\gamma(\Omega)$ is a Fréchet space with respect to the family of semi-norms

$$a \mapsto \|\phi a\|_{\mathcal{FL}_\omega^p S_\gamma}, \quad \phi \in C_0^\infty(\Omega). \quad (59)$$

Let us point out that any assumption is made about the ξ -derivatives of the symbol $a(x, \xi)$ in the above definition: the weight function γ only measures the ξ -decay at infinity of the symbol itself. It is clear that $\mathcal{FL}_\omega^p S_{\gamma_1} \equiv \mathcal{FL}_\omega^p S_{\gamma_2}$, whenever $\gamma_1 \sim \gamma_2$ (the same applies of course to the corresponding local classes on an open set). When the weight function γ is an arbitrary positive constant, the related symbol class $\mathcal{FL}_\omega^p S_\gamma$ will be simply denoted as $\mathcal{FL}_\omega^p S$, and its symbols (and related pseudo-differential operators) will be referred to as *zero-th order symbols* (and *zero-th order operators*). Finally, we notice that for every weight function $\omega = \omega(\xi)$ and $p \in [1, +\infty]$, the inclusion $\mathcal{FL}_\omega^p(\mathbb{R}^n) \subset \mathcal{FL}_\omega^p S$ holds true (the elements of $\mathcal{FL}_\omega^p(\mathbb{R}^n)$ being regarded as ξ -independent zero-th order symbols).

Proposition 6 For $p \in [1, +\infty]$ let the weight functions $\omega, \omega_1, \omega_2$ and γ satisfy

$$C_q := \sup_{\xi \in \mathbb{R}^n} \left\| \frac{\omega_2(\xi)\gamma(\cdot)}{\omega_1(\cdot)\omega(\xi - \cdot)} \right\|_{L^q} < +\infty, \quad (60)$$

where q is the conjugate exponent of p . Then the following hold true.

- (i) For every $a(x, \xi) \in \mathcal{FL}_\omega^p S_\gamma$ the pseudodifferential operator $a(x, D)$ extends to a unique linear bounded operator

$$a(x, D): \mathcal{FL}_{\omega_1}^p(\mathbb{R}^n) \rightarrow \mathcal{FL}_{\omega_2}^p(\mathbb{R}^n).$$

- (ii) For every $a(x, \xi) \in \mathcal{FL}_\omega^p S_\gamma(\Omega)$, with Ω open subset of \mathbb{R}^n , the pseudodifferential operator $a(x, D)$ extends to a unique linear bounded operator

$$a(x, D): \mathcal{FL}_{\omega_1}^p(\mathbb{R}^n) \rightarrow \mathcal{FL}_{\omega_2, \text{loc}}^p(\Omega).$$

If in addition the pseudodifferential operator $a(x, D)$ is properly supported, then it extends to a linear bounded operator

$$a(x, D): \mathcal{FL}_{\omega_1, \text{loc}}^p(\Omega) \rightarrow \mathcal{FL}_{\omega_2, \text{loc}}^p(\Omega).$$

Proof The second part of the statement *ii* is an immediate consequence of the first one; indeed, since the operator $a(x, D)$ is properly supported, for every function $\phi \in C_0^\infty(\Omega)$ another function $\tilde{\phi} \in C_0^\infty(\Omega)$ can be chosen in such a way that $\phi a(\cdot, D)u = \phi a(\cdot, D)(\tilde{\phi}u)$, cf. (56).

The first part of the statement *ii* follows, in its turn, from the statement *i* by noticing that $\phi(x)a(x, \xi) \in \mathcal{FL}_\omega^p S_\gamma$ for every function $\phi \in C_0^\infty(\Omega)$ (cf. Definition 4).

As for the proof of the statement *i*, we first observe that for every $u \in \mathcal{S}(\mathbb{R}^n)$ one computes

$$a(\widehat{\cdot}, D)u(\eta) = (2\pi)^{-n} \int \widehat{a}(\eta - \xi, \xi) \widehat{u}(\xi) d\xi, \quad (61)$$

where $\widehat{a}(\eta, \xi) := \widehat{a(\cdot, \xi)}(\eta)$ denotes the partial Fourier transform of the symbol $a(x, \xi)$ with respect to x . From (61) we find the following integral representation

$$\begin{aligned} \omega_2(\eta)\widehat{a(\cdot, D)}u(\eta) &= (2\pi)^{-n} \int \omega_2(\eta)\widehat{a}(\eta - \xi, \xi)\widehat{u}(\xi) d\xi \\ &= (2\pi)^{-n} \int \frac{\omega_2(\eta)\gamma(\xi)}{\omega(\eta - \xi)\omega_1(\xi)} \frac{\omega(\eta - \xi)\widehat{a}(\eta - \xi, \xi)}{\gamma(\xi)} \omega_1(\xi)\widehat{u}(\xi) d\xi \\ &= (2\pi)^{-n} \int F(\eta, \xi)f(\eta - \xi, \xi)g(\xi) d\xi, \end{aligned} \tag{62}$$

where it is set

$$f(\zeta, \xi) = \frac{\omega(\zeta)\widehat{a}(\zeta, \xi)}{\gamma(\xi)}, \quad F(\zeta, \xi) = \frac{\omega_2(\zeta)\gamma(\xi)}{\omega(\zeta - \xi)\omega_1(\xi)}, \quad g(\xi) = \omega_1(\xi)\widehat{u}(\xi). \tag{63}$$

The assumptions of Proposition 6 [see (57), (60)] yield the following

$$\sup_{\zeta \in \mathbb{R}^n} \|F(\zeta, \cdot)\|_{L^q} = C_q, \quad \sup_{\xi \in \mathbb{R}^n} \|f(\cdot, \xi)\|_{L^p} = \|a\|_{\mathcal{FL}_{\omega}^p S_{\gamma}}, \quad g \in L^p(\mathbb{R}^n).$$

Now we apply to $\omega_2(\eta)\widehat{a(\cdot, D)}u(\eta)$, written as the integral operator in (62), the result of Lemma 1. Then we have

$$\|a(\cdot, D)u\|_{\mathcal{FL}_{\omega_2}^p} = \|\omega_2\widehat{a(\cdot, D)}u\|_{L^p} \leq (2\pi)^{-n}C_q \|a\|_{\mathcal{FL}_{\omega}^p S_{\gamma}} \|u\|_{\mathcal{FL}_{\omega_1}^p}.$$

Remark 12 It is clear that Proposition 6 provides a generalization of the result given by Proposition 3; indeed the multiplication by a given function $v = v(x) \in \mathcal{FL}_{\omega}^p(\mathbb{R}^n) \subset \mathcal{FL}_{\omega}^p S$ can be thought to as the zero-th order pseudodifferential operator with ξ -independent symbol $a(x, \xi) = v(x)$, cf. Remark 10.

5 Microlocal regularity in weighted Fourier Lebesgue spaces

This section is devoted to introduce a microlocal counterpart of the weighted Fourier Lebesgue spaces presented in Sect. 2.2 and to define corresponding classes of pseudodifferential operators, with finitely regular symbols, naturally acting on such spaces.

Because of the lack of homogeneity of a generic weight function $\omega = \omega(\xi)$, in order to perform a microlocal analysis in the framework of weighted Fourier Lebesgue spaces it is convenient to replace the usual conic neighborhoods (used in Pilipović et al. [26,27]) by a suitable notion of ε -neighborhood of a set, modeled on the weight function itself, following the approach of Rodino [28] and Garello [7].

In the following, let $\omega: \mathbb{R}^n \rightarrow]0, +\infty[$ be a weight function satisfying the subadditivity condition (SA) and

(SH): for a suitable constant $C \geq 0$

$$\omega(t\xi) \leq C\omega(\xi), \quad \forall \xi \in \mathbb{R}^n, |t| \leq 1. \tag{64}$$

Every weight function $\omega = \omega(\xi)$ satisfying (\mathcal{SA}) and (\mathcal{SH}) also obeys the following

$$\frac{1}{C} \leq \frac{\omega(\xi + t\theta)}{\omega(\xi)} \leq C, \quad \text{when } \omega(\theta) \leq \frac{1}{C}\omega(\xi), \quad |t| \leq 1, \quad (65)$$

for a suitable constant $C > 1$, cf. [7].

Throughout the whole section, the weight function $\omega = \omega(\xi)$ will be assumed to be continuous.⁴ Then an easy consequence of condition (\mathcal{SH}) is that $\omega(\xi)$ satisfies (11).

To every set $X \subset \mathbb{R}^n$ one may associate a one-parameter family of open sets by defining for any $\varepsilon > 0$

$$X_{[\varepsilon\omega]} := \bigcup_{\xi_0 \in X} \{\xi \in \mathbb{R}^n : \omega(\xi - \xi_0) < \varepsilon\omega(\xi_0)\}. \quad (66)$$

We call $X_{[\varepsilon\omega]}$ the $[\omega]$ -neighborhood of X of size ε .

Remark 13 Since $\{\xi \in \mathbb{R}^n; \omega(\xi - \xi_0) < \varepsilon\omega(\xi_0)\} = \emptyset$ when $\omega(\xi_0) \leq c/\varepsilon$, where c is the constant in (11), we effectively have

$$X_{[\varepsilon\omega]} = \bigcup_{\xi_0 \in X : \omega(\xi_0) > \frac{c}{\varepsilon}} \{\omega(\xi - \xi_0) < \varepsilon\omega(\xi_0)\},$$

and for X bounded a constant $\varepsilon_0 = \varepsilon_0(X) > 0$ exists such that $X_{[\varepsilon\omega]} = \emptyset$ when $0 < \varepsilon < \varepsilon_0$.

As a consequence of (\mathcal{SA}) , (\mathcal{SH}) and (65), the $[\omega]$ -neighborhoods of a set X fulfil the following lemma.

Lemma 2 *Given $\varepsilon > 0$, there exists $0 < \varepsilon' < \varepsilon$ such that for every $X \subset \mathbb{R}^n$*

$$(X_{[\varepsilon'\omega]})_{[\varepsilon'\omega]} \subset X_{[\varepsilon\omega]}; \quad (67)$$

$$(\mathbb{R}^n \setminus X_{[\varepsilon\omega]})_{[\varepsilon'\omega]} \subset \mathbb{R}^n \setminus X_{[\varepsilon'\omega]}. \quad (68)$$

Moreover there exist constants $\widehat{c} > 0$ and $0 < \widehat{\varepsilon} < 1$ such that for all $X \subset \mathbb{R}^n$ and $0 < \varepsilon \leq \widehat{\varepsilon}$

$$\xi \in X_{[\varepsilon\omega]} \quad \text{yields} \quad \omega(\xi) > \frac{\widehat{c}}{\varepsilon}. \quad (69)$$

Proof Equations (67) and (68) are direct consequences of (\mathcal{SA}) , (\mathcal{SH}) and (65), see [7] for details.

⁴ This assumption is not as much restrictive, since it can be shown (see e.g. [32]) that for any weight function ω an equivalent weight function ω_0 exists such that $\omega_0 \in C^\infty(\mathbb{R}^n)$ and $\partial^\alpha \omega_0 / \omega_0 \in L^\infty(\mathbb{R}^n)$ for each multi-index α .

If $\xi \in X_{[\varepsilon\omega]}$ then $\xi_0 \in X$ exists such that

$$\omega(\xi - \xi_0) < \varepsilon\omega(\xi_0), \tag{70}$$

hence $\omega(\xi - \xi_0) \geq c$ implies $\omega(\xi_0) > \frac{c}{\varepsilon}$, cf. (11) and Remark 13.

In view of (65)

$$\omega(\xi) = \omega(\xi_0 + (\xi - \xi_0)) \geq \frac{1}{C}\omega(\xi_0) > \frac{c}{C\varepsilon}$$

follows from (70), provided that $\varepsilon \leq \frac{1}{C}$ [where C is the same constant involved in (65)].

We use the notion of $[\omega]$ -neighborhood of a set to define a microlocal version of the weighted Fourier Lebesgue spaces.

Definition 5 We say that a distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ belongs microlocally to \mathcal{FL}_ω^p at $X \subset \mathbb{R}^n$, writing $u \in \mathcal{FL}_{\omega, \text{mcl}}^p(X)$, $p \in [1, +\infty]$, if there exists $\varepsilon > 0$ such that

$$|u|_{X_{[\varepsilon\omega]}}^p := \int_{X_{[\varepsilon\omega]}} \omega(\xi)^p |\widehat{u}(\xi)|^p d\xi < +\infty \tag{71}$$

(with obvious modification for $p = +\infty$).

For Ω open subset of \mathbb{R}^n , $x_0 \in \Omega$ and $X \subset \mathbb{R}^n$, we say that a distribution $u \in \mathcal{D}'(\Omega)$ belongs microlocally to \mathcal{FL}_ω^p on the set X at the point x_0 , writing $u \in \mathcal{FL}_{\omega, \text{mcl}}^p(x_0 \times X)$, if there exists a function $\phi \in C_0^\infty(\Omega)$ such that $\phi(x_0) \neq 0$ and $\phi u \in \mathcal{FL}_{\omega, \text{mcl}}^p(X)$.

Remark 14 In view of Remark 13, condition (71) is meaningful only for *unbounded* X .

We can say that $u \in \mathcal{FL}_{\omega, \text{mcl}}^p(X)$ and $u \in \mathcal{FL}_{\omega, \text{mcl}}^p(x_0 \times X)$ if respectively

$$\chi_{[\varepsilon\omega]}(\xi)\omega(\xi)\widehat{u}(\xi) \in L^p(\mathbb{R}^n) \tag{72}$$

and

$$\chi_{[\varepsilon\omega]}(\xi)\omega(\xi)\widehat{\phi u}(\xi) \in L^p(\mathbb{R}^n), \tag{73}$$

where $\chi_{[\varepsilon\omega]} = \chi_{[\varepsilon\omega]}(\xi)$ denotes the characteristic function of the set $X_{[\varepsilon\omega]}$ and $\varepsilon > 0$, $\phi = \phi(x)$ are given as in Definition 5.

According to Definition 5 one can introduce the notion of *filter of Fourier Lebesgue singularities*, which is in some way the extension of the wave front set of Fourier Lebesgue singularities when we lack the homogeneity properties necessary to use effectively conic neighborhoods.

Definition 6 Assume that $u \in \mathcal{D}'(\Omega)$, $x_0 \in \Omega$, $p \in [1, +\infty]$. Then the filter of \mathcal{FL}_ω^p -singularities of u at the point x_0 is the class of all sets $X \subset \mathbb{R}^n$ such that $u \in \mathcal{FL}_{\omega, \text{mcl}}^p(x_0 \times (\mathbb{R}^n \setminus X))$. It may be easily verified that

$$\mathfrak{E}_{\mathcal{FL}_\omega^p, x_0} u := \bigcup_{\phi \in C_0^\infty(\Omega), \phi(x_0) \neq 0} \mathfrak{E}_{\mathcal{FL}_\omega^p} \phi u, \quad (74)$$

where for every $v \in \mathcal{S}'(\mathbb{R}^n)$, $\mathfrak{E}_{\mathcal{FL}_\omega^p} v$ is the class of all sets $X \subset \mathbb{R}^n$ such that $v \in \mathcal{FL}_{\omega, \text{mcl}}^p(\mathbb{R}^n \setminus X)$.

$\mathfrak{E}_{\mathcal{FL}_\omega^p} v$ and $\mathfrak{E}_{\mathcal{FL}_\omega^p, x_0} u$ defined above are $[\omega]$ -filters, in the sense that they satisfy the standard filter properties and moreover for all $X \in \mathfrak{E}_{\mathcal{FL}_\omega^p} v$ (respectively $X \in \mathfrak{E}_{\mathcal{FL}_\omega^p, x_0} u$) there exists $\varepsilon > 0$ such that $\mathbb{R}^n \setminus (\mathbb{R}^n \setminus X)_{[\varepsilon\omega]} \in \mathfrak{E}_{\mathcal{FL}_\omega^p} v$ (respectively $\mathbb{R}^n \setminus (\mathbb{R}^n \setminus X)_{[\varepsilon\omega]} \in \mathfrak{E}_{\mathcal{FL}_\omega^p, x_0} u$), see e.g. [33] for the definition and properties of a filter.

5.1 Symbols with microlocal regularity in spaces of Fourier Lebesgue type

Throughout the whole Section, we assume that $\lambda = \lambda(\xi)$ and $\Lambda = \Lambda(\xi)$ are two continuous weight functions, such that λ satisfies condition (11) and Λ conditions (\mathcal{SA}) and (\mathcal{SH}) .

For given $p \in [1, +\infty]$ and $X \subset \mathbb{R}^n$, the space $\mathcal{FL}_\lambda^p(\mathbb{R}^n) \cap \mathcal{FL}_{\Lambda, \text{mcl}}^p(X)$ is provided with the *inductive limit* locally convex topology defined on it by the family of subspaces

$$\mathcal{FL}_\lambda^p(\mathbb{R}^n) \cap \mathcal{FL}_{\Lambda, \varepsilon}^p(X) := \{u \in \mathcal{FL}_\lambda^p(\mathbb{R}^n) : |u|_{X_{[\varepsilon\Lambda]}} < +\infty\}$$

(cf. (71)), endowed with their natural semi-norm

$$\|u\|_{\mathcal{FL}_\lambda^p} + |u|_{X_{[\varepsilon\Lambda]}}, \quad \varepsilon > 0.$$

Analogously for every $x_0 \in \Omega$, the space $\mathcal{FL}_{\lambda, \text{loc}}^p(x_0) \cap \mathcal{FL}_{\Lambda, \text{mcl}}^p(x_0 \times X)$ is provided with the inductive limit topology defined by the subspaces

$$\mathcal{FL}_{\lambda, \phi}^p \cap \mathcal{FL}_{\Lambda, \varepsilon}^p(X) := \{u \in \mathcal{D}'(\Omega) : \phi u \in \mathcal{FL}_\lambda^p(\mathbb{R}^n) \cap \mathcal{FL}_{\Lambda, \varepsilon}^p(X)\},$$

endowed with the natural semi-norms

$$\|\phi u\|_{\mathcal{FL}_\lambda^p} + |\phi u|_{X_{[\varepsilon\Lambda]}}, \quad \phi \in C_0^\infty(\Omega), \phi(x_0) \neq 0, \varepsilon > 0.$$

From the general properties of the inductive limit topology (see e.g. [33]), it follows that a sequence $\{u_\nu\}$ converges to u in $\mathcal{FL}_\lambda^p(\mathbb{R}^n) \cap \mathcal{FL}_{\Lambda, \text{mcl}}^p(X)$ (resp. $\mathcal{FL}_{\lambda, \text{loc}}^p(x_0) \cap \mathcal{FL}_{\Lambda, \text{mcl}}^p(x_0 \times X)$) if and only if there exists some $\varepsilon > 0$ such that

$$\|u_\nu - u\|_{\mathcal{FL}_\lambda^p} \rightarrow 0 \quad \text{and} \quad |u_\nu - u|_{X_{[\varepsilon\Lambda]}} \rightarrow 0, \quad \text{as } \nu \rightarrow +\infty$$

(resp. there exist $\phi \in C_0^\infty(\Omega)$, with $\phi(x_0) \neq 0$, and $\varepsilon > 0$ such that

$$\|\phi(u_v - u)\|_{\mathcal{FL}_\lambda^p} \rightarrow 0 \quad \text{and} \quad |\phi(u_v - u)|_{X_{[\varepsilon\Lambda]}} \rightarrow 0, \quad \text{as } v \rightarrow +\infty).$$

Definition 7 Let $\lambda = \lambda(\xi)$, $\Lambda = \Lambda(\xi)$ be two weight functions as above and $\gamma = \gamma(\xi)$ a further continuous weight function, $x_0 \in \Omega$, $X \subset \mathbb{R}^n$ and $p \in [1, +\infty]$. We say that a distribution $a(x, \xi) \in \mathcal{D}'(\Omega \times \mathbb{R}^n)$ belongs to $\mathcal{FL}_{\lambda, \Lambda}^p S_\gamma(x_0 \times X)$ if the function $\xi \mapsto a(\cdot, \xi)$ takes values in $\mathcal{FL}_{\lambda, \text{loc}}^p(x_0) \cap \mathcal{FL}_{\Lambda, \text{mcl}}^p(x_0 \times X)$ and for some $\phi \in C_0^\infty(\Omega)$ such that $\phi(x_0) \neq 0$ and $\varepsilon > 0$ there holds

$$\begin{aligned} \|a\|_{\phi, \lambda, \gamma} &:= \sup_{\xi \in \mathbb{R}^n} \left\| \frac{\lambda(\cdot) \widehat{\phi a}(\cdot, \xi)}{\gamma(\xi)} \right\|_{L^p} < +\infty \quad \text{and} \\ |a|_{\phi, \Lambda, \gamma, \varepsilon, X} &:= \sup_{\xi \in \mathbb{R}^n} \left\| \frac{\Lambda(\cdot) \chi_{\varepsilon, \Lambda}(\cdot) \widehat{\phi a}(\cdot, \xi)}{\gamma(\xi)} \right\|_{L^p} < +\infty, \end{aligned} \tag{75}$$

where $\widehat{\phi a}(\eta, \xi) := \mathcal{F}_{x \rightarrow \eta}(\phi(x)a(x, \xi))$ (η) denotes the partial Fourier transform of $\phi(x)a(x, \xi)$ with respect to x .

Theorem 1 For $p \in [1, +\infty]$, $x_0 \in \Omega$, $X \subset \mathbb{R}^n$, let $\lambda = \lambda(\xi)$, $\Lambda = \Lambda(\xi)$, $\gamma = \gamma(\xi)$, $\sigma = \sigma(\xi)$ be weight functions such that λ obeys condition (43), where q is the conjugate exponent of p , Λ conditions (SA), (SH), $1/\sigma \in L^q(\mathbb{R}^n)$ and

$$\sigma(\xi) \leq \lambda(\xi) \leq \Lambda(\xi) \leq \frac{\lambda(\xi)^2}{\sigma(\xi)}. \tag{76}$$

(i) If $a(x, \xi) \in \mathcal{FL}_{\lambda, \Lambda}^p S_\gamma(x_0 \times X)$ then the corresponding pseudodifferential operator $a(x, D)$ extends to a bounded linear operator

$$\mathcal{FL}_{\lambda\gamma}^p(\mathbb{R}^n) \cap \mathcal{FL}_{\Lambda\gamma, \text{mcl}}^p(X) \rightarrow \mathcal{FL}_{\lambda, \text{loc}}^p(x_0) \cap \mathcal{FL}_{\Lambda, \text{mcl}}^p(x_0 \times X). \tag{77}$$

(ii) If in addition $a(x, D)$ is properly supported, then it extends to a bounded linear operator

$$\mathcal{FL}_{\lambda\gamma, \text{loc}}^p(x_0) \cap \mathcal{FL}_{\Lambda\gamma, \text{mcl}}^p(x_0 \times X) \rightarrow \mathcal{FL}_{\lambda, \text{loc}}^p(x_0) \cap \mathcal{FL}_{\Lambda, \text{mcl}}^p(x_0 \times X). \tag{78}$$

Proof The statement (ii) follows at once from (i) in view of the definition of a properly supported operator. Thus, let us focus on the proof of (i).

In view of Definition 7, there exist $\varepsilon > 0$ and $\phi \in C_0^\infty(\Omega)$, with $\phi(x_0) \neq 0$, such that conditions in (75) are satisfied. We are going first to prove that

$$\phi(x)a(x, D)u \in \mathcal{FL}_\lambda^p(\mathbb{R}^n), \tag{79}$$

as long as $u \in \mathcal{FL}_{\lambda\gamma}^p(\mathbb{R}^n)$. Let us denote for short

$$a_\phi(x, \xi) := \phi(x)a(x, \xi).$$

In order to check (79) it is enough to apply the result of Proposition 6 to the symbol $a_\phi(x, \xi) \in \mathcal{FL}_\lambda^p S_\gamma$ (cf. Definition 4) where, restoring the notations used there, we set

$$\omega_1(\zeta) = \lambda(\zeta)\gamma(\zeta), \quad \omega(\zeta) = \omega_2(\zeta) = \lambda(\zeta).$$

Under the previous positions, the condition (60) of Proposition 6 reduces to require that $\lambda = \lambda(\zeta)$ satisfies (43). From Proposition 6 we also deduce the continuity of $a(x, D)$ as a linear map from $\mathcal{FL}_{\lambda\gamma}^p(\mathbb{R}^n)$ into $\mathcal{FL}_{\lambda, \text{loc}}^p(x_0)$.

It remains to show that

$$a_\phi(x, D)u \in \mathcal{FL}_{\Lambda, \text{mcl}}^p(X), \quad (80)$$

when $u \in \mathcal{FL}_{\lambda\gamma}^p(\mathbb{R}^n) \cap \mathcal{FL}_{\Lambda\gamma, \text{mcl}}^p(X)$, as well as the continuity of $a(x, D)$ as an operator acting on the aforementioned spaces. Throughout the rest of the proof, we will denote by C some positive constant that is independent of the symbol $a(x, \xi)$ and the function $u(x)$ and may possibly differ from an occurrence to another.

In view of Lemma 2, there exists some $0 < \varepsilon' < \varepsilon$ such that

$$(\mathbb{R}^n \setminus X_{[\varepsilon\Lambda]})_{[\varepsilon'\Lambda]} \subset \mathbb{R}^n \setminus X_{[\varepsilon'\Lambda]}.$$

Let us denote for short

$$\chi(\zeta) := \chi_{[\varepsilon'\Lambda]}(\zeta), \quad \chi_1(\zeta) := \chi_{[\varepsilon\Lambda]}(\zeta), \quad \chi_2(\zeta) := 1 - \chi_{[\varepsilon\Lambda]}(\zeta) \quad (81)$$

and write

$$\widehat{a}_\phi(\zeta, \xi)\widehat{u}(\xi) = \sum_{i,j=1,2} \chi_i(\zeta)\widehat{a}_\phi(\zeta, \xi)\chi_j(\xi)\widehat{u}(\xi).$$

Then in view of (61) and condition (SA) for Λ , we find

$$\begin{aligned} & |\chi(\eta)\Lambda(\eta)\widehat{a_\phi(\cdot, D)u}(\eta)| \\ & \leq C(2\pi)^{-n} \int \chi(\eta) \{\Lambda(\eta - \xi) + \Lambda(\xi)\} |\widehat{a_\phi}(\eta - \xi, \xi)| |\widehat{u}(\xi)| d\xi \\ & \leq C(2\pi)^{-n} \int \chi(\eta) \sum_{i,j=1,2} \chi_i(\eta - \xi)\Lambda(\eta - \xi) |\widehat{a_\phi}(\eta - \xi, \xi)| \chi_j(\xi) |\widehat{u}(\xi)| d\xi \\ & \quad + C(2\pi)^{-n} \int \chi(\eta) \sum_{i,j=1,2} \chi_i(\eta - \xi) |\widehat{a_\phi}(\eta - \xi, \xi)| \chi_j(\xi) \Lambda(\xi) |\widehat{u}(\xi)| d\xi \\ & = \mathcal{I}_1 u(\eta) + \mathcal{I}_2 u(\eta). \end{aligned} \quad (82)$$

Let us set

$$\begin{aligned} g_1(\zeta, \xi) &= \chi_1(\zeta)\Lambda(\zeta)\gamma(\xi)^{-1} |\widehat{a_\phi}(\zeta, \xi)|; \\ g_2(\zeta, \xi) &= \chi_2(\zeta)\sigma(\zeta)\Lambda(\zeta)\gamma(\xi)^{-1} \Lambda(\xi)^{-1} |\widehat{a_\phi}(\zeta, \xi)|; \end{aligned}$$

$$\begin{aligned}
 \tilde{g}_2(\zeta, \xi) &= \chi_2(\zeta)\sigma(\zeta)^{1/2}\Lambda(\zeta)^{1/2}\gamma(\xi)^{-1}|\widehat{a}_\phi(\zeta, \xi)|; \\
 v_1(\xi) &= \chi_1(\xi)\gamma(\xi)\sigma(\xi)|\widehat{u}(\xi)|; \quad \tilde{v}_1(\xi) = \chi_1(\xi)\gamma(\xi)\Lambda(\xi)|\widehat{u}(\xi)|; \\
 v_2(\xi) &= \chi_2(\xi)\gamma(\xi)\sigma(\xi)|\widehat{u}(\xi)|; \\
 \tilde{v}_2(\zeta, \xi) &= \chi_2(\xi)\sigma(\xi)^{1/2}\Lambda(\zeta)^{1/2}\gamma(\xi)|\widehat{u}(\xi)|.
 \end{aligned} \tag{83}$$

Then the first integral in the right-hand side of (82) can be rewritten as

$$\begin{aligned}
 \mathcal{I}_1 u(\eta) &= \int \chi(\eta) \frac{1}{\sigma(\xi)} g_1(\eta - \xi, \xi) v_1(\xi) d\xi \\
 &+ \int \chi(\eta) \frac{1}{\sigma(\xi)} g_1(\eta - \xi, \xi) v_2(\xi) d\xi + \int \chi(\eta) \frac{1}{\sigma(\eta - \xi)} g_2(\eta - \xi, \xi) \tilde{v}_1(\xi) d\xi \\
 &+ \int \chi(\eta) \frac{1}{\sqrt{\sigma(\xi)\sigma(\eta - \xi)}} \tilde{g}_2(\eta - \xi, \xi) \tilde{v}_2(\eta - \xi, \xi) d\xi.
 \end{aligned} \tag{84}$$

In view of the assumptions in (76) it is easy to see that all the above functions v_1 , v_2 , \tilde{v}_1 defined in (83) belong to $L^p(\mathbb{R}^n)$, if $u \in \mathcal{F}L_{\lambda\gamma}^p(\mathbb{R}^n) \cap \mathcal{F}L_{\Lambda\gamma, \text{mcl}}^p(X)$, with the following estimates

$$\|v_1\|_{L^p} \leq \|u\|_{X_{[\varepsilon\Lambda\gamma]}}, \quad \|\tilde{v}_1\|_{L^p} \leq \|u\|_{X_{[\varepsilon\Lambda\gamma]}}, \quad \|v_2\|_{L^p} \leq \|u\|_{\mathcal{F}L_{\lambda\gamma}^p}. \tag{85}$$

Moreover the functions

$$F_1(\eta, \xi) := \frac{\chi(\eta)}{\sigma(\xi)}, \quad F_2(\eta, \xi) := \frac{\chi(\eta)}{\sigma(\eta - \xi)}, \quad F_3(\eta, \xi) := \frac{\chi(\eta)}{\sqrt{\sigma(\xi)\sigma(\eta - \xi)}} \tag{86}$$

belong to the space $\mathcal{L}_2^{\infty, q}(\mathbb{R}^{2n})$, with the estimates

$$\|F_i\|_{\mathcal{L}_2^{\infty, q}} \leq \|1/\sigma\|_{L^q}, \quad i = 1, 2, 3. \tag{87}$$

Again from (76) and (75) we easily obtain that $g_1(\zeta, \xi) \in \mathcal{L}_1^{p, \infty}(\mathbb{R}^{2n})$ and satisfies the estimate

$$\|g_1\|_{\mathcal{L}_1^{p, \infty}} \leq |a|_{\phi, \Lambda, \gamma, \varepsilon, X}.$$

In view of the previous analysis, the first two integral operators in (84) have the form of the operator considered in Lemma 1. Thus from Lemma 1 and the estimates collected above we get

$$\begin{aligned}
 &\left\| \int \chi(\cdot) \frac{1}{\sigma(\xi)} g_1(\cdot - \xi, \xi) v_1(\xi) d\xi \right\|_{L^p} + \left\| \int \chi(\cdot) \frac{1}{\sigma(\xi)} g_1(\cdot - \xi, \xi) v_2(\xi) d\xi \right\|_{L^p} \\
 &\leq \|1/\sigma\|_{L^q} |a|_{\phi, \Lambda, \gamma, \varepsilon, X} \left\{ \|u\|_{X_{[\varepsilon\Lambda\gamma]}} + \|u\|_{\mathcal{F}L_{\lambda\gamma}^p} \right\}.
 \end{aligned} \tag{88}$$

Concerning the third integral operator in (84), we notice that the involved function $g_2(\zeta, \xi)$ vanishes when $\zeta + \xi \notin X_{[\varepsilon'\Lambda]}$, due to the presence of the characteristic

function χ . For $\zeta + \xi \in X_{[\varepsilon'\Lambda]}$ and $\zeta \in \mathbb{R}^n \setminus X_{[\varepsilon\Lambda]}$ it follows that $\Lambda(\zeta) \leq \frac{1}{\varepsilon'}\Lambda(\xi)$; indeed the converse inequality $\Lambda((\zeta + \xi) - \zeta) = \Lambda(\xi) < \varepsilon'\Lambda(\zeta)$ should mean that $\zeta + \xi \in (\mathbb{R}^n \setminus X_{[\varepsilon\Lambda]})_{[\varepsilon'\Lambda]} \subset \mathbb{R}^n \setminus X_{[\varepsilon'\Lambda]}$. Hence we get

$$|g_2(\zeta, \xi)| \leq \frac{1}{\varepsilon'} \chi_2(\zeta) \sigma(\zeta) \gamma(\xi)^{-1} |\widehat{a}_\phi(\zeta, \xi)|, \quad (89)$$

and, using also $\sigma \leq \lambda$,

$$\|g_2(\cdot, \xi)\|_{L^p} \leq \frac{1}{\varepsilon'} \|\chi_2(\cdot) \sigma(\cdot) \gamma(\xi)^{-1} |\widehat{a}_\phi(\cdot, \xi)|\|_{L^p} \leq \frac{C}{\varepsilon'} \|a\|_{\phi, \lambda, \gamma}.$$

This yields that $g_2(\zeta, \xi) \in \mathcal{L}_1^{p, \infty}(\mathbb{R}^{2n})$ with norm bounded by

$$\|g_2\|_{\mathcal{L}_1^{p, \infty}} \leq \frac{C}{\varepsilon'} \|a\|_{\phi, \lambda, \gamma}.$$

Hence we may apply again Lemma 1 to the third operator in (84), and using also the estimates (85), (87) we find

$$\left\| \int \chi(\cdot) \frac{1}{\sigma(\cdot - \xi)} g_2(\cdot - \xi, \xi) \tilde{v}_1(\xi) d\xi \right\|_{L^p} \leq \frac{C}{\varepsilon'} \|1/\sigma\|_{L^q} \|a\|_{\phi, \lambda, \gamma} \|u\|_{X_{[\varepsilon\Lambda\gamma]}}. \quad (90)$$

Let us consider now the fourth integral operator in (84). Applying the same argument used to provide the estimate (89), we obtain

$$|\tilde{v}_2(\zeta, \xi)| \leq \frac{1}{\varepsilon'} \chi_2(\xi) \sigma(\xi)^{1/2} \Lambda(\xi)^{1/2} \gamma(\xi) |\widehat{u}(\xi)|. \quad (91)$$

Thanks to (76), $\sigma^{1/2} \Lambda^{1/2} \leq \lambda$, then

$$\begin{aligned} & \left| \int \frac{\chi(\eta)}{\sqrt{\sigma(\xi)\sigma(\eta - \xi)}} \tilde{g}_2(\eta - \xi, \xi) \tilde{v}_2(\eta - \xi, \xi) d\xi \right| \\ & \leq \frac{C}{\varepsilon'} \int \frac{\chi(\eta)}{\sqrt{\sigma(\xi)\sigma(\eta - \xi)}} |\tilde{g}_2(\eta - \xi, \xi)| \lambda(\xi) \gamma(\xi) |\widehat{u}(\xi)| d\xi. \end{aligned} \quad (92)$$

On the other hand, using again $\sigma^{1/2} \Lambda^{1/2} \leq \lambda$ and $a_\phi(\cdot, \xi)/\gamma(\xi) \in \mathcal{FL}_\lambda^p(\mathbb{R}^n)$, uniformly with respect to ξ , we establish that $\tilde{g}_2(\zeta, \xi)$ belongs to $\mathcal{L}_1^{p, \infty}(\mathbb{R}^{2n})$ and satisfies the estimate

$$\|\tilde{g}_2\|_{\mathcal{L}_1^{p, \infty}} \leq \|\sigma(\cdot)^{1/2} \Lambda(\cdot)^{1/2} \gamma(\xi)^{-1} \widehat{a}_\phi(\cdot, \xi)\|_{L^p} \leq C \|a\|_{\phi, \lambda, \gamma}. \quad (93)$$

Since $\lambda(\xi) \gamma(\xi) |\widehat{u}(\xi)| \in L^p(\mathbb{R}^n)$ (as $u \in \mathcal{FL}_{\lambda\gamma}^p(\mathbb{R}^n)$) and $F_3(\eta, \xi) = \frac{\chi(\eta)}{\sqrt{\sigma(\xi)\sigma(\eta - \xi)}}$ belongs to $\mathcal{L}_2^{\infty, q}(\mathbb{R}^{2n})$, the integral operator in the right-hand side of (92) satisfies the assumptions of Lemma 1, then from (87) and (89) we find

$$\begin{aligned} & \left\| \int \frac{\chi(\cdot)}{\sqrt{\sigma(\xi)\sigma(\cdot - \xi)}} \tilde{g}_2(\cdot - \xi, \xi) \tilde{v}_2(\cdot - \xi, \xi) d\xi \right\|_{L^p} \\ & \leq \frac{C}{\varepsilon'} \|1/\sigma\|_{L^q} \|a\|_{\phi, \lambda, \gamma} \|u\|_{\mathcal{FL}_{\lambda, \gamma}^p}. \end{aligned} \tag{94}$$

Summing up the estimates (88), (90), (94) the L^p -norm of $\mathcal{I}_1 u$ in the right-hand side of (82) is estimated by

$$\|\mathcal{I}_1 u\|_{L^p} \leq \frac{C}{\varepsilon'} \|1/\sigma\|_{L^q} (|a|_{\phi, \Lambda, \gamma, \varepsilon, X} + \|a\|_{\phi, \lambda, \gamma}) \left(|u|_{X_{[\varepsilon\Lambda\gamma]}} + \|u\|_{\mathcal{FL}_{\lambda, \gamma}^p} \right). \tag{95}$$

The second integral $\mathcal{I}_2 u(\eta)$ in (82) can be handled similarly as before to provide for its L^p -norm the same bound as in (95). From (82) we then get

$$\begin{aligned} & \|\chi \Lambda a_{\phi}(\widehat{\cdot, D})u\|_{L^p} \\ & \leq \frac{C}{\varepsilon'} \|1/\sigma\|_{L^q} (|a|_{\phi, \Lambda, \gamma, \varepsilon, X} + \|a\|_{\phi, \lambda, \gamma}) \left(|u|_{X_{[\varepsilon\Lambda\gamma]}} + \|u\|_{\mathcal{FL}_{(\lambda, \gamma)}^p} \right) \end{aligned} \tag{96}$$

which proves (80) and shows the continuity of $a(x, D)$ as a linear map from $\mathcal{FL}_{\lambda, \gamma}^p(\mathbb{R}^n) \cap \mathcal{FL}_{\Lambda, \text{mcl}}^p(X)$ into $\mathcal{FL}_{\lambda, \text{loc}}^p(x_0) \cap \mathcal{FL}_{\Lambda, \text{mcl}}^p(x_0 \times X)$.

Remark 15 Let the same hypotheses of Theorem 1 be satisfied. Clearly every $v = v(x) \in \mathcal{FL}_{\lambda, \text{loc}}^p(x_0) \cap \mathcal{FL}_{\Lambda, \text{mcl}}^p(x_0 \times X)$ is a ξ -independent symbol in the class $\mathcal{FL}_{\lambda, \Lambda}^p S_{\gamma}(x_0 \times X)$ corresponding to the weight function $\gamma(\xi) \equiv 1$, and the product of smooth functions by the multiplier v defines a properly supported zero-th order operator. Therefore we find that the product of any two elements $u, v \in \mathcal{FL}_{\lambda, \text{loc}}^p(x_0) \cap \mathcal{FL}_{\Lambda, \text{mcl}}^p(x_0 \times X)$ still belongs to the same space (giving a continuous bilinear mapping), as a direct application of Theorem 1. Similarly as in the proof of Corollary 2, see also the subsequent Remark 6, one can deduce that the composition of a vector-valued distribution $u = (u_1, \dots, u_N) \in \left(\mathcal{FL}_{\lambda, \text{loc}}^p(x_0) \cap \mathcal{FL}_{\Lambda, \text{mcl}}^p(x_0 \times X)\right)^N$ with some nonlinear function $F = F(x, \zeta)$ of $x \in \mathbb{R}^n$ and $\zeta \in \mathbb{C}^N$, which is locally smooth with respect to x on some neighborhood of x_0 and entire analytic with respect to ζ in the sense of Remark 6, is again a distribution in $\mathcal{FL}_{\lambda, \text{loc}}^p(x_0) \cap \mathcal{FL}_{\Lambda, \text{mcl}}^p(x_0 \times X)$.

Let us even point out that in the particular case where $\lambda \equiv \Lambda$ the assumption (76) in Theorem 1 reduces to $\sigma \leq \lambda$. In such a case $\mathcal{FL}_{\lambda, \text{loc}}^p(x_0) \cap \mathcal{FL}_{\Lambda, \text{mcl}}^p(x_0 \times X) \equiv \mathcal{FL}_{\lambda, \text{loc}}^p(x_0)$ and $\mathcal{FL}_{\lambda, \Lambda}^p S_{\gamma}(x_0 \times X) \equiv \mathcal{FL}_{\lambda}^p S_{\gamma}(V_{x_0})$ for a suitable neighborhood V_{x_0} of x_0 , see Definition 4, hence the statement of Theorem 1 reduces to a particular case of the statement of Proposition 6 (where $\omega_1 = \gamma\lambda$, $\omega = \omega_2 = \lambda$) under slightly more restrictive assumptions; indeed a sub-additive weight function λ satisfying $\sigma \leq \lambda$ for $1/\sigma \in L^q(\mathbb{R}^n)$ also fulfils condition (43) with the same q (that is the assumption required by Proposition 6), in view of Proposition 2.ii.

6 Propagation of singularities

In this section, we give some applications to the local and microlocal regularity of semilinear partial(pseudo)differential equations in weighted Fourier Lebesgue spaces.

The smooth symbols we consider in this section are related to a suitable subclass of the weight functions introduced in Sect. 2.1. More precisely, we consider a continuous function $\lambda : \mathbb{R}^n \rightarrow]0, +\infty[$ satisfying the following:

$$\lambda(\xi) \geq \frac{1}{C}(1 + |\xi|)^{\nu}, \quad \forall \xi \in \mathbb{R}^n; \quad (97)$$

$$\frac{1}{C} \leq \frac{\lambda(\xi)}{\lambda(\eta)} \leq C, \quad \text{as long as } |\xi - \eta| \leq \frac{1}{C}\lambda(\eta)^{1/\mu}, \quad (98)$$

for suitable constants $C \geq 1, 0 < \nu \leq \mu$.

Thanks to Proposition 2, it is clear that $\lambda(\xi)$ is a weight function; indeed it also satisfies the temperance condition (T) for $N = \mu$.

All the weight functions described in the examples 1–3 given in Sect. 2.1 obey the assumptions (97), (98).

For $r \in \mathbb{R}, \rho \in]0, 1/\mu]$, we define $S_{\rho, \lambda}^r$ as the class of smooth functions $a(x, \xi) \in C^\infty(\mathbb{R}^{2n})$ whose derivatives decay according to the following estimates

$$|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \leq C_{\alpha, \beta} \lambda(\xi)^{r - \rho|\alpha|}, \quad \forall (x, \xi) \in \mathbb{R}^{2n}. \quad (99)$$

If Ω is an open subset of \mathbb{R}^n , the local class $S_{\rho, \lambda}^r(\Omega)$ is the set of functions $a(x, \xi) \in C^\infty(\Omega \times \mathbb{R}^n)$ such that $\phi(x)a(x, \xi) \in S_{\rho, \lambda}^r$ for all $\phi \in C_0^\infty(\Omega)$. We will adopt the shortcut

$$S_\lambda^r := S_{1/\mu, \lambda}^r, \quad S_\lambda^r(\Omega) := S_{1/\mu, \lambda}^r(\Omega).$$

Hereafter, we will denote by $\text{Op } S_{\rho, \lambda}^r(\Omega)$ the class of properly supported pseudodifferential operators with symbols in $S_{\rho, \lambda}^r(\Omega)$ and, according to the above, we set

$$\text{Op } S_\lambda^r(\Omega) := \text{Op } S_{1/\mu, \lambda}^r(\Omega).$$

A symbol $a(x, \xi) \in S_\lambda^r(\Omega)$ (and the related pseudodifferential operator) is said to be λ -*elliptic* if for every compact subset K of Ω some positive constants c_K and $R_K > 1$ exist such that

$$|a(x, \xi)| \geq c_K \lambda(\xi)^r, \quad \forall x \in K \text{ and } |\xi| \geq R_K. \quad (100)$$

Let us also observe that $\bigcap_{r \in \mathbb{R}} S_{\rho, \lambda}^r(\Omega) = S^{-\infty}(\Omega)$, where in the classic terms $S^{-\infty}(\Omega)$ is the class of symbols $a(x, \xi) \in C^\infty(\Omega \times \mathbb{R}^n)$ such that for arbitrarily large $\theta > 0$, for all multi-indices $\alpha, \beta \in \mathbb{Z}_+^n$ and every compact set $K \subset \Omega$ there holds

$$|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \leq C_{\alpha, \beta, \theta} (1 + |\xi|)^{-\theta}, \quad \forall x \in K, \quad \forall \xi \in \mathbb{R}^n.$$

Pseudodifferential operators with symbols $a(x, \xi) \in S^{-\infty}(\Omega)$ are *regularizing* operators in the sense that they define linear bounded operators $a(x, D) : \mathcal{E}'(\Omega) \rightarrow C^\infty(\Omega)$.

The weighted symbol classes $S_{\rho, \lambda}^r(\Omega)$ considered above are a special case of the more general classes $S_{m, \Lambda}(\Omega)$, associated to the weight function $m(\xi) = \lambda(\xi)^r$ and the weight vector $\Lambda(\xi) = (\lambda(\xi)^\rho, \dots, \lambda(\xi)^\rho)$, as defined and studied in [18, Definition 1.1]. For the weighted symbol classes $S_{\rho, \lambda}^r(\Omega)$, a complete symbolic calculus is available, cf. [18, Sect.1]; in particular, the existence of a *parametrix* of any elliptic pseudodifferential operator is guaranteed.

Proposition 7 *Let $a(x, \xi)$ be a λ -elliptic symbol in $S_{\rho, \lambda}^r(\Omega)$. Then a symbol $b(x, \xi) \in S_{\rho, \lambda}^{-r}(\Omega)$ exists such that the operator $b(x, D)$ is properly supported and satisfies*

$$b(x, D)a(x, D) = \text{Id} + c(x, D),$$

where Id denotes the identity operator and $c(x, D)$ is a regularizing pseudodifferential operator.

The following inclusion

$$S_{\rho, \lambda}^r(\Omega) \subset \mathcal{FL}_\omega^p S_{\lambda^r}(\Omega) \tag{101}$$

holds true, with continuous imbedding, for all $r \in \mathbb{R}$, $\rho \in]0, 1/\mu]$, $p \in [1, +\infty]$ and any weight function $\omega(\xi)$. As a consequence of Proposition 6 we then obtain the following continuity result.

Proposition 8 *Let $\omega(\xi)$ be any weight function and $p \in [1, +\infty]$. Then every pseudodifferential operator with symbol $a(x, \xi) \in S_{\rho, \lambda}^r(\Omega)$ extends to a linear bounded operator*

$$a(x, D) : \mathcal{FL}_{\lambda^r \omega}^p(\mathbb{R}^n) \rightarrow \mathcal{FL}_{\omega, \text{loc}}^p(\Omega).$$

If in addition $a(x, D)$ is properly supported, then the latter extends to a linear bounded operator

$$a(x, D) : \mathcal{FL}_{\lambda^r \omega, \text{loc}}^p(\Omega) \rightarrow \mathcal{FL}_{\omega, \text{loc}}^p(\Omega).$$

Proof In view of (101), it is enough to observe that for any weight function $\omega(\xi)$, another weight function $\tilde{\omega}(\xi)$ can be found in such a way that

$$\sup_{\xi \in \mathbb{R}^n} \left\| \frac{\omega(\xi)}{\omega(\cdot) \tilde{\omega}(\xi - \cdot)} \right\|_{L^q} < +\infty, \tag{102}$$

where $q \in [1, +\infty]$ is the conjugate exponent of p ; for instance, one can take $\tilde{\omega}(\xi) = (1 + |\xi|)^{\tilde{N}}$, with $\tilde{N} > 0$ sufficiently large. Then the result follows at once, by noticing that $a(x, \xi)$ belongs to $\mathcal{FL}_\omega^p S_{\lambda^r}(\Omega)$ and (102) is nothing but condition (60), where γ , ω_1 , ω_2 and ω in Proposition 6 are replaced respectively by λ^r , $\lambda^r \omega$, ω and $\tilde{\omega}$.

6.1 Local regularity results

Let $\lambda = \lambda(\xi)$ be a given continuous weight function satisfying the assumptions (97) and (98). We consider a nonlinear pseudodifferential equation of the following type

$$a(x, D)u + F(x, b_i(x, D)u)_{1 \leq i \leq M} = f(x), \quad (103)$$

where $u = u(x)$ is defined on some open set $\Omega \subseteq \mathbb{R}^n$ and $a(x, D)$ is a properly supported pseudodifferential operator with symbol $a(x, \xi) \in S_\lambda^r(\Omega)$ for given $r > 0$. $F(x, b_i(x, D)u)_{1 \leq i \leq M}$ stands for a nonlinear function of $x \in \Omega$ and $b_1(x, D)u, b_2(x, D)u, \dots, b_M(x, D)u$ where $b_i(x, D)$ are still properly supported pseudodifferential operators, and $f = f(x)$ is a given forcing term. We require the equation (103) to be *semilinear* by assuming that the operators involved in the nonlinear part $F(x, b_i(x, D)u)$ have order strictly smaller than the order of the linear part $a(x, D)u$, that is

$$b_i(x, \xi) \in S_\lambda^{r-\varepsilon}(\Omega) \quad \text{for } i = 1, \dots, M, \quad (104)$$

for suitable $0 < \varepsilon < r$.

For $s \in \mathbb{R}$, $p \in [1, +\infty]$, let us set

$$\mathcal{FL}_{s,\lambda}^p(\mathbb{R}^n) := \mathcal{FL}_{\lambda^s}^p(\mathbb{R}^n), \quad \mathcal{FL}_{s,\lambda,\text{loc}}^p(\Omega) := \mathcal{FL}_{\lambda^s,\text{loc}}^p(\Omega).$$

The following regularity result can be proved.

Proposition 9 *Let the symbol $a(x, \xi) \in S_\lambda^r(\Omega)$ be λ -elliptic and the function $F = F(x, \zeta)$ obey the assumptions collected in Remark 6. For a given $p \in [1, +\infty]$, take a real number t such that $\lambda^{t-r+\varepsilon}$ fulfils condition (43) with q the conjugate exponent of p . If $u \in \mathcal{FL}_{t,\lambda,\text{loc}}^p(\Omega)$ is any solution of the equation (103), with forcing term $f \in \mathcal{FL}_{s-r,\lambda,\text{loc}}^p(\Omega)$ for some $s > t$, then $u \in \mathcal{FL}_{s,\lambda,\text{loc}}^p(\Omega)$.*

If in particular $u \in \mathcal{FL}_{t,\lambda,\text{loc}}^p(\Omega)$ solves the equation (103) with $f = 0$ (that is the equation (103) is homogeneous) then $u \in C^\infty(\Omega)$.

Proof Because of Proposition 8 and the assumption (104), we have $b_i(x, D)u \in \mathcal{FL}_{t-r+\varepsilon,\lambda,\text{loc}}^p(\Omega)$ for all $i = 1, \dots, M$. Since $\lambda^{t-r+\varepsilon}$ satisfies (43), Corollary 2 also implies $F(x, b_i(x, D)u) \in \mathcal{FL}_{t-r+\varepsilon,\lambda,\text{loc}}^p(\Omega)$ (cf. Remark 6).

If $t + \varepsilon \geq s$ then $a(x, D)u = -F(x, b_i(x, D)u) + f \in \mathcal{FL}_{s-r,\lambda,\text{loc}}^p(\Omega)$ hence $u \in \mathcal{FL}_{s,\lambda,\text{loc}}^p(\Omega)$ because of the λ -ellipticity of $a(x, D)$.

If on the contrary $t + \varepsilon < s$, applying again the λ -ellipticity of $a(x, D)$, from $a(x, D)u = -F(x, b_i(x, D)u) + f \in \mathcal{FL}_{t-r+\varepsilon,\lambda,\text{loc}}^p(\Omega)$ we derive $u \in \mathcal{FL}_{t+\varepsilon,\lambda,\text{loc}}^p(\Omega)$. In the latter case, we may repeat the same arguments above, where now t is replaced by $t + \varepsilon$.⁵ After that we obtain $F(x, b_i(x, D)u) \in \mathcal{FL}_{t-r+2\varepsilon,\lambda,\text{loc}}^p(\Omega)$

⁵ Let us notice in particular that if the weight function $\lambda^{t-r+\varepsilon}$ satisfies condition (43), then the same is true for any power of λ with exponent greater than $t - r + \varepsilon$, in view of Proposition 5.

and, provided that $t + 2\varepsilon < s$, $u \in \mathcal{FL}_{t+2\varepsilon,\lambda,\text{loc}}^p(\Omega)$. It is now clear that the second part of the argument above can be iterated N times, up to get $F(x, b_i(x, D)u) \in \mathcal{FL}_{t-r+N\varepsilon,\lambda,\text{loc}}^p(\Omega)$ with $t + N\varepsilon \geq s$; hence $a(x, D)u = -F(x, b_i(x, D)u) + f \in \mathcal{FL}_{s-r,\lambda,\text{loc}}^p(\Omega)$ implies $u \in \mathcal{FL}_{s,\lambda,\text{loc}}^p(\Omega)$ from the λ -ellipticity of $a(x, D)$.

The second part of the theorem, concerning the case $f = 0$, follows at once from the first one; in this case the argument above can be applied for arbitrarily large s , thus $u \in \bigcap_{s \geq t} \mathcal{FL}_{s,\lambda,\text{loc}}^p(\Omega) \subset C^\infty(\Omega)$.

Remark 16 Let us suppose that the weight function $\lambda = \lambda(\xi)$ fulfils condition (\mathcal{SA}) (respectively condition (\mathcal{G})), besides (97) and (98). Then $\lambda^{t-r+\varepsilon}$ satisfies condition (43) if $t > r + \frac{n}{\nu q} - \varepsilon$ (respectively $t > r + \frac{n}{(1-\delta)\nu q} - \varepsilon$) is assumed.

6.2 Microlocal regularity results

The results presented in this section apply to a class of weight functions which is smaller than the one considered in Sect. 6.1. More precisely here we deal with a continuous function $\lambda : \mathbb{R}^n \rightarrow]0, +\infty[$ which satisfies (\mathcal{SA}) , (\mathcal{SH}) and obeys the following

(\mathcal{PG}) polynomial growth conditions for suitable constants $C \geq 1, 0 < \nu \leq \mu$.

$$\frac{1}{C}(1 + |\xi|)^\nu \leq \lambda(\xi) \leq C(1 + |\xi|)^\mu, \quad \forall \xi \in \mathbb{R}^n. \tag{105}$$

Remark 17 It is known from the previous section that such a function λ also satisfies condition (65). Then it can be shown that (65), together with (105), also implies that λ obeys the slowly varying condition (98).⁶ Thus the class of weight functions considered in this section is a proper subclass of that considered in Sect. 6.1. It is worthy to be noticed that weight functions described in the examples 1, 2, given in Sect. 2.1, are included in the class of weight functions that we are considering here, whereas the multi-quasi-elliptic weight function illustrated in the example 3 does not meet all the assumptions required here, precisely the sub-additivity (\mathcal{SA}) is not satisfied unless the complete polyhedron \mathcal{P} gives rise to a quasi-homogeneous weight function of type (16). Additional examples of weight functions obeying conditions (\mathcal{SA}) , (\mathcal{SH}) and (\mathcal{PG}) are provided by the following

$$\lambda_{r,s}(\xi) = \langle \xi \rangle^s [\log(2 + \langle \xi \rangle)]^r, \quad \text{for } r, s \in]0, +\infty[,$$

which were studied by Triebel [34] (see also [13]), or even by such functions as

$$\langle \xi \rangle_{\mu,\nu}^2 = 1 + \sum_{j=1}^n |\xi_j|^{\mu_j} [\log(2 + |\xi_j|)]^{\nu_j},$$

for $\mu = (\mu_1, \dots, \mu_n), \nu = (\nu_1, \dots, \nu_n) \in]0, +\infty[^n$,

⁶ More precisely, from Sect. 6.1 we know that conditions (98) and the second inequality in (105) are equivalent under the assumptions (65) and (97).

or

$$\Lambda_{s,\mathcal{P}}(\xi) = \langle \xi \rangle^s + \log(\lambda_{\mathcal{P}}(\xi)), \quad \text{for } s \in]0, +\infty[,$$

being $\lambda_{\mathcal{P}}(\xi)$ the multi-quasi-elliptic weight associated to a complete polyhedron \mathcal{P} , as it was introduced in Example 3 of Sect. 2.1 [see (20)].

In order to take advantage of the slowly varying condition (98) (which allows in particular the symbolic calculus for smooth classes $S'_{\rho,\lambda}(\Omega)$, see Sect. 6), it is convenient to introduce here another family of neighborhoods of an arbitrary set X (in the frequency space \mathbb{R}^n_ξ), associated to the weight function λ , besides the $[\lambda]$ -neighborhoods $X_{[\varepsilon\lambda]}$ already defined as in (66). For arbitrary $X \subset \mathbb{R}^n$ and $\varepsilon > 0$ we set

$$X_{\varepsilon\lambda} := \bigcup_{\xi_0 \in X} \left\{ \xi \in \mathbb{R}^n : |\xi - \xi_0| < \varepsilon\lambda(\xi_0)^{1/\mu} \right\}, \quad (106)$$

where $\mu > 0$ is the same exponent involved in (105) (hence in (98) according to Remark 17); we will refer to the set $X_{\varepsilon\lambda}$ as the λ -neighborhood of X of size ε .

In the following for an open set $\Omega \subset \mathbb{R}^n$ and $x_0 \in \Omega$, we also set for short $X_{\varepsilon\lambda}(x_0) := B_\varepsilon(x_0) \times X_{\varepsilon\lambda}$, where $B_\varepsilon(x_0)$ denotes the open ball in Ω centered at x_0 with radius ε .

Compared to the case of $[\lambda]$ -neighborhoods of a set X , to define the corresponding λ -neighborhoods the weight function λ is replaced by the Euclidean norm, as the measure of the distance from points in $X_{\varepsilon\lambda}$ to points in X . This reflects into a slightly different behavior of λ -neighborhoods: it is clear (just from the definition) that for $\varepsilon > 0$ arbitrarily small the set $X_{\varepsilon\lambda}$ is never empty (unless $X = \emptyset$), cf. Remark 13; it is also clear that $X_{\varepsilon\lambda}$ is open, for it is the union of a family of open balls in \mathbb{R}^n (centered at points of X).

The same set inclusions as given in Lemma 2 remain true also when the $[\lambda]$ -neighborhoods of a set are replaced by the λ -neighborhoods, see [28], [16] for the proof.

Lemma 3 *Given $\varepsilon > 0$, there exists $0 < \varepsilon' < \varepsilon$ such that for every $X \subset \mathbb{R}^n$*

1. $(X_{\varepsilon'\lambda})_{\varepsilon'\lambda} \subset X_{\varepsilon\lambda}$;
2. $(\mathbb{R}^n \setminus X_{\varepsilon\lambda})_{\varepsilon'\lambda} \subset \mathbb{R}^n \setminus X_{\varepsilon'\lambda}$.

A significant relation between $[\lambda]$ - and λ -neighborhoods is established by the next two results.

Lemma 4 *Let $c > 0$ be arbitrarily fixed. For every $\varepsilon > 0$ there exists $0 < \varepsilon' < \varepsilon$ such that the set inclusion*

$$(X \cap \{\lambda(\xi) > c/\varepsilon'\})_{\varepsilon'\lambda} \subset X_{[\varepsilon\lambda]} \cap \{\lambda(\xi) > c/\varepsilon\} \quad (107)$$

holds true for every $X \subset \mathbb{R}^n$.

Proof Let $0 < \varepsilon' < \min\{1, \varepsilon\}$ be such that $X \cap \{\lambda(\xi) > c/\varepsilon'\}$ be nonempty and take an arbitrary $\xi \in (X \cap \{\lambda(\xi) > c/\varepsilon'\})_{\varepsilon'\lambda}$; ⁷ then there exists some $\xi_0 \in X$ such that

$$|\xi - \xi_0| < \varepsilon'\lambda(\xi_0)^{1/\mu} \quad \text{and} \quad \lambda(\xi_0) > c/\varepsilon'. \tag{108}$$

From (105) and (108) we get

$$\begin{aligned} \lambda(\xi - \xi_0) &\leq C(1 + |\xi - \xi_0|)^\mu \leq C2^{\mu-1}(1 + |\xi - \xi_0|^\mu) \\ &< C2^{\mu-1}(1 + \varepsilon'^\mu\lambda(\xi_0)) < C2^{\mu-1}(\varepsilon'/c\lambda(\xi_0) + \varepsilon'^\mu\lambda(\xi_0)) \\ &< C2^{\mu-1}\varepsilon'(1/c + 1)\lambda(\xi_0), \end{aligned} \tag{109}$$

hence $\lambda(\xi - \xi_0) < \varepsilon\lambda(\xi_0)$ provided that ε' is such that

$$C2^{\mu-1}\varepsilon'(1/c + 1) < \varepsilon.$$

Thus $\xi \in X_{[\varepsilon\lambda]}$ provided that $0 < \varepsilon' < \min\left\{1, \frac{\varepsilon}{C2^{\mu-1}(1/c+1)}\right\}$.

Let us now prove that $\lambda(\xi) > c/\varepsilon$ up to a further shrinking of ε' . We use again conditions (\mathcal{SA}) , (\mathcal{SH}) , (\mathcal{PG}) and (109) to find

$$\begin{aligned} \lambda(\xi) &\geq 1/C\lambda(\xi_0) - \lambda(\xi - \xi_0) \geq 1/C\lambda(\xi_0) - C(1 + |\xi - \xi_0|)^\mu \\ &\geq 1/C\lambda(\xi_0) - C2^{\mu-1}(1 + |\xi - \xi_0|^\mu) > 1/C\lambda(\xi_0) - C2^{\mu-1}(1 + \varepsilon'^\mu\lambda(\xi_0)) \\ &= (1/C - C2^{\mu-1}\varepsilon'^\mu)\lambda(\xi_0) - C2^{\mu-1}, \end{aligned}$$

from which we deduce, using also (108),

$$\lambda(\xi) > \frac{1}{2C}\lambda(\xi_0) - C2^{\mu-1} > \frac{c}{2C\varepsilon'} - C2^{\mu-1} > \frac{c}{4C\varepsilon'} > \frac{c}{\varepsilon},$$

provided that $\varepsilon' > 0$ is chosen such that

$$\varepsilon' < \min\left\{\frac{1}{2C2^{\mu-1}}, \frac{c}{2^{\mu+1}C^2}, \frac{\varepsilon}{4C}\right\}.$$

This ends the proof that $\xi \in X_{[\varepsilon\lambda]} \cap \{\lambda(\xi) > c/\varepsilon\}$.

Remark 18 If X is bounded, the set $X \cap \{\lambda(\xi) > c/\varepsilon'\}$ (hence the neighborhood $(X \cap \{\lambda(\xi) > c/\varepsilon'\})_{\varepsilon'\lambda}$) is empty for $\varepsilon' > 0$ sufficiently small, thus the inclusion (107) becomes trivial. However, thanks to (107), this never occurs when X is unbounded; in such a case the set $X \cap \{\lambda(\xi) > c/\varepsilon'\}$ is nonempty for arbitrarily small $\varepsilon' > 0$, since λ is unbounded on X as a consequence of the left inequality in (105). This yields in particular that, for an unbounded set X the $[\lambda]$ -neighborhood $X_{[\varepsilon\lambda]}$ is nonempty with size $\varepsilon > 0$ arbitrarily small, cf. Remark 13.

⁷ If X is unbounded then $X \cap \{\lambda(\xi) > c/\varepsilon'\} \neq \emptyset$ for $\varepsilon' > 0$ arbitrarily small, because of the left inequality of (105).

Corollary 4 For every $\varepsilon > 0$ there exists $0 < \varepsilon' < \varepsilon$ such that for all $X \subset \mathbb{R}^n$

$$(X_{[\varepsilon'\lambda]})_{\varepsilon'\lambda} \subset X_{[\varepsilon\lambda]}. \quad (110)$$

Proof In view of Lemma 2 we first notice that for arbitrary $\varepsilon > 0$ we may find $0 < \varepsilon^* < \varepsilon$ sufficiently small such that

$$(X_{[\varepsilon^*\lambda]})_{[\varepsilon^*\lambda]} \subset X_{[\varepsilon\lambda]}.$$

Then combining the results of Lemmas 2 and 4, with $X_{[\varepsilon^*\lambda]}$ instead of X , another $0 < \varepsilon' < \varepsilon^*$ sufficiently small can be chosen such that

$$\begin{aligned} (X_{[\varepsilon'\lambda]})_{\varepsilon'\lambda} &\equiv (X_{[\varepsilon'\lambda]} \cap \{\lambda(\xi) > \hat{c}/\varepsilon'\})_{\varepsilon'\lambda} \\ &\subset (X_{[\varepsilon^*\lambda]} \cap \{\lambda(\xi) > \hat{c}/\varepsilon'\})_{\varepsilon'\lambda} \subset (X_{[\varepsilon^*\lambda]})_{[\varepsilon^*\lambda]} \subset X_{[\varepsilon\lambda]}, \end{aligned}$$

where $\hat{c} > 0$ is given in Lemma 2. The proof is complete.

In order to perform the subsequent analysis, the next technical lemma will be useful; for its proof, the reader is addressed to [28, Lemma 1.10], see also [16, Lemma 1].

Proposition 10 For arbitrary $\varepsilon > 0$ and $X \subset \mathbb{R}^n$ there exists a symbol $\sigma = \sigma(\xi) \in S_\lambda^0$ such that $\text{supp } \sigma \subset X_{\varepsilon\lambda}$ and $\sigma(\xi) = 1$ if $\xi \in X_{\varepsilon'\lambda}$, for a suitable $\varepsilon' > 0$, with $0 < \varepsilon' < \varepsilon$, depending only on ε and λ . Moreover for every $x_0 \in \Omega$, where $\Omega \subset \mathbb{R}^n$ is an open set, there exists a symbol $\tau_0(x, \xi) \in S_\lambda^0(\Omega)$ such that $\text{supp } \tau_0 \subset X_{\varepsilon\lambda}(x_0)$ and $\tau_0(x, \xi) = 1$, for $(x, \xi) \in X_{\varepsilon^*\lambda}(x_0)$, with a suitable ε^* satisfying $0 < \varepsilon^* < \varepsilon$.

Remark 19 As an application of Corollary 4, one can easily see that a statement similar to Proposition 10 also holds when λ -neighborhoods are replaced with the corresponding $[\lambda]$ -neighborhoods; indeed for arbitrary $X \subset \mathbb{R}^n$ and $\varepsilon > 0$, take $0 < \tilde{\varepsilon} < \varepsilon$ such that $(X_{[\tilde{\varepsilon}\lambda]})_{\tilde{\varepsilon}\lambda} \subset X_{[\varepsilon\lambda]}$ and apply the result of Proposition 10, where X is replaced by $X_{[\tilde{\varepsilon}\lambda]}$. Then some numbers $0 < \varepsilon'' < \varepsilon' < \tilde{\varepsilon}$ and a symbol $\sigma = \sigma(\xi) \in S_\lambda^0$ exist such that $\text{supp } \sigma \subset (X_{[\tilde{\varepsilon}\lambda]})_{\varepsilon'\lambda} \subset (X_{[\tilde{\varepsilon}\lambda]})_{\tilde{\varepsilon}\lambda} \subset X_{[\varepsilon\lambda]}$ and $\sigma \equiv 1$ on $(X_{[\tilde{\varepsilon}\lambda]})_{\varepsilon''\lambda}$ (hence on $X_{[\tilde{\varepsilon}\lambda]}$). As for the construction of a counterpart of the variable coefficients symbol $\tau_0(x, \xi) \in S_\lambda^0(\Omega)$ in the second part of the statement above, it comes from the use of the symbol $\sigma(\xi)$ by following the same lines as in Proposition 10, see [16, Lemma 1].

Definition 8 Let us consider a symbol $a(x, \xi) \in S_{\rho, \lambda}^r(\Omega)$, $x_0 \in \Omega$ and $X \subset \mathbb{R}^n$. We say that $a(x, \xi)$ (or the corresponding pseudodifferential operator) is microlocally $[\lambda]$ -elliptic in X at point x_0 , writing $a(x, \xi) \in \text{mce}_{r, [\lambda]} X(x_0)$, if there exist constants $c_0 > 0$ and $\varepsilon > 0$ sufficiently small such that

$$|a(x_0, \xi)| \geq c_0 \lambda(\xi)^r, \quad \text{for } \xi \in X_{[\varepsilon\lambda]}. \quad (111)$$

Remark 20 Let us remark that in the above definition we do not explicitly require that frequencies ξ , for which (111) holds true, are larger than some positive constant (that is usual when defining an ellipticity condition, cf. (100)); indeed, because of Lemma 2,

$\xi \in X_{[\varepsilon\lambda]}$ yields $\lambda(\xi) > \hat{c}/\varepsilon$ and, for sufficiently small $\varepsilon > 0$, the latter turns out to be a largeness condition on ξ , in view of the polynomial growth condition $(\mathcal{P}\mathcal{G})$.

Let us recall the following notion, providing a microlocal counterpart of the notion of regularizing symbol

Definition 9 We say that a symbol $a(x, \xi) \in S^r_{\rho, \lambda}(\Omega)$ is rapidly decreasing in $\Theta \subset \Omega \times \mathbb{R}^n$ if there exists $a_0(x, \xi) \in S^r_{\rho, \lambda}(\Omega)$ such that $a(x, \xi) - a_0(x, \xi) \in S^{-\infty}(\Omega)$ and $a_0(x, \xi) = 0$ in Θ .

The following notion is a natural substitute of that of *characteristic set* of a symbol, in the absence of any homogeneity property.

Definition 10 We define the characteristic filter of $a(x, \xi) \in S^r_{\rho, \lambda}(\Omega)$ at a point $x_0 \in \Omega$ to be the set

$$\Sigma_{[\lambda], x_0} a := \{X \subset \mathbb{R}^n : a(x, \xi) \in \text{mce}_{r, [\lambda]}(\mathbb{R}^n \setminus X)(x_0)\}. \tag{112}$$

Using Lemma 2, it is easy to check that $\Sigma_{[\lambda], x_0} a$ is a $[\lambda]$ -filter.

The reader is addressed to [18, 28] where analogous notions as above are stated in a more general setting.

Arguing on the properties of λ -neighborhoods of a set and the slowly varying condition (98) as in the proof of [18, Lemma 4.3], one can prove that $a(x, \xi) \in S^s_{\rho, \lambda}(\Omega)$ is microlocally $[\lambda]$ -elliptic in X at point x_0 if and only if

$$|a(x, \xi)| \geq c^* \lambda(\xi)^r, \quad \text{for } (x, \xi) \in B_{\tilde{\varepsilon}}(x_0) \times (X_{[\tilde{\varepsilon}\lambda]})_{\tilde{\varepsilon}\lambda}, \tag{113}$$

for suitable constants $c^* > 0$ and sufficiently small $\tilde{\varepsilon} > 0$.

Then following the same lines of the proof of [18, Theorem 4.6] one can prove the following

Proposition 11 *For every symbol $a(x, \xi) \in S^r_{\rho, \lambda}(\Omega)$ microlocally $[\lambda]$ -elliptic in $\{x_0\} \times X$, there exists a symbol $b(x, \xi) \in S^{-r}_{\rho, \lambda}(\Omega)$ such that the associated operator $b(x, D)$ is properly supported and*

$$b(x, D)a(x, D) = \text{Id} + c(x, D), \tag{114}$$

where $c(x, \xi) \in S^0_{\rho, \lambda}(\Omega)$ is rapidly decreasing in $B_{\tilde{\varepsilon}}(x_0) \times (X_{[\tilde{\varepsilon}\lambda]})_{\tilde{\varepsilon}\lambda}$ for a suitable small $\tilde{\varepsilon} > 0$.

For $s \in \mathbb{R}$, $p \in [1, +\infty]$, U open neighborhood of $x_0 \in \mathbb{R}^n$ and $X \subset \mathbb{R}^n$ given, let $\mathcal{FL}^p_{s, \lambda, \text{loc}}(U)$ and $\mathcal{FL}^p_{s, \lambda, \text{mcl}}(x_0 \times X)$ denote the local and microlocal Fourier Lebesgue classes corresponding to the weight function λ^s , according to Definitions 3, 5. Agreeing with these notations, we denote by $\Xi_{[\lambda], \mathcal{FL}^p_{s, \lambda, x_0}}$ the related $[\lambda]$ -filter of Fourier Lebesgue singularities. By resorting to Proposition 10 and arguing similarly as in the proof of [17, Proposition 4.5] and [18, Proposition 4.10], we are able to prove the following characterization of microlocal Fourier Lebesgue spaces.

Proposition 12 *Let $x_0 \in \Omega$, $\Omega \subset \mathbb{R}^n$ open set, and $X \subset \mathbb{R}^n$ be given. A distribution $u \in \mathcal{D}'(\Omega)$ belongs to $\mathcal{FL}_{s,\lambda,\text{mcl}}^p(x_0 \times X)$ if and only if one of the following two equivalent conditions is satisfied:*

- (i) *there exist constants $0 < \varepsilon' < \varepsilon$ sufficiently small and $\phi \in C_0^\infty(\Omega)$, with $\phi(x_0) \neq 0$, such that*

$$\sigma(D)(\phi u) \in \mathcal{FL}_{s,\lambda}^p(\mathbb{R}^n), \quad (115)$$

where $\sigma = \sigma(\xi) \in S_\lambda^0$ is some symbol satisfying $\text{supp } \sigma \subset X_{[\varepsilon\lambda]}$ and $\sigma \equiv 1$ on $X_{[\varepsilon'\lambda]}$;

- (ii) *There exist an operator $\tau(x, D) \in \text{Op } S_\lambda^0(\Omega)$ microlocally $[\lambda]$ -elliptic in $\{x_0\} \times X$ such that*

$$\tau(x, D)u \in \mathcal{FL}_{s,\lambda,\text{loc}}^p(\Omega). \quad (116)$$

Following similar arguments to those in [17, Propositions 5.1, 5.2] we give the following results.

Proposition 13 *Let $s \in \mathbb{R}$, $r > 0$, $x_0 \in \Omega$, $X \subset \mathbb{R}^n$, $a(x, D) \in \text{Op}S_{\rho,\lambda}^r(\Omega)$ be given. Then for $p \in [1, \infty]$ and $u \in \text{mcl}\mathcal{FL}_{s,\lambda}^p(x_0 \times X)$ one has $a(x, D)u \in \text{mcl}\mathcal{FL}_{s-r,\lambda}^p(x_0 \times X)$.*

Proposition 14 *For $s \in \mathbb{R}$, $r > 0$, $x_0 \in \Omega$, $X \subset \mathbb{R}^n$, let $a(x, D) \in \text{Op}S_{\rho,\Delta}^r(\Omega)$ be microlocally $[\lambda]$ -elliptic in $\{x_0\} \times X$. Then for every $p \in [1, \infty]$ and $u \in \mathcal{D}'(\Omega)$ such that $a(x, D)u \in \text{mcl}\mathcal{FL}_{s-r,\lambda}^p(x_0 \times X)$ one has $u \in \text{mcl}\mathcal{FL}_{s,\lambda}^p(x_0 \times X)$.*

It is also straightforward to show that the results of Propositions 13, 14 can be restated in terms of the filter of Fourier Lebesgue singularities and characteristic filter of a symbol as follows.

Proposition 15 *Let $s \in \mathbb{R}$, $r > 0$ be arbitrary real numbers, $a(x, D) \in \text{Op } S_{\rho,\lambda}^r(\Omega)$, $x_0 \in \Omega$ and $p \in [1, \infty]$. Then the following inclusions are satisfied for every $u \in \mathcal{D}'(\Omega)$:*

$$\Xi_{[\lambda], \mathcal{FL}_{s-r,\lambda}^p, x_0} a(x, D)u \cap \Sigma_{[\lambda], x_0} a \subset \Xi_{[\lambda], \mathcal{FL}_{s,\lambda}^p, x_0} u \subset \Xi_{[\lambda], \mathcal{FL}_{s-r,\lambda}^p, x_0} a(x, D)u.$$

6.3 Semilinear equations

Gathering the results collected in the preceding Sects. 4 and 6.2, we prove here a result of microlocal regularity in Fourier Lebesgue spaces for solutions to semilinear partial differential equations of type (103) already considered in Sect. 6.1. Throughout this section, we assume that the weight function λ satisfies the conditions (\mathcal{SA}) , (\mathcal{SH}) and (\mathcal{PG}) . Moreover the operators $a(x, D)$, $b_i(x, D)$, for $1 \leq i \leq M$ in (103), are properly supported with symbols $S_\lambda^r(V_{x_0})$ and $S_\lambda^{r-\varepsilon}(V_{x_0})$ on some open bounded neighborhood V_{x_0} of a point x_0 , where $0 < \varepsilon < r$ are given as in Sect. 4; the nonlinear function $F = F(x, \zeta)$, depending as a C^∞ -function on its first argument $x \in V_{x_0}$ and entire

analytic on its second argument $\zeta = (\zeta_i)_{1 \leq i \leq M} \in \mathbb{C}^M$, satisfies the same requirement made in Sect. 3 (see also Remark 15).

The following result was originally proved in [7].

Theorem 2 For $0 < \varepsilon < r$ and $1 \leq p \leq +\infty$ given as above, let τ, \tilde{t}, s be positive real numbers such that

$$\tau + r - \varepsilon \leq \tilde{t} < s \tag{117}$$

and $\lambda^{-\tau} \in L^q(\mathbb{R}^n)$, where q is the conjugate exponent of p . As for the semilinear equation (103), let us assume that the pseudodifferential operator $a(x, D)$ is $[\lambda]$ -microlocally elliptic in $X \subset \mathbb{R}^n$ at the point x_0 and the source term $f = f(x)$ belongs to $\mathcal{FL}_{s-r, \lambda, \text{mcl}}^p(x_0 \times X)$. Then every solution $u \in \mathcal{FL}_{\tilde{t}, \lambda, \text{loc}}^p(x_0)$ to the Eq. (103) with source term f also satisfies

$$u \in \mathcal{FL}_{t, \lambda, \text{mcl}}^p(x_0 \times X), \quad \text{for all } t \leq \min \left\{ s, \tilde{t} + \left(E \left(\frac{\tilde{t} - r - \tau}{\varepsilon} \right) + 2 \right) \varepsilon \right\}, \tag{118}$$

where $E(\theta)$ is the greatest integer less than or equal to $\theta \in \mathbb{R}$.

Proof The proof relies on a bootstrapping argument similar to the one used to prove Proposition 9. So let $u \in \mathcal{FL}_{\tilde{t}, \lambda, \text{loc}}^p(x_0)$ be a solution to equation (103). From Propositions 8 we get

$$b_i(x, D)u \in \mathcal{FL}_{\tilde{t}-r+\varepsilon, \lambda, \text{loc}}^p(x_0), \quad i = 1, \dots, M. \tag{119}$$

In view of the assumptions (117), $\lambda^{-\tau} \in L^q(\mathbb{R}^n)$ and the sub-additivity of λ we may apply the result of Theorem 1 and its consequences stated in Remark 15; with reference to the statement of that theorem, here λ^τ plays the role of the weight functions σ whereas $\lambda^{\tilde{t}-r+\varepsilon}$ plays the role of both the weight functions λ, Λ .⁸ Thus it follows from (119) that

$$F(x, b_i(x, D)u)_{1 \leq i \leq M} \in \mathcal{FL}_{\tilde{t}-r+\varepsilon, \lambda, \text{loc}}^p(x_0). \tag{120}$$

If $s \leq \tilde{t} + \varepsilon$, we derive from (103) that $a(x, D)u \in \mathcal{FL}_{s-r, \lambda, \text{mcl}}^p(x_0 \times X)$ hence $u \in \mathcal{FL}_{s, \lambda, \text{mcl}}^p(x_0 \times X)$ in view of the $[\lambda]$ -microellipticity of $a(x, D)$ in X at point x_0 and Proposition 14 (notice that $\tilde{t} + \varepsilon \leq 2\tilde{t} - r - \tau + 2\varepsilon$ under the assumption (117) then $s \leq \tilde{t} + \varepsilon$ implies $s = \min\{s, 2\tilde{t} - r - \tau + 2\varepsilon\}$); if on the contrary $s > \tilde{t} + \varepsilon$ again from (103) we derive that $a(x, D)u \in \mathcal{FL}_{\tilde{t}-r+\varepsilon, \lambda, \text{mcl}}^p(x_0 \times X)$ hence $u \in \mathcal{FL}_{\tilde{t}+\varepsilon, \lambda, \text{mcl}}^p(x_0 \times X)$ by the same arguments as before. In this latter case, using once again Propositions 8 and 13 we get

$$b_i(x, D)u \in \mathcal{FL}_{\tilde{t}-r+\varepsilon, \lambda, \text{loc}}^p(x_0) \cap \mathcal{FL}_{\tilde{t}-r+2\varepsilon, \lambda, \text{mcl}}^p(x_0 \times X), \quad 1 \leq i \leq M.$$

⁸ Notice in particular that from the sub-additivity of λ and $\lambda^\tau \leq \lambda^{\tilde{t}-r+\varepsilon}$ (following from (117)), we derive that $\lambda^{\tilde{t}-r+\varepsilon}$ satisfies condition (43) with the conjugate exponent of p (that is required to apply Theorem 1).

Now we would like to apply Theorem 1 where the role of the weight functions σ , λ and Λ is covered respectively by λ^τ , $\lambda^{\tilde{t}-r+\varepsilon}$ and $\lambda^{\tilde{t}-r+2\varepsilon}$; the only assumption to check is $\Lambda \leq \lambda^2/\sigma$ which amounts to have that $\tilde{t} \geq \tau + r$ (cf. (76)). If this is the case, Theorem 1 applies to find

$$F(x, b_i(x, D)u) \in \mathcal{FL}_{\tilde{t}-r+\varepsilon, \lambda, \text{loc}}^p(x_0) \cap \mathcal{FL}_{\tilde{t}-r+2\varepsilon, \lambda, \text{mcl}}^p(x_0 \times X)$$

hence from (103) and Proposition 14

$$a(x, D)u \in \mathcal{FL}_{s-r, \lambda, \text{mcl}}^p(x_0 \times X) \Rightarrow u \in \mathcal{FL}_{s, \lambda, \text{mcl}}^p(x_0 \times X), \quad \text{if } s \leq \tilde{t} + 2\varepsilon$$

or

$$a(x, D)u \in \mathcal{FL}_{\tilde{t}-r+2\varepsilon, \lambda, \text{mcl}}^p(x_0 \times X) \Rightarrow u \in \mathcal{FL}_{\tilde{t}+2\varepsilon, \lambda, \text{mcl}}^p(x_0 \times X), \quad \text{otherwise.}$$

In the latter case

$$b_i(x, D)u \in \mathcal{FL}_{\tilde{t}-r+\varepsilon, \lambda, \text{loc}}^p(x_0) \cap \mathcal{FL}_{\tilde{t}-r+3\varepsilon, \lambda, \text{mcl}}^p(x_0 \times X), \quad 1 \leq i \leq M$$

and, provided that $\tilde{t} \geq \tau + r + \varepsilon$, we are still in the position to apply Theorem 1 where λ^τ and $\lambda^{\tilde{t}-r+\varepsilon}$ play again the role of σ and λ , while $\lambda^{\tilde{t}-r+3\varepsilon}$ plays the role of Λ . For $\tilde{t} \geq \tau + r + \varepsilon$ Theorem 1 yields

$$F(x, b_i(x, D)u) \in \mathcal{FL}_{\tilde{t}-r+\varepsilon, \lambda, \text{loc}}^p(x_0) \cap \mathcal{FL}_{\tilde{t}-r+3\varepsilon, \lambda, \text{mcl}}^p(x_0 \times X)$$

and again

$$a(x, D)u \in \mathcal{FL}_{s-r, \lambda, \text{mcl}}^p(x_0 \times X) \Rightarrow u \in \mathcal{FL}_{s, \lambda, \text{mcl}}^p(x_0 \times X), \quad \text{if } s \leq \tilde{t} + 3\varepsilon$$

or

$$a(x, D)u \in \mathcal{FL}_{\tilde{t}-r+3\varepsilon, \lambda, \text{mcl}}^p(x_0 \times X) \Rightarrow u \in \mathcal{FL}_{\tilde{t}+3\varepsilon, \lambda, \text{mcl}}^p(x_0 \times X), \quad \text{otherwise.}$$

By an iteration of the above procedure we find that if the integer $j \geq 0$ is such that

$$\tilde{t} < \tau + r + j\varepsilon \tag{121}$$

then

$$u \in \mathcal{FL}_{\min\{\tilde{t}+(j+1)\varepsilon, s\}, \lambda, \text{mcl}}^p(x_0 \times X).$$

The smallest nonnegative integer j satisfying (121) is $\tilde{j} = E\left(\frac{\tilde{t}-r-\tau}{\varepsilon}\right) + 1$ (from (117) $E\left(\frac{\tilde{t}-\tau-r}{\varepsilon}\right) \geq -1$ follows), hence $\tilde{t} + (\tilde{j} + 1)\varepsilon = \tilde{t} + \left(E\left(\frac{\tilde{t}-r-\tau}{\varepsilon}\right) + 2\right)\varepsilon$. This gives

$$u \in \mathcal{FL}_{\min\{\tilde{t} + (E(\frac{\tilde{t}-r-\tau}{\varepsilon}) + 2)\varepsilon, s\}, \lambda, \text{mcl}}^p(x_0 \times X),$$

which completes the proof.

In terms of the filter of Fourier Lebesgue singularities and characteristic filter of a symbol, the result of Theorem 2 can be restated as follows: for every solution $u \in \mathcal{FL}_{\tilde{t}, \lambda, \text{loc}}^p(x_0)$ to equation (103) one has

$$\Xi_{[\lambda], \mathcal{FL}_{s-r, \lambda}, x_0}^p f \cap \Sigma_{[\lambda], x_0} a \subset \Xi_{[\lambda], \mathcal{FL}_{\tilde{t}, \lambda}, x_0}^p u,$$

for all $t \leq \min\left\{s, \tilde{t} + \left(E\left(\frac{\tilde{t}-r-\tau}{\varepsilon}\right) + 2\right)\varepsilon\right\}$.

Remark 21 Because of the lower estimate of condition $(\mathcal{P}\mathcal{G})$, a sufficient condition for $\lambda^{-\tau} \in L^q(\mathbb{R}^n)$ is $\tau > \frac{n}{vq}$.

6.4 The case of quasi-homogeneous equations

In this section, we deal with pseudodifferential operators whose smooth symbols are associated to a quasi-homogeneous weight as defined in the Example 2 of Section 2.1. We recall that for $M = (m_1, \dots, m_n) \in \mathbb{N}^n$, with $m_* := \min_{1 \leq j \leq n} m_j \geq 1$, the *quasi-homogeneous weight* is defined as

$$\langle \xi \rangle_M := \left(1 + \sum_{j=1}^n \xi_j^{2m_j}\right)^{1/2}. \tag{122}$$

Throughout the rest of this Section, we will make use of the following notations. We set $m^* := \max_{1 \leq j \leq n} m_j$, $\frac{1}{M} := \left(\frac{1}{m_1}, \dots, \frac{1}{m_n}\right)$ and define the *quasi-homogeneous norm* as

$$|\xi|_M^2 := \sum_{j=1}^n \xi_j^{2m_j}. \tag{123}$$

Clearly the usual Euclidean norm $|\xi|$ corresponds to the quasi-homogeneous norm in the case of $M = (1, \dots, 1)$. For every $\alpha \in \mathbb{Z}_+^n$, $\xi \in \mathbb{R}^n$ and $t > 0$ we also set $\langle \alpha, \frac{1}{M} \rangle := \sum_{j=1}^n \frac{\alpha_j}{m_j}$ and $t^{1/M} \xi := (t^{1/m_1} \xi_1, \dots, t^{1/m_n} \xi_n)$. It is worth to notice that, in spite of the terminology, the quasi-homogeneous norm $|\cdot|_M$ is not a norm; instead of the homogeneity and the triangle inequality, required for norms, the quasi-homogeneous norm enjoys the following properties:

(i) *Quasi-Homogeneity*: for all $t > 0$, $\xi \in \mathbb{R}^n$

$$|t^{1/M}\xi|_M = t|\xi|_M;$$

(ii) *Sub-additivity*: a constant $C \geq 1$ depending only on M exists such that

$$|\xi + \eta|_M \leq C(|\xi|_M + |\eta|_M), \quad \forall \xi, \eta \in \mathbb{R}^n.$$

For $R > 0$ and $x_0 \in \mathbb{R}^n$, the M -open ball centered at x_0 with radius R is defined to be the set

$$B_M(x_0; R) := \{x \in \mathbb{R}^n : |x - x_0|_M < R\}.$$

The set

$$\mathbb{S}_M := \{x \in \mathbb{R}^n : |x|_M = 1\} \quad (124)$$

is the *unit M -sphere* (centered at the origin). For further details and properties of quasi-homogeneous norm and weight, we address the reader to [14, 15].

According to the behavior of the weight (122) expressed by the estimates (18), we introduce suitable classes of smooth symbols displaying a decaying behavior of *quasi-homogeneous* type.

Definition 11 Given $r \in \mathbb{R}$, S_M^r will be the class of functions $a(x, \xi) \in C^\infty(\mathbb{R}^{2n})$ such that for all multi-indices $\alpha, \beta \in \mathbb{Z}_+^n$ there exists $C_{\alpha, \beta} > 0$ such that:

$$|\partial_x^\beta \partial_\xi^\alpha a(x, \xi)| \leq C_{\alpha, \beta} \langle \xi \rangle_M^{r - (\alpha, \frac{1}{M})}, \quad \forall x, \xi \in \mathbb{R}^n. \quad (125)$$

If Ω is an arbitrary open subset of \mathbb{R}^n , we denote by $S_M^r(\Omega)$ the local class of functions $a(x, \xi) \in C^\infty(\Omega \times \mathbb{R}^n)$ such that $\phi(x)a(x, \xi) \in S_M^r$ for all $\phi \in C_0^\infty(\Omega)$.

Due to the underlying quasi-homogeneous structure, in the present framework the whole theory of propagation of singularities can be based upon a suitable notion of “conical” set in frequency space adapted to this structure.

Let us recall below some basic notions, see [15] for more details. Later on it is set for short $T^\circ\mathbb{R}^n := \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$.

Definition 12 We say that a set $\Gamma \subset \mathbb{R}^n \setminus \{0\}$ is an M -cone (or is M -conic), if

$$\xi \in \Gamma \Rightarrow t^{1/M}\xi \in \Gamma, \quad \forall t > 0.$$

For $\eta \in \mathbb{R}^n$ and $R > 0$ the set

$$\Gamma_M(\eta; R) := \left\{ t^{1/M}\xi : \xi \in B_M(\eta; R), t > 0 \right\} \cap (\mathbb{R}^n \setminus \{0\}) \quad (126)$$

is M -conic; it is called the M -cone generated by $B_M(\eta; R)$.

Since (122) also belongs to the class of weight functions considered in Sects. 6.1, 6.2, the results considered there, based upon the notion of $[\lambda]$ -filter, could be applied to the quasi-homogeneous setting (that is $\lambda(\xi) = \langle \xi \rangle_M$). The next results of this Section will provide some evidences that these two alternative approaches are essentially equivalent.

Proposition 16 *There exist constants $\hat{c} > 0$ and $\hat{\varepsilon} > 0$ sufficiently small such that for all $0 < \varepsilon \leq \hat{\varepsilon}$ another $0 < \varepsilon' < \varepsilon$ exists such that for all M -conic sets $X \subset \mathbb{R}^n \setminus \{0\}$*

$$X_{[\varepsilon' \langle \cdot \rangle_M]} \subset \bigcup_{\eta \in X \cap \mathbb{S}_M} \Gamma_M(\eta; \varepsilon) \cap \{\langle \xi \rangle_M > \hat{c}/\varepsilon\} \quad (127)$$

and, conversely,

$$\bigcup_{\eta \in X \cap \mathbb{S}_M} \Gamma_M(\eta; \varepsilon') \cap \{\langle \xi \rangle_M > \hat{c}/\varepsilon'\} \subset X_{[\varepsilon \langle \cdot \rangle_M]}. \quad (128)$$

Proof For a M -conic set X and an arbitrary $\tilde{\varepsilon} > 0$, take $\xi \in X_{[\tilde{\varepsilon} \langle \cdot \rangle_M]}$ and let $\eta \in X$ such that

$$\langle \xi - \eta \rangle_M < \tilde{\varepsilon} \langle \eta \rangle_M. \quad (129)$$

Making use of the trivial inequalities

$$1/\sqrt{2}(1 + |\zeta|_M) \leq \langle \zeta \rangle_M \leq 1 + |\zeta|_M, \quad \forall \zeta \in \mathbb{R}^n,$$

(129) implies

$$1 + |\xi - \eta|_M < \sqrt{2}\tilde{\varepsilon}(1 + |\eta|_M),$$

hence

$$|\xi - \eta|_M < \sqrt{2}\tilde{\varepsilon}|\eta|_M \quad (130)$$

provided that $0 < \tilde{\varepsilon} < 1/\sqrt{2}$. Since in particular (129) implies $\eta \neq 0$, because of the quasi-homogeneity of $|\cdot|_M$, condition (130) can be reformulated as follows

$$\begin{aligned} |\xi - \eta|_M < \sqrt{2}\tilde{\varepsilon}|\eta|_M &\Leftrightarrow |\eta|_M^{1/M}(\zeta - \tilde{\eta})|_M = |\eta|_M|\zeta - \tilde{\eta}|_M < \sqrt{2}\tilde{\varepsilon}|\eta|_M \\ &\Leftrightarrow |\zeta - \tilde{\eta}|_M < \sqrt{2}\tilde{\varepsilon}, \end{aligned}$$

where $\zeta := |\eta|_M^{-1/M}\xi$ and $\tilde{\eta} := |\eta|_M^{-1/M}\eta \in \mathbb{S}_M \cap X$ because X is M -conic. The last inequality above means that ζ belongs to the M -open ball centered at $\tilde{\eta}$ with radius $\sqrt{2}\tilde{\varepsilon}$, thus $\xi = |\eta|_M^{1/M}\zeta \in \Gamma_M(\tilde{\eta}; \sqrt{2}\tilde{\varepsilon})$ cf. (126). Since in view of Lemma 2, $\eta \in X_{[\tilde{\varepsilon} \langle \cdot \rangle_M]}$ also implies

$$\langle \eta \rangle_M > \hat{c}/\tilde{\varepsilon},$$

for suitable $\hat{c} > 0$ independent of X and $\tilde{\varepsilon} > 0$, the inclusion (127) follows taking $\hat{\varepsilon} = \frac{1}{\sqrt{2}C}$ and choosing for each $0 < \varepsilon \leq \hat{\varepsilon}$, $0 < \varepsilon' \leq \varepsilon/\sqrt{2}$.

Conversely, let $\xi \in \bigcup_{\eta \in X \cap \mathbb{S}_M} \Gamma_M(\eta; \tilde{\varepsilon}) \cap \{ \langle \xi \rangle_M > \hat{c}/\tilde{\varepsilon} \}$, where $\tilde{\varepsilon} > 0$ is still arbitrary and is chosen sufficiently small such that

$$1 + |\xi|_M \geq \langle \xi \rangle_M > \hat{c}/\tilde{\varepsilon} \Rightarrow |\xi|_M > \hat{c}/\tilde{\varepsilon} - 1 \geq \hat{c}/(2\tilde{\varepsilon}),$$

that is $0 < \tilde{\varepsilon} \leq \hat{c}/2$. By definition (see (126)) there exist $t > 0$ and $\tilde{\eta} \in X \cap \mathbb{S}_M$ such that

$$\xi = t^{1/M} \zeta \quad \text{for some } \zeta \in B_M(\tilde{\eta}; \tilde{\varepsilon}).$$

Therefore, in view of the M -homogeneity, we get

$$|\xi - t^{1/M} \tilde{\eta}|_M = |t^{1/M} \zeta - t^{1/M} \tilde{\eta}|_M = t |\zeta - \tilde{\eta}|_M < \tilde{\varepsilon} t = \tilde{\varepsilon} t |\tilde{\eta}|_M = \tilde{\varepsilon} |t^{1/M} \tilde{\eta}|_M.$$

Since X is M -conic, $\eta := t^{1/M} \tilde{\eta} \in X$; hence with such an $\eta \in X$ we have

$$|\xi - \eta|_M < \tilde{\varepsilon} |\eta|_M. \quad (131)$$

On the other hand, from the sub-additivity (ii), $|\xi|_M > \hat{c}/(2\tilde{\varepsilon})$ and (131) we derive

$$|\eta|_M \geq \frac{1}{C} |\xi|_M - |\xi - \eta|_M \geq \left(\frac{1}{C} - \tilde{\varepsilon} \right) |\xi|_M \geq \frac{1}{2C} |\xi|_M \geq \frac{\hat{c}}{2C\tilde{\varepsilon}}, \quad (132)$$

provided that $0 < \tilde{\varepsilon} \leq \frac{1}{2C}$. Summing up (131), (132) then gives

$$\frac{\hat{c}}{2C} + |\xi - \eta|_M < 2\tilde{\varepsilon} |\eta|_M < 2\tilde{\varepsilon} \langle \eta \rangle_M.$$

Combining the latter inequality with

$$\frac{\hat{c}}{2C} + |\xi - \eta|_M \geq (1 + |\xi - \eta|_M) \min \left\{ 1, \frac{\hat{c}}{2C} \right\} \geq \frac{1}{\sqrt{2}} \min \left\{ 1, \frac{\hat{c}}{2C} \right\} \langle \xi - \eta \rangle_M$$

we finally obtain

$$\langle \xi - \eta \rangle_M < \widehat{C} \tilde{\varepsilon} \langle \eta \rangle_M,$$

with a suitable constant $\widehat{C} > 0$ independent of $\tilde{\varepsilon}$, hence $\xi \in X_{[\widehat{C}\tilde{\varepsilon}\langle \cdot \rangle_M]}$. From the previous argument, we conclude that the second inclusion (128) holds true.

Essentially the result above tells that a $[\langle \cdot \rangle_M]$ -neighborhood of a M -conic set X is made by an arbitrary union of open M -cones “outgoing from points of $X \cap \mathbb{S}_M$, truncated near their vertex”.

Remark 22 It is worthwhile noticing that the quasi-homogeneous symbols considered in Definition 11 are related to the weighted smooth symbols introduced in Section 6 by the following inclusion

$$S_M^r \subset S_{1/m_*, \langle \cdot \rangle_M}^r$$

(a similar inclusion being valid for the corresponding classes of local symbols).

6.5 Example

For $M = (1, 2)$, let us consider in \mathbb{R}^2 the quasi-homogeneous weight function

$$\langle \xi \rangle_M = \left(1 + \xi_1^2 + \xi_2^4\right)^{1/2}. \tag{133}$$

We introduce the following operator

$$P(x, \partial) = x_1 \partial_{x_1} + i \partial_{x_1} - \partial_{x_2}^2. \tag{134}$$

Its symbol $P(x, \xi) = ix_1 \xi_1 - \xi_1 + \xi_2^2$ belongs to the local class $S_M^1(\Omega)$ where $\Omega = \mathbb{R}^2$.

Introducing the M -characteristic set of $P(x, \partial)$ as

$$\text{Char } P = \left\{ (x, \xi) \in \mathbb{R}_x^2 \times \mathbb{R}_\xi^2 \setminus \{0\}, P(x, \xi) = 0 \right\}, \tag{135}$$

we have

$$\text{Char } P = \left\{ (0, x_2, \xi_1, \xi_2); x_2 \in \mathbb{R}, \xi_1 = \xi_2^2, \xi_2 \neq 0 \right\} = \{0\} \times \mathbb{R} \times \bigcap_{0 < k < 1} (\mathbb{R}^2 \setminus X_k),$$

where

$$X_k = \left\{ (\xi_1, \xi_2) \in \mathbb{R}^2; \xi_1 \leq (1 - k)\xi_2^2 \text{ or } \xi_1 \geq \frac{1}{1 - k}\xi_2^2 \right\}, \quad 0 < k < 1. \tag{136}$$

Notice also that $P(x, \xi)$ is *quasi-homogeneous of degree one*, in the sense that

$$P(x, t^{1/M}\xi) = P(x, t\xi_1, t^{1/2}\xi_2) = tP(x, \xi), \quad \forall t > 0. \tag{137}$$

The properties collected above yield that the symbol $P(x, \xi)$ is $\langle \cdot \rangle_M$ -elliptic at every point $x^0 = (x_1^0, x_2^0) \in \mathbb{R}^2$ with $x_1^0 \neq 0$; indeed since $|P|$ does not vanish at each point of the compact set $\{x^0\} \times \mathbb{S}_M$, being $\mathbb{S}_M = \{\eta = (\eta_1, \eta_2) \in \mathbb{R}^2 : \eta_1^2 + \eta_2^4 = 1\}$ the unit M -sphere, by continuity

$$c_0 := \inf_{\eta \in \mathbb{S}_M} |P(x^0, \eta)| > 0.$$

Hence the quasi-homogeneity of P yields for $|\xi|_M \geq 1$:

$$|P(x^0, \xi)| = |\xi|_M |P(x^0, \eta)| \geq c_0 |\xi|_M \geq c_0 / \sqrt{2} \langle \xi \rangle_M,$$

where $\eta := |\xi|_M^{-1/M} \xi \in \mathbb{S}_M$.

By resorting to Proposition 16, we show now that at every point $x^0 = (0, x_2^0)$, with an arbitrary $x_2^0 \in \mathbb{R}$, $P(x, \xi)$ is $[\langle \cdot \rangle_M]$ -microlocally elliptic in any set of the family $\{X_k\}_{0 < k < 1}$ defined by (136). So, let us take arbitrary $x^0 = (0, x_2^0)$ and $0 < k < 1$; since $|P|$ is different from zero and continuous (hence uniformly continuous) on the compact subset $\{x^0\} \times (X_k \cap \mathbb{S}_M)$ of $T^o\mathbb{R}^n \setminus \text{Char} P$, some constants $c_k > 0$ and $0 < \tilde{\varepsilon} < 1$ sufficiently small can be found such that

$$|P(x^0, \eta)| \geq c_k, \quad (138)$$

for η ranging on the covering of $X_k \cap \mathbb{S}_M$ made by the open M -balls $B_M(\tilde{\eta}; \tilde{\varepsilon})$ centered at points $\tilde{\eta}$ of $X_k \cap \mathbb{S}_M$ with radius $\tilde{\varepsilon}$. Take now an arbitrary point $\xi \in \bigcup_{\tilde{\eta} \in X_k \cap \mathbb{S}_M} \Gamma_M(\tilde{\eta}; \tilde{\varepsilon})$ such that $|\xi|_M > c/\tilde{\varepsilon}$ with suitable $c > 0$; then $\tilde{\eta} \in X_k \cap \mathbb{S}_M$ and $t > 0$ exist such that $\xi = t^{1/M} \eta$ for some $\eta \in B_M(\tilde{\eta}; \tilde{\varepsilon})$. Since $|\tilde{\eta}|_M = 1$, we may take $\tilde{\varepsilon}$ so small that $|\eta|_M \leq \hat{c}$ for some positive constant \hat{c} (independent of η and $\tilde{\eta}$). Exploiting again the quasi-homogeneity of P and the quasi-norm $|\cdot|_M$ we get

$$|P(x^0, \xi)| = t |P(x^0, \eta)| \geq c_k t = \frac{c_k}{\hat{c}} t |\eta|_M = \frac{c_k}{\hat{c}} |\xi|_M \geq \tilde{c}_k \langle \xi \rangle_M,$$

with suitable $\tilde{c}_k > 0$. Since the set X_k is M -conic, in view of Proposition 16 there exists $0 < \varepsilon' < \tilde{\varepsilon}$ (up to shrink $\tilde{\varepsilon}$ if necessary) such that

$$(X_k)_{[\varepsilon' \langle \cdot \rangle_M]} \subset \bigcup_{\tilde{\eta} \in X_k \cap \mathbb{S}_M} \Gamma_M(\tilde{\eta}; \tilde{\varepsilon}) \cap \{|\xi|_M > c/\tilde{\varepsilon}\}.$$

This shows that P is microlocally $[\langle \cdot \rangle_M]$ -elliptic in X_k at the point $x^0 = (0, x_2^0)$.

Since $P(x, \xi) \in S_M^1(\mathbb{R}^2)$ and in view of Remark 22, the results of propagation of Fourier Lebesgue singularities for linear and semilinear equations, collected in the preceding Sects. 6.2, 6.3, can be applied to the operator $P(x, \partial)$.

Let $u \in \mathcal{D}'(\mathbb{R}^2)$ be a solution to the linear equation

$$P(x, \partial)u = f(x), \quad (139)$$

with a given forcing term f . Applying to (139) the result of Proposition 15 (with $r = 1$), we obtain at once that the following inclusions

$$\Xi_{[\langle \cdot \rangle_M], \mathcal{F}L_{s-1, M}^p, x^0} f \cap \Sigma_{[\langle \cdot \rangle_M], x^0} P \subset \Xi_{[\langle \cdot \rangle_M], \mathcal{F}L_{s, M}^p, x^0} u \subset \Xi_{[\langle \cdot \rangle_M], \mathcal{F}L_{s-1, M}^p, x^0} f$$

hold true for all $s \in \mathbb{R}$ and $p \in [1, +\infty]$.

Consider now the following semilinear equation

$$P(x, \partial)u + F(x, u, \partial_{x_2}u) = f(x), \tag{140}$$

where $F = F(x, \zeta)$ is a nonlinear function of $x = (x_1, x_2)$ and $\zeta = (\zeta_1, \zeta_2)$ fulfilling the regularity assumptions stated in Theorem 2, and $f = f(x)$ some given forcing term. With respect to the quasi-homogeneous weight (133), the order of derivatives of the unknown function u involved in the nonlinearity in (140) is easily seen to be $\leq 1/2$ (that is such derivatives are properly supported operators in S_M^l with order $l \leq 1/2$). Then we may apply to (140) the result of Theorem 2 (with $r = 1$ and $\varepsilon = 1/2$) to prove the following statement.

Proposition 17 *Given $x^0 = (x_1^0, x_2^0) \in \mathbb{R}^2$, $p \in [1, +\infty]$ and $s > \tilde{t} > \frac{2}{q} + \frac{1}{2}$, with $\frac{1}{p} + \frac{1}{q} = 1$, let $u \in \mathcal{FL}_{\tilde{t}, M, \text{loc}}^p(x^0)$ be a solution to (140).*

(a) *If $2\tilde{t} - 2 - \frac{4}{q} \notin \mathbb{Z}$ then*

$$\Xi_{[\langle \cdot \rangle_M], \mathcal{FL}_{s-1, M, x^0}^p} f \cap \Sigma_{[\langle \cdot \rangle_M], x^0} P \subset \Xi_{[\langle \cdot \rangle_M], \mathcal{FL}_{\tilde{t}, M, x^0}^p} u, \tag{141}$$

holds true for all $t \leq \min \left\{ s, \tilde{t} + 1 + \frac{1}{2}E \left(2\tilde{t} - 2 - \frac{4}{q} \right) \right\}$;

(b) *if $2\tilde{t} - 2 - \frac{4}{q} \in \mathbb{Z}$ then the previous inclusion (141) holds true for all $t \leq \min \left\{ s, \tilde{t} + \frac{1}{2} + \frac{1}{2}E \left(2\tilde{t} - 2 - \frac{4}{q} \right) \right\}$.*

Proof For any $\tilde{t} > \frac{2}{q} + \frac{1}{2}$, let $\tau > 0$ be chosen such that $\tilde{t} - \frac{1}{2} \geq \tau > \frac{2}{q}$, then Theorem 2 can be directly applied to equation (140) (where, according to the observations above, it is set $r = 1$, $\varepsilon = 1/2$, and we also make use of Remark 21 for $\lambda = \langle \cdot \rangle_M$ and τ as above) to find that inclusion (141) holds true for all $t \leq \min \left\{ s, \tilde{t} + 1 + \frac{1}{2}E(2\tilde{t} - 2 - 2\tau) \right\}$. To conclude, it is enough to observe that for $2\tilde{t} - 2 - \frac{4}{q} \notin \mathbb{Z}$ we can take τ sufficiently close to $\frac{2}{q}$ to have⁹

$$E(2\tilde{t} - 2 - 2\tau) = E \left(2\tilde{t} - 2 - \frac{4}{q} \right);$$

this proves the statement *a*.

If, on the contrary, one has $2\tilde{t} - 2 - \frac{4}{q} \in \mathbb{Z}$ then $2\tilde{t} - 2 - 2\tau < 2\tilde{t} - 2 - \frac{4}{q} = E(2\tilde{t} - 2 - \frac{4}{q})$ whenever τ is taken as above; then for τ sufficiently close to $\frac{2}{q}$ we get

$$E(2\tilde{t} - 2 - 2\tau) = E \left(2\tilde{t} - 2 - \frac{4}{q} \right) - 1$$

which gives the statement *b*.

⁹ According to the result of Theorem 2, this corresponds to the best possible range for t .

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