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On a particular family of Fourier integral operators

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Abstract Modern analysis, and in particular microlocal analysis, without *Fourier integral operator*s (FIOs) and *pseudo-differential operator*s (ψDOs) seems to have difficulty in acting. In these theories, *symbol*s lie in a certain category of infinitely differentiable functions. Here, we introduce a class of *supersymbol*s which generalize the usual concept of symbols. In this way, we can introduce particular families of ψDO (denoted by SψDO) and FIO (denoted by SFIO) with supersymbols and a kind of *phase function*. Finally, we try to make a SψDO from a SFIO under some additional conditions.

Keywords Microlocal analysis · Fourier integral operator · Pseudo-differential operator · Supersymbol

Mathematics Subject Classification Primary 35S30; Secondary 35S05

1 Introduction

From 1950s which the modern techniques of analysis began to develop, Fourier analysis became one of the main branches. This area, particularly relates to the study of Fourier integral operators, pseudo-differential operators, generalized functions, and the class of symbols. In the microlocal analysis, usually symbols lie in a certain class of infinitely differentiable functions [5,6]. But in the physics observations we often

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deal with functions which vary with respect to time and thus make *net*s of functions. Here, we deal with a class of symbols that vary in respect of time and are integrable with respect to a given measure introduced in [2]. Using this class of symbols we obtain generalized notion of supersymbols and supersingular pseudo-differential operators. In the sequel, suppose that N is a fixed natural number.

Definition 1.1 Let $\psi(x,\theta) \in C^\infty(\mathbb{R}^N \times \mathbb{R}^N)$. By ψ lies in $S^m_{\rho,\delta}$ we mean that for all $(\alpha, \beta, n) \in \mathbb{Z}^N_+ \times \mathbb{Z}^N_+ \times \mathbb{Z}_+$,

$$\|\psi : \alpha, \beta, n : S^m_{\rho,\delta}\| := \sup_{|x|\leq n} \sup_{\theta\in\mathbb{R}^N} |\partial^\alpha_\theta \partial^\beta_x \psi(x,\theta)| \langle\theta\rangle^{-m+\rho|\alpha|-\delta|\beta|} < \infty,$$

where $\langle\theta\rangle = (1+|\theta|^2)^{\frac{1}{2}}$. When $\rho = 1$ and $\delta = 0$, we simply write S^m instead of $S^m_{1,0}$.

Pseudo-differential operators (ψDOs) are generated from $S^m_{\rho,\delta}$ symbols by the formula

$$\mathrm{OP}\psi : \mathcal{E}'(\mathbb{R}^N) \longrightarrow \mathcal{D}'(\mathbb{R}^N),$$
$$f \longmapsto \int e^{ix\theta}\psi(x,\theta)\hat{f}(\theta)d\theta.$$

(Recall that $f(\theta) = \int e^{-iy\theta} f(y)dy$ is the *Fourier transform* of f). A ψDO is continuous on Sobolev and Lebesgue spaces:

$$\mathrm{OP} : S^m \to \mathcal{L}(H^s_{comp}(\mathbb{R}^N), H^{s-m}_{loc}(\mathbb{R}^N)), \quad s \in \mathbb{R},$$
$$\mathrm{OP} : S^0 \to \mathcal{L}(L^p_{comp}(\mathbb{R}^N), H^p_{loc}(\mathbb{R}^N)), \quad 1/p \in (0,1).$$

Moreover, both of OP above are continuous [4], the former on all Freshet spaces S^m, $m \in \mathbb{R}$, with seminorms $\|\psi : \alpha, \beta, n : S^m_{\rho,\delta}\|$.

Definition 1.2 Let σ be a complex (possibly infinite) measure on $\mathbb{R}^N$. A map $\psi : \mathbb{R}^N \to S^m_{\rho,\delta}$ for $m \in \mathbb{R}$, is called a supersymbol if for all $(\alpha, \beta, n) \in \mathbb{Z}^N_+ \times \mathbb{Z}^N_+ \times \mathbb{Z}_+$,

$$\int \|\psi(t) : \alpha, \beta, n; S^m_{\rho,\delta}\| d\sigma(t) < \infty.$$

Then we write $\psi \in SS^m_{\rho,\delta}(\sigma)$, or $(\psi, \sigma) \in SS^m_{\rho,\delta}$. Each supersymbol together with the measure generates a supersingular pseudo-differential operator (abbr. sψDO),

$$T(\psi,\sigma) := \int \mathrm{OP}\psi(t) \circ Sh_t d\sigma(t).$$

(Here, $Sh_t : f(x) \mapsto f(x-t)$ is the translation or shift operator).

As usual, notations $\mathrm{OP}SS^m_{\rho,\delta}(\sigma)$ and $\mathrm{OP}SS^{-\infty}$ etc. stand for the spaces of operators generated by the corresponding spaces of supersymbols, i.e., $SS^m_{\rho,\delta}(\sigma)$ and $SS^{-\infty} \equiv \cap_m SS^m_{\rho,\delta}$.

As a trivial case, when σ is the unit measure $\delta(t)$ supported at the origin, $T(\psi, \sigma)$ is the pseudo-differential operator $\mathrm{OP}\psi(0)$.

Theorem 1.3 *Each* SψDO $T(\psi, \sigma)$ *is continuous on Sobolev and Lebesgue spaces;*

$$T(\psi, \sigma) : H^s_{comp}(\mathbb{R}^N) \to H^{s-m}_{loc}(\mathbb{R}^N), \qquad \psi \in SS^m(\sigma), \quad \in \mathbb{R},$$
$$T(\psi, \sigma) : L^p_{comp}(\mathbb{R}^N) \to L^p_{loc}(\mathbb{R}^N), \qquad \psi \in SS^0(\sigma), \quad 1/p \in (0, 1).$$

Applying the SψDO $T(\psi, \sigma)$ on a smooth function f,

$$\begin{aligned} T(\psi, \sigma)(f) &= \int e^{ix\theta} \psi(t)(x, \theta)(\widehat{Sh_t f})(\theta) d\theta d\sigma(t) \\ &= \int e^{ix\theta} \psi(t)(x, \theta) e^{-iy\theta} f(y - t) dy d\theta d\sigma(t) \end{aligned}$$

which with a change of variable, it is equal to

$$\int e^{i(x-y-t)\theta} \psi(t)(x, \theta) f(y) dy d\theta d\sigma(t),$$

which is another form of SψDO.

2 Main results

Recall that the Fourier integral operators are of the form

$$\mathrm{F}(f)(x) = \int e^{i\phi(x,y,\theta)} \psi(x, \theta) f(y) dy d\theta,$$

where $\phi(x, y, \theta)$ is a *phase function* and $f(y)$ belongs to the *Schwartz space* $\mathcal{S}(\mathbb{R}^N)$ [1].

In a special case, it is well known that Fourier integral operator can be a pseudo-differential operator under some conditions. Our aim is to do for a Fourier integral operator but with supersymbol.

Definition 2.1 Let ψ be a supersymbol and ϕ a phase function such that $Sh_t \circ \phi(x, y, \theta) = \phi(x - t, y, \theta)$. Then we define a Fourier integral operator with supersymbole ψ and shifted phase function ϕ (denoted by SFIO) as follow

$$\mathrm{SF}(f)(x) := \int e^{i.Sh_t \circ \phi(x,y,\theta)} \psi(t)(x, \theta) dy\, d\theta\, d\sigma(t).$$

Applying Lemma 4.1 of [3] in which Φ is replaced by the phase function $Sh_t \circ \phi$, the required result will be achieved. Here, we establishe that change of variable is well-defined.

Note that, it makes sense to say that $\psi(t)(x,\theta) \in S^m_{\rho,\delta}(U)$, where U is an arbitrary region in $\mathbb{R}^N \times \mathbb{R}^N$, which is conic with respect to θ. Indeed, $\psi(t)(x,\theta) \in S^m_{\rho,\delta}(U)$, if for any compact set $K \subset (\mathbb{R}^N \times S^{N-1}) \cap U$ (S^{N-1} is the unit sphere in $\mathbb{R}^N$) and for arbitrary multi-indices α and β there is a constant $C_{\alpha,\beta,K} > 0$ such that

$$|\partial_\theta^\alpha \partial_x^\beta \psi(t)(x,\theta)| \leq C_{\alpha,\beta,K} \langle\theta\rangle^{m-\rho|\alpha|+\delta|\beta|}$$

for $(x, \theta/|\theta|) \in K$ and $|\theta| \geq 1$. For a diffeomorphism from a conical region $V \subset \mathbb{R}^N \times \mathbb{R}^N$ onto the conical region $U \subset \mathbb{R}^N \times \mathbb{R}^N$, commutating with the natural action of the multiplication group $\mathbb{R}_+$ of positive numbers, i.e., the diffeomorphism maps a point $(y,\eta) \in V$ to another point

$$(x(y,\eta), \theta(y,\eta)) \in U, \tag{2.1}$$

where $x(y,\eta)$ and $\theta(y,\eta)$ are positively homogeneous in η of degree 0 and 1, respectively. Then changing the variables in $\psi(t)(x,\theta)$:

$$\zeta(t)(y,\eta) = \psi(t)(x(y,\eta), \theta(y,\eta)), \tag{2.2}$$

yields that $\zeta(t)(y,\eta)$ belongs to the class of $S^m_{\rho,\delta}(V)$. A similar argument could be given in the the case of $SS^m_{\rho,\delta}$.

Here is the main theorem in which we want to reduce SFIO to SψDO.

Theorem 2.2 *Let ϕ_t be a phase function in $X \times X \times \mathbb{R}^n$, such that*

1. *$\phi_t(x,y,\theta)$ is linear in θ,*
2. *$(\phi_t)'_\theta(x,y,\theta) = 0 \Leftrightarrow x - t = y$.*

Let F be a SFIO *with phase function $\phi_t(x,y,\theta)$ and $\psi(t)(x,\theta) \in SS^m_{\rho,\delta}(X \times \mathbb{R}^N)$ where*

$$1 - \rho \leq \delta < \rho. \tag{2.3}$$

Then F is also a SψDO *with symbol in $SS^m_{\rho,\delta}$.*

To prove this theorem first we need the following proposition which makes it easy to do. At the last section a review of the proof of the theorem could be found.

Proposition 2.3 *Let $\psi(t)(x,\theta) \in SS^m_{\rho,\delta}(U)$ and assume that one of the following three conditions holds:*

1. *$\rho + \delta \leq 1$,*
2. *$\rho + \delta \geq 1$, and $x = x(y)$ does not depend on η, where η is mentioned in (2.1),*
3. *$x = x(y), \theta = \theta(\eta)$.*

Then $\zeta(t)(y,\eta) \in SS^m_{\rho,\delta}(V)$, where ζ is mentioned in (2.2).

Proof First, suppose that $|\alpha+\beta| \le 1$. Differentiating $\zeta(t)(y,\eta)$, we get

$$\frac{\partial \zeta(t)}{\partial \eta_r} = \sum_i \psi^{(i)} \frac{\partial x_i}{\partial \eta_r} + \sum_j \psi_{(j)} \frac{\partial \theta_j}{\partial \eta_r},$$
$$\frac{\partial \zeta(t)}{\partial y_s} = \sum_i \psi^{(i)} \frac{\partial x_i}{\partial y_s} + \sum_j \psi_{(j)} \frac{\partial \theta_j}{\partial y_s},$$

where $\psi^{(i)} = \frac{\partial \psi(t)}{\partial x_i}(x(y,\eta),\theta(y,\eta))$ and $\psi_{(j)} = \frac{\partial \psi(t)}{\partial \theta_j}(x(y,\eta),\theta(y,\eta))$. The functions $\frac{\partial x_i}{\partial \eta_r}$, $\frac{\partial \theta_j}{\partial \eta_r}$, $\frac{\partial x_i}{\partial y_s}$, and $\frac{\partial \theta_j}{\partial y_s}$ are positively homogeneous in θ of degree -1, 0, 0, and, 1, respectively, so they belong to the classes S^{-1}, S^0, S^0, and, S^1, respectively in V (since $x(y,\eta)$ and $\theta(y,\eta)$ are homogeneous in η of degree 0 and 1, respectively, so they belong to the classes S^0 and S^1, respectively). We need to estimate the integral

$$\int \sup_{|y|<n} \sup_{\eta\in\mathbb{R}^N} |\partial_\eta^\alpha \partial_y^\beta \zeta(t)(y,\eta)| \langle\eta\rangle^{-m+\rho|\alpha|-\delta|\beta|} d\sigma(t).$$

Using the definition of S^m and estimating the derivatives of ψ, for $|\eta| \ge 1$ we easily obtain

$$\left|\frac{\partial \zeta(t)}{\partial \eta_r}\right| \le C_K \left(|\eta|^{m+\delta-1} + |\eta|^{m-\rho}\right), \qquad \left(y, \frac{\eta}{|\eta|}\right) \in K,$$
$$\left|\frac{\partial \zeta(t)}{\partial y_s}\right| \le C_K \left(|\eta|^{m+\delta} + |\eta|^{m-\rho+1}\right), \qquad \left(y, \frac{\eta}{|\eta|}\right) \in K,$$

where K is a compact set in V. If $m+\delta-1 \le m+\rho$, i.e., $\rho+\delta \le 1$, then $\left|\frac{\partial \zeta(t)}{\partial \eta_r}\right| \le 2C_K\langle\eta\rangle^{m-\rho}$. If $x = x(y)$ then $\frac{\partial x_k}{\partial \eta_r} = 0$ and we do not need the condition $\rho+\delta \le 1$.

Similarly, if $m-\rho+1 \le m+\delta$, i.e., $\rho+\delta \ge 1$, it follows that $\left|\frac{\partial \zeta(t)}{\partial y_s}\right| \le 2C_K\langle\eta\rangle^{m+\delta}$ and if $\theta = \theta(\eta)$, the estimate is obtained without another assumption. Thus $\zeta(t)(y,\eta)$ would lie in $S^m_{\rho,\delta}$ and it only remains to confirm the condition of being supersymbol. Since $\psi(t)(x,\theta) \in S^m$, we get $|\partial_\theta^\alpha \partial_x^\beta \psi(t)(x,\theta)| \le C(t)\langle\theta\rangle^{m-\rho|\alpha|+\delta|\beta|}$. But $\psi(t)(x,\theta) \in SS^m$, so $\int \|\psi(t) : \alpha,\beta,n; S^m_{\rho,\delta}\| d\sigma(t) < \infty$, in which by considering $C(t)$ as infimum of the upper bound of the norm,

$$\begin{aligned}
&\int \sup_{|x|<n} \sup_{\theta\in\mathbb{R}^N} |\partial_\theta^\alpha \partial_x^\beta \xi(t)(x,\theta)| \langle\theta\rangle^{-m+\rho|\alpha|-\delta|\beta|} d\sigma(t) \\
&\le \int C(t)\langle\theta\rangle^{m-\rho|\alpha|+\delta|\beta|}\langle\theta\rangle^{-m+\rho|\alpha|-\delta|\beta|} d\sigma(t) \\
&= \int C(t) d\sigma(t) < \infty.
\end{aligned}$$

There is some constant A such that

$$\left|\frac{\partial \zeta(t)}{\partial \eta_r}\right| \leq AC(t)\langle \eta \rangle^{m-\rho},$$

and also some constant B such that

$$\left|\frac{\partial \zeta(t)}{\partial y_s}\right| \leq BC(t)\langle \eta \rangle^{m+\delta}.$$

Thus $\zeta(t)$ belongs to class of SψDOs, which proves the statement in the case of $|\alpha+\beta| \leq 1$.

Now suppose that the estimates hold when $|\alpha+\beta| \leq k$ for arbitrary $\psi(t) \in SS^m_{\rho,\delta}(U)$. In particular, for the derivatives of order $\leq k$ of $\psi^{(i)}$ and $\psi_{(i)}$ the estimates of the classes $S^{m-\rho}_{\rho,\delta}(V)$ and $S^{m+\delta}_{\rho,\delta}(V)$ hold, respectively. By a similar argument, these estimates hold when derivatives of order $\leq (k+1)$ and for arbitrary $\psi(t) \in SS^m_{\rho,\delta}(U)$. □

Now it is easy to prove the main theorem by using the above statement.

3 Proof of Theorem 2.2

Proof In view of Proposition 2.1 and Exercise 2.4 of [3], we may assume that $\psi(t)(x,\theta) = 0$ and also $(x, y) \notin \Omega'$ where Ω' is any neighbourhood of the diagonal $\Delta \subset X \times X$. Making the change of variable $\theta = \rho(x, y)(\xi)$ in the Fourier integral operator, and replacing x by $x - t$, we have

$$F(f)(x) = \int e^{i(x-y-t)\xi} \psi(t)(x, \rho(x, y)\xi)|\det \rho(x, y)| f(y) dy d\xi d\sigma(t).$$

Finally, we have to show $\psi_1(t)(x,\xi) = \psi(t)(x, \rho(x, y)\xi)$ belongs to the class of supersymbols. Now by using Proposition 2.3 it follows from the condition (2.3) that $\psi_1(t)(x,\xi) \in SS^m_{\rho,\delta}$ and this proves the theorem. □

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