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A simple proof of Kotake–Narasimhan theorem in some classes of ultradifferentiable functions

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Abstract We give a simple proof of a general theorem of Kotake–Narasimhan for elliptic operators in the setting of ultradifferentiable functions in the sense of Braun, Meise and Taylor. We follow the ideas of Komatsu. Based on an example of Métivier, we also show that the ellipticity is a necessary condition for the theorem to be true.

Keywords Iterates of an operator · Theorem of Kotake–Narasimhan · Ultradifferentiable functions

Mathematics Subject Classification 46E10 · 46F05

1 Introduction and main result

The problem of iterates began when Komatsu [11] in 1960 characterized analytic functions f in terms of the behaviour of successive iterates $P(D)^j f$ of the function f for a linear partial differential elliptic operator $P(D)$ with constant coefficients. He proved that a C^∞ function f is real analytic in Ω if and only if for every compact set

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$K \subset\subset \Omega$ there is a constant $C > 0$ such that

$$\|P(D)^j f\|_{L^2(K)} \leq C^{j+1}(j!)^m, \quad \forall j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},$$

where m is the order of the operator and $\|\cdot\|_{L^2(K)}$ is the L^2 norm on K. This result was generalized to the case of elliptic linear partial differential operators $P(x, D)$ with real analytic coefficients in Ω by Kotake and Narasimhan [14], and is known as "the Theorem of Kotake–Narasimhan". Komatsu [13] gave a simpler proof. Similar results have been previously considered by Nelson [22]. Later these results were extended to Gevrey functions by Newberger and Zielezny [23] in the case of operators with constant coefficients. Lions and Magenes [20] considered the case of Denjoy-Carleman classes of Roumieu type for elliptic linear partial differential operators $P(x, D)$ with variable coefficients in the same Roumieu class, and Oldrich [24] treated the case of Denjoy–Carleman classes of Beurling type with some loss of regularity with respect to the coefficients. Métivier [21] proved that the result of Lions and Magenes for Gevrey classes is true only for elliptic operators in the case of real analytic coefficients. Spaces of Gevrey type given by the iterates of a differential operator are called *generalized Gevrey classes* and were used by Langenbruch [16–19] for different purposes.

More recently, Juan-Huguet [9] extended the results of Komatsu [11], Newberger and Zielezny [23] and Métivier [21] to the setting of non-quasianalytic classes in the sense of Braun, Meise and Taylor [6] for operators with constant coefficients. In [9], Juan-Huguet introduced the generalized spaces of ultradifferentiable functions $\mathcal{E}_*^P(\Omega)$ on an open subset Ω of $\mathbb{R}^n$ for a fixed linear partial differential operator P with constant coefficients, and proved that these spaces are complete if and only if P is hypoelliptic. Moreover, Juan-Huguet showed that, in this case, the spaces are nuclear. Later, the same author in [10] established a Paley-Wiener theorem for the classes $\mathcal{E}_*^P(\Omega)$, again under the hypothesis of the hypoellipticity of P.

We used in [3] and [2] the results of Juan Huguet to define and characterize a wave front set for the generalized spaces of ultradifferentiable functions $\mathcal{E}_*^P(\Omega)$ when P is hypoelliptic. In particular, for P elliptic we obtain a microlocal version of the theorem of Kotake and Narasimhan. In order to remove the assumption on the hypoellipticity of the operator, we considered in [1] a different setting of ultradifferentiable functions, following the ideas of [4].

Here, we give a simple proof of the theorem of Kotake–Narasimhan [14, Theorem 1] in the setting of ultradifferentiable functions as introduced by Braun, Meise and Taylor [6] for quasianalytic or non-quasianalytic weight functions. We will consider subadditive weight functions, or more generally, weight functions which satisfy condition (α_0), that we define later (see for example Petzsche and Vogt [25, p. 19] or Fernández and Galbis [7, p. 401]). We follow the lines of Komatsu [13].

Let us recall from [6] the definitions of weight functions ω and of the spaces of ultradifferentiable functions of Beurling and Roumieu type:

Definition 1.1 A non-quasianalytic *weight function* is a continuous increasing function $\omega : [0, +\infty[\to [0, +\infty[$ with the following properties:

(α) $\exists\, L > 0$ s.t. $\omega(2t) \leq L(\omega(t) + 1) \quad \forall t \geq 0;$
(β) $\int_1^{+\infty} \frac{\omega(t)}{t^2} dt < +\infty,$

(γ) $\log(t) = o(\omega(t))$ as $t \to +\infty$;
(δ) $\varphi_\omega : t \mapsto \omega(e^t)$ is convex.

We say that ω is *quasianalytic* if, instead of (β) it satisfies:

(β') $\displaystyle\int_1^{+\infty} \frac{\omega(t)}{t^2} dt = +\infty.$

We will consider also the following property:

(α_0) $\exists\, C > 0, \quad \exists\, t_0 > 0, \quad \forall\, \lambda \geq 1, \quad \forall\, t \geq t_0 : \omega(\lambda t) \leq \lambda C \omega(t).$

The property (α_0) above is used in [25, p. 19] and [7, p. 401], for instance. Moreover, a weight function ω satisfies (α_0) if and only if it is equivalent to a *subadditive* (or *concave*) weight function. In the following, we will assume that our weight functions satisfy (α_0), and there is no loss of generality to consider only subadditive weights. This condition should be compared with [20, (1.4), p. 3] or [24, (2), p. 1], which is a similar condition for Denjoy–Carleman classes.

Normally, we will denote φ_ω simply by φ.

For a weight function ω we define $\overline{\omega} : \mathbb{C}^n \to [0, +\infty[$ by $\overline{\omega}(z) := \omega(|z|)$ and again we denote this function by ω.

The *Young conjugate* $\varphi^* : [0, +\infty[\to [0, +\infty[$ is defined by

$$\varphi^*(s) := \sup_{t \geq 0}\{st - \varphi(t)\}.$$

There is no loss of generality to assume that ω vanishes on $[0, 1]$. Then φ^* has only non-negative values, it is convex, $\varphi^*(t)/t$ is increasing and tends to ∞ as $t \to \infty$, and $\varphi^{**} = \varphi$.

Example 1.2 The following functions are, after a change in some interval $[0, M]$, examples of weight functions:

(i) $\omega(t) = t^d$ for $0 < d < 1$.
(ii) $\omega(t) = (\log(1 + t))^s$, $s > 1$.
(iii) $\omega(t) = t(\log(e + t))^{-\beta}$, $\beta > 1$.
(iv) $\omega(t) = \exp(\beta(\log(1 + t))^\alpha)$, $0 < \alpha < 1$.

In what follows, Ω denotes an arbitrary subset of $\mathbb{R}^n$ and $K \subset\subset \Omega$ means that K is a compact subset in Ω.

Definition 1.3 Let ω be a weight function. For a compact subset K in $\mathbb{R}^n$ which coincides with the closure of its interior and $\lambda > 0$, we define the seminorm

$$p_{K,\lambda}(f) := \sup_{\alpha \in \mathbb{N}_0^n} \sup_{x \in K} \left| f^{(\alpha)}(x) \right| \exp\left(-\lambda \varphi^* \left(\frac{|\alpha|}{\lambda} \right) \right),$$

where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and set

$$\mathcal{E}_\omega^\lambda(K) := \left\{ f \in C^\infty(K) : p_{K,\lambda}(f) < \infty \right\},$$

which is a Banach space endowed with the $p_{K,\lambda}(\cdot)$-topology.

For an open subset Ω in $\mathbb{R}^n$, the class of *ω-ultradifferentiable functions of Beurling type* is defined by

$$\mathcal{E}_{(\omega)}(\Omega) := \left\{ f \in C^\infty(\Omega) : p_{K,\lambda}(f) < \infty, \text{ for every } K \subset\subset \Omega \text{ and every } \lambda > 0 \right\}.$$

The topology of this space is

$$\mathcal{E}_{(\omega)}(\Omega) = \underset{\underset{K\subset\subset\Omega}{\longleftarrow}}{\text{proj}}\ \underset{\underset{\lambda>0}{\longleftarrow}}{\text{proj}}\, \mathcal{E}^{\lambda}_{\omega}(K),$$

and one can show that $\mathcal{E}_{(\omega)}(\Omega)$ is a Fréchet space.

For an open subset Ω in $\mathbb{R}^n$, the class of *ω-ultradifferentiable functions of Roumieu type* is defined by:

$$\mathcal{E}_{\{\omega\}}(\Omega) := \left\{ f \in C^\infty(\Omega) : \forall K \subset\subset \Omega\ \exists \lambda > 0 \text{ such that } p_{K,\lambda}(f) < \infty \right\}.$$

Its topology is the following

$$\mathcal{E}_{\{\omega\}}(\Omega) = \underset{\underset{K\subset\subset\Omega}{\longleftarrow}}{\text{proj}}\ \underset{\underset{m\in\mathbb{N}}{\longrightarrow}}{\text{ind}}\ \mathcal{E}^{\frac{1}{m}}_{\omega}(K).$$

This is a complete PLS-space, that is, a complete space which is a projective limit of LB-spaces. Moreover, $\mathcal{E}_{\{\omega\}}(\Omega)$ is also a nuclear and reflexive locally convex space. In particular, $\mathcal{E}_{\{\omega\}}(\Omega)$ is an ultrabornological (hence barrelled and bornological) space.

The elements of $\mathcal{E}_{(\omega)}(\Omega)$ (resp. $\mathcal{E}_{\{\omega\}}(\Omega)$) are called ultradifferentiable functions of Beurling type (resp. Roumieu type) in Ω.

In the case that $\omega(t) := t^d$ $(0 < d < 1)$, the corresponding Roumieu class is the Gevrey class with exponent $1/d$. In the limit case $d = 1$, the corresponding Roumieu class $\mathcal{E}_{\{\omega\}}(\Omega)$ is the space of real analytic functions on Ω whereas the Beurling class $\mathcal{E}_{(\omega)}(\mathbb{R}^n)$ gives the entire functions. Observe that Gevrey weights satisfy (α_0).

Given a polynomial $P \in \mathbb{C}[z_1, \ldots, z_n]$ of degree m, $P(z) = \sum_{|\alpha|\leq m} a_\alpha z^\alpha$, the partial differential operator $P(D)$ is defined as $P(D) = \sum_{|\alpha|\leq m} a_\alpha D^\alpha$, where $D = \frac{1}{i}\partial$. Following [9], we consider smooth functions in an open set Ω such that there exists $C > 0$ verifying for each $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$,

$$\|P^j(D)f\|_{L^2(K)} \leq C \exp\left(\lambda\varphi^*(\frac{jm}{\lambda})\right),$$

where K is a compact subset in Ω, $\|\cdot\|_{L^2(K)}$ denotes the L^2-norm on K and $P^j(D)$ is the j-th iterate of the partial differential operator $P(D)$ of order m, i.e.,

$$P^j(D) = \underbrace{P(D) \circ \cdots \circ P(D)}_{j}.$$

If $j = 0$, then we set $P^0(D)f = f$.

The spaces of ultradifferentiable functions with respect to the successive iterates of P are defined as follows.

Let ω be a weight function. Given a polynomial P, an open set Ω of $\mathbb{R}^n$, a compact subset $K \subset\subset \Omega$ and $\lambda > 0$, we define the seminorm

$$\|f\|_{K,\lambda} := \sup_{j\in\mathbb{N}_0} \|P^j(D)f\|_{2,K} \exp\left(-\lambda\varphi^*(\frac{jm}{\lambda})\right) \tag{1.1}$$

and set

$$\mathcal{E}^{\lambda}_{P,\omega}(K) = \left\{f \in C^{\infty}(K) : \|f\|_{K,\lambda} < +\infty\right\}.$$

It is a normed space endowed with the $\|\cdot\|_{K,\lambda}$-norm.

The space of *ultradifferentiable functions of Beurling type with respect to the iterates of P* is:

$$\mathcal{E}^{P}_{(\omega)}(\Omega) = \left\{f \in C^{\infty}(\Omega) : \|f\|_{K,\lambda} < +\infty \text{ for each } K \subset\subset \Omega \text{ and } \lambda > 0\right\},$$

endowed with the topology given by

$$\mathcal{E}^{P}_{(\omega)}(\Omega) := \underset{\overleftarrow{K\subset\subset\Omega}}{\text{proj}}\ \underset{\overleftarrow{\lambda>0}}{\text{proj}}\ \mathcal{E}^{\lambda}_{P,\omega}(K).$$

If $\{K_n\}_{n\in\mathbb{N}}$ is a compact exhaustion of Ω we have

$$\mathcal{E}^{P}_{(\omega)}(\Omega) = \underset{\overleftarrow{n\in\mathbb{N}}}{\text{proj}}\ \underset{\overleftarrow{k\in\mathbb{N}}}{\text{proj}}\ \mathcal{E}^{k}_{P,\omega}(K_n) = \underset{\overleftarrow{n\in\mathbb{N}}}{\text{proj}}\ \mathcal{E}^{n}_{P,\omega}(K_n).$$

This is a metrizable locally convex topology defined by the fundamental system of seminorms $\left\{\|\cdot\|_{K_n,n}\right\}_{n\in\mathbb{N}}$.

The space of *ultradifferentiable functions of Roumieu type with respect to the iterates of P* is defined by:

$$\mathcal{E}^{P}_{\{\omega\}}(\Omega) = \left\{f \in C^{\infty}(\Omega) : \forall K \subset\subset \Omega\ \exists\lambda > 0 \text{ such that } \|f\|_{K,\lambda} < +\infty\right\}.$$

Its topology is defined by

$$\mathcal{E}^{P}_{\{\omega\}}(\Omega) := \underset{\overleftarrow{K\subset\subset\Omega}}{\text{proj}}\ \underset{\overrightarrow{\lambda>0}}{\text{ind}}\ \mathcal{E}^{\lambda}_{P,\omega}(K).$$

In the following, $*$ will denote either $\{\omega\}$ or (ω).

The inclusion map $\mathcal{E}_*(\Omega) \hookrightarrow \mathcal{E}^P_*(\Omega)$ is continuous (see [9, Theorem 4.1]). The space $\mathcal{E}^P_*(\Omega)$ is complete if and only if P is hypoelliptic (see [9, Theorem 3.3]). Moreover, under a mild condition on ω introduced by Bonet et al. [5, 16 Corollary (3)], $\mathcal{E}^P_*(\Omega)$ coincides with the class of ultradifferentiable functions $\mathcal{E}_*(\Omega)$ if and only if P is elliptic (see [9, Theorem 4.12]).

Now, let $P(x, D) = \sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha$ be a linear partial differential operator of order m with smooth coefficients in an open subset $\Omega \subseteq \mathbb{R}^n$, i.e. $a_\alpha \in C^\infty(\Omega)$ for all multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$. We consider the q-th iterates $P^q = P \circ \cdots \circ P$ of $P := P(x, D)$ and define the corresponding spaces of iterates as above:

$$\mathcal{E}^P_{(\omega)}(\Omega) := \{ u \in C^\infty(\Omega) : \forall K \subset\subset \Omega\, \forall k \in \mathbb{N}\, \exists c_k > 0 \text{ s.t.} \\ \|P^q u\|_{L^2(K)} \leq c_k e^{k\varphi^*(qm/k)}\ \forall q \in \mathbb{N}_0\} \tag{1.2}$$

for the Beurling case, and

$$\mathcal{E}^P_{\{\omega\}}(\Omega) := \{ u \in C^\infty(\Omega) : \forall K \subset\subset \Omega\, \exists k \in \mathbb{N},\ c > 0 \text{ s.t.} \\ \|P^q u\|_{L^2(K)} \leq c e^{\frac{1}{k}\varphi^*(qmk)}\ \forall q \in \mathbb{N}_0\} \tag{1.3}$$

for the Roumieu case. We generalize some results of Juan–Huguet [9] for operators with variable coefficients in the following way. First, we state our main result in the Roumieu case:

Theorem 1.4 *Let ω be a subadditive weight function, $\Omega \subseteq \mathbb{R}^n$ a domain, i.e. open and connected, and $P(x, D)$ a linear partial differential operator of order m with coefficients in $\mathcal{E}_{\{\omega\}}(\Omega)$. Then:*

(i) $\mathcal{E}_{\{\omega\}}(\Omega) \subseteq \mathcal{E}^P_{\{\omega\}}(\Omega)$;
(ii) if $P(x, D)$ is elliptic, then $\mathcal{E}_{\{\omega\}}(\Omega) = \mathcal{E}^P_{\{\omega\}}(\Omega)$.

In the Beurling case we lose some regularity; compare to Oldrich [24, Teorema 1]:

Theorem 1.5 *Let ω be a subadditive weight function, $\Omega \subseteq \mathbb{R}^n$ a domain and $P(x, D)$ a linear partial differential operator of order m with coefficients in $\mathcal{E}_{(\omega)}(\Omega)$. Then:*

(i) $\mathcal{E}_{(\omega)}(\Omega) \subseteq \mathcal{E}^P_{(\omega)}(\Omega)$;
(ii) if $P(x, D)$ is elliptic, then $\mathcal{E}^P_{(\omega)}(\Omega) \subseteq \mathcal{E}_{(\sigma)}(\Omega)$ for every subadditive weight function $\sigma(t) = o(\omega(t))$ as $t \to +\infty$.

Theorem 1.4 is the generalization to the class of ultradifferentiable functions $\mathcal{E}_{\{\omega\}}(\Omega)$ of the theorem of Kotake–Narasimhan for an elliptic linear partial differential operator $P(x, D)$ with coefficients in the same class $\mathcal{E}_{\{\omega\}}(\Omega)$. We observe that the ellipticity of P is not needed for the inclusion $\mathcal{E}_{\{\omega\}}(\Omega) \subseteq \mathcal{E}^P_{\{\omega\}}(\Omega)$. However, we show in Example 3.1 that the ellipticity is necessary for the equality $\mathcal{E}_{\{\omega\}}(\Omega) = \mathcal{E}^P_{\{\omega\}}(\Omega)$ for a large family of weights ω. We use the example of Metivier [21, p. 831] to show that for suitable weight functions, which *are not* of Gevrey type in general, indeed weights which are between two given concrete Gevrey weights, statement (ii) in Theorems 1.4 and 1.5 fails if P is not elliptic. Finally, we remark that there is no restriction to assume that the weight ω is quasianalytic, i.e. satisfies condition (β') and not (β), in Theorems 1.4 and 1.5. However, in Example 3.1 the weights are taken to be non-quasianalytic.

2 Preliminary results

In order to prove Theorems 1.4 and 1.5 we collect in this section some preliminary results. First of all, we shall prove some properties of the Young conjugate function φ^* defined in Sect. 1:

Proposition 2.1 *Let ω be a subadditive weight function and define, for $j \in \mathbb{N}_0$, $\lambda > 0$,*

$$a_{j,\lambda} := \frac{e^{\lambda\varphi^*(j/\lambda)}}{j!}.$$

Then the following properties are satisfied:

(a) $a_{j,\lambda} \cdot a_{h,\lambda} \leq a_{j+h,\lambda} \quad \forall j, h \in \mathbb{N}_0,\ \lambda > 0$*;*
(b) $a_{j,\lambda} \leq a_{j+1,\lambda} \quad \forall j \in \mathbb{N}_0,\ \lambda > 0$*;*
(c) $\lambda \mapsto a_{j,\lambda}$ *is decreasing for all* $j \in \mathbb{N}_0$*;*
(d) $a_{j+h,\lambda} \leq a_{j,\lambda/2} \cdot a_{h,\lambda/2} \quad \forall j, h \in \mathbb{N}_0,\ \lambda > 0$*;*
(e) for every $\rho, \lambda > 0$ *there exists* λ', $D_{\rho,\lambda} > 0$ *such that*

$$\rho^j e^{\lambda\varphi^*(j/\lambda)} \leq D_{\rho,\lambda} e^{\lambda'\varphi^*(j/\lambda')} \quad \forall j \in \mathbb{N}_0,$$

with $D_{\rho,\lambda} := \exp\{\lambda[\log\rho + 1]\}$*, where* $[\log\rho + 1]$ *is the integer part of* $\log\rho + 1$*;*
(f) for every $j, h, r \in \mathbb{N}_0$ *with* $0 \leq h \leq j$*, and for all* $\lambda > 0$*:*

$$\frac{j!}{h!} a_{j-h,\lambda} \leq \frac{e^{\lambda\varphi^*\left(\frac{j+r}{\lambda}\right)}}{e^{\lambda\varphi^*\left(\frac{h+r}{\lambda}\right)}};$$

(g) for every $j, h, r \in \mathbb{N}_0,\ \lambda > 0$*:*

$$e^{\lambda\varphi^*\left(\frac{j}{\lambda}\right)} e^{\lambda\varphi^*\left(\frac{r+h}{\lambda}\right)} \leq e^{\frac{\lambda}{2}\varphi^*\left(\frac{j+h}{\lambda/2}\right)} e^{\frac{\lambda}{2}\varphi^*\left(\frac{r}{\lambda/2}\right)}.$$

(h) for every $\lambda > 0$ *and* $q, r \in \mathbb{N}_0$ *with* $q \geq r$ *we have that*

$$\frac{e^{\lambda\varphi^*\left(\frac{q+1}{\lambda}\right)}}{e^{\lambda\varphi^*\left(\frac{q}{\lambda}\right)}} \geq \frac{e^{\lambda\varphi^*\left(\frac{r+1}{\lambda}\right)}}{e^{\lambda\varphi^*\left(\frac{r}{\lambda}\right)}}.$$

Proof (a) has been proved in Lema 3.2.3 of [8].

(b) follows from (a) since $a_{1,\lambda} = e^{\lambda\varphi^*(1/\lambda)} \geq 1$.

(c) follows from the fact that $\varphi^*(s)/s$ is increasing (cf. [6]).

(d) follows from the convexity of φ^*:

$$\begin{aligned} a_{j+h,\lambda} = \frac{e^{\lambda\varphi^*\left(\frac{j+h}{\lambda}\right)}}{(j+h)!} &\leq \frac{j!h!}{(j+h)!} \frac{e^{\frac{\lambda}{2}\varphi^*\left(\frac{2j}{\lambda}\right)}}{j!} \frac{e^{\frac{\lambda}{2}\varphi^*\left(\frac{2h}{\lambda}\right)}}{h!} \\ &= \frac{1}{\binom{j+h}{h}} a_{j,\frac{\lambda}{2}} a_{h,\frac{\lambda}{2}} \leq a_{j,\frac{\lambda}{2}} a_{h,\frac{\lambda}{2}}. \end{aligned}$$

Point (e) follows from the next property of [8, Prop. 0.1.5(2) (a)]: for each $y \geq 0$, $n \in \mathbb{N}$, and $\lambda > 0$,

$$\lambda L^n \varphi^* \left(\frac{y}{\lambda L^n}\right) + ny \leq \lambda \varphi^* \left(\frac{y}{\lambda}\right) + \lambda \sum_{h=1}^{n} L^h, \tag{2.1}$$

where $L > 0$ is such that $\omega(et) \leq L(1+\omega(t))$ for all $t \geq 0$ (in our case ω is increasing and subadditive, so that we can take $L = 3$). Indeed, from (2.1) with $y = jL^n$ and dividing by L^n:

$$\lambda \varphi^* \left(\frac{j}{\lambda}\right) + nj \leq \frac{\lambda}{L^n} \varphi^* \left(\frac{j}{\lambda / L^n}\right) + \lambda \sum_{h=1}^{n} L^{h-n}$$

and therefore

$$\rho^j e^{\lambda \varphi^* \left(\frac{j}{\lambda}\right)} \leq e^{\frac{\lambda}{L^n} \varphi^* \left(\frac{j}{\lambda / L^n}\right) + \lambda n - nj + j \log \rho}.$$

Choosing $n_\rho := [\log \rho + 1] \in \mathbb{N}$ so that $-n_\rho + \log \rho \leq 0$, for $\lambda' = \lambda / L^{n_\rho}$ we thus have that

$$\rho^j e^{\lambda \varphi^* \left(\frac{j}{\lambda}\right)} \leq e^{\lambda n_\rho} e^{\lambda' \varphi^* \left(\frac{j}{\lambda'}\right)} \tag{2.2}$$

so that (e) is proved.

In order to prove (f), let us first remark that

$$\frac{j!}{h!} a_{j-h,\lambda} \leq \frac{(j+r)!}{(h+r)!} a_{j-h,\lambda} \tag{2.3}$$

since $h \leq j$.

From (2.3) we have that

$$\begin{aligned}
\frac{j!}{h!} a_{j-h,\lambda} \leq \; & \frac{(j+r)!}{e^{\lambda \varphi^* \left(\frac{j+r}{\lambda}\right)}} \cdot \frac{e^{\lambda \varphi^* \left(\frac{h+r}{\lambda}\right)}}{(h+r)!} \cdot \frac{e^{\lambda \varphi^* \left(\frac{j+r}{\lambda}\right)}}{e^{\lambda \varphi^* \left(\frac{h+r}{\lambda}\right)}} a_{j-h,\lambda} \\
= \; & \frac{a_{h+r,\lambda}\, a_{j-h,\lambda}}{a_{j+r,\lambda}} \cdot \frac{e^{\lambda \varphi^* \left(\frac{j+r}{\lambda}\right)}}{e^{\lambda \varphi^* \left(\frac{h+r}{\lambda}\right)}} \leq \frac{e^{\lambda \varphi^* \left(\frac{j+r}{\lambda}\right)}}{e^{\lambda \varphi^* \left(\frac{h+r}{\lambda}\right)}}
\end{aligned}$$

by the already proved point (a). Therefore (f) holds true.

Property (g) follows from the convexity of φ^*. Indeed, from (a)

$$\begin{aligned} e^{\lambda\varphi^*\left(\frac{j}{\lambda}\right)} e^{\lambda\varphi^*\left(\frac{r+h}{\lambda}\right)} &= a_{j,\lambda}\, a_{r+h,\lambda}\, j!(r+h)! \\ &\le a_{j+r+h,\lambda}\, j!(r+h)! = e^{\lambda\varphi^*\left(2\frac{j+r+h}{2\lambda}\right)} \frac{j!(r+h)!}{(j+r+h)!} \\ &\le e^{\frac{\lambda}{2}\varphi^*\left(\frac{j+h}{\lambda/2}\right)+\frac{\lambda}{2}\varphi^*\left(\frac{r}{\lambda/2}\right)} \frac{1}{\binom{j+r+h}{j}} \\ &\le e^{\frac{\lambda}{2}\varphi^*\left(\frac{j+h}{\lambda/2}\right)} e^{\frac{\lambda}{2}\varphi^*\left(\frac{r}{\lambda/2}\right)}. \end{aligned}$$

Let us finally prove (h). We first remark that, by the convexity of φ^*,

$$2\varphi^*\left(\frac{r+1}{\lambda}\right) = 2\varphi^*\left(\frac{r}{2\lambda}+\frac{r+2}{2\lambda}\right) \le \varphi^*\left(\frac{r}{\lambda}\right) + \varphi^*\left(\frac{r+2}{\lambda}\right)$$

i.e.

$$\varphi^*\left(\frac{r+1}{\lambda}\right) - \varphi^*\left(\frac{r}{\lambda}\right) \le \varphi^*\left(\frac{r+2}{\lambda}\right) - \varphi^*\left(\frac{r+1}{\lambda}\right).$$

Arguing recursively we get

$$\varphi^*\left(\frac{r+1}{\lambda}\right) - \varphi^*\left(\frac{r}{\lambda}\right) \le \varphi^*\left(\frac{q+1}{\lambda}\right) - \varphi^*\left(\frac{q}{\lambda}\right) \tag{2.4}$$

for every $q \in \mathbb{N}$ with $q \ge r$.

Clearly (2.4) implies (h) and the proof is complete. □

Remark 2.2 Note that we did not use the subadditivity of the weight ω to prove points (c), (d), (e), (h) of Proposition 2.1.

For the proof of Theorem 1.4 we shall follow the ideas of [13], so we define, for a domain $\Omega \subseteq \mathbb{R}^n$, $q \in \mathbb{N}_0$, $\delta > 0$ and $f \in C^\infty(G)$, with G a relatively compact subdomain of Ω,

$$\|\nabla^q f\|_\delta = \sum_{|\alpha|=q} \|D^\alpha f\|_{L^2(G_\delta)},$$

where

$$G_\delta := \{x \in G :\ \operatorname{dist}(x, \partial G) > \delta\}$$

and $\|\cdot\|_{L^2(G_\delta)} = 0$ if $G_\delta = \emptyset$.

If $P = P(x, D)$ is an elliptic linear partial differential operator of order m with C^∞ coefficients, then the following a priori estimates, for $\delta, \sigma > 0$ and $0 \le r \le m$, have been proved in [12]:

$$\|\nabla^m f\|_{\delta+\sigma} \le C(\|Pf\|_\sigma + \delta^{-m}\|f\|_\sigma) \tag{2.5}$$
$$\|\nabla^{m-r} f\|_{\delta+\sigma} \le C\varepsilon^r(\|\nabla^m f\|_\sigma + (\delta^{-m} + \varepsilon^{-m})\|f\|_\sigma), \tag{2.6}$$

for arbitrary $\varepsilon > 0$, where the constant $C > 0$ depends only on the operator P and the set G.

Then we define the semi-norm $N^{pm}(u)$ by

$$N^{pm}(u) := \sup_{0<\delta\le 1} \delta^{pm}\|\nabla^{pm}u\|_\delta.$$

The following inequality holds:

Proposition 2.3 *Let $\Omega \subseteq \mathbb{R}^n$ be a domain and $P(x, D)$ an elliptic linear partial differential operator of order m with coefficients in $\mathcal{E}_{\{\omega\}}(\Omega)$. For $u \in C^\infty(\Omega)$, there exist $k \in \mathbb{N}$ and a positive constant C_0 such that*

$$N^{pm}(u) \le C_0 \left\{ N^{(p-1)m}(Pu) + \sum_{q=0}^{p-1} \frac{e^{\frac{1}{k}\varphi^*(pmk)}}{e^{\frac{1}{k}\varphi^*(qmk)}} N^{qm}(u) \right\}. \tag{2.7}$$

for every $p \in \mathbb{N}$.

Proof By definition of the semi-norm $N^{(p+1)m}(u)$ and by (2.5) we have

$$\begin{aligned} N^{(p+1)m}(u) = & \sup_{(p+2)\delta\le 1} ((p+2)\delta)^{(p+1)m}\|\nabla^{(p+1)m}u\|_{(p+2)\delta} \\ \le & \sup_{(p+2)\delta\le 1} \left(\frac{p+2}{p}\right)^{(p+1)m} (p\delta)^{(p+1)m} C(\|P\nabla^{pm}u\|_{(p+1)\delta} \\ & \quad + \delta^{-m}\|\nabla^{pm}u\|_{(p+1)\delta}) \\ \le & \ 9^m C \sup_{(p+2)\delta\le 1} \{(p\delta)^{(p+1)m}\|P\nabla^{pm}u\|_{(p+1)\delta} \\ & \quad + p^m(p\delta)^{pm}\|\nabla^{pm}u\|_{(p+1)\delta}\}, \end{aligned} \tag{2.8}$$

since $\left(\frac{p+2}{p}\right)^{p+1} \le 9$.

We set $P^{[r]} := \sum_{|\alpha|=r} \sup_G |D_x^\alpha P|$. Since $\|\cdot\|_{(p+1)\delta} \leq \|\cdot\|_{p\delta}$ and $p^m(pm)! \leq ((p+1)m)!$, from (2.8) and Leibniz' formula we get:

$$\begin{aligned} N^{(p+1)m}(u) \leq\ & 9^m C \sup_{(p+2)\delta\leq 1} \Bigg\{ (p\delta)^{(p+1)m} \Bigg[\|\nabla^{pm} Pu\|_{(p+1)\delta} \\ & + \sum_{r=1}^{pm} \binom{pm}{r} \|P^{[r]} \nabla^{pm-r} u\|_{(p+1)\delta} \Bigg] + p^m (p\delta)^{pm} \|\nabla^{pm} u\|_{p\delta} \Bigg\} \\ \leq\ & 9^m C \sup_{(p+2)\delta\leq 1} \Bigg\{ \left(\frac{p}{p+1}\right)^{pm} [(p+1)\delta]^{pm} \\ & \times \left(\frac{p}{p+2}\right)^{m} [(p+2)\delta]^m \|\nabla^{pm} Pu\|_{(p+1)\delta} \\ & + (p\delta)^{(p+1)m} \sum_{r=1}^{pm} \binom{pm}{r} \|P^{[r]} \nabla^{pm-r} u\|_{(p+1)\delta} \\ & + \frac{((p+1)m)!}{(pm)!} N^{pm}(u) \Bigg\} \\ \leq\ & 9^m C \Bigg\{ N^{pm}(Pu) \\ & + \sup_{(p+2)\delta\leq 1} (p\delta)^{(p+1)m} \sum_{r=1}^{pm} \binom{pm}{r} \|P^{[r]} \nabla^{pm-r} u\|_{(p+1)\delta} \\ & + \frac{((p+1)m)!}{(pm)!} N^{pm}(u) \Bigg\}. \end{aligned} \tag{2.9}$$

Taking into account that the coefficients of $P(x, D)$ are in $\mathcal{E}_{\{\omega\}}(\Omega)$, we can write the following estimates, for $(p+2)\delta \leq 1$ and for some $k \in \mathbb{N}$ and $c > 0$:

$$\begin{aligned} \sum_{r=1}^{pm} \binom{pm}{r} \|P^{[r]} \nabla^{pm-r} u\|_{(p+1)\delta} & \leq c \sum_{r=1}^{pm} \binom{pm}{r} e^{\frac{1}{k}\varphi^*(rk)} \sum_{s=0}^{m} \|\nabla^{pm+s-r} u\|_{(p+1)\delta} \\ & \leq c \sum_{r=1}^{pm} \frac{(pm)!}{(pm-r)!} a_{r,\frac{1}{k}} \sum_{s=0}^{m} \|\nabla^{pm+s-r} u\|_{(p+1)\delta}. \end{aligned} \tag{2.10}$$

By the change of indexes $r = (p-q)m + t$ we obtain that (cf. also [13])

$$\begin{aligned} \sum_{r=1}^{pm} \binom{pm}{r} \|P^{[r]} \nabla^{pm-r} u\|_{(p+1)\delta} \leq\ & c(m+1) \sum_{q=1}^{p} \sum_{t=1}^{m} \frac{(pm)!}{(qm-t)!} a_{(p-q)m+t,\frac{1}{k}} \\ & \times \|\nabla^{(q+1)m-t} u\|_{(p+1)\delta} \\ & + cm \sum_{t=1}^{m} (pm)! a_{pm,\frac{1}{k}} \|\nabla^{m-t} u\|_{(p+1)\delta} \end{aligned}$$

$$
\begin{aligned}
&= c(m+1)\sum_{t=1}^{m}\frac{(pm)!}{(pm-t)!}a_{t,\frac{1}{k}}\\
&\quad\times\|\nabla^{(p+1)m-t}u\|_{(p+1)\delta}\\
&\quad+c(m+1)\sum_{q=1}^{p-1}\sum_{t=1}^{m}\frac{(pm)!}{(qm-t)!}a_{(p-q)m+t,\frac{1}{k}}\\
&\quad\times\|\nabla^{(q+1)m-t}u\|_{(p+1)\delta}\\
&\quad+cm\sum_{t=1}^{m}(pm)!a_{pm,\frac{1}{k}}\|\nabla^{m-t}u\|_{(p+1)\delta}. \qquad (2.11)
\end{aligned}
$$

From (2.11), by properties (b) and (d) of Proposition 2.1 we get:

$$
\sum_{r=1}^{pm}\binom{pm}{r}\|P^{[r]}\nabla^{pm-r}u\|_{(p+1)\delta}\leq S_1+S_2+S_3 \qquad (2.12)
$$

with

$$
\begin{aligned}
S_1 &:= c(m+1)\sum_{t=1}^{m}\frac{(pm)!}{(pm-t)!}a_{m,\frac{1}{k}}\|\nabla^{(p+1)m-t}u\|_{(p+1)\delta}\\
S_2 &:= ca_{m,\frac{1}{2k}}(m+1)\sum_{q=1}^{p-1}\sum_{t=1}^{m}\frac{(pm)!}{(qm-t)!}a_{(p-q)m,\frac{1}{2k}}\|\nabla^{(q+1)m-t}u\|_{(p+1)\delta}\\
S_3 &:= cm\sum_{t=1}^{m}(pm)!a_{pm,\frac{1}{k}}\|\nabla^{m-t}u\|_{(p+1)\delta}.
\end{aligned}
$$

By property (c) of Proposition 2.1 and by (2.6), setting

$$
C_2 := 9^m cC(m+1)a_{m,\frac{1}{2k}},
$$

we have the estimate

$$
\begin{aligned}
9^m C(p\delta)^{(p+1)m}S_1 &\leq C_2\sum_{t=1}^{m}\frac{(pm)!}{(pm-t)!}(p\delta)^{(p+1)m}\|\nabla^{(p+1)m-t}u\|_{(p+1)\delta}\\
&\leq C_2C\sum_{t=1}^{m}(pm)^t(p\delta)^{(p+1)m}\varepsilon^t(\|\nabla^{(p+1)m}u\|_{p\delta}\\
&\quad+(\delta^{-m}+\varepsilon^{-m})\|\nabla^{pm}u\|_{p\delta})\\
&= C_2C\sum_{t=1}^{m}(pm)^t\varepsilon^t\big\{(p\delta)^{(p+1)m}\|\nabla^{(p+1)m}u\|_{p\delta}\\
&\quad+(p^m+(p\delta)^m\varepsilon^{-m})(p\delta)^{pm}\|\nabla^{pm}u\|_{p\delta}\big\},
\end{aligned}
$$

since $(pm)!\leq(pm-t)!(pm)^t$.

Therefore, for $\varepsilon = (pm)^{-1}(2mCC_2)^{-1/t}$ and $(p+2)\delta \le 1$:

$$\begin{aligned} 9^m C(p\delta)^{(p+1)m} S_1 \le\ & \sum_{t=1}^{m} \frac{1}{2m} \Big\{ N^{(p+1)m}(u) \\ & + \Big(p^m + \Big(\frac{p}{p+2}\Big)^m [(p+2)\delta]^m \\ & \times (pm)^m (2mCC_2)^{m/t} \Big) N^{pm}(u) \Big\} \\ \le\ & \sum_{t=1}^{m} \frac{1}{2m} \Big\{ N^{(p+1)m}(u) \\ & + \left(p^m + (pm)^m (2mCC_2)^{m/t} \right) N^{pm}(u) \Big\} \\ \le\ & \frac{1}{2} N^{(p+1)m}(u) + C_3 p^m N^{pm}(u) \\ \le\ & \frac{1}{2} N^{(p+1)m}(u) + C_3 \frac{((p+1)m)!}{(pm)!} N^{pm}(u) \end{aligned} \tag{2.13}$$

for some $C_3 > 0$, because of $p^m (pm)! \le ((p+1)m)!$.

In order to estimate S_2, let us first prove the following estimate, for $1 \le q \le p-1$, $(p+1)\delta = (q+1)\delta'$ and $(p+2)\delta \le 1$:

$$(p\delta)^{(p+1)m} \le (2e)^m (q\delta')^{(q+1)m}. \tag{2.14}$$

Indeed,

$$\begin{aligned} (p\delta)^{(p+1)m} =\ & \frac{p^{(p+1)m}\delta^{(p+1)m}}{q^{(q+1)m}\left(\frac{p+1}{q+1}\right)^{(q+1)m}\delta^{(q+1)m}} \cdot (q\delta')^{(q+1)m} \\ =\ & \left(\frac{p}{p+1}\frac{q+1}{q}\right)^{(q+1)m} (p\delta)^{(p-q)m}(q\delta')^{(q+1)m} \\ \le\ & \left(1+\frac{1}{q}\right)^{qm}\left(1+\frac{1}{q}\right)^{m}\left(\frac{p}{p+2}\right)^{(p-q)m} \\ & \times [(p+2)\delta]^{(p-q)m}(q\delta')^{(q+1)m} \\ \le\ & e^m 2^m (q\delta')^{(q+1)m}. \end{aligned}$$

Therefore (2.14) is proved and, for $1 \le q \le p-1$, $(p+1)\delta = (q+1)\delta'$ and $(p+2)\delta \le 1$:

$$\begin{aligned} & 9^m C(p\delta)^{(p+1)m} S_2 \\ & \le C_2 \sum_{q=1}^{p-1}\sum_{t=1}^{m} \frac{(pm)!}{(qm-t)!} a_{(p-q)m,\frac{1}{2k}} (p\delta)^{(p+1)m} \|\nabla^{(q+1)m-t} u\|_{(p+1)\delta} \end{aligned}$$

$$\le (2e)^m \sum_{q=1}^{p-1} \frac{(pm)!}{(qm)!} a_{(p-q)m,\frac{1}{2k}}$$
$$\times C_2 \sum_{t=1}^{m} \frac{(qm)!}{(qm-t)!} (q\delta')^{(q+1)m} \|\nabla^{(q+1)m-t} u\|_{(q+1)\delta'}.$$

By (2.13) with q and δ' instead of p and δ respectively, and because of properties (f) and (b) of Proposition 2.1 we finally get the following estimate for S_2:

$$9^m C(p\delta)^{(p+1)m} S_2$$
$$\le D \sum_{q=1}^{p-1} \frac{(pm)!}{(qm)!} a_{(p-q)m,\frac{1}{2k}} \left\{ \frac{1}{2} N^{(q+1)m}(u) + C_3' \frac{((q+1)m)!}{(qm)!} N^{qm}(u) \right\}$$
$$\le D' \sum_{q=1}^{p-1} \left(\frac{e^{\frac{1}{2k}\varphi^*(2(p+1)mk)}}{e^{\frac{1}{2k}\varphi^*(2(q+1)mk)}} N^{(q+1)m} + \frac{e^{\frac{1}{2k}\varphi^*(2(p+1)mk)}}{e^{\frac{1}{2k}\varphi^*(2qmk)}} N^{qm}(u) \right)$$
$$\le 2D' \sum_{q=1}^{p-1} \frac{e^{\frac{1}{2k}\varphi^*(2(p+1)mk)}}{e^{\frac{1}{2k}\varphi^*(2qmk)}} N^{qm}(u) + D' \frac{e^{\frac{1}{2k}\varphi^*(2(p+1)mk)}}{e^{\frac{1}{2k}\varphi^*(2pmk)}} N^{pm}(u) \tag{2.15}$$

for some C_3', D, $D' > 0$.

Let us now estimate S_3. By (2.6) with $\varepsilon = 1$ and because of properties (e), (f) (with $h = 0$) and (b) of Proposition 2.1, for $(p+2)\delta \le 1$:

$$9^m C(p\delta)^{(p+1)m} S_3$$
$$\le C_2 \sum_{t=1}^{m} (pm)! a_{pm,\frac{1}{k}} (p\delta)^{(p+1)m} \|\nabla^{m-t} u\|_{(p+1)\delta}$$
$$\le CC_2 \sum_{t=1}^{m} (pm)!(p\delta)^{pm} a_{pm,\frac{1}{k}} \left((p\delta)^m \|\nabla^m u\|_{p\delta} + p^m(1+\delta^m)\|u\|_{p\delta} \right)$$
$$\le CC_2 \sum_{t=1}^{m} (pm)! a_{pm,\frac{1}{k}} \left(N^m(u) + 2p^m N^0(u) \right)$$
$$\le CC_2 m(pm)! a_{pm,\frac{1}{k}} N^m(u) + 2CC_2 m((p+1)m)! a_{pm,\frac{1}{k}} N^0(u)$$
$$\le CC_2 m \frac{e^{\frac{1}{k}\varphi^*((p+1)mk)}}{e^{\frac{1}{k}\varphi^*(mk)}} N^m(u) + 2CC_2 m((p+1)m)! a_{(p+1)m,\frac{1}{k}} N^0(u)$$
$$\le \tilde{D} e^{\frac{1}{k}\varphi^*((p+1)mk)} \left(N^m(u) + N^0(u) \right), \tag{2.16}$$

for some $\tilde{D} > 0$.

Substituting (2.13), (2.15) and (2.16) in (2.12) and then in (2.9) and applying (b) of Proposition 2.1, we finally get:

$$N^{(p+1)m}(u) \le C_5 N^{pm}(Pu) + \frac{1}{2} N^{(p+1)m}(u) + C_5 \sum_{q=0}^{p} \frac{e^{\frac{1}{k'}\varphi^*((p+1)mk')}}{e^{\frac{1}{k'}\varphi^*(qmk')}} N^{qm}(u),$$

for some $k' \in \mathbb{N}$ and $C_5 > 0$, concluding the proof. □

We shall also need, in the following, the next result:

Proposition 2.4 *Let $P(x, D)$ be an elliptic linear partial differential operator of order m with coefficients in $\mathcal{E}_{\{\omega\}}(\Omega)$. For $u \in C^\infty(\Omega)$, there are $k \in \mathbb{N}$ and a positive constant $C_1 > 0$ such that*

$$N^{pm}(u) \le C_1^p \sum_{q=0}^{p} \binom{p}{q} \frac{e^{\frac{1}{k}\varphi^*(pmk)}}{e^{\frac{1}{k}\varphi^*(qmk)}} N^0(P^q u) \tag{2.17}$$

for every $p \in \mathbb{N}_0$.

Proof Let us proceed by induction on p. For $p = 0$ it's trivial. Let us assume (2.17) to be true for $0, 1, \ldots, p-1$ and let us prove it for p.

Applying (2.7) for $q \in \{1, \ldots, p-1\}$ instead of p, we have that

$$N^m(u) \le C_0 \left\{ N^0(Pu) + e^{\frac{1}{k}\varphi^*(mk)} N^0(u) \right\}$$

$$\vdots$$

$$N^{(p-1)m}(u) \le C_0 \left\{ N^{(p-2)m}(Pu) + \sum_{q=0}^{p-2} \frac{e^{\frac{1}{k}\varphi^*((p-1)mk)}}{e^{\frac{1}{k}\varphi^*(qmk)}} N^{qm}(u) \right\}.$$

Substituting in (2.7) and taking into account (b) of Proposition 2.1:

$$\begin{aligned}
&N^{pm}(u)\\
&\le C_0 \left\{ N^{(p-1)m}(Pu) + \frac{e^{\frac{1}{k}\varphi^*(pmk)}}{e^{\frac{1}{k}\varphi^*((p-1)mk)}} N^{(p-1)m}(u) + \cdots + e^{\frac{1}{k}\varphi^*(pmk)} N^0(u) \right\}\\
&\le C_0 \left\{ N^{(p-1)m}(Pu) + \frac{e^{\frac{1}{k}\varphi^*(pmk)}}{e^{\frac{1}{k}\varphi^*((p-1)mk)}} C_0 \left[N^{(p-2)m}(Pu) \right.\right.\\
&\quad \left. + \frac{e^{\frac{1}{k}\varphi^*((p-1)mk)}}{e^{\frac{1}{k}\varphi^*((p-2)mk)}} N^{(p-2)m}(u) + \cdots + e^{\frac{1}{k}\varphi^*((p-1)mk)} N^0(u) \right]\\
&\quad \left. + \cdots + e^{\frac{1}{k}\varphi^*(pmk)} N^0(u) \right\}\\
&\le C_0 N^{(p-1)m}(Pu) + C_0^2 \frac{e^{\frac{1}{k}\varphi^*(pmk)}}{e^{\frac{1}{k}\varphi^*((p-1)mk)}} N^{(p-2)m}(Pu)\\
&\quad + C_0^2 \frac{e^{\frac{1}{k}\varphi^*(pmk)}}{e^{\frac{1}{k}\varphi^*((p-2)mk)}} N^{(p-2)m}(u) + \cdots + C_0(C_0+1) e^{\frac{1}{k}\varphi^*(pmk)} N^0(u)
\end{aligned}$$

$$\begin{aligned}
&\vdots\\
&\leq \sum_{q=0}^{p-1} \frac{e^{\frac{1}{k}\varphi^*(pmk)}}{e^{\frac{1}{k}\varphi^*((q+1)mk)}} C_0^{p-q} N^{qm}(Pu) + (C_0+1)^p e^{\frac{1}{k}\varphi^*(pmk)} N^0(u)\\
&\leq \sum_{q=0}^{p-1} \frac{e^{\frac{1}{k}\varphi^*(pmk)}}{e^{\frac{1}{k}\varphi^*((q+1)mk)}} C_1^{p-q} N^{qm}(Pu) + C_1^p e^{\frac{1}{k}\varphi^*(pmk)} N^0(u)
\end{aligned}$$

with $C_1 := C_0 + 1$.

Therefore, by the induction assumption and because of property (h) of Proposition 2.1,

$$\begin{aligned}
N^{pm}(u) \leq\; & \sum_{q=0}^{p-1} \frac{e^{\frac{1}{k}\varphi^*(pmk)}}{e^{\frac{1}{k}\varphi^*((q+1)mk)}} C_1^{p-q} C_1^q \sum_{r=0}^{q} \binom{q}{r} \frac{e^{\frac{1}{k}\varphi^*(qmk)}}{e^{\frac{1}{k}\varphi^*(rmk)}} N^0(P^r Pu)\\
& + C_1^p e^{\frac{1}{k}\varphi^*(pmk)} N^0(u)\\
\leq\; & C_1^p \sum_{r=0}^{p-1} \sum_{q=r}^{p-1} \frac{e^{\frac{1}{k}\varphi^*(pmk)}}{e^{\frac{1}{k}\varphi^*((r+1)mk)}} \binom{q}{r} N^0(P^{r+1}u)\\
& + C_1^p e^{\frac{1}{k}\varphi^*(pmk)} N^0(u).
\end{aligned} \tag{2.18}$$

Let us now remark that $\sum_{q=r}^{p-1} \binom{q}{r} = \binom{p}{r+1}$ and hence substituting in (2.18), we finally have:

$$\begin{aligned}
N^{pm}(u) \leq\; & C_1^p \sum_{r=0}^{p-1} \binom{p}{r+1} \frac{e^{\frac{1}{k}\varphi^*(pmk)}}{e^{\frac{1}{k}\varphi^*((r+1)mk)}} N^0(P^{r+1}u) + C_1^p e^{\frac{1}{k}\varphi^*(pmk)} N^0(u)\\
=\; & C_1^p \sum_{r'=0}^{p} \binom{p}{r'} \frac{e^{\frac{1}{k}\varphi^*(pmk)}}{e^{\frac{1}{k}\varphi^*(r'mk)}} N^0(P^{r'}u),
\end{aligned}$$

so that (2.17) is valid with $C_1 = 1 + C_0$. □

3 Proof of Theorems 1.4 and 1.5

We can now proceed with the

Proof of Theorem 1.4 Let us first prove that if $P(x, D)$ is elliptic then $\mathcal{E}^P_{\{\omega\}}(\Omega) \subseteq \mathcal{E}_{\{\omega\}}(\Omega)$. Let $u \in C^\infty(\Omega)$ satisfy (1.3) for every $K \subset\subset \Omega$. In particular it satisfies (1.3) for every relatively compact subdomain $G \subset \Omega$. From Proposition 2.4, for every fixed $\delta > 0$ and for all $p \in \mathbb{N}_0$

$$\begin{aligned}
\|\nabla^{pm}u\|_\delta \le\ & \delta^{-pm}N^{pm}(u) \le \delta^{-pm}C_1^p\sum_{q=0}^{p}\binom{p}{q}\frac{e^{\frac{1}{k}\varphi^*(pmk)}}{e^{\frac{1}{k}\varphi^*(qmk)}}N^0(P^q u)\\
\le\ & \delta^{-pm}C_1^p\sum_{q=0}^{p}\binom{p}{q}\frac{e^{\frac{1}{k}\varphi^*(pmk)}}{e^{\frac{1}{k}\varphi^*(qmk)}}\|P^q u\|_{L^2(G)}\\
\le\ & \delta^{-pm}C_1^p\sum_{q=0}^{p}\binom{p}{q}\frac{e^{\frac{1}{k}\varphi^*(pmk)}}{e^{\frac{1}{k}\varphi^*(qmk)}}ce^{\frac{1}{k}\varphi^*(qmk)}\\
\le\ & c(\delta^{-1}C_1^{1/m}2^{1/m})^{pm}e^{\frac{1}{k}\varphi^*(pmk)}\\
\le\ & cD_\delta\, e^{\frac{1}{k'}\varphi^*(pmk')} = \tilde{C}e^{\frac{1}{k'}\varphi^*(pmk')}
\end{aligned} \tag{3.1}$$

for some $k' \in \mathbb{N}$, D_δ , $\tilde{C} > 0$, because of (e) of Proposition 2.1.

By (2.6) (with $\sigma = \delta$, $\varepsilon = 1$, $f = \nabla^{pm}u$), and by (3.1), for all $1 \le t \le m-1$, $t' = m-t$, $q = pm+t$ we have, by the convexity of φ^*:

$$\begin{aligned}
\|\nabla^q u\|_{2\delta} =\ & \|\nabla^{pm+t}u\|_{2\delta} = \|\nabla^{m-t'}\nabla^{pm}u\|_{2\delta}\\
\le\ & C\left(\|\nabla^{(p+1)m}u\|_\delta + (\delta^{-m}+1)\|\nabla^{pm}u\|_\delta\right)\\
\le\ & C\tilde{C}\left[e^{\frac{1}{k'}\varphi^*((p+1)mk')} + (\delta^{-m}+1)e^{\frac{1}{k'}\varphi^*(pmk')}\right]\\
\le\ & C\tilde{C}(2+\delta^{-m})e^{\frac{1}{k'}\varphi^*(((p+1)m+t)k')}\\
\le\ & C\tilde{C}(2+\delta^{-m})e^{\frac{1}{2k'}\varphi^*(2(pm+t)k')}e^{\frac{1}{2k'}\varphi^*(2mk')}\\
=\ & C_\delta e^{\frac{1}{k''}\varphi^*(qk'')}
\end{aligned} \tag{3.2}$$

for $C_\delta = C\tilde{C}(2+\delta^{-m})e^{\frac{1}{2k'}\varphi^*(2mk')}$ and $k'' = 2k'$.

From (3.1) and (3.2), and by Sobolev inequality (cf. [15, Lemma 2.5]), we thus have that $u \in \mathcal{E}_{\{\omega\}}(G_{2\delta})$ for every fixed $\delta > 0$ and hence $u \in \mathcal{E}_{\{\omega\}}(\Omega)$.

Let us now show (i). Let $u \in \mathcal{E}_{\{\omega\}}(\Omega)$ and prove by induction on p that there exists $k \in \mathbb{N}$ such that for every $q \in \mathbb{N}_0$ there is $C_q > 0$ such that for every $K \subset\subset \Omega$

$$\|\nabla^q P^p u\|_{L^2(K)} \le C_q e^{\frac{1}{k}\varphi^*((q+pm)k)} \quad \forall p, q \in \mathbb{N}_0. \tag{3.3}$$

Indeed, for $p = 0$ (3.3) is valid because $u \in \mathcal{E}_{\{\omega\}}(\Omega)$. Let us assume (3.3) to be true for p, and all $q \in \mathbb{N}_0$, and prove it for $p+1$:

$$\|\nabla^q P^{p+1}u\|_{L^2(K)} = \|\nabla^q[P(P^p u)]\|_{L^2(K)} = \sum_{r=0}^{q}\binom{q}{r}\|P^{[r]}\nabla^{q-r}P^p u\|_{L^2(K)}$$

$$\le \sum_{r=0}^{q}\binom{q}{r}ce^{\frac{1}{k}\varphi^*(rk)}\sum_{s=0}^{m}\|\nabla^{q+s-r}(P^p u)\|_{L^2(K)}$$

$$= c\sum_{r=0}^{q}\frac{q!}{(q-r)!}a_{r,\frac{1}{k}}\|\nabla^{q+m-r}(P^p u)\|_{L^2(K)}$$
$$+ c\sum_{r=0}^{q}\frac{q!}{r!(q-r)!}e^{\frac{1}{k}\varphi^*(rk)}\sum_{s=0}^{m-1}\|\nabla^{q+s-r}(P^p u)\|_{L^2(K)} \tag{3.4}$$

for some $c > 0$ since $P(x, D)$ has coefficients in $\mathcal{E}_{\{\omega\}}(\Omega)$. By property (b) of Proposition 2.1 we have that, for $0 \le r \le q$,

$$\frac{q!}{(q-r)!}a_{r,\frac{1}{k}} \le \frac{q!}{(q-r)!}a_{q,\frac{1}{k}} \le q!a_{q,\frac{1}{k}}$$

and hence, substituting in (3.4) and separating the derivatives $\nabla^\sigma(P^p u)$ for $\sigma \ge m$ and $0 \le \sigma \le m-1$:

$$\begin{aligned}\|\nabla^q P^{p+1}u\|_{L^2(K)} &\le c\sum_{r=0}^{q}\frac{q!}{(q-r)!}a_{r,\frac{1}{k}}\|\nabla^{q+m-r}(P^p u)\|_{L^2(K)}\\ &\quad + mc\sum_{r=0}^{q}\frac{q!}{(q-r)!}a_{r,\frac{1}{k}}\|\nabla^{q+m-r}(P^p u)\|_{L^2(K)}\\ &\quad + mcq!a_{q,\frac{1}{k}}\sum_{\sigma=0}^{m-1}\|\nabla^\sigma P^p u\|_{L^2(K)}\\ &= (m+1)c\sum_{r=0}^{q}\frac{q!}{(q-r)!}a_{r,\frac{1}{k}}\|\nabla^{q+m-r}(P^p u)\|_{L^2(K)}\\ &\quad + mcq!a_{q,\frac{1}{k}}\sum_{\sigma=0}^{m-1}\|\nabla^\sigma(P^p u)\|_{L^2(K)}.\end{aligned}$$

By the inductive assumption (3.3) and by property (a) of Proposition 2.1 we have therefore that

$$\begin{aligned}&\|\nabla^q P^{p+1}u\|_{L^2(K)} \le (m+1)c\sum_{r=0}^{q}\frac{q!}{(q-r)!}a_{r,\frac{1}{k}}C_q e^{\frac{1}{k}\varphi^*((q+m-r+pm)k)}\\ &+ mcq!a_{q,\frac{1}{k}}\sum_{\sigma=0}^{m-1}C_q e^{\frac{1}{k}\varphi^*((\sigma+pm)k)}\\ &= (m+1)cC_q\Bigg[\sum_{r=0}^{q}\frac{q!}{(q-r)!}(q+(p+1)m-r)!a_{r,\frac{1}{k}}a_{q+(p+1)m-r,\frac{1}{k}}\\ &+ \sum_{\sigma=0}^{m-1}q!(\sigma+pm)!a_{q,\frac{1}{k}}a_{\sigma+pm,\frac{1}{k}}\Bigg]\end{aligned}$$

$$
\begin{aligned}
&\leq (m+1)cC_q\Bigg[\sum_{r=0}^{q}\frac{q!}{(q-r)!}(q+(p+1)m-r)!a_{q+(p+1)m,\frac{1}{k}}\\
&\quad+\sum_{\sigma=0}^{m-1}q!(\sigma+pm)!a_{q+\sigma+pm,\frac{1}{k}}\Bigg]\\
&=(m+1)cC_q\Bigg[\sum_{r=0}^{q}\frac{q!}{(q-r)!}\frac{(q+(p+1)m-r)!}{(q+(p+1)m)!}\\
&\quad+\sum_{\sigma=0}^{m-1}\frac{q!(\sigma+pm)!}{(q+(p+1)m)!}\Bigg]e^{\frac{1}{k}\varphi^*((q+(p+1)m)k)}\\
&\leq cC_q(m+1)(m+q)e^{\frac{1}{k}\varphi^*((q+(p+1)m)k)},
\end{aligned}
$$

since

$$
\frac{q!}{(q-r)!}\frac{(q+(p+1)m-r)!}{(q+(p+1)m)!}=\frac{\binom{q}{r}}{\binom{q+(p+1)m}{r}}\leq 1,
$$

and

$$
\frac{q!(\sigma+pm)!}{(q+(p+1)m)!}\leq\frac{1}{\binom{q+(p+1)m}{q}}\leq 1.
$$

Therefore (3.3) is proved by induction and, in particular, (1.3) holds true for $q=0$. The proof of Theorem 1.4 is therefore complete. □

Proof of Theorem 1.5 The proof of (i) is similar to the Roumieu case, Theorem 1.4(i), for $C_{q,k}$ and c_k instead of C_q and c.

However, since the constant C_1 of (2.17) depends on k, we cannot deduce formula (3.1) from (e) of Proposition 2.1. To prove (ii) we first remark that $\mathcal{E}_{\{\omega\}}(\Omega)\subseteq\mathcal{E}_{(\sigma)}(\Omega)$ for $\sigma(t)=o(\omega(t))$ as $t\to\infty$ by [6, Prop. 4.7]. Therefore by Theorem 1.4(ii) we have

$$
\mathcal{E}^P_{(\omega)}(\Omega)\subseteq\mathcal{E}^P_{\{\omega\}}(\Omega)\subseteq\mathcal{E}_{\{\omega\}}(\Omega)\subseteq\mathcal{E}_{(\sigma)}(\Omega)
$$

which concludes the proof in the Beurling case. □

We conclude proving that ellipticity is necessary in Theorems 1.4(ii) and 1.5(ii):

Example 3.1 Let $P(x,D)$ be a linear partial differential operator with real analytic coefficients of order m not elliptic in $(x_0,\xi_0)\in\Omega\times\mathbb{R}^n$, for a domain $\Omega\subseteq\mathbb{R}^n$ and $\|\xi_0\|=1$, i.e.

$$
P_m(x_0,\xi_0)=0,
$$

where P_m is the principal part of P. We are going to prove that there exist a function u and a subadditive weight ω, which is not a Gevrey weight in general and is between two given Gevrey weights, and such that $u \in \mathcal{E}^P_{\{\omega\}}(\Omega)\setminus\mathcal{E}_{\{\omega\}}(\Omega)$, and that $u \in \mathcal{E}^P_{(\omega)}(\Omega)\setminus\mathcal{E}_{(\sigma)}(\Omega)$, for some subadditive weight function $\sigma = o(\omega)$. Consequently, the ellipticity of P is needed for statement (ii) of Theorems 1.4 and 1.5. To construct ω and the function u we follow [21]: for any fixed $s > 1$ we choose $\sigma \in (1, s)$ and $\varepsilon > 0$ such that

$$0 < \varepsilon < \frac{m(s-\sigma)}{2ms-\sigma} < \frac{1}{2}.$$

Then we take $\delta > 0$ so that $B(x_0, 2\delta) \subset\subset \Omega$ and $\varphi \in \mathcal{E}_{(t^{1/\sigma})}(\mathbb{R}^n)$ with $\operatorname{supp}\varphi \subset B(0, 2\delta)$. For $\eta = \frac{m-\varepsilon}{ms}$ we finally define, as in [21],

$$u(x) := \int_1^{+\infty} \varphi\big(\rho^\varepsilon(x-x_0)\big)e^{-\rho^\eta}e^{i\rho\langle x-x_0,\xi_0\rangle}d\rho\,.$$

It was proved in [21] that

$$(D^\alpha_{\xi_0}u)(x_0) = \frac{1}{\eta}\Gamma\left(\frac{\alpha+1}{\eta}\right) + o(1), \tag{3.5}$$

where Γ is the gamma function, so that $u \notin \mathcal{E}_{\{t^{1/s'}\}}(U)$ in any neighborhood U of x_0 for any $s' < 1/\eta$ (nor, in particular, for $s' = s$), but $u \in \mathcal{E}_{\{t^\eta\}}(\mathbb{R}^n)$. Moreover, it was proved in [21] that $u \in \mathcal{E}^P_{\{t^{1/s}\}}(\Omega)$.

Let us now consider any subadditive weight function $\omega(t)$ such that $\omega(t) = o(t^{1/s})$ and $t^{1/s'} = o(\omega(t))$ for $s' > s > 1$. For instance, $\omega(t) = t^{1/s}/\log t$. In general, such a weight exists by [6, Proposition 1.9].

We have that $\mathcal{E}_{(\omega)}(\Omega) \subseteq \mathcal{E}_{\{\omega\}}(\Omega) \subseteq \mathcal{E}_{\{t^{1/s'}\}}(\Omega)$ and $\mathcal{E}_{\{t^{1/s}\}}(\Omega) \subseteq \mathcal{E}_{(\omega)}(\Omega) \subseteq \mathcal{E}_{\{\omega\}}(\Omega)$ by [6, Prop. 4.7]. Analogously $\mathcal{E}^P_{\{t^{1/s}\}}(\Omega) \subseteq \mathcal{E}^P_{(\omega)}(\Omega) \subseteq \mathcal{E}^P_{\{\omega\}}(\Omega)$, so that $u \in \mathcal{E}^P_{\{\omega\}}(\Omega)\setminus\mathcal{E}_{\{\omega\}}(\Omega)$ and ellipticity is necessary in Theorem 1.4 (ii).

Moreover, if $\sigma(t) := t^{1/s'}$ we clearly have $u \in \mathcal{E}^P_{(\omega)}(\Omega)\setminus\mathcal{E}_{(\sigma)}(\Omega)$. Since $\sigma(t) = o(\omega(t))$ as $t \to \infty$, this proves that ellipticity is necessary in Theorem 1.5 (ii).

References

1. Boiti, C., Jornet, D.: The problem of iterates in some classes of ultradifferentiable functions. Oper. Theory Adv. Appl. Birkhauser Basel **245**, 21–33 (2015)
2. Boiti, C., Jornet, D.: A characterization of the wave front set defined by the iterates of an operator with constant coefficients. arXiv:1412.4954
3. Boiti, C., Jornet, D., Juan-Huguet, J.: Wave front set with respect to the iterates of an operator with constant coefficients. Abstr. Appl. Anal. **2014**, 1–17 Article ID 438716 (2014). doi:10.1155/2014/438716
4. Bolley, P., Camus, J., Mattera, C.: Analyticité microlocale et itérés d'operateurs hypoelliptiques. Séminaire Goulaouic-Schwartz, 1978–1979, Exp No. 13, École Polytech, Palaiseau
5. Bonet, J., Meise, R., Melikhov, S.N.: A comparison of two different ways of define classes of ultradifferentiable functions. Bull. Belg. Math. Soc. Simon Stevin **14**, 425–444 (2007)
6. Braun, R.W., Meise, R., Taylor, B.A.: Ultradifferentiable functions and Fourier analysis. Result. Math. **17**, 206–237 (1990)

7. Fernández, C., Galbis, A.: Superposition in classes of ultradifferentiable functions. Publ. Res. I Math. Sci. **42**(2), 399–419 (2006)
8. Jornet Casanova, D.: Operadores Pseudodiferenciales en Clases no Casianalíticas de Tipo Beurling. Universitat Politècnica de València (2004). doi:10.4995/Thesis/10251/54953
9. Juan-Huguet, J.: Iterates and hypoellipticity of partial differential operators on non-quasianalytic classes. Integr. Equ. Oper. Theory **68**, 263–286 (2010)
10. Juan-Huguet, J.: A Paley–Wiener type theorem for generalized non-quasianalytic classes. Stud. Math. **208**(1), 31–46 (2012)
11. Komatsu, H.: A characterization of real analytic functions. Proc. Jpn Acad. **36**, 90–93 (1960)
12. Komatsu, H.: On interior regularities of the solutions of principally elliptic systems of linear partial differential equations. J. Fac. Sci. Univ. Tokyo Sect. 1, **9**, 141–164 (1961)
13. Komatsu, H.: A proof of Kotaké and Narasimhan's theorem. Proc. Jpn Acad. **38**(9), 615–618 (1962)
14. Kotake, T., Narasimhan, M.S.: Regularity theorems for fractional powers of a linear elliptic operator. Bull. Soc. Math. Fr. **90**, 449–471 (1962)
15. Kumano-Go, H.: Pseudo-Differential Operators. The MIT Press, Cambridge, London (1982)
16. Langenbruch, M.: P-Funktionale und Randwerte zu hypoelliptischen Differentialoperatoren. Math. Ann. **239**(1), 55–74 (1979)
17. Langenbruch, M.: Fortsetzung von Randwerten zu hypoelliptischen Differentialoperatoren und partielle Differentialgleichungen. J. Reine Angew. Math. 311/312, 57–79 (1979)
18. Langenbruch, M.: On the functional dimension of solution spaces of hypoelliptic partial differential operators. Math. Ann. **272**, 217–229 (1985)
19. Langenbruch, M.: Bases in solution sheaves of systems of partial differential equations. J. Reine Angew. Math. **373**, 1–36 (1987)
20. Lions, J.L., Magenes, E.: Problèmes aux limites non homogènes et applications, vol. 3. Dunod, Paris (1970)
21. Métivier, G.: Propriété des itérés et ellipticité. Commun. Part. Differ. Eq. **3**(9), 827–876 (1978)
22. Nelson, E.: Analytic vectors. Ann. Math. **70**, 572–615 (1959)
23. Newberger, E., Zielezny, Z.: The growth of hypoelliptic polynomials and Gevrey classes. Proc. Am. Math. Soc. **39**(3), 547–552 (1973)
24. Oldrich, J.: Sulla regolarità delle soluzioni delle equazioni lineari ellittiche nelle classi di Beurling. (Italian) Boll. Un. Mat. Ital. (4) **2**, 183–195 (1969)
25. Petzsche, H.-J., Vogt, D.: Almost analytic extension of ultradifferentiable functions and the boundary values of holomorphic functions. Math. Ann. **267**(1), 17–35 (1984)