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Traces of pseudo-differential operators on $\mathbb{S}^{n-1}$

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Abstract We give a characterization of and a trace formula for trace class pseudo-differential operators on the unit sphere $\mathbb{S}^{n-1}$ centered at the origin in $\mathbb{R}^n$.

Keywords Pseudo-differential operators on $\mathbb{S}^{n-1}$ · Hilbert–Schmidt operators · Trace class operators · Traces · Spherical harmonics

Mathematics Subject Classification (2010) Primary 47G30; Secondary 42A45

1 Introduction

In the paper [5] and the book [9], a characterization of Hilbert–Schmidt pseudo-differential operators on the unit circle $\mathbb{S}^1$ centered at the origin is given. The condition given on the symbol is L^2 in nature and is different from Hörmander's S^m condition that is prevalent in the study of partial differential equations [4,8]. The L^2 and the related L^p, $1 \leq p \leq \infty$, conditions on the symbols allow singularities and are ideal for a broad spectrum of disciplines ranging from functional analysis to operator algebras and to time-frequency analysis. Trace class pseudo-differential operators on $L^p(\mathbb{S}^1)$ and their traces can be found in the paper [3].

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The aim of this paper is to give a characterization of trace class pseudo-differential operators on the unit sphere $\mathbb{S}^{n-1}$ with center at the origin in $\mathbb{R}^n$ and we give a formula for the trace of each such trace class operator. Spherical harmonics on $\mathbb{S}^{n-1}$ are the natural analogs of Fourier series when we pass from the circle $\mathbb{S}^1$ to the sphere $\mathbb{S}^{n-1}$. The main technique is to obtain a formula for the symbol of the product of two pseudo-differential operators T_σ and T_τ on $\mathbb{S}^{n-1}$.

We give a brief recall of Hilbert–Schmidt operators and trace class operators in Sect. 2. In Sect. 3 is a summary of results on spherical harmonics on $\mathbb{S}^{n-1}$ that we need in this paper. We give a product formula for pseudo-differential operators, a characterization of trace class pseudo-differential operators and then a trace formula for pseudo-differential operators on $\mathbb{S}^1$ in Sect. 4. The same is done for $\mathbb{S}^{n-1}$ in Sect. 5.

The study of trace class operators and their traces can best be put in the context of noncommutative measure theory in which trace class operators are the noncommutative analogs of measurable functions and traces are noncommutative analogs of integrals. See the book [2] by Alain Connes in this connection.

2 Hilbert–Schmidt operators and trace class operators

Let X be a complex and separable Hilbert space in which the inner product and norm are denoted by $(\ ,\ )_X$ and $\|\ \|_X$ respectively. Let $A : X \to X$ be a compact operator. Then the absolute value $|A|$ of the operator A is defined by

$$|A| = (A^*A)^{1/2}.$$

The operator $|A|$ is compact and positive. So, by the spectral theorem, there exists an orthonormal basis $\{\varphi_j\}_{j=1}^{\infty}$ of eigenvectors of $|A|$. For $j = 1, 2, \ldots,$ let s_j be the eigenvalue of $|A|$ corresponding to the eigenvectors φ_j. We call the numbers $s_j,\ j = 1, 2, \ldots,$ the singular values of A. The operator is said to be in the Hilbert–Schmidt class S_2 if

$$\sum_{j=1}^{\infty} s_j^2 < \infty$$

and in the trace class S_1 if

$$\sum_{j=1}^{\infty} s_j < \infty.$$

We need the following three theorems on Hilbert–Schmidt operators and trace class operators, which can be found in, say, the book [6] by Reed and Simon.

Theorem 2.1 *Let $A : X \to X$ be a bounded linear operator. Then $A \in S_2$ if and only if there exists an orthonormal basis $\{\varphi_j\}$ for X such that*

$$\sum_{j=1}^{\infty} \|A\varphi_j\|_X^2 < \infty.$$

Theorem 2.2 *If $A : X \to X$ is in S_1, then for every orthonormal basis $\{\varphi_j\}_{j=1}^{\infty}$ for X, $\sum_{j=1}^{\infty}(A\varphi_j, \varphi_j)_X$ converges absolutely and the limit is independent of the choice of the orthonormal basis.*

Let $A : X \to X$ be in the trace class. Then we define the trace $\operatorname{tr}(A)$ by

$$\operatorname{tr}(A) = \sum_{j=1}^{\infty}(A\varphi_j, \varphi_j)_X,$$

where $\{\varphi_j\}_{j=1}^{\infty}$ is any orthonormal basis for X.

Theorem 2.3 *Let $A : X \to X$ be a bounded linear operator. Then $A \in S_1$ if and only if $A = BC$, where B and C are in S_2.*

Let σ be a measurable function on $\mathbb{S}^1 \times \mathbb{Z}$. Then for all f in $L^2(\mathbb{S}^1)$, we define the function $T_\sigma f$ on $\mathbb{S}^1$ formally by

$$(T_\sigma f)(\theta) = \sum_{n=-\infty}^{\infty} e^{in\theta}\sigma(\theta, n)\hat{f}(n), \quad \theta \in [-\pi, \pi].$$

We call T_σ the pseudo-differential operator associated to the symbol σ.
The following result can be found in [5,9].

Theorem 2.4 *The pseudo-differential operator $T_\sigma : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is in S_2 if and only if $\sigma \in L^2(\mathbb{S}^1 \times \mathbb{Z})$. Moreover, if $T_\sigma : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is in S_2, then*

$$\|T_\sigma\|_{S_2} = (2\pi)^{-1/2}\|\sigma\|_{L^2(\mathbb{S}^1\times\mathbb{Z})},$$

where $\| \; \|_{S_2}$ is the Hilbert–Schmidt norm in S_2.

3 Spherical harmonics

Let P be a homogeneous polynomial on $\mathbb{R}^n$ of degree k, which is harmonic on $\mathbb{R}^n$, $i.e.$,

$$P(x) = \sum_{|\alpha|=m} a_\alpha x^\alpha$$

and

$$(\Delta P)(x) = 0, \quad x \in \mathbb{R}^n,$$

where

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}.$$

A spherical harmonic Y_k of degree k is the restriction to $\mathbb{S}^{n-1}$ of a harmonic and homogeneous polynomial of degree k. Let $\mathcal{H}_k$ denote the space of all spherical harmonics of degree k. The collection of all finite linear combinations of elements in $\bigcup_{k=1}^{\infty} \mathcal{H}_k$ is dense in $L^2(\mathbb{S}^{n-1})$. If $\{Y_1^{(k)}, Y_2^{(k)}, \ldots, Y_{a_k}^{(k)}\}$ is an orthonormal basis for $\mathcal{H}_k$, then $\bigcup_{k=1}^{\infty}\{Y_1^{(k)}, Y_2^{(k)}, \ldots, Y_{a_k}^{(k)}\}$ forms an orthonormal basis for $L^2(\mathbb{S}^{n-1})$. Thus, for $f \in L^2(\mathbb{S}^{n-1})$, f can be expanded into a Fourier series to the effect that

$$f = \sum_{k=0}^{\infty} \sum_{j=1}^{a_k} (f, Y_j^{(k)}) Y_j^{(k)},$$

where $(\ ,\ )$ is the inner product in $L^2(\mathbb{S}^{n-1})$. In the case when $n = 2$, the expansion of a function in $L^2(\mathbb{S}^1)$ into spherical harmonics on $\mathbb{S}^1$ coincides with the usual Fourier series expansion when polar coordinates are used. See, for instance, the books [1,7] for more details on spherical harmonics.

Let σ be a measurable function on $\mathbb{S}^{n-1} \times \mathbb{N} \times \mathbb{N}$. For all functions f in $L^2(\mathbb{S}^{n-1})$, we define the function $T_\sigma f$ on $\mathbb{S}^{n-1}$ *formally* by

$$(T_\sigma f)(\omega) = \sum_{k=0}^{\infty} \sum_{j=1}^{a_k} \sigma(\omega, k, j)(f, Y_j^{(k)}) Y_j^{(k)}(\omega), \quad \omega \in \mathbb{S}^{n-1}.$$

We call T_σ the pseudo-differential operator corresponding to the symbol σ.

4 Pseudo-differential operators on $\mathbb{S}^1$

We first derive a formula for the product of two pseudo-differential operators T_σ and T_τ on $\mathbb{S}^1$, where σ and τ are suitable measurable functions on $\mathbb{S}^1 \times \mathbb{Z}$. Indeed, let f be a suitable measurable function on $\mathbb{S}^1$. Then for all $\theta \in [-\pi, \pi]$,

$$\begin{aligned}
(T_\sigma T_\tau f)(\theta) &= \sum_{n=-\infty}^{\infty} \sigma(\theta, n)(T_\tau f)^\wedge(n) e^{in\theta} \\
&= \sum_{n=-\infty}^{\infty} \sigma(\theta, n) \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} e^{-in\phi} (T_\tau f)(\phi)\, d\phi \right) e^{in\theta}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=-\infty}^{\infty} \sigma(\theta, n) \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} e^{-in\phi} \sum_{m=-\infty}^{\infty} \tau(\phi, m) \hat{f}(m) e^{im\phi} d\phi \right) e^{in\theta} \\
&= \sum_{m=-\infty}^{\infty} \frac{1}{2\pi} \left(\sum_{n=-\infty}^{\infty} \sigma(\theta, n) \int_{-\pi}^{\pi} e^{-i(n-m)\phi} e^{i(n-m)\theta} \tau(\phi, m)\, d\phi \right) \hat{f}(m) e^{im\theta} \\
&= \sum_{m=-\infty}^{\infty} \frac{1}{2\pi} \left(\sum_{n=-\infty}^{\infty} \sigma(\theta, n) \int_{-\pi}^{\pi} e^{-(m-n)(\theta-\phi)} \tau(\phi, m)\, d\phi \right) \hat{f}(m) e^{im\theta} \\
&= \sum_{m=-\infty}^{\infty} \frac{1}{2\pi} \left(\sum_{n=-\infty}^{\infty} \sigma(\theta, n) \int_{-\pi}^{\pi} e^{-i(m-n)\phi} \tau(\theta - \phi, m)\, d\phi \right) \hat{f}(m) e^{im\theta} \\
&= \sum_{m=-\infty}^{\infty} \lambda(\theta, m) \hat{f}(m) e^{im\theta},
\end{aligned}
$$

where

$$\lambda(\theta, m) = (\sigma \circledast \tau)(\theta, m),$$

and

$$(\sigma \circledast \tau)(\theta, m) = \frac{1}{2\pi} \left(\sum_{n=-\infty}^{\infty} \sigma(\theta, n) \int_{-\pi}^{\pi} e^{-i(m-n)\phi} \tau(\theta - \phi, m) d\phi \right).$$

Thus,

$$T_\sigma T_\tau = T_{\sigma \circledast \tau}.$$

For $n \in \mathbb{Z}$, we define the function e_n on $[-\pi, \pi]$ by

$$e_n(\theta) = (2\pi)^{-1/2} e^{in\theta}, \quad \theta \in [-\pi, \pi].$$

Then $\{e_n : n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{S}^1)$. Furthermore, let $\sigma \in L^2(\mathbb{S}^1 \times \mathbb{Z})$. Then for $n \in \mathbb{Z}$,

$$\begin{aligned}(T_\sigma e_n)(\theta) &= \sum_{k=-\infty}^{\infty} e^{ik\theta}\sigma(\theta,k)\widehat{e_n}(k)\\ &= \sum_{k=-\infty}^{\infty} e^{ik\theta}\sigma(\theta,k)(2\pi)^{-3/2}\int_{-\pi}^{\pi} e^{-ik\phi}e^{in\phi}d\phi\\ &= (2\pi)^{-1/2}e^{in\theta}\sigma(\theta,n)\\ &= \sigma(\theta,n)e_n(\theta)\end{aligned}$$

for all θ in $[-\pi,\pi]$. Thus, if $\sigma \in L^2(\mathbb{S}^1 \times \mathbb{Z})$ is such that $T_\sigma \in S_1$, then the trace $\mathrm{tr}(T_\sigma)$ of T_σ is given by

$$\mathrm{tr}(T_\sigma) = \sum_{n=-\infty}^{\infty} (T_\sigma e_n, e_n). \tag{4.1}$$

The following theorem is an immediate consequence of Theorems 2.2 and 2.3.

Theorem 4.1 *Let λ be a measurable function on $\mathbb{S}^1 \times \mathbb{Z}$. Then T_λ is in S_1 if and only if there exist symbols σ and τ in $L^2(\mathbb{S}^1 \times \mathbb{Z})$ such that*

$$\lambda = \sigma \circledast \tau.$$

We can now give a trace formula for trace class pseudo-differential operators on $\mathbb{S}^1$.

Theorem 4.2 *Let $\lambda \in L^2(\mathbb{S}^1 \times \mathbb{Z})$ be such that there exist symbols σ and τ in $L^2(\mathbb{S}^1 \times \mathbb{Z})$ for which*

$$\lambda = \sigma \circledast \tau.$$

Then $T_\lambda \in S_1$ and

$$\mathrm{tr}(T_\lambda) = \sum_{n=-\infty}^{\infty}\int_{-\pi}^{\pi} \lambda(\theta,n)\,d\theta = 2\pi \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \hat{\sigma}(m-n,n)\hat{\tau}(n-m,m).$$

Proof We begin with a check on the absolute convergence of the series $\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \hat{\sigma}(m-n,n)\hat{\tau}(n-m,m)$. Indeed, using the Schwarz inequality and the Plancherel's theorem for Fourier series, we have

$$\begin{aligned}&\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} |\hat{\sigma}(m-n,n)\hat{\tau}(n-m,m)|\\ &\quad\le \left(\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} |\hat{\sigma}(m-n,n)|^2\right)^{1/2}\left(\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} |\hat{\tau}(n-m,m)|^2\right)^{1/2}\end{aligned}$$

$$= \left(\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} |\hat{\sigma}(m,n)|^2 \right)^{1/2} \left(\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |\hat{\tau}(n,m)|^2 \right)^{1/2}$$

$$= \frac{1}{4\pi^2} \left(\sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} |\sigma(\theta,n)|^2 d\theta \right)^{1/2} \left(\sum_{m=-\infty}^{\infty} \int_{-\pi}^{\pi} |\tau(\theta,m)|^2 d\theta \right)^{1/2}$$

$$= \frac{1}{4\pi^2} \|\sigma\|_{L^2(\mathbb{S}^1 \times \mathbb{Z})} \|\tau\|_{L^2(\mathbb{S}^1 \times \mathbb{Z})} < \infty.$$

Since for $n \in \mathbb{Z}$,

$$\begin{aligned}(T_\sigma e_n, e_n) &= (\sigma(\cdot,n)e_n, e_n) \\ &= \int_{-\pi}^{\pi} \sigma(\theta,n) e_n(\theta) \overline{e_n(\theta)}\, d\theta \\ &= \int_{-\pi}^{\pi} \sigma(\theta,n)\, d\theta,\end{aligned}$$

it follows from (4.1) that

$$\operatorname{tr}(T_\sigma) = \sum_{n=-\infty}^{\infty} (T_\sigma e_n, e_n) = \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} \sigma(\theta,n)\, d\theta.$$

Now,

$$\begin{aligned}\operatorname{tr}(T_\lambda) &= \sum_{m=-\infty}^{\infty} \int_{-\pi}^{\pi} \lambda(\theta,m)\, d\theta \\ &= \sum_{m=-\infty}^{\infty} \int_{-\pi}^{\pi} \frac{1}{2\pi} \left(\sum_{n=-\infty}^{\infty} \sigma(\theta,n) \int_{-\pi}^{\pi} e^{-i(m-n)\phi} \tau(\theta-\phi,m)\, d\phi \right) d\theta \\ &= \sum_{m=-\infty}^{\infty} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} \sigma(\theta,n) e^{-i(m-n)\theta} \hat{\tau}(n-m,m)\, d\theta \\ &= 2\pi \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \hat{\sigma}(m-n,n) \hat{\tau}(n-m,m).\end{aligned}$$

□

5 Pseudo-differential operators on $\mathbb{S}^{n-1}$

We first give a result on the L^2-boundedness of pseudo-differential operators on $\mathbb{S}^{n-1}$.

Theorem 5.1 *Let σ be a measurable function on $\mathbb{S}^{n-1} \times \mathbb{N} \times \mathbb{N}$ such that*

$$\sum_{k=0}^{\infty} \sum_{j=1}^{a_k} \int_{\mathbb{S}^{n-1}} |\sigma(\omega, k, j) Y_j^{(k)}(\omega)|^2 d\omega < \infty.$$

Then $T_\sigma : L^2(\mathbb{S}^{n-1}) \to L^2(\mathbb{S}^{n-1})$ is a bounded linear operator and

$$\|T_\sigma\|_* \leq \left(\sum_{k=0}^{\infty} \sum_{j=1}^{a_k} \int_{\mathbb{S}^{n-1}} |\sigma(\omega, k, j) Y_j^{(k)}(\omega)|^2 d\omega \right)^{1/2},$$

where $\| \, \|_*$ denotes the norm in the C^*-algebra of all bounded linear operators from $L^2(\mathbb{S}^{n-1})$ into $L^2(\mathbb{S}^{n-1})$.

Proof Let $f \in L^2(\mathbb{S}^{n-1})$. By Minkowski's inequality and the Schwarz inequality,

$$\begin{aligned}
&\|T_\sigma f\|_{L^2(\mathbb{S}^{n-1})} \\
&\quad = \left(\int_{\mathbb{S}^{n-1}} \left| \sum_{k=0}^{\infty} \sum_{j=1}^{a_k} \sigma(\omega, k, j)(f, Y_j^{(k)}) Y_j^{(k)}(\omega) \right|^2 d\omega \right)^{1/2} \\
&\quad \leq \sum_{k=0}^{\infty} \left(\int_{\mathbb{S}^{n-1}} \left| \sum_{j=1}^{a_k} \sigma(\omega, k, j)(f, Y_j^{(k)}) Y_j^{(k)}(\omega) \right|^2 d\omega \right)^{1/2} \\
&\quad \leq \sum_{k=0}^{\infty} \left(\int_{\mathbb{S}^{n-1}} \left(\sum_{j=1}^{a_k} |(f, Y_j^{(k)})|^2 \right) \left(\sum_{j=1}^{a_k} |\sigma(\omega, k, j) Y_j^{(k)}(\omega)|^2 \right) dw \right)^{1/2} \\
&\quad \leq \sum_{k=0}^{\infty} \left(\sum_{j=1}^{a_k} |(f, Y_j^{(k)})|^2 \right)^{1/2} \left(\int_{\mathbb{S}^{n-1}} \sum_{j=1}^{a_k} |\sigma(\omega, k, j) Y_j^{(k)}(\omega)|^2 d\omega \right)^{1/2} \\
&\quad \leq \left(\sum_{k=0}^{\infty} \sum_{j=1}^{a_k} |(f, Y_j^{(k)})|^2 \right)^{1/2} \left(\sum_{k=0}^{\infty} \int_{\mathbb{S}^{n-1}} \sum_{j=1}^{a_k} |\sigma(\omega, k, j) Y_j^{(k)}(\omega)|^2 d\omega \right)^{1/2}
\end{aligned}$$

$$= \left(\sum_{k=1}^{\infty}\sum_{j=1}^{a_k} |(f, Y_j^{(k)})|^2\right)^{1/2} \left(\sum_{k=1}^{\infty}\sum_{j=1}^{a_k} \int_{\mathbb{S}^{n-1}} |\sigma(\omega, k, j) Y_j^{(k)}(\omega)|^2 d\omega\right)^{1/2}$$

$$= \|f\|_{L^2(\mathbb{S}^{n-1})} \left(\sum_{k=0}^{\infty}\sum_{j=1}^{a_k} \int_{\mathbb{S}^{n-1}} |\sigma(\omega, k, j) Y_j^{(k)}(\omega)|^2 d\omega\right)^{1/2}.$$

□

We denote by $L^2_Y(\mathbb{S}^{n-1}\times\mathbb{N}\times\mathbb{N})$ the set of measurable functions σ on $\mathbb{S}^{n-1}\times\mathbb{N}\times\mathbb{N}$ such that

$$\sum_{k=0}^{\infty}\sum_{j=1}^{a_k} \int_{\mathbb{S}^{n-1}} |\sigma(\omega, k, j) Y_j^{(k)}(\omega)|^2 d\omega < \infty.$$

It coincides with $L^2(\mathbb{S}^1 \times \mathbb{Z})$ when $n = 2$.

Theorem 5.2 *Let σ be a measurable function on $\mathbb{S}^{n-1} \times \mathbb{N} \times \mathbb{N}$. Then $T_\sigma \in S_2$ if and only if $\sigma \in L^2_Y(\mathbb{S}^{n-1} \times \mathbb{N} \times \mathbb{N})$.*

Proof For $k_0 = 0, 1, 2, \ldots,$ and $j_0 = 1, 2, \ldots, a_{k_0}$, we get for all ω in $\mathbb{S}^{n-1}$,

$$(T_\sigma Y_{j_0}^{(k_0)})(\omega) = \sum_{k=0}^{\infty}\sum_{j=1}^{a_k} \sigma(\omega, k, j)(Y_{j_0}^{k_0}, Y_j^{(k)}) Y_j^{(k)}(\omega) = \sigma(\omega, k_0, j_0) Y_{j_0}^{(k_0)}(\omega).$$

Therefore

$$\begin{aligned}
&\sum_{k=0}^{\infty}\sum_{j=1}^{a_k} \|T_\sigma Y_j^{(k)}\|^2_{L^2(\mathbb{S}^{n-1})} \\
&\quad = \sum_{k=0}^{\infty}\sum_{j=1}^{a_k} \left(\int_{\mathbb{S}^{n-1}} |T_\sigma Y_j^{(k)}|^2 d\omega\right) \\
&\quad = \sum_{k=0}^{\infty}\sum_{j=1}^{a_k} \int_{\mathbb{S}^{n-1}} |\sigma(\omega, k, j) Y_j^{(k)}(\omega)|^2 d\omega
\end{aligned}$$

and the proof is complete. □

As in the case of the circle, we first derive a formula for the symbol λ of the product of two pseudo-differential operators T_σ and T_τ on $\mathbb{S}^{n-1}$. Indeed,

$$
\begin{aligned}
&(T_\sigma T_\tau f)(\xi)\\
&= (T_\sigma (T_\tau f))(\xi)\\
&= \sum_{k=0}^{\infty}\sum_{j=1}^{a_k} \sigma(\xi,k,j)(T_\tau f, Y_j^{(k)})Y_j^{(k)}(\xi)\\
&= \sum_{k=0}^{\infty}\sum_{j=1}^{a_k} \sigma(\xi,k,j)\left(\int_{\mathbb{S}^{n-1}} (T_\tau f)(\omega)\overline{Y_j^{(k)}(\omega)}\,d\omega\right) Y_j^{(k)}(\xi)\\
&= \sum_{k=0}^{\infty}\sum_{j=1}^{a_k} \sigma(\xi,k,j)\left(\int_{\mathbb{S}^{n-1}} \sum_{l=0}^{\infty}\sum_{i=1}^{a_l} \tau(\omega,l,i)(f,Y_i^l)Y_i^{(l)}(\omega)\overline{Y_j^{(k)}(\omega)}\,d\omega\right)\\
&\qquad\qquad Y_j^{(k)}(\xi)\\
&= \sum_{k=0}^{\infty}\sum_{j=1}^{a_k} \sigma(\xi,k,j)\left(\sum_{l=0}^{\infty}\sum_{i=1}^{a_l}(f,Y_i^{(l)})\int_{\mathbb{S}^{n-1}} \tau(\omega,l,i)Y_i^{(l)}(\omega)\overline{Y_j^{(k)}(\omega)}\,d\omega\right)\\
&\qquad\qquad Y_j^{(k)}(\xi)\\
&= \sum_{l=0}^{\infty}\sum_{i=1}^{a_l}\left(\int_{\mathbb{S}^{n-1}} \tau(\omega,l,i)Y_i^{(l)}(\omega)\sum_{k=0}^{\infty}\sum_{j=1}^{a_k}\sigma(\xi,k,j)\overline{Y_j^{(k)}(\omega)}Y_j^{(k)}(\xi)\,d\omega\right)\\
&\qquad\qquad (Y_i^{(l)}(\xi))^{-1}(f,Y_i^{(l)})Y_i^{(l)}(\xi)\\
&= \sum_{l=0}^{\infty}\sum_{i=1}^{a_l}\lambda(\xi,l,i)(f,Y_i^{(l)})Y_i^{(l)}(\xi)
\end{aligned}
$$

for all ξ in $\mathbb{S}^{n-1}$, where

$$\lambda(\xi,l,i) = (\sigma \circledast \tau)(\xi,l,i)$$

and

$$
\begin{aligned}
&(\sigma \circledast \tau)(\xi,l,i)\\
&\quad= \left(\int_{\mathbb{S}^{n-1}} \tau(w,l,i)Y_i^{(l)}(\omega)\sum_{k=0}^{\infty}\sum_{j=1}^{a_k}\sigma(\xi,k,j)\overline{Y_j^{(k)}(\omega)}Y_j^{(k)}(\xi)\,d\omega\right)(Y_i^{(l)}(\xi))^{-1}.
\end{aligned}
$$

Thus,

$$T_\lambda = T_\sigma T_\tau,$$

where

$$\lambda = \sigma \circledast \tau.$$

We can now give a characterization of trace class pseudo-differential operators on $\mathbb{S}^{n-1}$.

Theorem 5.3 *Let λ be a measurable function on $\mathbb{S}^{n-1} \times \mathbb{N} \times \mathbb{N}$. Then $T_\lambda \in S_1$ if and only if there exists symbols σ and τ in $L^2_Y(\mathbb{S}^{n-1} \times \mathbb{N} \times \mathbb{N})$ such that*

$$\lambda = \sigma \circledast \tau.$$

A trace formula for trace class pseudo-differential operators on $\mathbb{S}^{n-1}$ is given in the following theorem.

Theorem 5.4 *Let $\lambda \in L^2_Y(\mathbb{S}^{n-1} \times \mathbb{N} \times \mathbb{N})$ be such that there exist symbols σ and τ in $L^2_Y(\mathbb{S}^{n-1}) \times \mathbb{N} \times \mathbb{N}$ for which*

$$\lambda = \sigma \circledast \tau.$$

Then $T_\lambda \in S_1$ and

$$\begin{aligned}\operatorname{tr}(T_\lambda) &= \sum_{l=0}^{\infty}\sum_{i=1}^{a_l}\int_{\mathbb{S}^{n-1}} \lambda(\xi, l, i)|Y_i^{(l)}(\xi)|^2 d\xi \\ &= \sum_{l=0}^{\infty}\sum_{i=1}^{a_l}\sum_{k=0}^{\infty}\sum_{j=1}^{a_k}(\sigma(\cdot, k, j)Y_j^{(k)}, Y_i^{(l)})(\tau(\cdot, l, i)Y_i^{(l)}, Y_j^{(k)}).\end{aligned}$$

Proof The absolute convergence of the series

$$\sum_{l=0}^{\infty}\sum_{i=1}^{a_l}\sum_{k=0}^{\infty}\sum_{j=1}^{a_k}(\sigma(\cdot, k, j)Y_j^{(k)}, Y_i^{(l)})(\tau(\cdot, l, i)Y_i^{l)}, Y_j^{(k)})$$

can be proved as in Theorem 4.2. Since $\bigcup_{k=1}^{\infty}\{Y_1^k, Y_2^{(k)}, \ldots Y_{a_k}^{(k)}\}$ is an orthonormal basis for $L^2(\mathbb{S}^{n-1})$, it follows that

$$\begin{aligned}&\operatorname{tr}(T_\lambda) \\ &= \sum_{l=0}^{\infty}\sum_{i=1}^{a_l}(T_\lambda Y_i^{(l)}, Y_i^{(l)}) \\ &= \sum_{l=0}^{\infty}\sum_{i=1}^{a_l}\int_{\mathbb{S}^{n-1}} \lambda(\xi, l, i)Y_i^{(l)}(\xi)\overline{Y_i^{(l)}(\xi)}d\xi\end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=0}^{\infty} \sum_{i=1}^{a_l} \int_{\mathbb{S}^{n-1}} \lambda(\xi, l, i) |Y_i^{(l)}(\xi)|^2 d\xi \\
&= \sum_{l=0}^{\infty} \sum_{i=1}^{a_l} \int_{\mathbb{S}^{n-1}} \\
&\left(\int_{\mathbb{S}^{n-1}} \tau(\omega, l, i) Y_i^{(l)}(\omega) \sum_{k=0}^{\infty} \sum_{j=1}^{a_k} \sigma(\xi, k, j) \overline{Y_j^{(k)}(\omega)} Y_j^{(k)}(\xi)\, d\omega \right) \\
&\qquad \overline{Y_i^{(l)}(\xi)}\, d\xi \\
&= \sum_{l=0}^{\infty} \sum_{i=1}^{a_l} \sum_{k=0}^{\infty} \sum_{j=1}^{a_k} \int_{\mathbb{S}^{n-1}} \tau(\omega, l, i) Y_i^{(l)}(\omega) \overline{Y_j^{(k)}(\omega)}\, d\omega \int_{\mathbb{S}^{n-1}} \sigma(\xi, k, j) \\
&\qquad Y_j^{(k)}(\xi) \overline{Y_i^{(l)}(\xi)}\, d\xi \\
&= \sum_{l=0}^{\infty} \sum_{i=1}^{a_l} \sum_{k=0}^{\infty} \sum_{j=1}^{a_k} (\sigma(\cdot, k, j) Y_j^{(k)}, Y_i^{(l)}) (\tau(\cdot, l, j) Y_i^{(l)}, Y_j^{(k)})
\end{aligned}
$$

□

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