J. Pseudo-Differ. Oper. Appl. (2012) 3:329–349
DOI 10.1007/s11868-012-0046-9

Global existence and nonexistence for semilinear parabolic equations with conical degeneration

Hua Chen · Gongwei Liu

Received: 6 January 2012 / Revised: 27 January 2012 / Accepted: 30 January 2012 /
Published online: 11 February 2012

Abstract In this article we study the initial boundary value problem of semilinear parabolic equations $u_t - \triangle_{\mathbb{B}} u = |u|^{p-1}u$ on a manifold with conical singularity, where $\triangle_{\mathbb{B}}$ is Fuchsian type Laplace operator investigated in Chen et al. (Calc Var 43:463–484, 2012) with totally characteristic degeneracy on the boundary $x_1 = 0$. By using a family of potential wells, we obtain existence theorem of global solutions with exponential decay and show the blow-up in finite time of solutions. Especially, the relation between the above two phenomena is derived as a sharp condition.

Keywords Semilinear parabolic equations · Cone Laplacian · Totally characteristic degeneracy · Global solution · Blow-up

Mathematics Subject Classification (2000) 35K10 · 35B40 · 58J45

1 Introduction and main results

In this paper, we study the following initial-boundary value problems for a class of degenerate parabolic type equations

$$\begin{cases} \partial_t u - \triangle_{\mathbb{B}} u = |u|^{p-1}u & x \in int\mathbb{B},\ t > 0, \\ u(0,x) = u_0(x) & x \in int\mathbb{B}, \\ u(t,x) = 0 & x \in \partial\mathbb{B},\ t \geq 0 \end{cases} \tag{1.1}$$

H. Chen (✉) · G. Liu
School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China
e-mail: chenhua@whu.edu.cn

where $2 < p+1 < \frac{2n}{n-2} = 2^*$, and 2^* is the critical cone Sobolev exponents. Here $\mathbb{B} = [0,1) \times X$, X is an $(n-1)$-dimensional closed compact manifold, which is regarded as the local model near the conical points, and $\partial\mathbb{B} = \{0\} \times X$. Moreover the operator $\triangle_{\mathbb{B}}$ in (1.1) is defined by $(x_1\partial_{x_1})^2 + \partial_{x_2}^2 + \cdots + \partial_{x_n}^2$, which is an elliptic operator with conical degeneration on the boundary $x_1 = 0$ (we also called it Fuchsian type Laplace operator), and the corresponding gradient operator is denoted by $\nabla_{\mathbb{B}} = (x_1\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n})$. Near $\partial\mathbb{B}$ we will often use coordinates $(x_1, x') = (x_1, x_2, \ldots, x_n)$ for $0 \le x_1 < 1, x' \in X$. In this paper, we shall prove the existence and blow-up theorem for the solutions of problem (1.1).

In the classical case, we have

$$\begin{cases} \partial_t u - \triangle u = |u|^{p-1}u & x \in \Omega,\ t > 0, \\ u(0,x) = u_0(x) & x \in \Omega, \\ u(t,x) = 0 & x \in \partial\Omega,\ t \ge 0 \end{cases} \tag{1.2}$$

where Ω is an open bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and $\triangle$ is the standard Laplace operator. It's well known that problem (1.2) has been studied by many authors. A powerful technique for treating problem (1.2) is the so called "potential well method", which was established by Sattinger [9], Payne and Sattinger [8]. Moreover, potential well method was greatly improved by [7]. Recently, there are some interesting results about the global existence and blow-up in [5] in which the authors proved for low energy data u_0, i.e. $J(u_0) < d$, problem (1.2) has a global solution in a standard Sobolev space which will be vanishing if $u_0 \in \mathcal{N}_+$ and blow up if $u_0 \in \mathcal{N}_-$. Here $\mathcal{N}_+ = \{u \in H_0^1(\Omega); K(u) > 0\}$ and $\mathcal{N}_- = \{u \in H_0^1(\Omega); K(u) < 0\}$. However they have not proved the asymptotic behavior for the global solution.

In this paper, we shall consider the corresponding problem (1.1) on the manifold with conical singularities. Similar to the classical case, we introduced the following functionals on cone Sobolev space $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$:

$$J(u) = \frac{1}{2}\int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{dx_1}{x_1} dx' - \frac{1}{p+1}\int_{\mathbb{B}} |u|^{p+1}\frac{dx_1}{x_1}dx', \tag{1.3}$$

$$K(u) = \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{dx_1}{x_1} dx' - \int_{\mathbb{B}} |u|^{p+1}\frac{dx_1}{x_1}dx'. \tag{1.4}$$

Here the weighted Sobolev space $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ will be introduced in the next section. Then $J(u)$ and $K(u)$ are well-defined and belong to $C^1(\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}), \mathbb{R})$. Now we define

$$\mathcal{N} = \left\{ u \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) \Big| K(u) = 0, \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{dx_1}{x_1}dx' \ne 0 \right\},$$

$$d = \inf\left\{\sup_{\lambda\geq 0} J(\lambda u), u \in \mathcal{H}^{1,\frac{n}{2}}_{2,0}, \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{dx_1}{x_1} dx' \neq 0\right\}.$$

Thus, similar to the results in [7] and [8], one has $0 < d = \inf_{u\in\mathcal{N}} J(u)$.

Next, we give the following notations with $\delta > 0$,

$$K_\delta(u) = \delta \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{dx_1}{x_1} dx' - \int_{\mathbb{B}} |u|^{p+1} \frac{dx_1}{x_1} dx',$$

$$\mathcal{N}_\delta = \left\{u \in \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}) \middle| K_\delta(u) = 0, \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{dx_1}{x_1} dx' \neq 0\right\}, \tag{1.5}$$

$$d(\delta) = \inf_{u\in\mathcal{N}_\delta} J(u).$$

By making use of the functionals above, we introduce the following potential wells

$$W = \left\{u \in \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}) \middle| K(u) > 0, J(u) < d\right\} \cup \{0\},$$

$$W_\delta = \left\{u \in \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}) \middle| K_\delta(u) > 0, J(u) < d(\delta)\right\} \cup \{0\}, \quad 0 < \delta < \frac{p+1}{2},$$

and the outside sets of the corresponding potential wells are defined as follows

$$V = \left\{u \in \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}) \middle| K(u) < 0, J(u) < d\right\},$$

$$V_\delta = \left\{u \in \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}) \middle| K_\delta(u) < 0, J(u) < d(\delta)\right\}, \quad 0 < \delta < \frac{p+1}{2}.$$

Therefore we can construct the relation between the properties of the solution (global existence and blow-up in finite time) and the initial datum u_0 via the method of the potential wells as introduced above.

First we introduce the following definition of the weak solution:

Definition 1.1 $u = u(t, x)$ is called a weak solution of problem (1.1) on $int\mathbb{B}\times [0, T)$, with $0 < T \leq +\infty$ being the maximal existence time, if $u \in L^\infty(0, T; \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}))$ with $u_t \in L^2(0, T; L^{\frac{n}{2}}_2(\mathbb{B}))$ and satisfies problem (1.1) in the distribution sense, i.e.

$$(u_t, v)_2 + (\nabla_{\mathbb{B}} u, \nabla_{\mathbb{B}} v)_2 = (|u|^{p-1}u, v)_2, \quad \forall\, v \in \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}), \quad t \in (0, T) \tag{1.6}$$

with $u(0, x) = u_0(x)$ in $\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$ and $\int_0^t \|u_\tau\|^2_{L^{\frac{n}{2}}_2(\mathbb{B})} d\tau + J(u(t)) \leq J(u_0)$, for $0 \leq t < T$.

We are now in a position to state our main results. Our first result is concerned with the global existence and the asymptotic behavior of problem (1.1) with low initial energy data, i.e. $J(u_0) < d$.

Theorem 1.1 *Let* $u_0 \in \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$. *Suppose* $J(u_0) < d$ *and* $K(u_0) > 0$ *or* $\|\nabla_{\mathbb{B}} u_0\|_{L_2^{\frac{n}{2}}(\mathbb{B})} = 0$. *Then problem* (1.1) *has a global weak solution* $u \in L^\infty(0, \infty; \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}))$ *with* $u_t \in L^2(0, \infty; L_2^{\frac{n}{2}}(\mathbb{B}))$. *Moreover* $u(t) \in W$ *for* $0 \le t < \infty$, *and there exist constants* $C > 0$ *and* $\lambda > 0$ *such that*

$$\|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})} \le Ce^{-\lambda t}, \quad 0 \le t < \infty.$$

The following result shows the finite time blow-up for certain solutions of problem (1.1) with low initial energy data, i.e. $J(u_0) < d$.

Theorem 1.2 *Let* $u_0 \in \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$. *Suppose* $J(u_0) < d$ *and* $K(u_0) < 0$. *Then the existence time of the weak solution for problem* (1.1) *is finite, i.e. there exists a* $T > 0$ *such that*

$$\lim_{t \to T^-} \int_0^t \|u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} d\tau = +\infty. \tag{1.7}$$

In the next two theorems we will show the similar results as in Theorem 1.1 and Theorem 1.2 with critical initial energy data, i.e. $J(u_0) = d$.

Theorem 1.3 *Let* $u_0 \in \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$. *Suppose* $J(u_0) = d$ *and* $K(u_0) \ge 0$. *Then problem* (1.1) *has a global weak solution* $u \in L^\infty(0, \infty; \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}))$ *with* $u_t \in L^2(0, \infty; L_2^{\frac{n}{2}}(\mathbb{B}))$. *Moreover the global solution* $u(t) \in \overline{W}$ *for* $0 \le t < \infty$, *and there exist* $C > 0$, $\lambda > 0$ *and* $t_1 > 0$ *such that*

$$\|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})} \le Ce^{-\lambda t}, \quad t_1 \le t < \infty. \tag{1.8}$$

Theorem 1.4 *Let* $u_0 \in \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$. *Suppose* $J(u_0) = d$ *and* $K(u_0) < 0$. *Then the existence time of the weak solution for the problem* (1.1) *is finite, i.e. there exists a* $T > 0$ *such that*

$$\lim_{t \to T^-} \int_0^t \|u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} d\tau = +\infty.$$

This paper is organized as follows. In Sect. 2 we will introduce the cone Sobolev spaces and the corresponding properties (more details can been seen [1–3]). In Sect. 3 we will give some properties of potential wells for problem (1.1) on the manifold with conical singularity, which is very useful in the process of our main results, moreover we introduce the lemma of Komornik [6] which plays a critical role in the study of exponential asymptotic behavior of the global solution here. In Sect. 4, we give the proofs for the results of global existence, exponential decay and finite time blowing-up for our problems.

2 Cone Sobolev spaces

In this section we introduce the manifold with conical singularities and the corresponding cone Sobolev spaces.

Let X be a closed, compact, C^∞ manifold. We set $X^\Delta = (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$, as a local model interpreted as a cone with the base X. Next, we Denote $X^\wedge = \mathbb{R}_+ \times X$ as the corresponding open stretched cone with the base X.

An n-dimensional manifold B with conical singularities is a topological space with a finite subset $B_0 = \{b_1, \ldots, b_M\} \subset B$ of conical singularities, with the following two properties:

1. $B \backslash B_0$ is a C^∞ manifold.
2. Every $b \in B_0$ has an open neighhourhood U in B, such that there is a homeomorphism $\varphi : U \to X^\Delta$ for some closed compact C^∞ manifold $X = X(b)$, and φ restricts to a diffeomorphism $\varphi' : U \backslash \{b\} \to X^\wedge$.

For simplicity, we assume that the manifold B has only one conical point on the boundary. Thus, near the conical point, we have a stretched manifold $\mathbb{B}$, associated with B. Here $\mathbb{B} = [0, 1) \times X$, $\partial\mathbb{B} = \{0\} \times X$ and X is a closed compact manifold of dimension $n - 1$. Also, near the conical point, we use the coordinates $(x_1, x') = (x_1, x_2, \ldots, x_n)$ for $0 \leq x_1 < 1$, $x' \in X$.

Recently, the authors in [1] introduced a class of weighted Sobolev spaces (also see [3,10,11]), and proved the corresponding cone Sobolev inequality and Poincaré inequality. Also in [1] and [2], by applying these inequalities, the authors proved the existence theorems for a class of semilinear equations with subcritical and critical cone Sobolev exponents on the manifolds with conical singularities.

In order to make the paper readable, we shall give some definitions and properties of the cone Sobolev spaces as follows:

Definition 2.1 Let $\mathbb{B} = [0, 1) \times X$ be the stretched manifold of the manifold B with conical singularity. Then the cone Sobolev space $\mathcal{H}_p^{m,\gamma}(\mathbb{B})$, for $m \in \mathbb{N}, \gamma \in \mathbb{R}$ and $1 < p < +\infty$, is defined as

$$\mathcal{H}_p^{m,\gamma}(\mathbb{B}) = \{u \in W_{\text{loc}}^{m,p}(\text{int}\mathbb{B}) \mid \omega u \in \mathcal{H}_p^{m,\gamma}(X^\wedge)\},$$

for any cut-off function ω, supported by a collar neighborhood of $(0, 1) \times \partial\mathbb{B}$. Moreover, the subspace $\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{B})$ of $\mathcal{H}_p^{m,\gamma}(\mathbb{B})$ is defined by

$$\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{B}) := [\omega]\mathcal{H}_{p,0}^{m,\gamma}(X^\wedge) + [1 - \omega]W_0^{m,p}(\text{int}\mathbb{B}),$$

where $W_0^{m,p}(\text{int}\mathbb{B})$ denotes the closure of $C_0^\infty(\text{int}\mathbb{B})$ in Sobolev spaces $W^{m,p}(\tilde{X})$ when $\tilde{X}$ is a closed compact C^∞ manifold of dimension n that containing $\mathbb{B}$ as a submanifold with boundary.

Definition 2.2 Let $\mathbb{B} = [0, 1) \times X$. We say $u(x) \in L_p^\gamma(\mathbb{B})$ with $1 < p < +\infty$ and $\gamma \in \mathbb{R}$ if

$$\|u\|^p_{L_p^\gamma(\mathbb{B})} = \int_{\mathbb{B}} x_1^n \mid x_1^{-\gamma} u(x) \mid^p \frac{dx_1}{x_1} dx' < +\infty.$$

Observe that if $u(x) \in L_p^{\frac{n}{p}}(\mathbb{B})$, $v(x) \in L_q^{\frac{n}{q}}(\mathbb{B})$ with $p, q \in (1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then we have the following Hölder's inequality

$$\int_{\mathbb{B}} \mid u(x)v(x) \mid \frac{dx_1}{x_1} dx' \leq \left(\int_{\mathbb{B}} \mid u(x) \mid^p \frac{dx_1}{x_1} dx' \right)^{\frac{1}{p}} \left(\int_{\mathbb{B}} \mid v(x) \mid^q \frac{dx_1}{x_1} dx' \right)^{\frac{1}{q}}. \tag{2.1}$$

In the sequel, for convenience we denote

$$(u, v)_2 = \int_{\mathbb{B}} u(x)v(x) \frac{dx_1}{x_1} dx', \quad \|u\|^p_{L_p^{\frac{n}{p}}(\mathbb{B})} = \int_{\mathbb{B}} |u(x)|^p \frac{dx_1}{x_1} dx'.$$

Proposition 2.1 (cf. [1]) [Poincaré Inequality] *Let* $\mathbb{B} = [0, 1) \times X$ *be a bounded subspace in* $\mathbb{R}^n_+$ *with* $X \subset \mathbb{R}^{n-1}$, *and* $1 < p < +\infty$, $\gamma \in \mathbb{R}$. *If* $u(x) \in \mathcal{H}^{1,\gamma}_{p,0}(\mathbb{B})$, *then*

$$\|u(x)\|_{L_p^\gamma(\mathbb{B})} \leq c \|\nabla_{\mathbb{B}} u(x)\|_{L_p^\gamma(\mathbb{B})}, \tag{2.2}$$

where $\nabla_{\mathbb{B}} = (x_1\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n})$, *and the constant* c *depending only on* $\mathbb{B}$.

Proposition 2.2 (cf. [2], Prop. 3.3) *For* $1 < p < 2^*$, *the embedding* $\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}) \hookrightarrow \mathcal{H}^{0,\frac{n}{p}}_{p,0}(\mathbb{B})$ *is continuous.*

Proposition 2.3 (cf. [2], Prop. 3.4) *There exist* $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots$, *and* $\lambda_k \to +\infty$, *such that for each* $k \geq 1$, *the following Dirichlet problem*

$$\begin{cases} -\triangle_{\mathbb{B}} \phi_k = \lambda_k \phi_k, & x \in int(\mathbb{B}), \\ \phi_k = 0 & \text{on } \partial\mathbb{B}, \end{cases} \tag{2.3}$$

admits non-trivial solution in $\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$. *Moreover, we can choose positive* $\{\phi_k\}_{k\geq 1}$ *constitute an orthonormal basis of Hilbert space* $\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$, *and the inequality*

$$\lambda_1^{\frac{1}{2}} \|u(x)\|_{L_2^{\frac{n}{2}}(\mathbb{B})} \leq \|\nabla_{\mathbb{B}} u(x)\|_{L_2^{\frac{n}{2}}(\mathbb{B})}, \tag{2.4}$$

holds.

3 Some auxiliary results

This section is devoted to introduce a family of potential wells for problem (1.1) and to prove a series of properties which are useful in the proof of our main results listed in Sect. 4.

The results in the following lemmas give the relations between the functional $K_\delta(u)$ and $\|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})}$. First, let $1 < p < \frac{n+2}{n-2}$, $n \geq 3$. Assume that $u \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$, $u \neq 0$, we denote $C_* = \sup\left\{\|u\|_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})} / \|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})}\right\}$, the constant C_* can be obtained from Proposition 2.1 and Proposition 2.2. Thus we have

Lemma 3.1 *If* $0 < \|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})} < r(\delta)$, *then* $K_\delta(u) > 0$. *In particular, if* $0 < \|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})} < r(1)$, *then* $K(u) > 0$, *where* $r(\delta) = \left(\frac{\delta}{C_*^{p+1}}\right)^{\frac{1}{p-1}}$.

Proof From $0 < \|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})} < r(\delta)$, we get

$$\begin{aligned}\int_{\mathbb{B}} |u|^{p+1} \frac{dx_1}{x_1} dx' &\leq C_*^{p+1} \|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})}^{p+1} \\ &= C_*^{p+1} \|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})}^{p-1} \|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})}^2 < \delta \|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})}^2.\end{aligned}$$

Here we use the embedding $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) \hookrightarrow \mathcal{H}_{p+1,0}^{0,\frac{n}{p+1}}(\mathbb{B})$ is continuous and $K_\delta(u) > 0$. □

Lemma 3.2 *If* $K_\delta(u) < 0$, *then* $\|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})} > r(\delta)$. *In particular, if* $K(u) < 0$, *then* $\|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})} > r(1)$.

Proof Notice that $K_\delta(u) < 0$ gives $\|\nabla_{\mathbb{B}} u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \neq 0$ and

$$\begin{aligned}\delta \|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})}^2 &< \int_{\mathbb{B}} |u|^{p+1} \frac{dx_1}{x_1} dx' \leq C_*^{p+1} \|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})}^{p+1} \\ &= C_*^{p+1} \|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})}^{p-1} \|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})}^2.\end{aligned}$$

Hence $\|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})}^{p-1} > \frac{\delta}{C_*^{p+1}} = r^{p-1}(\delta)$. □

If $u \in \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$ and $\|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})} \neq 0$. From (1.3), we deduce that

$$\lim_{\lambda\to+\infty} J(\lambda u) = -\infty, \qquad \frac{\partial J(\lambda u)}{\partial \lambda} = \lambda \|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})} - \lambda^p \int_{\mathbb{B}} |u|^{p+1} \frac{dx_1}{x_1} dx'.$$

Let $\frac{\partial J(\lambda u)}{\partial \lambda}\big|_{\lambda=\lambda^*} = 0$, then $\lambda^* = \left(\frac{\|\nabla_{\mathbb{B}} u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}}{\int_{\mathbb{B}} |u|^{p+1} \frac{dx_1}{x_1} dx'} \right)^{\frac{1}{p-1}}$. Also,

$$\frac{\partial^2 J(\lambda u)}{\partial \lambda^2} = \|\nabla_{\mathbb{B}} u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - p\lambda^{p-1} \int_{\mathbb{B}} |u|^{p+1} \frac{dx_1}{x_1} dx'.$$

Hence

$$\left.\frac{\partial^2 J(\lambda u)}{\partial \lambda^2}\right|_{\lambda=\lambda^*} = \|\nabla_{\mathbb{B}} u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - p\|\nabla_{\mathbb{B}} u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} < 0, \text{ as } p > 1.$$

Therefore we have the following lemma:

Lemma 3.3 (i) *When* $\lambda = \lambda^*$, $J(\lambda u)$ *takes the maximum value. Also,* $J(\lambda u)$ *is strictly increasing if* $0 \leq \lambda \leq \lambda^*$, *and is strictly decreasing if* $\lambda^* < \lambda < +\infty$.

(ii) $K(\lambda^* u) = 0$, *and* $K(\lambda u) > 0$ *for* $0 < \lambda < \lambda^*$, $K(\lambda u) < 0$ *for* $\lambda^* < \lambda$.

(iii) $d = \inf\{\sup_{\lambda\geq 0} J(\lambda u), u \in \mathcal{H}^{1,\frac{n}{2}}_{2,0}, \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{dx_1}{x_1} dx' \neq 0\} = \frac{p-1}{2(p+1)} C_*^{-\frac{2(p+1)}{p-1}}$.

Proof From the discussion above we only need to show (iii). While $\sup_{\lambda\geq 0} J(\lambda u) = J(\lambda^* u)$, we complete the proof from the definition of C^* directly, □

Lemma 3.4 $d(\delta) \geq a(\delta) r^2(\delta)$ *for* $a(\delta) = \frac{1}{2} - \frac{\delta}{p+1}$ *and* $0 < \delta < \frac{p+1}{2}$. *Moreover we have*

$$d(\delta) = \inf_{u\in\mathcal{N}_\delta} J(u) = \delta^{\frac{2}{p-1}} \left(\frac{1}{2} - \frac{\delta}{p+1}\right) \frac{2(p+1)}{p-1} d, \quad 0 < \delta < \frac{p+1}{2}, \tag{3.1}$$

where $\mathcal{N}_\delta$ *is defined by* (1.5).

Proof Let $u \in \mathcal{N}_\delta$. Similar to Lemma 3.2, we have $\|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})} \geq r(\delta)$. Thus

$$\begin{aligned} J(u) &= \frac{1}{2}\|\nabla_{\mathbb{B}} u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - \frac{1}{p+1} \int_{\mathbb{B}} |u|^{p+1} \frac{dx_1}{x_1} dx' \\ &= \left(\frac{1}{2} - \frac{\delta}{p+1}\right) \|\nabla_{\mathbb{B}} u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + \frac{1}{p+1} K_\delta(u) = a(\delta) \|\nabla_{\mathbb{B}} u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \geq a(\delta) r^2(\delta). \end{aligned}$$

The first part of this lemma is proved. Now let us prove Eq. (3.1).

(1) If $\delta > 0$, $\bar{u} \in \mathcal{N}_\delta$ is a minimizer of $d(\delta) = \inf_{u \in \mathcal{N}_\delta} J(u)$, i.e. $J(\bar{u}) = d(\delta)$. In this case we define $\lambda = \lambda(\delta)$ by $\|\nabla_{\mathbb{B}}(\lambda\bar{u})\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} = \int_{\mathbb{B}} |\lambda\bar{u}|^{p+1} \frac{dx_1}{x_1} dx'$. Then for each $\delta > 0$, there exists a unique λ which satisfies

$$\lambda = \left(\frac{\|\nabla_{\mathbb{B}}\bar{u}\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}}{\int_{\mathbb{B}} |\bar{u}|^{p+1} \frac{dx_1}{x_1} dx'} \right)^{\frac{1}{p-1}} = \left(\frac{1}{\delta}\right)^{\frac{1}{p-1}}.$$

Thus for such λ and $\lambda\bar{u} \in \mathcal{N}$, we get from the definition of d

$$\begin{aligned} d \le J(\lambda\bar{u}) &= \frac{\lambda^2}{2}\|\nabla_{\mathbb{B}}\bar{u}\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - \frac{\lambda^{p+1}}{p+1}\int_{\mathbb{B}} |\bar{u}|^{p+1}\frac{dx_1}{x_1}dx' \\ &= \frac{1}{2}\left(\frac{1}{\delta}\right)^{\frac{2}{p-1}}\|\nabla_{\mathbb{B}}\bar{u}\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - \frac{1}{p+1}\left(\frac{1}{\delta}\right)^{\frac{p+1}{p-1}}\int_{\mathbb{B}} |\bar{u}|^{p+1}\frac{dx_1}{x_1}dx' \\ &= \left(\frac{1}{\delta}\right)^{\frac{2}{p-1}}\left(\frac{1}{2}\|\nabla_{\mathbb{B}}\bar{u}\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - \frac{1}{(p+1)\delta}\int_{\mathbb{B}} |\bar{u}|^{p+1}\frac{dx_1}{x_1}dx'\right) \\ &= \left(\frac{1}{\delta}\right)^{\frac{2}{p-1}}\frac{p-1}{2(p+1)}\|\nabla_{\mathbb{B}}\bar{u}\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}. \end{aligned}$$

Notice that $d(\delta) = J(\bar{u}) = \left(\frac{1}{2} - \frac{\delta}{p+1}\right)\|\nabla_{\mathbb{B}}\bar{u}\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}$, we get

$$d \le \left(\frac{1}{\delta}\right)^{\frac{2}{p-1}}\frac{p-1}{2(p+1)}\left(\frac{1}{2} - \frac{\delta}{p+1}\right)^{-1} d(\delta),$$

which implies

$$d(\delta) \ge \delta^{\frac{2}{p-1}}\left(\frac{1}{2} - \frac{\delta}{p+1}\right)\frac{2(p+1)}{p-1}d, \quad 0 < \delta < \frac{p+1}{2}. \tag{3.2}$$

(2) If $\delta > 0$, and $\tilde{u} \in \mathcal{N}$ is a minimizer of $d = \inf_{u \in \mathcal{N}} J(u)$, i.e. $J(\tilde{u}) = d$. In this case we define $\lambda = \lambda(\delta)$ by $\delta\|\nabla_{\mathbb{B}}(\lambda\tilde{u})\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} = \int_{\mathbb{B}} |\lambda\tilde{u}|^{p+1}\frac{dx_1}{x_1}dx'$. Then for each $\delta > 0$, there exists a unique λ satisfying

$$\lambda = \left(\frac{\delta\|\nabla_{\mathbb{B}}\tilde{u}\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}}{\int_{\mathbb{B}} |\tilde{u}|^{p+1} \frac{dx_1}{x_1} dx'} \right)^{\frac{1}{p-1}} = \delta^{\frac{1}{p-1}}.$$

Thus, for $\lambda\tilde{u} \in \mathcal{N}_\delta$, we get from the definition of $d(\delta)$ that

$$d(\delta) \leq J(\lambda\tilde{u}) = \delta^{\frac{2}{p-1}} \left(\frac{1}{2} - \frac{\delta}{p+1}\right) \|\nabla_{\mathbb{B}}\tilde{u}\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}.$$

Notice that $d = J(\tilde{u}) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\|\nabla_{\mathbb{B}}\tilde{u}\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} = \frac{p-1}{2(p+1)}\|\nabla_{\mathbb{B}}\tilde{u}\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}$, we get

$$d(\delta) \leq \delta^{\frac{2}{p-1}} \left(\frac{1}{2} - \frac{\delta}{p+1}\right) \frac{2(p+1)}{p-1} d, \quad 0 < \delta < \frac{p+1}{2}. \tag{3.3}$$

From (3.2) and (3.3) we obtain the conclusion (3.1). □

Remark 3.1 We deduce immediately the following results from (3.1).

(a) $\lim_{\delta\to 0} d(\delta) = 0$, $\lim_{\delta\to\frac{p+1}{2}} d(\delta) = 0$;

(b) $d(\delta)$ is strictly increasing for $0 < \delta \leq 1$, is strictly decreasing for $1 < \delta < \frac{p+1}{2}$.

(c) $d(\delta)$ takes the maximum at $\delta = 1$, since $d'(\delta) = \frac{2(p+1)d}{(p-1)^2}\delta^{\frac{2}{p-1}-1}(1-\delta)$.

Lemma 3.5 *Let $0 < J(u) < d$ for some $u \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$, $\delta_1 < \delta_2$ (in fact $\delta_1 < 1 < \delta_2$) are the two roots of the equation $d(\delta) = J(u)$. Then the sign of $K_\delta(u)$ are not changed for $\delta_1 < \delta < \delta_2$.*

Proof We can deduce $\|\nabla_{\mathbb{B}}u\|_{L_2^{\frac{n}{2}}(\mathbb{B})} \neq 0$ from $J(u) > 0$. If the sign of $K_\delta(u)$ are changed for $\delta_1 < \delta < \delta_2$, then there exists a $\delta_0 \in (\delta_1, \delta_2)$ such that $K_{\delta_0}(u) = 0$. Thus by the definition of $d(\delta)$ we have $J(u) \geq d(\delta_0)$, which contradicts with $J(u) = d(\delta_1) = d(\delta_2) < d(\delta_0)$. □

Now we discuss the invariance of some sets under the flows (1.1). Here we use the similar methods in [7] and [8].

Lemma 3.6 *Suppose $u_0(x) \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$, $0 < e < d$. Let $\delta_1 < \delta_2$ are the two roots of the equation $d(\delta) = e$. Then we have*

(1) *All solutions of problem (1.1) with $0 < J(u_0) \leq e$ belong to W_δ for $\delta_1 < \delta < \delta_2$, provided $K(u_0) > 0$ or $\|\nabla_{\mathbb{B}}u_0\|_{L_2^{\frac{n}{2}}(\mathbb{B})} = 0$;*

(2) *All solutions of problem (1.1) with $0 < J(u_0) \leq e$ belong to V_δ for $\delta_1 < \delta < \delta_2$, provided $K(u_0) < 0$.*

Proof (1) Let $u(t)$ be a solution of problem (1.1) with initial datum $u_0(x)$ satisfying $0 < J(u_0) \leq e < d$, $K(u_0) > 0$ or $\|\nabla_{\mathbb{B}}u\|_{L_2^{\frac{n}{2}}(\mathbb{B})} \neq 0$. T is the existence time of $u(t)$. If $\|\nabla_{\mathbb{B}}u_0\|_{L_2^{\frac{n}{2}}(\mathbb{B})} = 0$, then $u_0(x) \in W_\delta$. If $K(u_0) > 0$, then from Lemma 3.5 and

$$0 < J(u_0) \leq e = d(\delta_1) = d(\delta_2) < d(\delta) \leq d, \quad \delta_1 < \delta < \delta_2,$$

it follows that $K_\delta(u_0) > 0$. That means $u_0(x) \in W_\delta$ for $\delta_1 < \delta < \delta_2$. Next we should prove $u(t) \in W_\delta$ for $\delta_1 < \delta < \delta_2$ and $0 < t < T$. Otherwise, we can find $t_0 \in (0, T)$ such that $u(t_0) \in \partial W_\delta$ for some $\delta_1 < \delta < \delta_2$. That implies that $K_\delta(u(t_0)) = 0$ and $\|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})} \neq 0$ or $J(u(t_0)) = d(\delta)$. Since the energy inequality

$$\int_0^t \|u_\tau\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} d\tau + J(u(t)) \leq J(u_0) \leq e < d(\delta),$$
$$\text{for} \quad \delta_1 < \delta < \delta_2, \quad \text{and} \quad 0 < t < T. \tag{3.4}$$

Thus it is not possible to get the result $J(u(t_0)) = d(\delta)$. On the other hand, if $K_\delta(u(t_0)) = 0$, $\|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})} \neq 0$, then by the definition of $d(\delta)$ we have $J(u(t_0)) \geq d(\delta)$ which is contradictive with the energy inequality (3.4) again.

(2) Similar to the first part, we can easily deduce that $u_0(x) \in V_\delta$ for $\delta_1 < \delta < \delta_2$ provided $K(u_0(x)) < 0$. Next we should prove $u(t) \in V_\delta$ for $\delta_1 < \delta < \delta_2$ and $0 < t < T$. Otherwise, if $t_0 \in (0, T)$ such that $u(t) \in V_\delta$ for $0 \leq t < t_0$ and $u(t_0) \in \partial V_\delta$, i.e. $K_\delta(u(t_0)) = 0$ or $J(u(t_0)) = d(\delta)$ for some $\delta_1 < \delta < \delta_2$. We can deduce that $J(u(t_0)) = d(\delta)$ is impossible from (3.4). If $K_\delta(u(t_0)) = 0$, then $K_\delta(u(t)) < 0$ for $0 < t < t_0$ and Lemma 3.2 show that $\|\nabla_{\mathbb{B}} u(t)\|_{L_2^{\frac{n}{2}}(\mathbb{B})} > r(\delta)$ and $\|\nabla_{\mathbb{B}} u(t_0)\|_{L_2^{\frac{n}{2}}(\mathbb{B})} \geq r(\delta) \neq 0$. Hence by the definition of $d(\delta)$ we get $J(u(t_0)) \geq d(\delta)$ which contradicts with (3.4) again. □

Remark 3.2 (a) Let $u_0(x)$, e and δ be the same as those in Lemma 3.6. Then both sets W_δ and V_δ are invariant for any $\delta \in (\delta_1, \delta_2)$. Moreover both sets

$$W_{\delta_1\delta_2} = \bigcup_{\delta_1 < \delta < \delta_2} W_\delta, \quad \text{and} \quad V_{\delta_1\delta_2} = \bigcup_{\delta_1 < \delta < \delta_2} V_\delta$$

are invariant respectively under the flow of (1.1), provided $0 < J(u_0) \leq e$.

(b) By the definition of d and the proof of Lemma 3.6, we see that $K(u_0) = 0$ and $\|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})} \neq 0$ is impossible provided $0 < J(u_0) \leq e < d$. Thus the result of Lemma 3.6 shows that for any solution of problem (1.1) with $0 < J(u_0) \leq e$ which would be not in the region $U_e = \mathcal{N}_{\delta_1\delta_2} = \bigcup_{\delta_1 < \delta < \delta_2} \mathcal{N}_\delta$.

4 Proofs of the main results

In this section we prove the main results by making use of the family of potential wells introduced above. First we have the following lemma which will be used in the proof of the asymptotic behavior for global solutions of problem (1.1).

Lemma 4.1 *Let $y(t)$: $\mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function, and assume that there is a constant $A > 0$ such that*

$$\int_s^{+\infty} y(t)dt \le Ay(s), \quad 0 \le s < +\infty,$$

then $y(t) \le y(0)e^{1-t/A}$, for all $t \ge 0$.

Proof of Theorem 1.1 We divide the proof in two steps.

Step 1 Proof of global existence.

From Proposition 2.3, we can choose $\{w_j(x)\}$ as the orthonormal basis of $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$). Now we Construct the following approximate solutions $u_m(t,x)$ of problem (1.1) as done in [4] and [7]:

$$u_m(t,x) = \sum_{j=1}^{m} d_{jm}(t)w_j(x), \quad m = 1, 2, \dots,$$

which satisfy

$$(u_{mt}, w_k)_2 + (\nabla_{\mathbb{B}}u_m, \nabla_{\mathbb{B}}w_k)_2 = (u_m|u_m|^{p-1}, w_k)_2, \quad k = 1, 2, \dots, \tag{4.1}$$

$$u_m(0,x) = \sum_{j=1}^{m} d_{jm}(0)w_j(x) \to u_0(x) \quad \text{in } \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})). \tag{4.2}$$

Multiplying (4.1) and (4.2) by $d'_{km}(t)$ and summing for k we can deduce that

$$\|u_{mt}\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + \frac{1}{2}\frac{d}{dt}\|\nabla_{\mathbb{B}}u_m\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} = \frac{1}{p+1}\frac{d}{dt}\int_{\mathbb{B}} |u_m|^{p+1}\frac{dx_1}{x_1}dx'. \tag{4.3}$$

Integrating (4.3) with respect to t we obtain

$$\int_0^t \|u_{m\tau}\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} d\tau + \frac{1}{2}\|\nabla_{\mathbb{B}}u_m\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}$$
$$-\frac{1}{p+1}\int_{\mathbb{B}} |u_m|^{p+1}\frac{dx_1}{x_1}dx' = J(u_{m0}) < d, \tag{4.4}$$

and $u_m \in W$ for sufficiently large m and $0 \le t < \infty$ by Lemma 3.6. Hence from (4.4) and

$$\begin{aligned} J(u_m) &= \frac{1}{2}\|\nabla_{\mathbb{B}} u_m\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - \frac{1}{p+1}\int_{\mathbb{B}} |u_m|^{p+1}\frac{dx_1}{x_1}dx' \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\|\nabla_{\mathbb{B}} u_m\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + \frac{1}{p+1}K(u_m) \\ &\ge \frac{p-1}{2(p+1)}\|\nabla_{\mathbb{B}} u_m\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}, \end{aligned}$$

it follows that

$$\int_0^t \|u_{m\tau}\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} d\tau + \frac{p-1}{2(p+1)}\|\nabla_{\mathbb{B}} u_m\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} < d, \quad 0 \le t < \infty, \tag{4.5}$$

for sufficiently large m. Thus from (4.5) we can deduce that

$$\|\nabla_{\mathbb{B}} u_m\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} < \tfrac{2(p+1)}{p-1}d, \quad 0 \le t < \infty, \tag{4.6}$$

$$\int_0^t \|u_{m\tau}\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} d\tau < d, \quad 0 \le t < \infty, \tag{4.7}$$

$$\begin{aligned} &\textstyle\int_{\mathbb{B}} \left||u_m|^{p-1}u_m\right|^{\frac{p+1}{p}} \frac{dx_1}{x_1}dx' = \int_{\mathbb{B}} |u_m|^{p+1}\frac{dx_1}{x_1}dx' \\ &\le C_*^{p+1}\|\nabla_{\mathbb{B}} u_m\|^{p+1}_{L_2^{\frac{n}{2}}(\mathbb{B})} < C_*^{p+1}\left(\tfrac{2(p+1)}{p-1}d\right)^{\frac{p+1}{2}}. \end{aligned} \tag{4.8}$$

From (4.6), (4.7) and (4.8) it follows that there exist u and a subsequence still denoted as $\{u_m\}$ such that, as $m \to \infty$,

$u_m \to u \quad in \quad L^\infty(0, \infty; \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}))$ weakly star and a.e. in $int\mathbb{B} \times [0, \infty)$,

$u_{mt} \to u_t \quad in \quad L^2(0, \infty; L_2^{\frac{n}{2}}(\mathbb{B}))$ weakly star,

$|u_m|^{p-1}u_m \to |u|^{p-1}u \quad in \quad L^\infty(0, \infty; L^{\frac{pn}{p+1}}_{\frac{p+1}{p}}(\mathbb{B}))$ weakly star and a.e. in $int\mathbb{B} \times [0, \infty)$.

Hence in (4.1), for k fixed and $m \to \infty$, we have

$$(u_t, w_k)_2 + (\nabla_{\mathbb{B}} u, \nabla_{\mathbb{B}} w_k)_2 = (u|u|^{p-1}, w_k)_2.$$

On the other hand from (4.2) we obtain $u(0, x) = u_0(x)$ in $\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$. By density we obtain $u \in L^\infty(0, \infty; \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}))$ with $u_t \in L^2(0, \infty; L_2^{\frac{n}{2}}(\mathbb{B}))$ is a global weak solution of problem (1.1). It is obvious that $u(t) \in W$ for $0 \le t < \infty$.

Step 2 By Step 1, $K(u(t)) \geq 0$ for all $t \geq 0$. Thus

$$J(u_0) \geq J(u(t)) = \left(\frac{1}{2} - \frac{1}{p+1}\right) \|\nabla_{\mathbb{B}} u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + \frac{1}{p+1} K(u) \geq \frac{p-1}{2(p+1)} \|\nabla_{\mathbb{B}} u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}. \tag{4.9}$$

Hence the definition of C_* or Proposition 2.2 implies

$$\int_{\mathbb{B}} |u|^{p+1} \frac{dx_1}{x_1} dx' \leq C_*^{p+1} \left(\frac{2(p+1)}{p-1} J(u_0)\right)^{\frac{p-1}{2}} \|\nabla_{\mathbb{B}} u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}.$$

Let $C_*^{p+1}\left(\frac{2(p+1)}{p-1} J(u_0)\right)^{\frac{p-1}{2}} = \sigma$ ($0 < \sigma < 1$ by (iii) of Lemma 3.3 and $J(u_0) < d$), $\gamma = 1 - \sigma$, then

$$\int_{\mathbb{B}} |u|^{p+1} \frac{dx_1}{x_1} dx' \leq (1-\gamma) \|\nabla_{\mathbb{B}} u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}, \quad i.e. \quad \gamma \|\nabla_{\mathbb{B}} u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \leq K(u(t)). \tag{4.10}$$

Let $T > 0$ be a fixed time, we have $\frac{d}{dt} \int_{\mathbb{B}} |u(t)|^2 \frac{dx_1}{x_1} dx' = -2K(u(t))$, thus from (2.2) we obtain

$$\int_t^T K(u(\tau)) d\tau = \frac{1}{2} \int_{\mathbb{B}} |u(t)|^2 \frac{dx_1}{x_1} dx' - \frac{1}{2} \int_{\mathbb{B}} |u(T)|^2 \frac{dx_1}{x_1} dx' \leq C(\mathbb{B}) \|\nabla_{\mathbb{B}} u(t)\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}. \tag{4.11}$$

By (4.9) and (4.11) we have

$$\int_t^T K(u(\tau)) d\tau \leq C(\mathbb{B}) \left(\frac{2(p+1)}{p-1} J(u(t))\right) = C_1 J(u(t)), \text{ for } t \in [0, T].$$

Furthermore, (4.9) and (4.10) imply that

$$J(u(t)) \leq \left(\frac{p-1}{2\gamma(p+1)} + \frac{1}{p+1}\right) K(u(t)).$$

Hence on $[0, T]$, we obtain $\int_t^T K(u(\tau)) d\tau \leq AK(u(t))$ with constant $A = C_1\left(\frac{p-1}{2\gamma(p+1)} + \frac{1}{p+1}\right)$. Then by the arbitrariness of $T > 0$, it follows that

$$\int_t^\infty K(u(\tau))d\tau \le AK(u(t)).$$

That means, from Lemma 4.1, that

$$K(u(t)) \le K(u_0)e^{1-t/A}, \quad t \ge 0.$$

Thus we can deduce the asymptotic behavior of the solution immediately from (4.9). □

Proof of Theorem 1.2 We shall employ the classical concavity method. Let $u(t)$ be any solution of problem (1.1) with $J(u_0) < d$ and $K(u_0) < 0$, T being the existence time of $u(t)$. We should prove $T < \infty$ by contradiction. Let

$$M(t) = \int_0^t \|u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} d\tau.$$

Then $\dot{M}(t) = \|u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}$ and

$$\ddot{M}(t) = 2(u_t, u)_2 = 2((u|u|^{p-1}, u)_2 - \|\nabla_{\mathbb{B}} u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}) = -2K(u). \tag{4.12}$$

We can obtain

$$\begin{aligned}
\ddot{M}(t) &= -2\left(\|\nabla_{\mathbb{B}} u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - \int_{\mathbb{B}} |u|^{p+1}\frac{dx_1}{x_1}dx'\right) \\
&= -2(p+1)J(u) + (p-1)\|\nabla_{\mathbb{B}} u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \\
&\ge 2(p+1)\int_0^t \|u_\tau\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} d\tau + (p-1)\lambda_1 \dot{M}(t) - 2(p+1)J(u_0), \qquad (4.13)
\end{aligned}$$

where λ_1 is constructed by (2.4). Since

$$\begin{aligned}
\left(\int_0^t (u_\tau, u)_2 d\tau\right)^2 &= \left(\frac{1}{2}\int_0^t \frac{d}{d\tau}\|u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} d\tau\right)^2 \\
&= \frac{1}{4}\left(\dot{M}(t) - 2\|u_0\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}\dot{M}(t) + \|u_0\|^4_{L_2^{\frac{n}{2}}(\mathbb{B})}\right),
\end{aligned}$$

we obtain

$$M\ddot{M}-\frac{p+1}{2}\dot{M}^2 \geq 2(p+1)\left\{\int_0^t \|u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}d\tau \int_0^t \|u_\tau\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}d\tau-\left(\int_0^t (u_\tau, u)_2 d\tau\right)^2\right\}$$
$$+(p-1)\lambda_1 M\dot{M}-(p+1)\|u_0\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}\dot{M}$$
$$-2(p+1)J(u_0)M+\frac{p+1}{2}\|u_0\|^4_{L_2^{\frac{n}{2}}(\mathbb{B})}.$$

By cone Hölder inequality (2.1) we get

$$M\ddot{M}-\frac{p+1}{2}\dot{M}^2 \geq (p-1)\lambda_1 M\dot{M}-(p+1)\|u_0\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}\dot{M}-2(p+1)J(u_0)M. \tag{4.14}$$

(a) If $J(u_0) \leq 0$, then $M\ddot{M}-\frac{p+1}{2}\dot{M}^2 \geq (p-1)\lambda_1 M\dot{M}-(p+1)\|u_0\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}\dot{M}$. First we prove $K(u) < 0$ for $t > 0$. Otherwise, we must have $t_0 > 0$ such that $K(u(t_0)) = 0$ and $K(u(t)) < 0$ for all $0 \leq t < t_0$. Hence from Lemma 3.2 we have $\|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})} > r(1)$ for $0 \leq t < t_0$. Then $\|\nabla_{\mathbb{B}} u(t_0)\|_{L_2^{\frac{n}{2}}(\mathbb{B})} \geq r(1)$ and $J(u(t_0)) \geq d$, which contradicts with the energy inequality. So we obtain $\ddot{M}(t) > 0$ from (4.12) immediately. Since $\dot{M}(0) = \|u_0\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} \geq 0$, then there exists a $t_0 \geq 0$ such that $\dot{M}(t_0) > 0$ and

$$M(t) \geq \dot{M}(t_0)+\dot{M}(t_0)(t-t_0) > \dot{M}(t_0)(t-t_0), \quad t \geq t_0.$$

Thus for sufficiently large t we have $(p-1)\lambda_1 M > (p+1)\|u_0\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}$ and

$$M\ddot{M}-\frac{p+1}{2}\dot{M}^2 > 0. \tag{4.15}$$

(b) If $0 < J(u_0) < d$, we can obtain $u(t) \in V_\delta$ for $1 < \delta < \delta_2$ and $t > 0$ by Lemma 3.6. Here δ_2 is the larger root of equation $d(\delta) = J(u_0)$. From the result of Lemma 3.2 we have $\|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})} > r(\delta)$ for $1 < \delta < \delta_2$. So we get $K_{\delta_2}(u) \leq 0$ and $\|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})} \geq r(\delta_2)$ for $t > 0$. By (4.12), we obtain

$$\ddot{M}(t)=-2K(u)=2(\delta_2-1)\|\nabla_{\mathbb{B}} u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}-2K_{\delta_2}(u) \geq 2(\delta_2-1)r^2(\delta_2), \quad t \geq 0,$$

$$\dot{M}(t) \geq 2(\delta_2-1)r^2(\delta_2)t+\dot{M}(0) \geq 2(\delta_2-1)r^2(\delta_2)t, \quad t \geq 0,$$

$$M(t) \geq (\delta_2-1)r^2(\delta_2)t^2+tM(0) \geq (\delta_2-1)r^2(\delta_2)t^2, \quad t \geq 0.$$

Hence for sufficiently large t, we have

$$\frac{1}{2}(p-1)\lambda_1 M(t) > (p+1)\|u_0\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}, \quad \frac{1}{2}(p-1)\lambda_1 \dot{M}(t) > (p+1)J(u_0).$$

Combining with (4.14) we get (4.15) immediately for sufficient large t again.

By a directly computation we can see that

$$(M^{-\alpha})'' = -\alpha M^{-\alpha-2}(M\ddot{M} - (\alpha+1)\dot{M}^2).$$

Let $\alpha = \frac{p-1}{2}$, (4.15) implies $(M^{-\frac{p-1}{2}})'' < 0$ for sufficiently large t. That means, for $t > \tilde{t}$, that

$$M^{-\frac{p-1}{2}}(t) \leq M^{-\frac{p-1}{2}}(\tilde{t})\left(1 - \left(\frac{p-1}{2}\right)\frac{\dot{M}(\tilde{t})}{M(\tilde{t})}(t-\tilde{t})\right),$$

which implies that there exist a $T > 0$ such that

$$\lim_{t \to T^-} M^{-\frac{p-1}{2}}(t) = 0.$$

Theorem 1.2 is proved. □

Proof of Theorem 1.3 Let $\mu_m = 1 - \frac{1}{m}$ and $u_{0m} = \mu_m u_0$, $m = 2, 3, \ldots$. We consider the following problem

$$\begin{cases} \partial_t u - \triangle_{\mathbb{B}} u = |u|^{p-1}u & x \in int\mathbb{B}, t > 0, \\ u(0,x) = u_{0m}(x) & x \in \mathbb{B}, \\ u(t,x) = 0 & x \in \partial\mathbb{B}, t \geq 0. \end{cases} \tag{4.16}$$

From $K(u_0) \geq 0$ and Lemma 3.3, we have $\mu^* = \mu^*(u_0) \geq 1$. Therefore $K(u_{0m}) > 0$ and $J(u_{0m}) = J(\mu_m u_0) < J(u_0) = d$. Here

$$\begin{aligned} J(u_{0m}) &= \frac{1}{2}\|\nabla_{\mathbb{B}} u_{0m}\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} - \frac{1}{p+1}\int_{\mathbb{B}} |u_{0m}|^{p+1}\frac{dx_1}{x_1}dx' \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\|\nabla_{\mathbb{B}} u_{0m}\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} + \frac{1}{p+1}K(u_{0m}) > 0. \end{aligned}$$

So it follows from Theorem 1.1 that, for each m, problem (4.16) admits a global weak solution $u_m(t) \in L^\infty(0,\infty;\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}))$ with $u_{mt} \in L^2(0,\infty; L^{\frac{n}{2}}_2(\mathbb{B}))$ and $u_m(t) \in \overline{W}$ for $0 \le t < \infty$, satisfying

$$(u_{mt}, v)_2 + (\nabla_{\mathbb{B}} u_m, \nabla_{\mathbb{B}} v)_2 = (|u_m|^{p-1}u_m, v)_2,\ \text{for any } v \in \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}),\quad \text{and}\quad t \in (0,\infty),$$

$$\int_0^t \|u_{m\tau}\|^2_{L^{\frac{n}{2}}_2(\mathbb{B})}\, d\tau + J(u_m(t)) \le J(u_{0m}) < J(u_0) = d. \tag{4.17}$$

By a direct computation we can see that

$$\int_0^t \|u_{m\tau}\|^2_{L^{\frac{n}{2}}_2(\mathbb{B})}\, d\tau + \left(\frac{1}{2} - \frac{1}{p+1}\right)\|\nabla_{\mathbb{B}} u_m\|^2_{L^{\frac{n}{2}}_2(\mathbb{B})} + \frac{1}{p+1}K(u_m) < d.$$

Since $K(u_m) \ge 0$, we can deduce (4.6), (4.7) and (4.8) for each m. Hence there exist a u and a subsequence still denoted as $\{u_m\}$, such that, as $m \to \infty$,

$u_m \to u \quad in \quad L^\infty(0,\infty;\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}))$ weakly star and a.e. in $int\mathbb{B} \times [0,\infty)$,

$u_{mt} \to u_t \quad in \quad L^2(0,\infty; L^{\frac{n}{2}}_2(\mathbb{B}))$ weakly star,

$|u_m|^{p-1}u_m \ \to\ |u|^{p-1}u \quad in \quad L^\infty(0,\infty; L^{\frac{pn}{p+1}}_{\frac{p+1}{p}}(\mathbb{B}))$ weakly star and a.e. in int $\mathbb{B} \times [0,\infty)$.

The proof of global existence for the solution is the same as that in the first part of the Theorem 1.1.

Now it sufficient to show the asymptotic behavior of the solution. Let $u(t)$ be the global solution of problem (1.1) with $J(u_0) = d$, $K(u_0) > 0$, then we obtain $K(u(t)) \ge 0$ for $0 \le t < \infty$. Next we consider the following two cases

(i) Assume that $K(u(t)) > 0$ for $0 \le t < \infty$. Then from $(u_t, u)_2 = -K(u) < 0$, it follows that $\|u_t\|_{L^{\frac{n}{2}}_2(\mathbb{B})} > 0$ and $\int_0^t \|u_\tau\|^2_{L^{\frac{n}{2}}_2(\mathbb{B})}\, d\tau$ is strictly increasing for $0 \le t < \infty$. Taking any $t_1 > 0$ and letting

$$d_1 = J(u(t_1)) = J(u_0) - \int_0^t \|u_\tau\|^2_{L^{\frac{n}{2}}_2(\mathbb{B})}\, d\tau,$$

then by the energy inequality we get $0 < J(u) \le d_1 < d$ for $t_1 \le t < \infty$. Similar to the proof of Theorem 1.1, we can deduce the exponential decay (1.8) if we take $t = t_1$ as the initial time.

(ii) Assume that there exists a $t_1 > 0$ such that $K(u(t_1)) = 0$ and $K(u) > 0$ for $0 \le t < t_1$. We also have $\|u_t\|_{L^{\frac{n}{2}}_2(\mathbb{B})} > 0$ and $\int_0^t \|u_\tau\|^2_{L^{\frac{n}{2}}_2(\mathbb{B})}\, d\tau$ is strictly increasing for $0 \le t < t_1$. From the energy inequality we can also deduce that

$$J(u(t_1)) \le J(u_0) - \int_0^t \|u_\tau\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} d\tau < d,$$

which implies that the result $\|\nabla_{\mathbb{B}} u(t_1)\|_{L_2^{\frac{n}{2}}(\mathbb{B})} \neq 0$ is not true. If $\|\nabla_{\mathbb{B}} u(t_1)\|_{L_2^{\frac{n}{2}}(\mathbb{B})} = 0$, i.e. $J(u(t_1)) = 0$, then we get $J(u) \le 0$ for $t_1 \le t < \infty$ from

$$J(u(t_1)) \ge \int_{t_1}^t \|u_\tau\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} d\tau + J(u), \quad t_1 \le t < \infty.$$

Hence from

$$\begin{aligned}
\frac{1}{2}\|\nabla_{\mathbb{B}} u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} &\le \frac{1}{p+1}\int_{\mathbb{B}} |u|^{p+1}\frac{dx_1}{x_1}dx' \\
&\le \frac{1}{p+1}C_*^{p+1}\|\nabla_{\mathbb{B}} u\|^{p-1}_{L_2^{\frac{n}{2}}(\mathbb{B})}\|\nabla_{\mathbb{B}} u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})},
\end{aligned}$$

it follows that either $\|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})} = 0$ for $t_1 \le t < \infty$, hence (1.8) holds; or $\|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})} \ge (\frac{p+1}{2C_*^{p+1}})^{\frac{1}{p-1}}$, for $t_1 \le t < \infty$, which is impossible since $\|\nabla_{\mathbb{B}} u\|_{L_2^{\frac{n}{2}}(\mathbb{B})} = 0$. This completes the proof. □

Proof of Theorem 1.4 Let $u(t)$ be a solution of problem (1.1) with $J(u_0) = d$ and $K(u_0) < 0$, T is the existence time of $u(t)$. We need to prove that $T < +\infty$. From the continuities of $J(u)$ and $K(u)$ with respect to t, we know that there exists a sufficient small $t_1 > 0$ with $t_1 < T$ such that $J(u(t_1)) > 0$ and $K(u) < 0$ for $0 \le t \le t_1$. So we can deduce $(u_t, u)_2 = -K(u) > 0$ and $\|u_t\|_{L_2^{\frac{n}{2}}(\mathbb{B})} > 0$, for $0 \le t \le t_1$. Therefore $\int_0^t \|u_\tau\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} d\tau$ is strictly increasing for $0 \le t \le t_1$, and we can choose t_1 such that

$$0 < d_1 = d - \int_0^{t_1} \|u_\tau\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} d\tau < d.$$

From the energy inequality, we have

$$0 < J(u(t_1)) = J(u_0) - \int_0^{t_1} \|u_\tau\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})} d\tau < J(u_0) = d.$$

In Theorem 1.2, if we take $t = t_1$ as the initial time, then the Theorem 1.2 implies that the existence time T of $u(t)$ is finite and

$$\lim_{t\to T^-}\int_0^t \|u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}\,d\tau = +\infty.$$

□

Remark 4.1 From Theorem 1.1, Theorem 1.2, Theorem 1.3 and Theorem 1.4, we can moreover obtain the following exact conditions for the global existence of solutions for problem (1.1):

Let $u_0 \in \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$, and $J(u_0) \le d$. Then the sign of $K(u_0)$ makes critical role on the solutions of problem (1.1), namely

(1) When $K(u_0) \ge 0$, problem (1.1) admits a global weak solution $u \in L^\infty(0,\infty; \mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}))$ with $u_t \in L^2(0,\infty; L_2^{\frac{n}{2}}(\mathbb{B}))$.
(2) When $K(u_0) < 0$, there is no global weak solution for problem (1.1), that the solution of problem (1.1) blows up in finite time in the sense of

$$\lim_{t\to T^-}\int_0^t \|u\|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}\,d\tau = +\infty.$$

Therefore, by means of cone Poincaré inequality (2.2), we have

$$\limsup_{t\to T^-}\|\nabla_{\mathbb{B}}u\|_{L_2^{\frac{n}{2}}(\mathbb{B})} = +\infty.$$

Acknowledgments This work was partially supported by the NSFC grant 11131005 and the SECC grant 20090141110003. The initial version of this paper was started when the first author was visiting the University of Kyoto in May of 2011, He would like to thank Professor Yoshinori Morimoto for his invitation and facilities. Also, the authors would like to express their gratitude to the referees for helpful comments.

References

1. Chen, H., Liu, X., Wei, Y.: Cone Sobolev inequality and Dirichlet problem for nonlinear elliptic equations on manifold with conical singularities. Calc. Var. **43**, 463–484 (2012)
2. Chen, H., Liu, X., Wei, Y.: Existence theory for a class of semilinear totally characteristic elliptic equations with critical cone Sobolev exponents. Ann. Glob. Anal. Geom. **39**, 27–43 (2011)
3. Coriasco, S., Schrohe, E., Seiler, J.: Realizations of differential operators on conic manifolds with boundary. Ann. Glob. Anal. Geom. **31**, 223–285 (2007)
4. Evans, L.C.: Partial differential equations. Graduate Studies in Mathematics, vol. 19. American Mathematical Society (1998)
5. Gazzola, F., Weth, T.: Finite time blow-up and global solutions for semilinear parabolic equations with initial data at high energy level. Differ. Integr. Equ. **18**, 961–990 (2005)
6. Komornik, V.: Exact Controllability and Stabilization. The Multiplier Method. Mason-John Wiley, Paris (1994)
7. Liu, Y., Zhao, J.: On potential wells and applications to semilinear hyperbolic and parabolic equations. Nonlinear Anal. **64**, 2665–2687 (2006)

8. Payne, L.E., Sattinger, D.H.: Saddle points and instability of nonlinear hyperbolic equations. Isr. J. Math. **22**, 273–303 (1975)
9. Sattinger, D.H.: On global solutions of nonlinear hyperbolic equations. Arch. Ration. Mech. Anal. **30**, 148–172 (1968)
10. Schrohe, E., Seiler, J.: Ellipticity and invertibility in the cone algebra on L_p-Sobolev spaces. Integr. Equ. Oper. Theory **41**, 93–114 (2001)
11. Schulze, B.-W.: Boundary value problems and singular pseudo-differential operators. Wiley, Chichester (1998)