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Multiplication properties in pseudo-differential calculus with small regularity on the symbols

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Abstract We consider modulation space and spaces of Schatten–von Neumann symbols where corresponding pseudo-differential operators map one Hilbert space to another. We prove Hölder–Young and Young type results for such spaces under dilated convolutions and multiplications. We also prove continuity properties for such spaces under the twisted convolution, and the Weyl product. These results lead to continuity properties for twisted convolutions on Lebesgue spaces, e.g. $L^p_{(\omega)}$ is a twisted convolution algebra when $1 \leq p \leq 2$ and ω is an appropriate weight.

Keywords Twisted · Convolution · Weyl product · Schatten–von Neumann · Modulation · Toeplitz

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0 Introduction

In the paper we consider pseudo-differential operators with small or minimal regularity conditions on the symbols. For such classes we establish continuity properties for different types of products which are important in the theory of pseudo-differential operators. Especially we establish Hölder and Young type results for the Weyl product, dilated convolution, twisted convolution and dilated multiplication on Schatten–von Neumann classes, (weighted) Lebesgue spaces and on (weighted) modulation spaces.

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We recall that the composition of two Weyl operators corresponds to the Weyl product of the two operator symbols on the symbol side, and the twisted convolution appears when Weyl product is conjugated by symplectic Fourier transform (see Sect. 1 for the details). Convolutions and multiplications appear when investigating Toeplitz operators (also known as localization operators) in the framework of pseudo-differential calculus. More precisely, each Toeplitz operator $\operatorname{Tp}(a)$, with (Toeplitz) symbol a, agrees with a pseudo-differential operator $\operatorname{Op}(b)$ with symbol b. The symbol b is obtained by an ordinary convolution of the Toeplitz symbol a and a symbol to an operator of rank one. This convolution corresponds to a multiplication on the Fourier transform side. We remark that Toeplitz operators can be used to approximate certain pseudo-differential operators with smooth symbols (see, e.g. [7,38]), and that Toeplitz operators might be convenient to use when estimating kinetic energy in mechanics (cf. [26]).

Assume that $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces of tempered distributions on $\mathbf{R}^d$ which contain $\mathcal{S}(\mathbf{R}^d)$ (we use the same notation for the usual function and distribution spaces as in [24]). Let $\mathcal{I}_p(\mathcal{H}_1,\mathcal{H}_2)$, $p\in[1,\infty]$, be the set of Schatten–von Neumann operators of order p from $\mathcal{H}_1$ to $\mathcal{H}_2$, and let $s_p^w(\mathcal{H}_1,\mathcal{H}_2)$ be the set of all distributions $a\in\mathcal{S}'(\mathbf{R}^{2d})$ such that the corresponding Weyl operators $\operatorname{Op}^w(a)$ belong to $\mathcal{I}_p(\mathcal{H}_1,\mathcal{H}_2)$ (see below or Sect. 1 for notations and strict definitions).

For spaces of the form $s_p^w = s_p^w(\mathcal{H}_1,\mathcal{H}_2)$ it is an easy task to establish continuity properties under the Weyl product and twisted convolution, because such questions can easily be reformulated into compositions for Schatten–von Neumann operators. For example, it is well-known that the Hölder condition $p^{-1}+q^{-1}=r^{-1}$ is sufficient for the embedding

$$\mathcal{I}_p(\mathcal{H}_2,\mathcal{H}_3)\circ\mathcal{I}_q(\mathcal{H}_1,\mathcal{H}_2)\subseteq\mathcal{I}_r(\mathcal{H}_1,\mathcal{H}_3)$$

to hold. Similar properties carry over to the case when the $\mathcal{I}_p$ spaces are replaced by s_p^w spaces, and the composition $\circ$ is replaced by the Weyl product or the twisted convolution.

It is more tricky to find continuity relations for dilated multiplications and convolutions on the s_p^w spaces, because such products take complicated forms on the operator side. In this situation we use certain Fourier techniques, similar to those in [38, Section 3], to get convenient integral formulas. By making appropriate estimates on these formulas in combination with duality and interpolation, we establish Young type results for s_p^w spaces under those dilated products.

However, when applying the Fourier technique and performing the estimates, we need some additional structure on the involved Hilbert spaces, and in our approach we assume that $\mathcal{H}_1$ and $\mathcal{H}_2$ are appropriate modulation spaces of Hilbert type. We remark that this should not be an essential restriction for the applications, since the familly of modulation spaces of Hilbert types contain every Sobolev space H_s^2, weighted Hilbert Lebesgue space and mixed versions of such spaces.

For Lebesgue and modulation spaces, the difficulties appear in the opposite situations. That is in contrast to spaces of Schatten symbols, it is tricky to find results under the Weyl product and the twisted convolution, while finding Hölder–Young results under convolutions and multiplications are straight-forward. Continuity properties for modulation spaces under the Weyl product have been investigated in, e.g. [19,23,25,32], and in Sect. 2 we use Theorem 0.3′ in [23] to prove continuity results for modulation and Lebesgue spaces under the twisted convolution. In particular we obtain a weighted version of the fact that L^2 is an algebra under the twisted convolution.

In order to be more specific, we recall some definitions. Assume that $t \in \mathbf{R}$ is fixed and that $a \in \mathcal{S}(\mathbf{R}^{2d})$. Then the pseudo-differential operator $\operatorname{Op}_t(a)$ with symbol a is the linear and continuous operator on $\mathcal{S}(\mathbf{R}^d)$, defined by the formula

$$(\operatorname{Op}_t(a)f)(x) = (2\pi)^{-d} \iint a((1-t)x + ty, \xi) f(y) e^{i\langle x-y,\xi\rangle}\, dy d\xi. \tag{0.1}$$

The definition of $\operatorname{Op}_t(a)$ extends to each $a \in \mathcal{S}'(\mathbf{R}^{2d})$, and then $\operatorname{Op}_t(a)$ is continuous from $\mathcal{S}(\mathbf{R}^d)$ to $\mathcal{S}'(\mathbf{R}^d)$ (cf., e.g. [24] or Sect. 1). If $t = 1/2$, then $\operatorname{Op}_t(a)$ is equal to the Weyl operator $\operatorname{Op}^w(a)$ for a. If instead $t = 0$, then the standard (Kohn–Nirenberg) representation $a(x, D)$ is obtained.

The modulation spaces was introduced by Feichtinger [11], and developed further and generalized in [12,14–16,18]. We are especially interested in the modulation spaces $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ and $W^{p,q}_{(\omega)}(\mathbf{R}^d)$ which are the sets of tempered distributions on $\mathbf{R}^d$ whose short-time Fourier transform (STFT) belong to the weighted and mixed Lebesgue spaces $L^{p,q}_{1,(\omega)}(\mathbf{R}^{2d})$ and $L^{p,q}_{2,(\omega)}(\mathbf{R}^{2d})$, respectively (cf. (1.16) and (1.17) below for the definition of the latter space norms). Here the weight function ω should belong to $\mathcal{P}(\mathbf{R}^{2d})$, the set of all polynomially moderated functions on the phase (or time-frequency shift) space $\mathbf{R}^{2d}$, and $p, q \in [1, \infty]$. It follows that ω, p and q to some extent quantify the degrees of asymptotic decay and singularity of the distributions in $M^{p,q}_{(\omega)}$ and $W^{p,q}_{(\omega)}$ (we refer to [13] for the most updated description of modulation spaces).

In the Weyl calculus of pseudo-differential operators, operator composition corresponds on the symbol level to the Weyl product #, sometimes also called the twisted product. A problem in this field is to find conditions on the weight functions ω_j and $p_j, q_j \in [1, \infty]$, for the map $(a_1, a_2) \mapsto a_1 \# a_2$ on $\mathcal{S}(\mathbf{R}^{2d})$ to be uniquely extendable to a continuous map from $\mathcal{M}^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d}) \times \mathcal{M}^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d})$ to $\mathcal{M}^{p_0,q_0}_{(\omega_0)}(\mathbf{R}^{2d})$. Here the modulation spaces $\mathcal{M}^{p,q}_{(\omega)}$ and $\mathcal{W}^{p,q}_{(\omega)}$ are obtained by replacing the usual STFT with the symplectic STFT in the definition of modulation space norms. Important contributions in this context can be found in [19,23,25,32,37], where Theorem 0.3′ in [23] seems to be the most general result so far (see also Theorem 2.2).

The Weyl product on the symplectic Fourier transform side corresponds to the twisted convolution $*_\sigma$. It follows that the continuity questions here above are equivalent to find appropriate conditions on ω_j and $p_j, q_j \in [1, \infty]$, in order to allow the map $(a_1, a_2) \mapsto a_1 *_\sigma a_2$ to be uniquely extendable to a map from $\mathcal{W}^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d}) \times \mathcal{W}^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d})$ to $\mathcal{W}^{p_0,q_0}_{(\omega_0)}(\mathbf{R}^{2d})$, which is continuous in the sense that

$$\|a_1 *_\sigma a_2\|_{\mathcal{W}^{p_0,q_0}_{(\omega_0)}} \le C \|a_1\|_{\mathcal{W}^{p_1,q_1}_{(\omega_1)}} \|a_2\|_{\mathcal{W}^{p_2,q_2}_{(\omega_2)}}, \tag{0.2}$$

should hold for some constant $C > 0$ which is independent of $a_1 \in \mathcal{W}^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d})$ and $a_2 \in \mathcal{W}^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d})$. Appropriate assumptions are then

$$\omega_0(X,Y) \le C\omega_1(X-Y+Z,Z)\omega_2(Y-Z,X+Z), \quad X,Y,Z \in \mathbf{R}^{2d}, \tag{0.3}$$

$$\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_0} = 1-\left(\frac{1}{q_1}+\frac{1}{q_2}-\frac{1}{q_0}\right) \tag{0.4}$$

and

$$0 \le \frac{1}{q_1}+\frac{1}{q_2}-\frac{1}{q_0} \le \frac{1}{p_j}, \frac{1}{q_j} \le \frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_0}, \quad j=0,1,2. \tag{0.5}$$

The results for twisted convolution which corresponds to Theorem 0.3′ in [23] is the following.

Theorem 0.1 *Assume that* $\omega_0,\omega_1,\omega_2 \in \mathcal{P}(\mathbf{R}^{4d})$ *satisfy* (0.3)*, and that* $p_j,q_j \in [1,\infty]$ *for* $j=0,1,2$*, satisfy* (0.4) *and* (0.5)*. Then the map* $(a_1,a_2)\mapsto a_1 *_\sigma a_2$ *on* $\mathcal{S}(\mathbf{R}^{2d})$ *extends uniquely to a continuous map from* $\mathcal{W}^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d}) \times \mathcal{W}^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d})$ *to* $\mathcal{W}^{p_0,q_0}_{(\omega_0)}(\mathbf{R}^{2d})$*, and for some constant* $C>0$*, the bound* (0.2) *holds for every* $a_1 \in \mathcal{W}^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d})$ *and* $a_2 \in \mathcal{W}^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d})$.

In the end of Sect. 2 we especially consider the case when $p_j=q_j=2$, and the involved weights $\omega_j(X,Y)$ are independent of the Y-variable, i.e. $\omega_j(X,Y)=\omega_j(X)$. In this case, $\mathcal{W}^{2,2}_{(\omega_j)}$ agrees with $L^2_{(\omega_j)}$, and the condition (0.3) is reduced into

$$\omega_0(X_1+X_2) \le C\omega_1(X_1)\omega_2(X_2). \tag{0.6}$$

By Theorem 0.1 it now follows that the map $(a_1,a_2)\mapsto a_1 *_\sigma a_2$ extends to a continuous mapping from $L^2_{(\omega_1)} \times L^2_{(\omega_2)}$ to $L^2_{(\omega_0)}$, and that

$$\|a_1 *_\sigma a_2\|_{L^2_{(\omega_0)}} \le C\|a_1\|_{L^2_{(\omega_1)}}\|a_2\|_{L^2_{(\omega_2)}}, \tag{0.7}$$

holds when $a_1 \in L^2_{(\omega_1)}(\mathbf{R}^{2d})$ and $a_2 \in L^2_{(\omega_2)}(\mathbf{R}^{2d})$. The latter continuity is a special case of the following result, also proved in Sect. 2.

Theorem 0.2 *Assume that* $\omega_j \in \mathcal{P}(\mathbf{R}^{2d})$ *and* $p_j \in [1,\infty]$ *for* $j=0,1,2$ *satisfy* (0.6) *and*

$$\max\left(\frac{1}{p_0},\frac{1}{p_0'}\right) \le \frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_0} \le 1.$$

Then the map $(a_1,a_2)\mapsto a_1 *_\sigma a_2$ *extends uniquely to a continuous mapping from* $L^{p_1}_{(\omega_1)}(\mathbf{R}^{2d}) \times L^{p_2}_{(\omega_2)}(\mathbf{R}^{2d})$ *to* $L^{p_0}_{(\omega_0)}(\mathbf{R}^{2d})$*, and*

$$\|a_1 *_\sigma a_2\|_{L^{p_0}_{(\omega_0)}} \le C\|a_1\|_{L^{p_1}_{(\omega_1)}}\|a_2\|_{L^{p_2}_{(\omega_2)}}, \tag{0.8}$$

holds for some constant C *which is independent of* $a_1 \in L^{p_1}_{(\omega_1)}(\mathbf{R}^{2d})$ *and* $a_2 \in L^{p_2}_{(\omega_2)}(\mathbf{R}^{2d})$.

We also prove a generalization of Theorem 0.2 in Sect. 2, which involve mixed weighted Lebesgue spaces, and use this generalization in Sect. 3 to extend the class of possible window functions in the definition of modulation space norms.

In Sect. 5 we establish Young type results for dilated multiplications and convolutions for the spaces $s_p^w(\omega_1,\omega_2) \equiv s_p^w(\mathcal{H}_1,\mathcal{H}_2)$, when $\mathcal{H}_j$ for $j=1,2$ is modulation space $M^{2,2}_{(\omega_j)}(\mathbf{R}^d) = M^2_{(\omega_j)}(\mathbf{R}^d)$ with appropriate weights ω_j. The involved Schatten exponents should satisfy the Young condition

$$p_1^{-1} + p_2^{-1} = 1 + r^{-1}, \qquad 1 \le p_1, p_2, r \le \infty, \tag{0.9}$$

and the involved dilation factors should satisfy

$$(-1)^{j_1} t_1^{-2} + (-1)^{j_2} t_2^{-2} = 1. \tag{0.10}$$

or

$$(-1)^{j_1} t_1^{2} + (-1)^{j_2} t_2^{2} = 1. \tag{0.11}$$

The conditions for the involved weight functions are

$$\begin{aligned} \vartheta(X_1+X_2) &\le C\vartheta_{j_1,1}(t_1X_1)\vartheta_{j_2,2}(t_2X_2),\\ \omega(X_1+X_2) &\le C\omega_{j_1,1}(t_1X_1)\omega_{j_2,2}(t_2X_2), \end{aligned} \tag{0.12}$$

where

$$\omega_{0,k}(X) = \vartheta_{1,k}(X) = \omega_k(X), \quad \vartheta_{0,k}(X) = \omega_{1,k}(X) = \vartheta_k(X). \tag{0.13}$$

With these conditions we prove

$$\|a_{1,t_1} * a_{2,t_2}\|_{s_r^w(1/\omega,\vartheta)} \le C^d \|a_1\|_{s_{p_1}^w(1/\omega_1,\vartheta_1)} \|a_2\|_{s_{p_2}^w(1/\omega_2,\vartheta_2)}, \tag{0.14}$$

$$\|a_{1,t_1} a_{2,t_2}\|_{s_r^w(1/\omega,\vartheta)} \le C^d \|a_1\|_{s_{p_1}^w(1/\omega_1,\vartheta_1)} \|a_2\|_{s_{p_2}^w(1/\omega_2,\vartheta_2)}, \tag{0.15}$$

holds for admissible a_1 and a_2. Here and in what follows we set $a_{j,t} = a_j(t\,\cdot\,)$, and we let $p' \in [1,\infty]$ be the conjugate exponent of $p \in [1,\infty]$, i.e. p and p' should satisfy $1/p + 1/p' = 1$. More precisely, in Sect. 5 we prove the following two theorems, as well as multi-linear extensions of these results (cf. Theorems $0.3'$ and $0.4'$). We remark that these multi-linear versions generalize Theorem 3.3, Theorem $3.3'$ and Corollary 3.5 in [38]. In fact, the latter results follow by letting $\mathcal{H}_1 = \mathcal{H}_2 = L^2$ in Theorems $0.3'$ and $0.4'$.

Theorem 0.3 *Assume that $p_1, p_2, r \in [1,\infty]$ satisfy (0.9), and that $t_1, t_2 \in \mathbf{R}\backslash 0$ satisfy (0.10), for some choices of $j_1, j_2 \in \{0,1\}$. Also assume that $\omega, \omega_j, \vartheta, \vartheta_j \in \mathcal{P}(\mathbf{R}^{2d})$ for $j=1,2$ satisfy (0.12) and (0.13). Then the map $(a_1,a_2) \mapsto a_{1,t_1} * a_{2,t_2}$ on $\mathcal{S}(\mathbf{R}^{2d})$, extends uniquely to a continuous mapping from*

$$s_{p_1}^w(1/\omega_1,\vartheta_1) \times s_{p_2}^w(1/\omega_2,\vartheta_2)$$

to $s^w_r(1/\omega, \vartheta)$*. Furthermore,* (0.14) *holds for some constant*

$$C = C_0^2|t_1|^{-2/p_1}|t_2|^{-2/p_2},$$

where C_0 *is independent of* $a_1 \in s^w_{p_1}(1/\omega_1, \vartheta_1)$*,* $a_2 \in s^w_{p_2}(1/\omega_2, \vartheta_2)$*,* t_1, t_2 *and* d*.*
Moreover, $\operatorname{Op}^w(a_{1,t_1} * a_{2,t_2}) \geq 0$ *when* $\operatorname{Op}^w(a_j) \geq 0$ *for each* $1 \leq j \leq 2$*.*

Theorem 0.4 *Assume that* $p_1, p_2, r \in [1, \infty]$ *satisfy* (0.9)*, and that* $t_1, t_2 \in \mathbf{R}\backslash 0$ *satisfy* (0.11)*, for some choices of* $j_1, j_2 \in \{0, 1\}$*. Also assume that* $\omega, \omega_j, \vartheta, \vartheta_j \in \mathcal{P}(\mathbf{R}^{2d})$ *for* $j = 1, 2$ *satisfy* (0.12) *and* (0.13)*. Then the map* $(a_1, a_2) \mapsto a_{1,t_1}a_{2,t_2}$ *on* $\mathcal{S}(\mathbf{R}^{2d})$*, extends uniquely to a continuous mapping from*

$$s^w_{p_1}(1/\omega_1, \vartheta_1) \times s^w_{p_2}(1/\omega_2, \vartheta_2)$$

to $s^w_r(1/\omega, \vartheta)$*. Furthermore,* (0.15) *holds for some constant*

$$C = C_0^2|t_1|^{-2/p_1'}|t_2|^{-2/p_2'},$$

where C_0 *is independent of* $a_1 \in s^w_{p_1}(1/\omega_1, \vartheta_1)$*,* $a_2 \in s^w_{p_2}(1/\omega_2, \vartheta_2)$*,* t_1, t_2 *and* d*.*

Some important preparations to the dilated convolution and multiplication results in Sect. 5 are given in Sect.4, where we consider dual properties for $s^w_p(\mathcal{H}_1, \mathcal{H}_2)$. Here $\mathcal{H}_1$ and $\mathcal{H}_2$ belong to a broad class of Hilbert spaces containing any $M^{2,2}_{(\omega)}$ space. More precisely, assume that $p < \infty$. Then we prove that the dual for $s^w_p(\mathcal{H}_1, \mathcal{H}_2)$ can be identified with $s^w_p(\mathcal{H}_1', \mathcal{H}_2')$ for appropriate Hilbert spaces $\mathcal{H}_1'$ and $\mathcal{H}_2'$ through a unique extension of the L^2 form on $\mathcal{S}$ (cf. Theorem 4.12).

In the last section we apply the results in Sect. 5 to prove that the class of trace-symbols is invariant under compositions with odd entire functions. Here we also show how Theorem 0.3 can be used to define Toeplitz operators with symbols in dilated s^w_p spaces, and that such operators fulfill certain Schatten–von Neumann properties.

1 Preliminaries

In this section we introduce some notations and discuss basic results. In the first part we recall some properties within the theory of pseudo-differential operators. Especially we discuss the Weyl product and twisted convolution. In the second part we recall some facts about modulation spaces. The proofs are in general omitted, since the results in pseudo-differential calculus can be found in [17,24,36], and the essential parts of modulation space theory can be found in [9,11,14,15,18].

In all these discussions, the Fourier transform $\mathcal{F}$ is essential. This is defined as the linear and continuous map on $\mathcal{S}'(\mathbf{R}^d)$, which takes the form

$$(\mathcal{F}f)(\xi) = \widehat{f}(\xi) = (2\pi)^{-d/2} \int f(x)e^{-i\langle x,\xi\rangle}\, dx, \tag{1.1}$$

when $f \in L^1(\mathbf{R}^d)$. It follows that $\mathcal{F}$ is a homeomorphism on $\mathcal{S}'(\mathbf{R}^d)$ which restricts to a homeomorphism on $\mathcal{S}(\mathbf{R}^d)$ and to a unitary operator on $L^2(\mathbf{R}^d)$.

Assume that $a \in \mathcal{S}(\mathbf{R}^{2d})$, and that $t \in \mathbf{R}$ is fixed. Then the pseudo-differential operator $\operatorname{Op}_t(a)$ in (0.1) is a linear and continuous operator on $\mathcal{S}(\mathbf{R}^d)$. For general $a \in \mathcal{S}'(\mathbf{R}^{2d})$, the pseudo-differential operator $\operatorname{Op}_t(a)$ is defined as the continuous operator from $\mathcal{S}(\mathbf{R}^d)$ to $\mathcal{S}'(\mathbf{R}^d)$ with distribution kernel given by

$$K_{a,t}(x,y) = (\mathcal{F}_2^{-1}a)((1-t)x+ty, x-y). \tag{1.2}$$

Here $\mathcal{F}_2 F$ is the partial Fourier transform of $F(x,y) \in \mathcal{S}'(\mathbf{R}^{2d})$ with respect to the y variable. This definition makes sense, since the mappings $\mathcal{F}_2$ and $F(x,y) \mapsto F((1-t)x+ty, y-x)$ are homeomorphisms on $\mathcal{S}'(\mathbf{R}^{2d})$. Furthermore, by Schwartz kernel theorem it follows that the map $a \mapsto \operatorname{Op}_t(a)$ is a bijection from $\mathcal{S}'(\mathbf{R}^{2d})$ to the set of linear and continuous operators from $\mathcal{S}(\mathbf{R}^d)$ to $\mathcal{S}'(\mathbf{R}^d)$.

In particular, for each $a \in \mathcal{S}'(\mathbf{R}^{2d})$ and $s, t \in \mathbf{R}$, there is a unique $b \in \mathcal{S}'(\mathbf{R}^{2d})$ such that $\operatorname{Op}_s(a) = \operatorname{Op}_t(b)$. The relation between a and b is given by

$$\operatorname{Op}_s(a) = \operatorname{Op}_t(b) \qquad \Longleftrightarrow \qquad b(x,\xi) = e^{i(t-s)\langle D_x, D_\xi\rangle} a(x,\xi). \tag{1.3}$$

(Cf. [24, Sect. 18.5].) Note here that the right-hand side makes sense, since $e^{i(t-s)\langle D_x, D_\xi\rangle}$ on the Fourier transform side is a multiplication by the bounded function $e^{i(t-s)\langle x,\xi\rangle}$.

Assume that $t \in \mathbf{R}$ and $a \in \mathcal{S}'(\mathbf{R}^{2d})$ are fixed. Then a is called a rank-one element with respect to t, if the corresponding pseudo-differential operator is of rank-one, i.e. for each $f \in \mathcal{S}(\mathbf{R}^d)$ we have

$$\operatorname{Op}_t(a)f = (f, f_2)f_1, \tag{1.4}$$

for some $f_1, f_2 \in \mathcal{S}'(\mathbf{R}^d)$. By straight-forward computations it follows that (1.4) is fulfilled, if and only if $a = (2\pi)^{d/2} W^t_{f_1,f_2}$, where the $W^t_{f_1,f_2}$ t-Wigner distribution, defined by the formula

$$W^t_{f_1,f_2}(x,\xi) \equiv \mathcal{F}(f_1(x+t\,\cdot)\overline{f_2(x-(1-t)\,\cdot)})(\xi), \tag{1.5}$$

which takes the form

$$W^t_{f_1,f_2}(x,\xi) = (2\pi)^{-d/2} \int f_1(x+ty)\overline{f_2(x-(1-t)y)} e^{-i\langle y,\xi\rangle}\, dy,$$

when $f_1, f_2 \in \mathcal{S}(\mathbf{R}^d)$. By combining these facts with (1.3), it follows that

$$W^t_{f_1,f_2} = e^{i(t-s)\langle D_x, D_\xi\rangle} W^s_{f_1,f_2}, \tag{1.6}$$

for each $f_1, f_2 \in \mathcal{S}'(\mathbf{R}^d)$ and $s, t \in \mathbf{R}$. Since the Weyl case is particulary important to us, we set $W^t_{f_1,f_2} = W_{f_1,f_2}$ when $t = 1/2$. It follows that W_{f_1,f_2} is the usual (cross-) Wigner distribution of f_1 and f_2.

Next we discuss the Weyl product, twisted convolution and related objects. Assume that $a, b \in \mathcal{S}'(\mathbf{R}^{2d})$ are appropriate. Then the Weyl product $a\#b$ between a and b is the function or distribution which fulfills $\operatorname{Op}^w(a\#b) = \operatorname{Op}^w(a) \circ \operatorname{Op}^w(b)$, provided the right-hand side makes sense. More general, if $t \in \mathbf{R}$, then the product $\#_t$ is defined by the formula

$$\operatorname{Op}_t(a\#_t b) = \operatorname{Op}_t(a) \circ \operatorname{Op}_t(b), \tag{1.7}$$

provided the right-hand side makes sense as a continuous operator from $\mathcal{S}(\mathbf{R}^d)$ to $\mathcal{S}'(\mathbf{R}^d)$.

The Weyl product can also, in a convenient way, be expressed in terms of the symplectic Fourier transform and twisted convolution. More precisely, the *symplectic Fourier transform* for $a \in \mathcal{S}(\mathbf{R}^{2d})$ is defined by the formula

$$(\mathcal{F}_\sigma a)(X) = \pi^{-d} \int a(Y) e^{2i\sigma(X,Y)}\, dY.$$

Here σ is the symplectic form, which is defined by

$$\sigma(X,Y) = \langle y, \xi\rangle - \langle x, \eta\rangle, \qquad X = (x,\xi) \in \mathbf{R}^{2d},\ Y = (y,\eta) \in \mathbf{R}^{2d},$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product on $\mathbf{R}^d$.

It follows that $\mathcal{F}_\sigma$ is continuous on $\mathcal{S}(\mathbf{R}^{2d})$, and extends as usual to a homeomorphism on $\mathcal{S}'(\mathbf{R}^{2d})$, and to a unitary map on $L^2(\mathbf{R}^{2d})$. Furthermore, $\mathcal{F}_\sigma^2$ is the identity operator.

Assume that $a, b \in \mathcal{S}(\mathbf{R}^{2d})$. Then the *twisted convolution* of a and b is defined by the formula

$$(a *_\sigma b)(X) = (2/\pi)^{d/2} \int a(X-Y) b(Y) e^{2i\sigma(X,Y)}\, dY. \tag{1.8}$$

The definition of $*_\sigma$ extends in different ways. For example, it extends to a continuous multiplication on $L^p(\mathbf{R}^{2d})$ when $p \in [1,2]$, and to a continuous map from $\mathcal{S}'(\mathbf{R}^{2d}) \times \mathcal{S}(\mathbf{R}^{2d})$ to $\mathcal{S}'(\mathbf{R}^{2d})$ (cf. [38]). If $a, b \in \mathcal{S}'(\mathbf{R}^{2d})$, then $a\#b$ makes sense if and only if $a *_\sigma \widehat{b}$ makes sense, and then

$$a\#b = (2\pi)^{-d/2} a *_\sigma (\mathcal{F}_\sigma b). \tag{1.9}$$

We also remark that for the twisted convolution we have

$$\mathcal{F}_\sigma(a *_\sigma b) = (\mathcal{F}_\sigma a) *_\sigma b = \check{a} *_\sigma (\mathcal{F}_\sigma b), \tag{1.10}$$

where $\check{a}(X) = a(-X)$ (cf. [36,38,39]). A combination of (1.9) and (1.10) gives

$$\mathcal{F}_\sigma(a\#b) = (2\pi)^{-d/2} (\mathcal{F}_\sigma a) *_\sigma (\mathcal{F}_\sigma b). \tag{1.11}$$

In the Weyl calculus it is many times convenient to use the operator A on $\mathcal{S}'(\mathbf{R}^{2d})$, defined by the formula

$$Aa(x,y) = (\mathcal{F}_2^{-1}a)((y-x)/2, -(x+y)), \quad a \in \mathcal{S}'(\mathbf{R}^{2d}). \tag{1.12}$$

We note that $Aa(x,y)$ agrees with $(2\pi)^{d/2}K_a^w(-x,y)$, where K_a^w is the distribution kernel to the Weyl operator $\operatorname{Op}^w(a)$. If $a \in L^1(\mathbf{R}^{2d})$, then Aa is given by

$$Aa(x,y) = (2\pi)^{-d/2} \int a((y-x)/2, \xi) e^{-i\langle x+y,\xi\rangle}\, dy.$$

The operator A is important when using the twisted convolution, because for each $a, b \in \mathcal{S}(\mathbf{R}^{2d})$ we have

$$A(a *_\sigma b) = Aa \circ Ab \tag{1.13}$$

(See [17,36,38,39]). Here and in what follows we have identified operators with their Schwartz kernels.

In the following lemma we list some facts about the operator A. The result is a consequence of Fourier's inversion formula, and the verifications are left for the reader.

Lemma 1.1 *Let A be as above and let $U = Aa$ where $a \in \mathcal{S}'(\mathbf{R}^{2d})$. Then the following is true*:

(1) $\check{U} = A\check{a}$, *if* $\check{a}(X) = a(-X)$;
(2) $J_{\mathcal{F}}U = A\mathcal{F}_\sigma a$, *where* $J_{\mathcal{F}}U(x,y) = U(-x,y)$;
(3) $A(\mathcal{F}_\sigma a) = (2\pi)^{d/2}\operatorname{Op}^w(a)$ *and* $(\operatorname{Op}^w(a)f, g) = (2\pi)^{-d/2}(Aa, \check{g} \otimes \overline{f})$ *when* $f, g \in \mathcal{S}(\mathbf{R}^d)$;
(4) *the Hilbert space adjoint of Aa equals $A\widetilde{a}$, where $\widetilde{a}(X) = \overline{a(-X)}$. Furthermore, if $a_1, a_2, b \in \mathcal{S}(\mathbf{R}^{2d})$, then*

$$(a_1 *_\sigma a_2, b) = (a_1, b *_\sigma \widetilde{a}_2) = (a_2, \widetilde{a}_1 *_\sigma b), \quad (a_1 *_\sigma a_2) *_\sigma b = a_1 *_\sigma (a_2 *_\sigma b).$$

A linear and continuous operator from $\mathcal{S}(\mathbf{R}^d)$ to $\mathcal{S}'(\mathbf{R}^n)$ is called positive semi-definite when $(Tf, f)_{L^2} \geq 0$ for every $f \in \mathcal{S}(\mathbf{R}^d)$. We write $T \geq 0$ when T is positive semi-definite. A distribution $a \in \mathcal{S}'(\mathbf{R}^{2d})$ is called σ-positive if Aa is a positive semi-definite operator. The set of all σ-positive distributions on $\mathbf{R}^{2d}$ is denoted by $\mathcal{S}'_+(\mathbf{R}^{2d})$.

The following result is an immediate consequence of Lemma 1.1.

Proposition 1.2 *Assume that $a \in \mathcal{S}'(\mathbf{R}^{2d})$. Then*

$$a \in \mathcal{S}'_+(\mathbf{R}^{2d}) \quad \Longleftrightarrow \quad Aa \geq 0 \text{ as operator} \quad \Longleftrightarrow \quad \operatorname{Op}^w(\mathcal{F}_\sigma a) \geq 0.$$

We refer to [38,39] for more facts about σ-positive functions and distributions.

In the end of Sect. 5 we also discuss continuity for Toeplitz operators. Assume that $a \in \mathcal{S}(\mathbf{R}^{2d})$ and $h_1, h_2 \in \mathcal{S}(\mathbf{R}^d)$. Then the Toeplitz operator $\mathrm{Tp}_{h_1,h_2}(a)$, with symbol a, and window functions h_1 and h_2, is defined by the formula

$$(\mathrm{Tp}_{h_1,h_2}(a)f_1, f_2) = (aV_{\check{h}_1}f_1, V_{\check{h}_2}f_2) = (a(2\,\cdot\,)W_{f_1,h_1}, W_{f_2,h_2})$$

when $f_1, f_2 \in \mathcal{S}(\mathbf{R}^d)$. The definition of $\mathrm{Tp}_{h_1,h_2}(a)$ extends in several ways (cf. e.g. [6,22,36,38,40,43,44]).

In most of these extensions as well as in Sect. 5, we interprete Toeplitz operators as pseudo-differential operators, using the fact that

$$\begin{aligned} \mathrm{Tp}_{h_1,h_2}(a) &= \mathrm{Op}_t(a * u) \quad \text{when} \\ u(X) &= (2\pi)^{-d/2} W^t_{h_2,h_1}(-X), \end{aligned} \tag{1.14}$$

h_1, h_2 are suitable window functions on $\mathbf{R}^d$ and a is an appropriate distribution on $\mathbf{R}^{2d}$. The relation (1.14) is well-known when $t = 0$ or $t = 1/2$ (cf. e.g. [6,8,30,36,38,40–42,44]). For general t, (1.14) is an immediate consequence of the case $t = 1/2$, (1.6), and the fact that

$$e^{i(t-s)\langle D_x, D_\xi\rangle}(a * u) = a * (e^{i(t-s)\langle D_x, D_\xi\rangle}u),$$

which follows by integration by parts.

Next we discuss basic properties for modulation spaces, and start by recalling the conditions for the involved weight functions. Assume that $0 < \omega, v \in L^\infty_{loc}(\mathbf{R}^d)$. Then ω is called v-moderate if

$$\omega(x + y) \le C\omega(x)v(y) \tag{1.15}$$

for some constant C which is independent of $x, y \in \mathbf{R}^d$. Here the function v is called submultiplicative, if (1.15) holds when $\omega = v$. If v in (1.15) can be chosen as a polynomial, then ω is called polynomially moderate. We let $\mathcal{P}(\mathbf{R}^d)$ be the set of all polynomially moderated functions on $\mathbf{R}^d$. If $\omega(x, \xi) \in \mathcal{P}(\mathbf{R}^{2d})$ is constant with respect to the x-variable (ξ-variable), then we write $\omega(\xi)$ ($\omega(x)$) instead of $\omega(x, \xi)$. In this case we consider ω as an element in $\mathcal{P}(\mathbf{R}^{2d})$ or in $\mathcal{P}(\mathbf{R}^d)$ depending on the situation.

We also remark that the polynomially moderate functions may be considered as a particular case of (σ, g)-temperate functions as defined in [24, Section 18.5], by Hörmander.

Let $\varphi \in \mathcal{S}'(\mathbf{R}^d)$ be fixed. Then the *short-time Fourier transform* $V_\varphi f$ of $f \in \mathcal{S}'(\mathbf{R}^d)$ with respect to the *window function* φ is the tempered distribution on $\mathbf{R}^{2d}$, defined by

$$V_\varphi f(x, \xi) \equiv \mathcal{F}(f\,\overline{\varphi(\,\cdot\, - x)})(\xi).$$

If $f,\varphi\in\mathcal{S}(\mathbf{R}^d)$, then it follows that

$$V_\varphi f(x,\xi)=(2\pi)^{-d/2}\int f(y)\overline{\varphi(y-x)}e^{-i\langle y,\xi\rangle}\,dy.$$

Assume that $\omega\in\mathcal{P}(\mathbf{R}^{2d})$, $p,q\in[1,\infty]$ and $\varphi\in\mathcal{S}(\mathbf{R}^d)\setminus 0$ are fixed. Then the mixed Lebesgue space $L^{p,q}_{1,(\omega)}(\mathbf{R}^{2d})$ consists of all $F\in L^1_{loc}(\mathbf{R}^{2d})$ such that $\|F\|_{L^{p,q}_{1,(\omega)}}<\infty$, and $L^{p,q}_{2,(\omega)}(\mathbf{R}^{2d})$ consists of all $F\in L^1_{loc}(\mathbf{R}^{2d})$ such that $\|F\|_{L^{p,q}_{2,(\omega)}}<\infty$. Here

$$\|F\|_{L^{p,q}_{1,(\omega)}}=\left(\int\left(\int|F(x,\xi)\omega(x,\xi)|^p\,dx\right)^{q/p}d\xi\right)^{1/q}, \tag{1.16}$$

and

$$\|F\|_{L^{p,q}_{2,(\omega)}}=\left(\int\left(\int|F(x,\xi)\omega(x,\xi)|^q\,d\xi\right)^{p/q}dx\right)^{1/p}, \tag{1.17}$$

with obvious modifications when $p=\infty$ or $q=\infty$. We note that these norms might attain $+\infty$.

The *modulation spaces* $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ and $W^{p,q}_{(\omega)}(\mathbf{R}^d)$ are the Banach spaces which consist of all $f\in\mathcal{S}'(\mathbf{R}^d)$ such that $\|f\|_{M^{p,q}_{(\omega)}}<\infty$ and $\|f\|_{W^{p,q}_{(\omega)}}<\infty$, respectively. Here

$$\|f\|_{M^{p,q}_{(\omega)}}\equiv\|V_\varphi f\|_{L^{p,q}_{1,(\omega)}},\ \text{and}\ \|f\|_{W^{p,q}_{(\omega)}}\equiv\|V_\varphi f\|_{L^{p,q}_{2,(\omega)}}. \tag{1.18}$$

We remark that the definitions of $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ and $W^{p,q}_{(\omega)}(\mathbf{R}^d)$ are independent of the choice of $\varphi\in\mathcal{S}(\mathbf{R}^d)\setminus 0$ in (1.18) and different φ gives rise to equivalent norms (see Proposition 1.3 below). By Fourier's inversion formula we get

$$V_{\widehat{\varphi}}\widehat{f}(\xi,-x)=e^{i\langle x,\xi\rangle}V_{\check{\varphi}}f(x,\xi),\qquad \check{\varphi}(x)=\varphi(-x), \tag{1.19}$$

which gives

$$f\in W^{q,p}_{(\omega)}(\mathbf{R}^d)\quad\Longleftrightarrow\quad\widehat{f}\in M^{p,q}_{(\omega_0)}(\mathbf{R}^d),\qquad \omega_0(\xi,-x)=\omega(x,\xi).$$

For convenience we set $M^p_{(\omega)}=M^{p,p}_{(\omega)}$, which agrees with $W^p_{(\omega)}=W^{p,p}_{(\omega)}$. Furthermore we set $M^{p,q}=M^{p,q}_{(\omega)}$ and $W^{p,q}=W^{p,q}_{(\omega)}$ when $\omega\equiv 1$.

The proof of the following proposition is omitted, since the results can be found in [10,11,14–16,18,40–43]. Here we recall that $p,p'\in[1,\infty]$ satisfy $1/p+1/p'=1$.

Proposition 1.3 *Assume that $p,q,p_j,q_j\in[1,\infty]$ for $j=1,2$, and $\omega,\omega_1,\omega_2,v\in\mathcal{P}(\mathbf{R}^{2d})$ are such that $v=\check{v}$, ω is v-moderate and $\omega_2\le C\omega_1$ for some constant $C>0$. Then the following is true*:

(1) *$f \in M^{p,q}_{(\omega)}(\mathbf{R}^d)$ if and only if (1.18) holds for any $\varphi \in M^1_{(v)}(\mathbf{R}^d)\backslash 0$. Moreover, $M^{p,q}_{(\omega)}$ is a Banach space under the norm in (1.18) and different choices of φ give rise to equivalent norms*;

(2) *if $p_1 \le p_2$ and $q_1 \le q_2$ then*

$$\mathcal{S}(\mathbf{R}^d) \hookrightarrow M^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^n) \hookrightarrow M^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^d) \hookrightarrow \mathcal{S}'(\mathbf{R}^d);$$

(3) *the L^2 product $(\,\cdot\,,\,\cdot\,) = (\,\cdot\,,\,\cdot\,)_{L^2}$ on $\mathcal{S}(\mathbf{R}^d)$ extends uniquely to a continuous map from $M^{p,q}_{(\omega)}(\mathbf{R}^n) \times M^{p',q'}_{(1/\omega)}(\mathbf{R}^d)$ to $\mathbf{C}$. On the other hand, if $\|a\| = \sup|(a,b)|$, where the supremum is taken over all $b \in \mathcal{S}(\mathbf{R}^d)$ such that $\|b\|_{M^{p',q'}_{(1/\omega)}} \le 1$, then $\|\cdot\|$ and $\|\cdot\|_{M^{p,q}_{(\omega)}}$ are equivalent norms*;

(4) *if $p, q < \infty$, then $\mathcal{S}(\mathbf{R}^d)$ is dense in $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ and the dual space of $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ can be identified with $M^{p',q'}_{(1/\omega)}(\mathbf{R}^d)$, through the L^2-form $(\,\cdot\,,\,\cdot\,)_{L^2}$. Moreover, $\mathcal{S}(\mathbf{R}^d)$ is weakly dense in $M^{\infty}_{(\omega)}(\mathbf{R}^d)$ with respect to the L^2-form.*

Similar facts hold if the $M^{p,q}_{(\omega)}$ spaces are replaced by $W^{p,q}_{(\omega)}$ spaces.

Proposition 1.3 (1) allows us be rather vague concerning the choice of $\varphi \in M^1_{(v)}\backslash 0$ in (1.18). For example, if $C > 0$ is a constant and $\mathcal{A}$ is a subset of $\mathcal{S}'$, then $\|a\|_{M^{p,q}_{(\omega)}} \le C$ for every $a \in \mathcal{A}$, means that the inequality holds for some choice of $\varphi \in M^1_{(v)}\backslash 0$ and every $a \in \mathcal{A}$. Evidently, a similar inequality is true for any other choice of $\varphi \in M^1_{(v)}\backslash 0$, with a suitable constant, larger than C if necessary.

Remark 1.4 Assume that $s, t \in \mathbf{R}$. In many applications it is common that functions of the form

$$\sigma_s(x) = \langle x\rangle^s \quad \text{and} \quad \sigma_{s,t}(x,\xi) \equiv \langle x\rangle^t \langle \xi\rangle^s,$$

are involved. Here and in what follows we let $\langle x\rangle = (1+|x|^2)^{1/2}$, when $x \in \mathbf{R}^d$. Then it easily follows that σ_s and $\sigma_{s,t}$ are $\sigma_{|s|}$-moderate and $\sigma_{|s|,|t|}$-moderate, respectively. For convenience we set

$$M^{p,q}_{s,t} = M^{p,q}_{(\sigma_{s,t})} \quad M^{p,q}_s = M^{p,q}_{(\sigma_s)},$$

when $\sigma_s(x,\xi) = \langle x,\xi\rangle^s$, and

$$L^p_s = L^p_{(\sigma_s)},$$

when $\sigma_s(x) = \langle x\rangle^s$. We note that for such weight functions we have

$$M^{p,q}_{s,t}(\mathbf{R}^d) = \{\, f \in \mathcal{S}'(\mathbf{R}^d)\,;\ \langle x\rangle^t \langle D\rangle^s f \in M^{p,q}(\mathbf{R}^d)\,\}, \quad s,t \in \mathbf{R}$$

and

$$M_s^{p,q}(\mathbf{R}^d) = M_{s,0}^{p,q}(\mathbf{R}^d) \cap M_{0,s}^{p,q}(\mathbf{R}^d), \quad s \ge 0$$

(cf. [41]). Since $M^2 = L^2$, we get $M_{0,s}^2(\mathbf{R}^d) = L_s^2(\mathbf{R}^d)$ and that $M_{s,0}^2(\mathbf{R}^d)$ agrees with the Sobolev space $H_s^2(\mathbf{R}^d)$ which consists of all $f \in \mathcal{S}'(\mathbf{R}^d)$ such that $\langle D\rangle^s f \in L^2$. We also get $M_s^2 = H_s^2 \cap L_s^2$, when $s \ge 0$ (cf. [18,20]).

The symplectic short-time Fourier transform of $a \in \mathcal{S}'(\mathbf{R}^{2d})$ with respect to the window function $\varphi \in \mathcal{S}'(\mathbf{R}^{2d})$ is defined by

$$\mathcal{V}_\varphi a(X,Y) = \mathcal{F}_\sigma\,(a\,\varphi(\,\cdot\, - X))\,(Y), \quad X,Y \in \mathbf{R}^{2d}.$$

Assume that $\omega \in \mathcal{P}(\mathbf{R}^{4d})$. Then $\mathcal{M}_{(\omega)}^{p,q}(\mathbf{R}^{2d})$ and $\mathcal{W}_{(\omega)}^{p,q}(\mathbf{R}^{2d})$ denote the modulation spaces, where the symplectic short-time Fourier transform is used instead of the usual short-time Fourier transform in the definitions of the norms. It follows that any property valid for $M_{(\omega)}^{p,q}(\mathbf{R}^{2d})$ or $W_{(\omega)}^{p,q}(\mathbf{R}^{2d})$ carry over to $\mathcal{M}_{(\omega)}^{p,q}(\mathbf{R}^{2d})$ and $\mathcal{W}_{(\omega)}^{p,q}(\mathbf{R}^{2d})$, respectively. For example, for the symplectic short-time Fourier transform we have

$$\mathcal{V}_{\mathcal{F}_\sigma\varphi}(\mathcal{F}_\sigma a)(X,Y) = e^{2i\sigma(Y,X)}\mathcal{V}_\varphi a(Y,X), \tag{1.20}$$

(cf. (1.19)) which implies that

$$\mathcal{F}_\sigma\mathcal{M}_{(\omega)}^{p,q}(\mathbf{R}^{2d}) = \mathcal{W}_{(\omega_0)}^{q,p}(\mathbf{R}^{2d}), \qquad \omega_0(X,Y) = \omega(Y,X). \tag{1.21}$$

2 Twisted convolution on modulation spaces and Lebesgue spaces

In this section we discuss algebraic properties of the twisted convolution when acting on modulation spaces of the form $\mathcal{W}_{(\omega)}^{p,q}$. The most general result corresponds to Theorem 0.3′ in [23], which concerns continuity for the Weyl product on modulation spaces of the form $\mathcal{M}_{(\omega)}^{p,q}$. Thereafter we use this result to establish continuity properties for the twisted convolution when acting on weighted Lebesgue spaces.

For completeness we write down the following lemma, where the first part agrees with Lemma 4.4 in [37] and Lemma 2.1 in [23], and was fundamental in the proofs of [37, Theorem 4.1] and for the Weyl product results in [23]. The second part follows from the first one, (1.9), (1.10) and (1.20).

Lemma 2.1 *Assume that $a_1 \in \mathcal{S}'(\mathbf{R}^{2d})$, $a_2 \in \mathcal{S}(\mathbf{R}^{2d})$, $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbf{R}^{2d})$ and $X, Y \in \mathbf{R}^{2d}$. Then the following is true*:

(1) *if $\varphi = \pi^d\varphi_1\#\varphi_2$, then $\varphi \in \mathcal{S}(\mathbf{R}^{2d})$, and the map*

$$Z \mapsto e^{2i\sigma(Z,Y)}(\mathcal{V}_{\chi_1}a_1)(X-Y+Z,Z)\,(\mathcal{V}_{\chi_2}a_2)(X+Z,Y-Z)$$

belongs to $L^1(\mathbf{R}^{2d})$*, and*

$$\begin{aligned}&\mathcal{V}_\varphi(a_1\# a_2)(X,Y)\\&\quad=\int e^{2i\sigma(Z,Y)}(\mathcal{V}_{\chi_1}a_1)(X-Y+Z,Z)\,(\mathcal{V}_{\chi_2}a_2)(X+Z,Y-Z)\,dZ;\end{aligned}$$

(2) *if* $\varphi=2^{-d}\varphi_1*_\sigma\varphi_2$*, then* $\varphi\in\mathcal{S}(\mathbf{R}^{2d})$*, and the map*

$$Z\mapsto e^{2i\sigma(X,Z-Y)}(\mathcal{V}_{\chi_1}a_1)(X-Y+Z,Z)\,(\mathcal{V}_{\chi_2}a_2)(Y-Z,X+Z)$$

belongs to $L^1(\mathbf{R}^{2d})$*, and*

$$\begin{aligned}&\mathcal{V}_\varphi(a_1*_\sigma a_2)(X,Y)\\&\quad=\int e^{2i\sigma(X,Z-Y)}(\mathcal{V}_{\chi_1}a_1)(X-Y+Z,Z)\,(\mathcal{V}_{\chi_2}a_2)(Y-Z,X+Z)\,dZ.\end{aligned}$$

For completeness we also write down the following restatement of Theorem 0.3′ in [23]. Here the involved weight functions should satisfy

$$\omega_0(X,Y)\le C\omega_1(X-Y+Z,Z)\omega_2(X+Z,Y-Z),\quad X,Y,Z\in\mathbf{R}^{2d},\tag{2.1}$$

for some constant $C>0$, and the exponent $p_j,q_j\in[1,\infty]$ satisfy (0.4) and

$$0\le\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_0}\le\frac{1}{p_j},\frac{1}{q_j}\le\frac{1}{q_1}+\frac{1}{q_2}-\frac{1}{q_0},\quad j=0,1,2.\tag{2.2}$$

Theorem 2.2 *Assume that* $\omega_0,\omega_1,\omega_2\in\mathcal{P}(\mathbf{R}^{4d})$ *satisfy* (2.1)*, and that* $p_j,q_j\in[1,\infty]$ *for* $j=0,1,2$*, satisfy* (0.4) *and* (2.2)*. Then the map* $(a_1,a_2)\mapsto a_1\# a_2$ *on* $\mathcal{S}(\mathbf{R}^{2d})$ *extends uniquely to a continuous map from* $\mathcal{M}^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d})\times\mathcal{M}^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d})$ *to* $\mathcal{M}^{p_0,q_0}_{(\omega_0)}(\mathbf{R}^{2d})$*, and for some constant* $C>0$*, the bound*

$$\|a_1\# a_2\|_{\mathcal{M}^{p_0,q_0}_{(\omega_0)}}\le C\|a_1\|_{\mathcal{M}^{p_1,q_1}_{(\omega_1)}}\|a_2\|_{\mathcal{M}^{p_2,q_2}_{(\omega_2)}},\tag{2.3}$$

holds for every $a_1\in\mathcal{M}^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d})$ *and* $a_2\in\mathcal{M}^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d})$.

We note that Theorem 0.1 is an immediate consequence of (1.21), (1.11) and Theorem 2.2. Another way to prove Theorem 0.1 is to use similar arguments as in the proof of [23, Theorem 0.3′], based on (2) instead of (1) in Lemma 2.1.

We are now able to state and prove mapping results for the twisted convolution on weighted Lebesgue spaces. We start with the proof of Theorem 0.2 from the introduction.

Proof of Theorem 0.2 From the assumptions it follows that at most one of p_1 and p_2 are equal to ∞. By reasons of symmetry we may therefore assume that $p_2<\infty$.

Since $\mathcal{W}^2_{(\omega)} = \mathcal{M}^2_{(\omega)} = L^2_{(\omega)}$ when $\omega(X,Y) = \omega(X)$, in view of Theorem 2.2 in [41], the result follows from Theorem 0.1 in the case $p_0 = p_1 = p_2 = 2$.

Now assume that $1/p_1 + 1/p_2 - 1/p_0 = 1$, $a_1 \in L^{p_1}(\mathbf{R}^{2d})$ and that $a_2 \in \mathcal{S}(\mathbf{R}^{2d})$. Then

$$\|a_1 *_\sigma a_2\|_{L^{p_0}_{(\omega_0)}} \le (2/\pi)^{d/2} \| \, |a_1| * |a_2| \, \|_{L^{p_0}_{(\omega_0)}} \le C\|a_1\|_{L^{p_1}_{(\omega_1)}} \|a_2\|_{L^{p_2}_{(\omega_2)}},$$

by Young's inequality. The result now follows in this case from the fact that $\mathcal{S}$ is dense in $L^{p_2}_{(\omega_2)}$, when $p_2 < \infty$.

For general p_0, p_1, p_2, the result follows by multi-linear interpolation between the case $p_0 = p_1 = p_2 = 2$ and the case $1/p_1 + 1/p_2 - 1/p_0 = 1$, using Theorem 4.4.1 in [1] and the fact that

$$(L^{q_1}_{(\omega)}(\mathbf{R}^{2d}), (L^{q_2}_{(\omega)}(\mathbf{R}^{2d}))_{[\theta]} = L^{q}_{(\omega)}(\mathbf{R}^{2d}), \quad \text{when} \quad \frac{1-\theta}{q_1} + \frac{\theta}{q_2} = \frac{1}{q}$$

(cf. Chapter 5 in [1]). The proof is complete. □

By letting $p_1 = p$ and $p_2 = q \le \min(p, p')$, or $p_2 = p$ and $p_1 = q \le \min(p, p')$ in Theorem 0.2, we get the following:

Corollary 2.3 *Assume that $\omega_j \in \mathcal{P}(\mathbf{R}^{2d})$ for $j = 0, 1, 2$ and $p, q \in [1, \infty]$ satisfy (0.6), and $q \le \min(p, p')$ for some constant C. Then the map $(a_1, a_2) \mapsto a_1 *_\sigma a_2$ extends uniquely to a continuous mapping from $L^p_{(\omega_1)}(\mathbf{R}^{2d}) \times L^q_{(\omega_2)}(\mathbf{R}^{2d})$ or $L^q_{(\omega_1)}(\mathbf{R}^{2d}) \times L^p_{(\omega_2)}(\mathbf{R}^{2d})$ to $L^p_{(\omega_0)}(\mathbf{R}^{2d})$.*

In particular, if $p \in [1, 2]$ and in addition ω_0 is submultitplicative, then $(L^p_{(\omega_0)}(\mathbf{R}^{2d}), *_\sigma)$ is an algebra.

In the next section we need the following refinement of Theorem 0.2 concerning mixed Lebesgue spaces.

Theorem 0.2′ *Assume that $k \in \{1, 2\}$, $\omega_j \in \mathcal{P}(\mathbf{R}^{2d})$ and $p_j, q_j \in [1, \infty]$ for $j = 0, 1, 2$ satisfy (0.6) and*

$$\max\left(\frac{1}{p_0}, \frac{1}{p_0'}, \frac{1}{q_0}, \frac{1}{q_0'}\right) \le \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_0}, \; \frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{q_0} \le 1.$$

Then the map $(a_1, a_2) \mapsto a_1 *_\sigma a_2$ extends uniquely to a continuous mapping from $L^{p_1,q_1}_{k,(\omega_1)}(\mathbf{R}^{2d}) \times L^{p_2,q_2}_{k,(\omega_2)}(\mathbf{R}^{2d})$ to $L^{p_0,q_0}_{k,(\omega_0)}(\mathbf{R}^{2d})$, and

$$\|a_1 *_\sigma a_2\|_{L^{p_0,q_0}_{k,(\omega_0)}} \le C\|a_1\|_{L^{p_1,q_1}_{k,(\omega_1)}} \|a_2\|_{L^{p_2,q_2}_{k,(\omega_2)}}, \qquad (0.8)'$$

holds for some constant C which is independent of $a_1 \in L^{p_1,q_1}_{k,(\omega_1)}(\mathbf{R}^{2d})$ and $a_2 \in L^{p_2,q_2}_{k,(\omega_2)}(\mathbf{R}^{2d})$.

Proof The result follows from Minkowski's inequality when $p_1 = q_1 = 1$ or $p_2 = q_2 = 1$. In the case $p_1 = p_2 = q_1 = q_2 = 2$ the result is an immediate consequence of Theorem 0.2. For general p_j and q_j, the result follows from these cases and multi-linear interpolation. □

3 Window functions in modulation space norms

In this section we use the results in the previous section to prove that the class of permitted windows in the modulation space norms can be extended. More precisely we have the following.

Theorem 3.1 *Assume that* $p, p_0, q, q_0 \in [1, \infty]$ *and* $\omega, v \in \mathcal{P}(\mathbf{R}^{2d})$ *are such that* $p_0, q_0 \le \min(p, p', q, q')$, $\check{v} = v$ *and* ω *is* v*-moderate. Also assume that* $f \in \mathcal{S}'(\mathbf{R}^d)$*. Then the following is true*:

(1) *if* $\varphi \in M^{p_0,q_0}_{(v)}(\mathbf{R}^d)\backslash 0$, *then* $f \in M^{p,q}_{(\omega)}(\mathbf{R}^d)$ *if and only if* $V_\varphi f \in L^{p,q}_{1,(\omega)}(\mathbf{R}^{2d})$. *Furthermore,* $\|f\| \equiv \|V_\varphi f\|_{L^{p,q}_{1,(\omega)}}$ *defines a norm for* $M^{p,q}_{(\omega)}(\mathbf{R}^d)$, *and different choices of* φ *give rise to equivalent norms*;

(2) *if* $\varphi \in W^{p_0,q_0}_{(v)}(\mathbf{R}^d)\backslash 0$, *then* $f \in W^{p,q}_{(\omega)}(\mathbf{R}^d)$ *if and only if* $V_\varphi f \in L^{p,q}_{2,(\omega)}(\mathbf{R}^{2d})$. *Furthermore,* $\|f\| \equiv \|V_\varphi f\|_{L^{p,q}_{2,(\omega)}}$ *defines a norm for* $W^{p,q}_{(\omega)}(\mathbf{R}^d)$, *and different choices of* φ *give rise to equivalent norms.*

For the proof we note that the relation between Wigner distributions (cf. (1.5) with $t = 1/2$) and short-time Fourier transform is given by

$$W_{f,g}(x, \xi) = 2^d e^{i\langle x,\xi\rangle/2} V_{\check{g}} f(2x, 2\xi),$$

which implies that

$$\|W_{f,\check{\varphi}}\|_{L^{p,q}_{k,(\omega_0)}} = 2^d \|V_\varphi f\|_{L^{p,q}_{k,(\omega)}}, \quad \text{when} \quad \omega_0(x, \xi) = \omega(2x, 2\xi) \tag{3.1}$$

for $k = 1, 2$.

Finally, by Fourier's inversion formula it follows that if $f_1, g_2 \in \mathcal{S}'(\mathbf{R}^d)$ and $f_1, g_2 \in L^2(\mathbf{R}^d)$, then

$$W_{f_1,g_1} *_\sigma W_{f_2,g_2} = (\check{f}_2, g_1)_{L^2} W_{f_1,g_2}. \tag{3.2}$$

Proof of Theorem 3.1 We may assume that $p_0 = q_0 = \min(p, p', q, q')$. Assume that $\varphi, \psi \in M^{p_0,q_0}_{(v)}(\mathbf{R}^d) \subseteq L^2(\mathbf{R}^d)$, where the inclusion follows from the facts that $p_0, q_0 \le 2$ and $v \ge c$ for some constant $c > 0$. Since $\|V_\varphi\psi\|_{L^{p_0,q_0}_{k,(v)}} = \|V_\psi\varphi\|_{L^{p_0,q_0}_{k,(v)}}$ when $\check{v} = v$, the result follows if we prove that

$$\|V_\varphi f\|_{L^{p,q}_{k,(\omega)}} \le C\|\psi\|^{-2}_{L^2}\|V_\psi f\|_{L^{p,q}_{k,(\omega)}}\|V_\varphi\psi\|_{L^{p_0,q_0}_{k,(v)}}, \tag{3.3}$$

for some constant C which is independent of $f \in \mathcal{S}'(\mathbf{R}^d)$ and $\varphi, \psi \in M^{p_0,q_0}_{(v)}(\mathbf{R}^d)$. For reasons of homogeneity, it is then no restriction to assume that $\|\psi\|_{L^2} = 1$.

If $p_1 = p$, $p_2 = p_0$, $q_1 = q$, $q_2 = q_0$, $\omega_0 = \omega(2\,\cdot)$ and $v_0 = v(2\,\cdot)$, then Theorem 0.2′ and (3.2) give

$$\begin{aligned}\|V_\varphi f\|_{L^{p,q}_{k,(\omega)}} &= C_1\|W_{f,\check\varphi}\|_{L^{p,q}_{k,(\omega_0)}} = C_2\|W_{f,\check\psi} *_\sigma W_{\psi,\check\varphi}\|_{L^{p,q}_{k,(\omega_0)}}\\ &\le C_3\|W_{f,\check\psi}\|_{L^{p,q}_{k,(\omega_0)}}\|W_{\psi,\check\varphi}\|_{L^{p_0,q_0}_{k,(v_0)}} = C_4\|V_\psi f\|_{L^{p,q}_{k,(\omega)}}\|V_\varphi\psi\|_{L^{p_0,q_0}_{k,(v)}},\end{aligned}$$

and (3.3) follows. The proof is complete. □

4 Schatten–von Neumann classes and pseudo-differential operators

In this section we discuss Schatten–von Neumann classes of pseudo-differential operators from a Hilbert space $\mathcal{H}_1$ to another Hilbert space $\mathcal{H}_2$. Schatten–von Neumann classes were introduced by Schatten in [28] in the case $\mathcal{H}_1 = \mathcal{H}_2$ (see also [31]). The general situation, when $\mathcal{H}_1$ is not necessarily equal to $\mathcal{H}_2$, has thereafter been considered in, e.g. [2,29,45].

Let $\mathrm{ON}(\mathcal{H}_j)$, $j = 1, 2$, denote the family of orthonormal sequences in $\mathcal{H}_j$, and assume that $T : \mathcal{H}_1 \to \mathcal{H}_2$ is linear, and that $p \in [1, \infty]$. Then we set

$$\|T\|_{\mathcal{I}_p} = \|T\|_{\mathcal{I}_p(\mathcal{H}_1,\mathcal{H}_2)} \equiv \sup\Big(\sum |(Tf_j, g_j)_{\mathcal{H}_2}|^p\Big)^{1/p}$$

(with obvious modifications when $p = \infty$). Here the supremum is taken over all $(f_j) \in \mathrm{ON}(\mathcal{H}_1)$ and $(g_j) \in \mathrm{ON}(\mathcal{H}_2)$. Then $\mathcal{I}_p = \mathcal{I}_p(\mathcal{H}_1, \mathcal{H}_2)$, the Schatten–von Neumann class of order p, consists of all linear and continuous operators T from $\mathcal{H}_1$ to $\mathcal{H}_2$ such that $\|T\|_{\mathcal{I}_p(\mathcal{H}_1,\mathcal{H}_2)}$ is finite. We note that $\mathcal{I}_\infty(\mathcal{H}_1, \mathcal{H}_2)$ agrees with $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, the set of linear and continuous operators from $\mathcal{H}_1$ to $\mathcal{H}_2$, with equality in norms. We also let $\mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$ be the set of all linear and compact operators from $\mathcal{H}_1$ to $\mathcal{H}_2$, and equip this space with the operator norm as usual (note that the notation $\mathcal{I}_\sharp(\mathcal{H}_1, \mathcal{H}_2)$ was used instead of $\mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$ in [43]). If $\mathcal{H}_1 = \mathcal{H}_2$, then the shorter notation $\mathcal{I}_p(\mathcal{H}_1)$ is used instead of $\mathcal{I}_p(\mathcal{H}_1, \mathcal{H}_2)$, and similarly for $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$.

Assume that (e_j) is an orthonormal basis in $\mathcal{H}_1$, and that $S \in \mathcal{I}_1(\mathcal{H}_1)$. Then the trace of S is defined as

$$\mathrm{tr}_{\mathcal{H}_1} S = \sum (Se_j, e_j)_{\mathcal{H}_1}.$$

For each pairs of operators $T_1, T_2 \in \mathcal{I}_\infty(\mathcal{H}_1, \mathcal{H}_2)$ such that $T_2^* \circ T_1 \in \mathcal{I}_1(\mathcal{H}_1)$, the sesqui-linear form

$$(T_1, T_2) = (T_1, T_2)_{\mathcal{H}_1,\mathcal{H}_2} \equiv \mathrm{tr}_{\mathcal{H}_1}(T_2^* \circ T_1)$$

of T_1 and T_2 is well-defined. Here we note that T belongs to $\mathcal{I}_p(\mathcal{H}_1, \mathcal{H}_2)$ if and only if T^* belongs to $\mathcal{I}_p(\mathcal{H}_2, \mathcal{H}_1)$ with equal norms. We refer to [2,31,45] for more facts about Schatten–von Neumann classes.

In order for discussing Schatten–von Neumann operators within the theory of pseudo-differential operators, we assume from now on that the Hilbert spaces $\mathcal{H}, \mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \dots$ are "tempered" in the following sense.

Definition 4.1 The Hilbert space $\mathcal{H} \subseteq \mathcal{S}'(\mathbf{R}^d)$ is called *tempered* (on $\mathbf{R}^d$), if $\mathcal{S}(\mathbf{R}^d)$ is contained and dense in $\mathcal{H}$.

Assume that $\mathcal{H}$ is a tempered Hilbert space on $\mathbf{R}^d$. Then we let $\check{\mathcal{H}}$ and $\mathcal{H}^\tau$ be the sets of all $f \in \mathcal{S}'(\mathbf{R}^d)$ such that $\check{f} \in \mathcal{H}$ and $\overline{f} \in \mathcal{H}$, respectively. Then $\check{\mathcal{H}}$ and $\mathcal{H}^\tau$ are tempered Hilbert spaces under the norms

$$\|f\|_{\check{\mathcal{H}}} \equiv \|\check{f}\|_{\mathcal{H}} \quad \text{and} \quad \|f\|_{\mathcal{H}^\tau} \equiv \|\overline{f}\|_{\mathcal{H}},$$

respectively.

The L^2-*dual*, $\mathcal{H}'$, of $\mathcal{H}$ is the set of all $\varphi \in \mathcal{S}'(\mathbf{R}^d)$ such that

$$\|\varphi\|_{\mathcal{H}'} \equiv \sup |(\varphi, f)_{L^2(\mathbf{R}^d)}|$$

is finite. Here the supremum is taken over all $f \in \mathcal{S}(\mathbf{R}^d)$ such that $\|f\|_{\mathcal{H}} \le 1$. Assume that $\varphi \in \mathcal{H}'$. Since $\mathcal{S}$ is dense in $\mathcal{H}$, it follows from the definitions that the map $f \mapsto (\varphi, f)_{L^2}$ from $\mathcal{S}(\mathbf{R}^d)$ to $\mathbf{C}$ extends uniquely to a continuous mapping from $\mathcal{H}$ to $\mathbf{C}$. The following version of Riesz lemma is useful for us. In order to be self-contained, we also give a proof.

Lemma 4.2 *Assume that $\mathcal{H} \subseteq \mathcal{S}'(\mathbf{R}^d)$ is a tempered Hilbert space with L^2-dual $\mathcal{H}'$. Then the following is true*:

(1) *$\mathcal{H}'$ is a tempered Hilbert space which can be identified with the dual space of $\mathcal{H}$ through the L^2-form*;
(2) *there is a unique map $T_{\mathcal{H}}$ from $\mathcal{H}$ to $\mathcal{H}'$ such that*

$$(f, g)_{\mathcal{H}} = (T_{\mathcal{H}} f, g)_{L^2(\mathbf{R}^d)}, \quad f, g \in \mathcal{H}; \tag{4.1}$$

(3) *if $T_{\mathcal{H}}$ is the map in* (2), *$(e_j)_{j\in I}$ is an orthonormal basis in $\mathcal{H}$ and $\varepsilon_j = T_{\mathcal{H}} e_j$, then $T_{\mathcal{H}}$ is isometric, $(\varepsilon_j)_{j\in I}$ is an orthonormal basis in $\mathcal{H}'$ and*

$$(\varepsilon_j, e_k)_{L^2(\mathbf{R}^d)} = \delta_{j,k}.$$

Proof We have that $\mathcal{S} \subseteq \mathcal{H}' \subseteq \mathcal{S}'$, and since $\mathcal{S}$ is dense in $\mathcal{H}$, it follows that $\mathcal{S}$ is dense also in $\mathcal{H}'$.

First assume that $f \in \mathcal{H}$, $g \in \mathcal{S}(\mathbf{R}^d)$, and let $T_{\mathcal{H}} f$ in $\mathcal{S}'(\mathbf{R}^d)$ be defined by (4.1). By the definitions it follows that $T_{\mathcal{H}} f \in \mathcal{H}'$, and that $T_{\mathcal{H}}$ from $\mathcal{H}$ to $\mathcal{H}'$ is isometric. Furthermore, since the dual space of $\mathcal{H}$ can be identified with itself, under the scalar product of $\mathcal{H}$, the asserted duality properties of $\mathcal{H}'$ follow.

Let $(e_j)_{j\in I}$ be an arbitrary orthonormal basis in $\mathcal{H}$, and let $\varepsilon_j = T_{\mathcal{H}} e_j$. Then it follows that $\|\varepsilon_j\|_{\mathcal{H}'} = 1$ and

$$(\varepsilon_j, e_k)_{L^2} = (e_j, e_k)_{\mathcal{H}} = \delta_{j,k}.$$

Furthermore, if

$$f = \sum \alpha_j e_j, \quad \varphi = \sum \alpha_j \varepsilon_j, \quad g = \sum \beta_j e_j \quad \text{and} \quad \gamma = \sum \beta_j \varepsilon_j$$

are finite sums, and we set $(\varphi, \gamma)_{\mathcal{H}'} \equiv (f, g)_{\mathcal{H}}$, then it follows that $(\,\cdot\,,\,\cdot\,)_{\mathcal{H}'}$ defines a scalar product on such finite sums in $\mathcal{H}'$, and that $\|\varphi\|^2_{\mathcal{H}'} = (\varphi, \varphi)_{\mathcal{H}'}$. By continuity extensions it now follows that $(\varphi, \gamma)_{\mathcal{H}'}$ extends uniquely to each $\varphi, \gamma \in \mathcal{H}'$, and that the identity $\|\varphi\|^2_{\mathcal{H}'} = (\varphi, \varphi)_{\mathcal{H}'}$ holds. This proves the result. □

In what follows we consider the basis (ε_j) in Lemma 4.2 as the *dual basis* of (e_j).

Example 4.3 Let $\mathcal{H}_1 = H^2_s(\mathbf{R}^d)$, $\mathcal{H}_2 = M^2_{(\omega_0)}(\mathbf{R}^d)$ where $\omega_0 \in \mathcal{P}(\mathbf{R}^{2d})$, and let $\mathcal{H}_3 = H(\omega, g)$ be the Sobolev space of Bony–Chemin type with admissible weight ω and metric g on $\mathbf{R}^{2d}$(cf. [3]). Then $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ are tempered Hilbert spaces with duals $\mathcal{H}'_1 = H^2_{-s}(\mathbf{R}^d)$, $\mathcal{H}'_2 = M^2_{(1/\omega_0)}(\mathbf{R}^d)$ and $\mathcal{H}'_3 = H(1/\omega, g)$, respectively. We remark that $H(\omega, g) = M^2_{(\omega)}(\mathbf{R}^d)$ when g is the constant euclidean metric on $\mathbf{R}^{2d}$ in view of [21]. If moreover $\omega_s(x, \xi) = \sigma_s(\xi) = \langle \xi \rangle^s$, then $H(\omega_s, g) = M^2_{(\omega_s)} = H^2_s$ (cf. Remark 1.4).

Corollary 4.4 *Assume that $\mathcal{H}$ is a tempered Hilbert space on $\mathbf{R}^d$. Then*

$$M^2_s(\mathbf{R}^d) \subseteq \mathcal{H}, \mathcal{H}' \subseteq M^2_{-s}(\mathbf{R}^d),$$

for some $s \geq 0$. Furthermore, $M^2_s(\mathbf{R}^d)$ is dense in $\mathcal{H}$ and $\mathcal{H}'$, which in turn are dense in $M^2_{-s}(\mathbf{R}^d)$. A similar fact holds when M^2_s and M^2_{-s} are replaced by $M^2_{s,s}$ and $M^2_{-s,-s}$, respectively.

Proof Since

$$M^2_{2s} \subseteq M^2_{s,s} \subseteq M^2_s \quad \text{and} \quad M^2_{-s} \subseteq M^2_{-s,-s} \subseteq M^2_{-2s}$$

when $s \geq 0$, it suffices to consider the case when the modulation spaces are of the form $M^2_{s,s}$ or $M^2_{-s,-s}$.

The topology in $\mathcal{S}$ can be obtained by using the semi-norms

$$\|f\|_{[s]} \equiv \sum_{|\alpha|,|\beta|\leq s} \|x^\alpha D^\beta f\|_{L^2}, \quad s = 0, 1, 2, \ldots.$$

From the fact that $\mathcal{S}$ is continuously embedded in $\mathcal{H}$ and in $\mathcal{H}'$, it therefore follows that

$$\|f\|_{\mathcal{H}} \leq C\|f\|_{[s]} \quad \text{and} \quad \|\varphi\|_{\mathcal{H}'} \leq C\|\varphi\|_{[s]},$$

when $f \in \mathcal{S}$, provided s is chosen large enough.

Since the completion of $\mathcal{S}(\mathbf{R}^d)$ under $\|\cdot\|_{[s]}$ is equal to $M^2_{s,s}(\mathbf{R}^d)$ in view of Remark 1.4, the result follows by a standard argument of approximation, using the duality properties in Proposition 1.3 (4), together with the facts that $\mathcal{S}$ is dense in $\mathcal{H}$, $\mathcal{H}'$, $M^2_{s,s}$ and $M^2_{-s,-s}$. The proof is complete. □

Assume that $\mathcal{H}_1, \mathcal{H}_2 \subseteq \mathcal{S}'(\mathbf{R}^d)$ are tempered Hilbert spaces, $t \in \mathbf{R}$ is fixed and that $p \in [1,\infty]$. Then we let $s^A_p(\mathcal{H}_1,\mathcal{H}_2)$ and $s_{t,p}(\mathcal{H}_1,\mathcal{H}_2)$ be the sets of all $a \in \mathcal{S}'(\mathbf{R}^{2d})$ such that $Aa \in \mathcal{I}_p(\mathcal{H}_1,\mathcal{H}_2)$ and $\operatorname{Op}_t(a) \in \mathcal{I}_p(\mathcal{H}_1,\mathcal{H}_2)$, respectively. We also let $s^A_\sharp(\mathcal{H}_1,\mathcal{H}_2)$ and $s_{t,\sharp}(\mathcal{H}_1,\mathcal{H}_2)$ be the set of all $a \in \mathcal{S}'(\mathbf{R}^{2d})$ such that $Aa \in \mathcal{K}(\mathcal{H}_1,\mathcal{H}_2)$ and $\operatorname{Op}_t(a) \in \mathcal{K}(\mathcal{H}_1,\mathcal{H}_2)$, respectively. These spaces are equipped by the norms

$$\|a\|_{s_{t,p}(\mathcal{H}_1,\mathcal{H}_2)} \equiv \|\operatorname{Op}_t(a)\|_{\mathcal{I}_p(\mathcal{H}_1,\mathcal{H}_2)}, \qquad \|a\|_{s^A_p(\mathcal{H}_1,\mathcal{H}_2)} \equiv \|Aa\|_{\mathcal{I}_p(\mathcal{H}_1,\mathcal{H}_2)},$$

$$\|a\|_{s_{t,\sharp}(\mathcal{H}_1,\mathcal{H}_2)} \equiv \|a\|_{s_{t,\infty}(\mathcal{H}_1,\mathcal{H}_2)}, \qquad \|a\|_{s^A_\sharp(\mathcal{H}_1,\mathcal{H}_2)} \equiv \|a\|_{s^A_\infty(\mathcal{H}_1,\mathcal{H}_2)}.$$

Since the mappings $a \mapsto Aa$ and $a \mapsto \operatorname{Op}_t(a)$ are bijections from $\mathcal{S}'(\mathbf{R}^{2d})$ to the set of linear and continuous operators from $\mathcal{S}(\mathbf{R}^d)$ to $\mathcal{S}'(\mathbf{R}^d)$, it follows that $a \mapsto Aa$ and $a \mapsto \operatorname{Op}_t(a)$ restrict to isometric bijections from $s^A_p(\mathcal{H}_1,\mathcal{H}_2)$ and $s_{t,p}(\mathcal{H}_1,\mathcal{H}_2)$, respectively, to $\mathcal{I}_p(\mathcal{H}_1,\mathcal{H}_2)$. Consequently, the properties for $\mathcal{I}_p(\mathcal{H}_1,\mathcal{H}_2)$ carry over to $s^A_p(\mathcal{H}_1,\mathcal{H}_2)$ and $s_{t,p}(\mathcal{H}_1,\mathcal{H}_2)$. In particular, elements in $s^A_1(\mathcal{H}_1,\mathcal{H}_2)$ of finite rank (i.e. elements of the form $a \in s^A_1(\mathcal{H}_1,\mathcal{H}_2)$ such that Aa is a finite rank operator) are dense in $s^A_\sharp(\mathcal{H}_1,\mathcal{H}_2)$ and $s^A_p(\mathcal{H}_1,\mathcal{H}_2)$ when $p < \infty$. Similar facts hold for $s_{t,\sharp}(\mathcal{H}_1,\mathcal{H}_2)$ and $s_{t,p}(\mathcal{H}_1,\mathcal{H}_2)$. Since the Weyl quantization is particularly important in our considerations we also set

$$s^w_p = s_{t,p} \quad \text{and} \quad s^w_\sharp = s_{t,\sharp}, \quad \text{when} \quad t = 1/2.$$

If $\omega_1, \omega_2 \in \mathcal{P}(\mathbf{R}^{2d})$, then we use the notation $s^A_p(\omega_1,\omega_2)$ instead of $s^A_p(M^2_{(\omega_1)}, M^2_{(\omega_2)})$. Furthermore we set $s^A_p(\omega_1,\omega_2) = s^A_p(\mathbf{R}^{2d})$ when $\omega_1 = \omega_2 = 1$. In the same way the notations for $s_{t,p}$, s^w_p, $s_{t,\sharp}$ and $s^w_\sharp$ are simplified.

Remark 4.5 Except for the Hilbert-Schmidt case ($p = 2$), it is in general a hard task to find simple characterizations of Schatten–von Neumann classes. Important questions therefore concern of finding embeddings between Schatten–von Neumann classes and well-known function and distribution spaces. Here we recall some of such embeddings:

(i) In Chapter 4 in [31], it is proved that if Q is a unit cube on $\mathbf{R}^d$, $1 \le p \le 2$ and f and g are measureable on $\mathbf{R}^d$ and satisfy

$$\left(\sum_{x_\alpha \in \mathbf{Z}^n} \left(\int_{x_\alpha + Q} |f(x)|^2\, dx \right)^{p/2} \right)^{1/p} < \infty,$$

and similarly for g, then $f(x)g(D) \in \mathcal{I}_p(L^2)$, or equivalently, $f(x)g(\xi) \in s_{t,p}(\mathbf{R}^{2d})$ when $t = 0$;

(ii) Let $B_s^{p,q}(\mathbf{R}^d)$ be the Besov space with parameters $p, q \in [1, \infty]$ and $s \in \mathbf{R}$ (cf. [38,40,42,43] for strict definitions). In [38] sharp embeddings of the form

$$B_{s_1}^{p,q_1}(\mathbf{R}^{2d}) \subseteq s_{t,p}(\mathbf{R}^{2d}) \subseteq B_{s_2}^{p,q_2}(\mathbf{R}^{2d})$$

is presented. Here

$$q_1 = \min(p, p') \quad \text{and} \quad q_2 = \max(p, p'). \tag{4.2}$$

We also remark that the sharp embedding $B_s^{\infty,1}(\mathbf{R}^{2d}) \subseteq s_{t,\infty}(\mathbf{R}^{2d})$ for certain choices of t was proved already in [4,5,27,33];

(iii) In [43, Theorem 4.13] it is proved that if $\omega_1, \omega_2 \in \mathcal{P}(\mathbf{R}^{2d})$ satisfy

$$\omega(x, \xi, \eta, y) = \omega_2(x - ty, \xi + (1 - t)\eta)/\omega_1(x + (1 - t)y, \xi - t\eta)$$

and $p, q_1, q_2 \in [1, \infty]$ satisfy (4.2), then

$$M_{(\omega)}^{p,q_1}(\mathbf{R}^{2d}) \subseteq s_{t,p}(\omega_1, \omega_2) \subseteq M_{(\omega)}^{p,q_2}(\mathbf{R}^{2d}). \tag{4.3}$$

In particular, (4.3) covers the Schatten–von Neumann results in [20,32,40], where similar questions are considered in the case $\omega_1 = \omega_2 = \omega = 1$. Furthermore, in [43], embeddings between $s_{t,p}(\omega_1, \omega_2)$ with $\omega_1 = \omega_2$ and Besov spaces are established.

Remark 4.6 Assume that $t, t_1, t_2 \in \mathbf{R}$, $p \in [1, \infty]$, $\mathcal{H}_1, \mathcal{H}_2$ are tempered Hilbert spaces on $\mathbf{R}^d$ and that $a, b \in \mathcal{S}'(\mathbf{R}^{2d})$. Then it follows by Fourier's inversion formula that the map $e^{it\langle D_x, D_\xi\rangle}$ is a homeomorphism on $\mathcal{S}(\mathbf{R}^{2d})$ which extends uniquely to a homeomorphism on $\mathcal{S}'(\mathbf{R}^{2d})$. Furthermore, by (1.3) it follows that $e^{i(t_2-t_1)\langle D_x, D_\xi\rangle}$ restricts to an isometric bijection from $s_{t_1,p}(\mathcal{H}_1, \mathcal{H}_2)$ to $s_{t_2,p}(\mathcal{H}_1, \mathcal{H}_2)$.

The following proposition shows how $s_{t,p}(\mathcal{H}_1, \mathcal{H}_2)$, $s_p^A(\mathcal{H}_1, \mathcal{H}_2)$ and other similar spaces are linked together. The proof is essentially the same as the proof of Proposition 5.1 in [45]. Here and in what follows we let $a^\tau(x, \xi) = \overline{a(x, -\xi)}$ be the "torsion" of $a \in \mathcal{S}'(\mathbf{R}^{2d})$.

Proposition 4.7 *Assume that $t \in \mathbf{R}$, $\mathcal{H}_1, \mathcal{H}_2$ are tempered Hilbert spaces in $\mathbf{R}^d$, $a \in \mathcal{S}'(\mathbf{R}^{2d})$, and that $p \in [1, \infty]$. Then $s_p^w(\mathcal{H}_1, \mathcal{H}_2) = s_p^A(\mathcal{H}_1, \check{\mathcal{H}}_2)$. Furthermore, the following conditions are equivalent*:

(1) $a \in s_p^w(\mathcal{H}_1, \mathcal{H}_2)$;

(2) $\mathcal{F}_\sigma a \in s_p^w(\mathcal{H}_1, \check{\mathcal{H}}_2) = s_p^A(\mathcal{H}_1, \mathcal{H}_2)$;

(3) $\overline{a} \in s_p^w(\mathcal{H}_2', \mathcal{H}_1')$;

(4) $a^\tau \in s_p^A(\mathcal{H}_1^\tau, \mathcal{H}_2^\tau)$;

(5) $\check{a} \in s_p^w(\check{\mathcal{H}}_1, \check{\mathcal{H}}_2)$;

(6) $\widetilde{a} \in s_p^w(\check{\mathcal{H}}_2', \check{\mathcal{H}}_1')$;

(7) $e^{i(t-1/2)\langle D_\xi, D_x\rangle} a \in s_{t,p}(\mathcal{H}_1, \mathcal{H}_2)$.

Proof Let $a_1 = \mathcal{F}_\sigma a$, $a_2 = \overline{a}$, $a_3 = a^\tau$, $a_4 = \check{a}$ and $a_5 = \widetilde{a}$. Then the equivalences follow immediately from Remark 4.6 and the equalities

$$
\begin{aligned}
(\mathrm{Op}^w(a)f, g) &= (\mathrm{Op}^w(a_1)f, \check{g}) = (f, \mathrm{Op}^w(a_2)g) \\
&= \overline{(\mathrm{Op}^w(a_3)(x, D)\overline{f}, \overline{g})} = (\mathrm{Op}^w(a_4)\check{f}, \check{g}) = (\check{f}, \mathrm{Op}^w(a_5)\check{g}),
\end{aligned}
$$

when $a \in \mathcal{S}'(\mathbf{R}^{2d})$ and $f, g \in \mathcal{S}(\mathbf{R}^d)$. Here the first equality follows from the fact that if $K(x, y)$ is the distribution kernel of $\mathrm{Op}^w(a)$, then $K(-x, y)$ is the distribution kernel of $\mathrm{Op}^w(\mathcal{F}_\sigma a) = (2\pi)^{-d/2}Aa$ (cf. [36,38]). The proof is complete. □

In Remarks 4.8 and 4.9 below we list some properties for $s_{t,p}(\mathcal{H}_1, \mathcal{H}_2)$ and $s_p^A(\mathcal{H}_1, \mathcal{H}_2)$. These properties follow from well-known Schatten–von Neumann results in [2,31,45], in combination with (1.7), (1.13) and the fact that the mappings $a \mapsto \mathrm{Op}_t(a)$ and $a \mapsto Aa$ are isometric bijections from $s_{t,p}(\mathcal{H}_1, \mathcal{H}_2)$ and $s_p^A(\mathcal{H}_1, \mathcal{H}_2)$, respectively, to $\mathcal{I}_p(\mathcal{H}_1, \mathcal{H}_2)$. Here the forms $(\,\cdot\,,\,\cdot\,)_{s_{t,2}(\mathcal{H}_1,\mathcal{H}_2)}$ and $(\,\cdot\,,\,\cdot\,)_{s_2^A(\mathcal{H}_1,\mathcal{H}_2)}$ are defined by the formula

$$
(a, b)_{s_{t,2}(\mathcal{H}_1,\mathcal{H}_2)} = (\mathrm{Op}_t(a), \mathrm{Op}_t(b))_{\mathcal{I}_2(\mathcal{H}_1,\mathcal{H}_2)}, \quad a, b \in s_{t,2}(\mathcal{H}_1, \mathcal{H}_2)
$$

and

$$
(a, b)_{s_2^A(\mathcal{H}_1,\mathcal{H}_2)} = (Aa, Ab)_{\mathcal{I}_2(\mathcal{H}_1,\mathcal{H}_2)}, \quad a, b \in s_2^A(\mathcal{H}_1, \mathcal{H}_2).
$$

Finally, the set l_0^∞ consists of all sequences in l^∞ which turns to zero at infinity, and l_0^1 consists of all sequences $(\lambda_j)_{j\in I}$ such that $\lambda_j = 0$ except for finite numbers of $j \in I$.

Remark 4.8 Assume that $p, p_j, q, r \in [1, \infty]$ for $1 \le j \le 2$, $t \in \mathbf{R}$, and that $\mathcal{H}_1, \dots, \mathcal{H}_4$ are tempered Hilbert spaces on $\mathbf{R}^d$. Then the following is true:

(1) the set $s_{t,p}(\mathcal{H}_1, \mathcal{H}_2)$ is a Banach space which increases with the parameter p. If in addition $p < \infty$ and $p_1 \le p_2$, then $s_{t,p}(\mathcal{H}_1, \mathcal{H}_2) \subseteq s_{t,\sharp}(\mathcal{H}_1, \mathcal{H}_2)$, $s_{t,1}(\mathcal{H}_1, \mathcal{H}_2)$ is dense in $s_{t,p}(\mathcal{H}_1, \mathcal{H}_2)$ and in $s_{t,\sharp}(\mathcal{H}_1, \mathcal{H}_2)$, and

$$
\|a\|_{s_{t,p_2}(\mathcal{H}_1,\mathcal{H}_2)} \le \|a\|_{s_{t,p_1}(\mathcal{H}_1,\mathcal{H}_2)}, \quad a \in s_{t,\infty}(\mathcal{H}_1, \mathcal{H}_2); \tag{4.4}
$$

(2) equality is attained in (4.4), if and only if a is of rank one, and then $\|a\|_{s_{t,p}(\mathcal{H}_1,\mathcal{H}_2)_p} = (2\pi)^{-d/2}\|f_0\|_{\mathcal{H}_1}\|g_0\|_{\mathcal{H}_2}$, when a is given by (1.5);

(3) if $1/p_1 + 1/p_2 = 1/r$, $a_1 \in s_{t,p_1}(\mathcal{H}_1, \mathcal{H}_2)$ and $a_2 \in s_{t,p_2}(\mathcal{H}_2, \mathcal{H}_3)$, then $a_2 \#_t a_1 \in s_{t,r}(\mathcal{H}_1, \mathcal{H}_3)$, and

$$
\|a_2 \#_t a_1\|_{s_{t,r}(\mathcal{H}_1,\mathcal{H}_3)} \le \|a_1\|_{s_{t,p_1}(\mathcal{H}_1,\mathcal{H}_2)}\|a_2\|_{s_{t,p_2}(\mathcal{H}_2,\mathcal{H}_3)}. \tag{4.5}
$$

On the other hand, for any $a \in s_{t,r}(\mathcal{H}_1, \mathcal{H}_3)$, there are elements $a_1 \in s_{t,p_1}(\mathcal{H}_1, \mathcal{H}_2)$ and $a_2 \in s_{t,p_2}(\mathcal{H}_2, \mathcal{H}_3)$ such that $a = a_2 \#_t a_1$ and equality holds in (4.5);

(4) if $\mathcal{H}_1 \subseteq \mathcal{H}_2$ and $\mathcal{H}_3 \subseteq \mathcal{H}_4$, then $s_{t,p}(\mathcal{H}_2, \mathcal{H}_3) \subseteq s_{t,p}(\mathcal{H}_1, \mathcal{H}_4)$.

Similar facts hold when the $s_{t,p}$ spaces and the product $\#_t$ are replaced by s_p^A spaces and $*_\sigma$.

Remark 4.9 Assume that $p, p_j, q, r \in [1, \infty]$ for $1 \le j \le 2$, $t \in \mathbf{R}$, and that $\mathcal{H}_1, \mathcal{H}_2$ are tempered Hilbert spaces on $\mathbf{R}^d$. Then the following is true:

(1) the form $(\,\cdot\,,\,\cdot\,)_{s_{t,2}(\mathcal{H}_1,\mathcal{H}_2)}$ on $s_{t,1}(\mathcal{H}_1, \mathcal{H}_2)$ extends uniquely to a sesqui-linear and continuous form on $s_{t,p}(\mathcal{H}_1, \mathcal{H}_2) \times s_{t,p'}(\mathcal{H}_1, \mathcal{H}_2)$, and for every $a_1 \in s_{t,p}(\mathcal{H}_1, \mathcal{H}_2)$ and $a_2 \in s_{t,p'}(\mathcal{H}_1, \mathcal{H}_2)$, it holds

$$\begin{aligned}
(a_1, a_2)_{s_{t,2}(\mathcal{H}_1,\mathcal{H}_2)} &= \overline{(a_2, a_1)_{s_{t,2}(\mathcal{H}_1,\mathcal{H}_2)}},\\
|(a_1, a_2)_{s_{t,2}(\mathcal{H}_1,\mathcal{H}_2)}| &\le \|a_1\|_{s_{t,p}(\mathcal{H}_1,\mathcal{H}_2)} \|a_2\|_{s_{t,p'}(\mathcal{H}_1,\mathcal{H}_2)} \quad \text{and}\\
\|a_1\|_{s_{t,p}(\mathcal{H}_1,\mathcal{H}_2)} &= \sup |(a_1, b)_{s_{t,2}(\mathcal{H}_1,\mathcal{H}_2)}|,
\end{aligned}$$

where the supremum is taken over all $b \in s_{t,p'}(\mathcal{H}_1, \mathcal{H}_2)$ such that $\|b\|_{s_{t,p'}(\mathcal{H}_1,\mathcal{H}_2)} \le 1$. If in addition $p < \infty$, then the dual space of $s_{t,p}(\mathcal{H}_1, \mathcal{H}_2)$ can be identified with $s_{t,p'}(\mathcal{H}_1, \mathcal{H}_2)$ through this form;

(2) if $a \in s_{t,\sharp}(\mathcal{H}_1, \mathcal{H}_2)$, then

$$\operatorname{Op}_t(a) f = \sum_{j=1}^{\infty} \lambda_j (f, f_j)_{\mathcal{H}_1} g_j, \tag{4.6}$$

holds for some $(f_j)_{j=1}^\infty \in \mathrm{ON}(\mathcal{H}_1)$, $(g_j)_{j=1}^\infty \in \mathrm{ON}(\mathcal{H}_2)$ and non-negative decreasing sequence $\lambda = (\lambda_j)_{j=1}^\infty \in l_0^\infty$, where the operator on the right-hand side of (4.6) convergences with respect to the operator norm. Moreover, $a \in s_{t,p}(\mathcal{H}_1, \mathcal{H}_2)$, if and only if $\lambda \in l^p$, and then

$$\|a\|_{s_{t,p}} = \|\lambda\|_{l^p}$$

and the operator on the right-hand side of (4.6) converges with respect to the norm $\|\cdot\|_{s_{t,p}(\mathcal{H}_1,\mathcal{H}_2)}$;

(3) If $0 \le \theta \le 1$ is such that $1/p = (1-\theta)/p_1 + \theta/p_2$, then the (complex) interpolation formula

$$(s_{t,p_1}(\mathcal{H}_1, \mathcal{H}_2), s_{t,p_2}(\mathcal{H}_1, \mathcal{H}_2))_{[\theta]} = s_{t,p}(\mathcal{H}_1, \mathcal{H}_2)$$

holds with equality in norms.

Similar facts hold when the $s_{t,p}$ spaces are replaced by s_p^A spaces.

We may now prove the following.

Proposition 4.10 *Assume that* $p \in [1,\infty)$*, and that* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *are tempered Hilbert spaces on* $\mathbf{R}^d$*. Then* $\mathcal{S}(\mathbf{R}^{2d})$ *is dense in* $s_{t,p}(\mathcal{H}_1,\mathcal{H}_2)$*,* $s_p^A(\mathcal{H}_1,\mathcal{H}_2)$*,* $s_{t,\sharp}(\mathcal{H}_1,\mathcal{H}_2)$ *and* $s_\sharp^A(\mathcal{H}_1,\mathcal{H}_2)$*. Furthermore,* $\mathcal{S}(\mathbf{R}^{2d})$ *is dense in* $s_{t,\infty}(\mathcal{H}_1,\mathcal{H}_2)$ *and* $s_\infty^A(\mathcal{H}_1,\mathcal{H}_2)$ *with respect to the weak* topology.*

Proof The result is an immediate consequence of Corollary 4.4 Remarks 4.5 (iii) and 4.8 (4). The proof is complete. □

A problem with the form $(\,\cdot\,,\,\cdot\,)_{s_{t,2}(\mathcal{H}_1,\mathcal{H}_2)}$ in Remark 4.9 is the somewhat complicated structure. In the following we show that there is a canonical way to replace this form with $(\,\cdot\,,\,\cdot\,)_{L^2}$. We start with the following result concerning polar decomposition of compact operators.

Proposition 4.11 *Assume that* $t \in \mathbf{R}$*,* $p \in [1,\infty]$*,* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *are tempered Hilbert spaces on* $\mathbf{R}^d$ *and that* $a \in s_{t,\sharp}(\mathcal{H}_1,\mathcal{H}_2)$ *(*$a \in s_\sharp^A(\mathcal{H}_1,\mathcal{H}_2)$*). Then*

$$a \equiv \sum_{j\in I} \lambda_j W^t_{g_j,\varphi_j} \quad \left(a \equiv \sum_{j\in I} \lambda_j W_{\check{g}_j,\varphi_j} \right)$$

(with norm convergence) for some orthonormal sequences $(\varphi_j)_{j\in I}$ *in* $\mathcal{H}_1'$ *and* $(g_j)_{j\in I}$ *in* $\mathcal{H}_2$*, and a sequence* $(\lambda_j)_{j\in I} \in l_0^\infty$ *of non-negative real numbers which decreases to zero at infinity. Furthermore,* $a \in s_{t,p}(\mathcal{H}_1,\mathcal{H}_2)$ *(*$a \in s_p^A(\mathcal{H}_1,\mathcal{H}_2)$*), if and only if* $(\lambda_j)_{j\in I} \in l^p$*, and*

$$\|a\|_{s_{t,p}(\mathcal{H}_1,\mathcal{H}_2)} = (2\pi)^{-d/2}\|(\lambda_j)_{j\in I}\|_{l^p} \quad (\|a\|_{s_p^A(\mathcal{H}_1,\mathcal{H}_2)} = \|(\lambda_j)_{j\in I}\|_{l^p}).$$

Proof By Remark 4.9 (2) it follows that if $f \in \mathcal{S}(\mathbf{R}^d)$, then

$$\operatorname{Op}_t(a)f(x) = \sum_{j\in I} \lambda_j (f, f_j)_{\mathcal{H}_1} g_j \tag{4.7}$$

for some orthonormal sequences (f_j) in $\mathcal{H}_1$ and (g_j) in $\mathcal{H}_2$, and a sequence $(\lambda_j) \in l_0^\infty$ of non-negative real numbers which decreases to zero at infinity. Now let $(\varphi_j)_{j\in I}$ be an orthonormal sequence in $\mathcal{H}_1'$ such that $(\varphi_j, f_k)_{L^2} = \delta_{j,k}$. Then $(f, f_j)_{\mathcal{H}_1} = (f,\varphi_j)_{L^2}$, and the result follows from (4.7), and the fact that

$$\operatorname{Op}_t(W^t_{g_j,\varphi_j})f = (2\pi)^{-d/2}(f,\varphi_j)_{L^2} g_j = (2\pi)^{-d/2}(f, f_j)_{\mathcal{H}_1} g_j.$$

The proof is complete. □

Next we prove that the duals for $s_{t,p}(\mathcal{H}_1,\mathcal{H}_2)$ and $s_p^A(\mathcal{H}_1,\mathcal{H}_2)$ can be identified with $s_{t,p'}(\mathcal{H}_1',\mathcal{H}_2')$ and $s_{p'}^A(\mathcal{H}_1',\mathcal{H}_2')$, respectively, through the form $(\,\cdot\,,\,\cdot\,)_{L^2}$.

Theorem 4.12 *Assume that* $t \in \mathbf{R}$, $p \in [1, \infty)$ *and that* $\mathcal{H}_1, \mathcal{H}_2$ *are tempered Hilbert spaces on* $\mathbf{R}^d$. *Then the* L^2 *form on* $\mathcal{S}(\mathbf{R}^{2d})$ *extends uniquely to a duality between* $s_{t,p}(\mathcal{H}_1, \mathcal{H}_2)$ *and* $s_{t,p'}(\mathcal{H}_1', \mathcal{H}_2')$, *and the dual space for* $s_{t,p}(\mathcal{H}_1, \mathcal{H}_2)$ *can be identified with* $s_{t,p'}(\mathcal{H}_1', \mathcal{H}_2')$ *through this form. Moreover, if* $\ell \in s_{t,p}(\mathcal{H}_1, \mathcal{H}_2)^*$ *and* $a \in s_{t,p'}(\mathcal{H}_1', \mathcal{H}_2')$ *are such that* $\overline{\ell(b)} = (a, b)_{L^2}$ *when* $b \in s_{t,p}(\mathcal{H}_1, \mathcal{H}_2)$, *then*

$$\|\ell\| = \|a\|_{s_{t,p'}(\mathcal{H}_1', \mathcal{H}_2')}.$$

The same is true if the $s_{t,p}(\mathcal{H}_1, \mathcal{H}_2)$ *spaces are replaced by* $s_p^A(\mathcal{H}_1, \mathcal{H}_2)$ *spaces.*

Proof We only prove the assertion in the case $t = 1/2$. The general case follows by similar arguments and is left for the reader. Assume that $\ell \in s_p^w(\mathcal{H}_1, \mathcal{H}_2)^*$. Since the map $b \mapsto \operatorname{Op}^w(b)$ is an isometric bijection from $s_p^w(\mathcal{H}_1, \mathcal{H}_2)$ to $\mathcal{I}_p(\mathcal{H}_1, \mathcal{H}_2)$, it follows from Remark 4.9 (1) that for some $S \in \mathcal{I}_{p'}(\mathcal{H}_1, \mathcal{H}_2)$ and each orthonormal basis $(f_j) \in \operatorname{ON}(\mathcal{H}_1)$ we have

$$\begin{aligned} &\ell(b) = \operatorname{tr}_{\mathcal{H}_1}(S^* \circ \operatorname{Op}^w(b)) = \sum (\operatorname{Op}^w(b) f_j, S f_j)_{\mathcal{H}_2} \quad \text{and} \\ &\|\ell\| = \|S\|_{\mathcal{I}_{p'}(\mathcal{H}_1, \mathcal{H}_2)}, \end{aligned} \tag{4.8}$$

when $b \in s_p^w(\mathcal{H}_1, \mathcal{H}_2)$.

Now let $b \in s_p^w(\mathcal{H}_1, \mathcal{H}_2)$ be an arbitrary finite rank element. Then

$$b = \sum \lambda_j W_{g_j, \varphi_j} \quad \text{and} \quad \|b\|_{s_p^w(\mathcal{H}_1, \mathcal{H}_2)} = (2\pi)^{-d/2} \|(\lambda_j)\|_{l^p},$$

for some orthonormal bases $(\varphi_j) \in \operatorname{ON}(\mathcal{H}_1')$ and $(g_j) \in \operatorname{ON}(\mathcal{H}_2)$, and some sequence $(\lambda_j) \in l_0^1$. We also let $(f_j) \in \operatorname{ON}(\mathcal{H}_1)$ be the dual basis of (φ_j) and a the Weyl symbol of the operator $T_{\mathcal{H}_2} \circ S \circ T_{\mathcal{H}_1'}$. Then $a \in s_{p'}^w(\mathcal{H}_1, \mathcal{H}_2)$ and $\|a\|_{s_{p'}^w(\mathcal{H}_1, \mathcal{H}_2)} = \|\ell\|$. By straight-forward computations we get

$$\begin{aligned} \ell(b) &= \operatorname{tr}_{\mathcal{H}_1}(S^* \circ \operatorname{Op}^w(b)) = \sum (b^w(x, D) f_j, S f_j)_{\mathcal{H}_2} \\ &= \sum \lambda_j (g_j, S f_j)_{\mathcal{H}_2} = \sum \lambda_j (g_j, \operatorname{Op}^w(a) \varphi_j)_{L^2(\mathbf{R}^d)} \\ &= (2\pi)^{-d/2} \sum \lambda_j (W_{g_j, \varphi_j}, a)_{L^2(\mathbf{R}^{2d})} = (2\pi)^{-d/2} (b, a)_{L^2(\mathbf{R}^{2d})}. \end{aligned}$$

Hence $\ell(b) = (2\pi)^{-d/2}(b, a)_{L^2(\mathbf{R}^{2d})}$. The result now follows from these identities and the fact that the set of finite rank elements are dense in $s_p^w(\mathcal{H}_1, \mathcal{H}_2)$. The proof is complete. □

5 Young inequalities for weighted Schatten–von Neumann classes

In this section we establish Young type results for dilated convolutions and multiplications on $s_p^w(\mathcal{H}_1, \mathcal{H}_2)$, when $\mathcal{H}_1$ and $\mathcal{H}_2$ are appropriate modulation spaces of Hilbert type. Especially we prove multi-linear versions of Theorems 0.3 and 0.4.

We need some preparations before stating the results. If we have N convolutions, then the corresponding condition comparing to (0.9) is

$$p_1{}^{-1} + \cdots + p_N{}^{-1} = N - 1 + r^{-1}, \qquad 1 \le p_1, \dots, p_N, r \le \infty. \tag{0.9$'$}$$

In the same way, (0.10) should be replaced by

$$(-1)^{j_1} t_1^{-2} + \cdots + (-1)^{j_N} t_N^{-2} = 1, \tag{0.10$'$}$$

and (0.11) by

$$(-1)^{j_1} t_1^{2} + \cdots + (-1)^{j_N} t_N^{2} = 1. \tag{0.11$'$}$$

The condition (0.12) of the involved weight functions is modified into

$$\begin{aligned} \vartheta(X_1 + \cdots + X_N) &\le C \vartheta_{j_1,1}(t_1 X_1) \cdots \vartheta_{j_N,N}(t_N X_N), \\ \omega(X_1 + \cdots + X_N) &\le C \omega_{j_1,1}(t_1 X_1) \cdots \omega_{j_N,N}(t_N X_N), \end{aligned} \tag{0.12$'$}$$

where

$$\omega_{0,k}(X) = \vartheta_{1,k}(X) = \omega_k(X), \quad \vartheta_{0,k}(X) = \omega_{1,k}(X) = \vartheta_k(X). \tag{0.13$'$}$$

With these conditions we shall essentially prove estimates of the form

$$\|a_{1,t_1} * \cdots * a_{N,t_N}\|_{s_r^w(1/\omega,\vartheta)} \le C^d \|a_1\|_{s_{p_1}^w(1/\omega_1,\vartheta_1)} \cdots \|a_N\|_{s_{p_N}^w(1/\omega_N,\vartheta_N)}, \tag{0.14$'$}$$

and

$$\|a_{1,t_1} \cdots a_{N,t_N}\|_{s_r^w(1/\omega,\vartheta)} \le C^d \|a_1\|_{s_{p_1}^w(1/\omega_1,\vartheta_1)} \cdots \|a_N\|_{s_{p_N}^w(1/\omega_N,\vartheta_N)}. \tag{0.15$'$}$$

Theorem 0.3$'$ *Assume that $p_1, \cdots, p_N, r \in [1,\infty]$ satisfy (0.9)$'$, and that $t_1, \dots, t_N \in \mathbf{R}\backslash 0$ satisfy (0.10)$'$, for some choices of $j_1, \dots, j_N \in \{0,1\}$. Also assume that $\omega, \omega_j, \vartheta, \vartheta_j \in \mathcal{P}(\mathbf{R}^{2d})$ for $j = 1, \dots, N$ satisfy (0.12)$'$ and (0.13)$'$. Then the map $(a_1, \dots, a_N) \mapsto a_{1,t_1} * \cdots * a_{N,t_N}$ on $\mathcal{S}(\mathbf{R}^{2d})$, extends uniquely to a continuous mapping from*

$$s_{p_1}^w(1/\omega_1, \vartheta_1) \times \cdots \times s_{p_N}^w(1/\omega_N, \vartheta_N)$$

to $s_r^w(1/\omega, \vartheta)$. Furthermore, (0.14)$'$ holds for some constant

$$C = C_0^N |t_1|^{-2/p_1} \cdots |t_N|^{-2/p_N},$$

where C_0 *is independent of* $a_1 \in s^w_{p_1}(1/\omega_1, \vartheta_1)$, ..., $a_N \in s^w_{p_N}(1/\omega_N, \vartheta_N)$, $t_1, \dots, t_N$ *and* d.

Moreover, $\operatorname{Op}^w(a_{1,t_1} * \cdots * a_{N,t_N}) \ge 0$ *when* $\operatorname{Op}^w(a_j) \ge 0$ *for each* $1 \le j \le N$.

Theorem 0.4′ *Assume that* $p_1, \dots, p_N, r \in [1, \infty]$ *satisfy* (0.9)′, *and that* $t_1, \dots, t_N \in \mathbf{R}\backslash 0$ *satisfy* (0.11)′, *for some choices of* $j_1, \dots, j_N \in \{0, 1\}$. *Also assume that* $\omega, \omega_j, \vartheta, \vartheta_j \in \mathcal{P}(\mathbf{R}^{2d})$ *for* $j = 1, \dots, N$ *satisfy* (0.12)′ *and* (0.13)′. *Then the map* $(a_1, \dots, a_N) \mapsto a_{1,t_1} \cdots a_{N,t_N}$ *on* $\mathcal{S}(\mathbf{R}^{2d})$, *extends uniquely to a continuous mapping from*

$$s^w_{p_1}(1/\omega_1, \vartheta_1) \times \cdots \times s^w_{p_N}(1/\omega_N, \vartheta_N)$$

to $s^w_r(1/\omega, \vartheta)$. *Furthermore,* (0.15)′ *holds for some constant*

$$C = C_0^N |t_1|^{-2/p_1'} \cdots |t_N|^{-2/p_N'},$$

where C_0 *is independent of* $a_1 \in s^w_{p_1}(1/\omega_1, \vartheta_1), \dots, a_N \in s^w_{p_N}(1/\omega_N, \vartheta_N)$, $t_1, \dots, t_N$ *and* d.

We need some preparations for the proof. First we observe that the roles of multiplications and convolutions are essentially interchanged on the symplectic Fourier transform side, because

$$\mathcal{F}_\sigma(a_1 * \cdots * a_N) = \pi^{dN}(\mathcal{F}_\sigma a_1) \cdots (\mathcal{F}_\sigma a_N), \tag{5.1}$$

holds when $a_1, \dots, a_N \in \mathcal{S}(\mathbf{R}^{2d})$. Hence it follows immediately from Lemma 1.1 and Proposition 4.7 that Theorems 0.3′ and 0.4′ are equivalent to the following two propositions. Here the condition (0.13)′ should be replaced by

$$\omega_{0,k}(X) = \vartheta_{1,k}(-X) = \omega_k(X), \quad \vartheta_{0,k}(X) = \omega_{1,k}(-X) = \vartheta_k(X). \tag{5.2}$$

We also recall that $a \in \mathcal{S}'_+(\mathbf{R}^{2d})$, if and only if the operator Aa is positive semi-definite (cf. Proposition 1.2).

Proposition 5.1 *Assume that* $p_1, \dots, p_N, r \in [1, \infty]$ *satisfy* (0.9)′, *and that* $t_1, \dots, t_N \in \mathbf{R}\backslash 0$ *satisfy* (0.10)′, *for some choices of* $j_1, \dots, j_N \in \{0, 1\}$. *Also assume that* $\omega, \omega_j, \vartheta, \vartheta_j \in \mathcal{P}(\mathbf{R}^{2d})$ *for* $j = 1, \dots, N$ *satisfy* (0.12)′ *and* (5.2). *Then the continuity assertions in Theorem 0.3′ holds after the* s^w_p *spaces have been replaced by* s^A_p *spaces.*

Proposition 5.2 *Assume that* $p_1, \dots, p_N, r \in [1, \infty]$ *satisfy* (0.9)′, *and that* $t_1, \dots, t_N \in \mathbf{R}\backslash 0$ *satisfy* (0.11), *for some choices of* $j_1, \dots, j_N \in \{0, 1\}$. *Also assume that* $\omega, \omega_j, \vartheta, \vartheta_j \in \mathcal{P}(\mathbf{R}^{2d})$ *for* $j = 1, \dots, N$ *satisfy* (0.12)′ *and* (5.2). *Then the continuity assertions in Theorem 0.4′ holds after the* s^w_p *spaces have been replaced by* s^A_p *spaces.*

Moreover, $a_{1,t_1} \cdots a_{N,t_N} \in \mathcal{S}'_+(\mathbf{R}^{2d})$ *when* $a_{j,t_j} \in \mathcal{S}'_+(\mathbf{R}^{2d})$ *for each* $j = 1, \dots, N$.

When proving Propositions 5.1 and 5.2 we need some technical lemmas, and start with the following classification of Hilbert modulation spaces.

Lemma 5.3 *Assume that* $\omega \in \mathcal{P}(\mathbf{R}^{4d})$ *is such that* $\omega(x,y,\xi,\eta) = \omega(x,\xi)$, $\chi \in \mathcal{S}(\mathbf{R}^d)\backslash 0$ *and that* $F \in \mathcal{S}'(\mathbf{R}^{2d})$. *Then* $F \in M^2_{(\omega)}$, *if and only if*

$$\|F\| \equiv \left(\iiint |V_\chi(F(\,\cdot\,,y))(x,\xi)\omega(x,\xi)|^2\, dxdyd\xi \right)^{1/2} < \infty. \tag{5.3}$$

Furthermore, $F \mapsto \|F\|$ *in* (5.3) *defines a norm which is equivalent to any* $M^2_{(\omega)}$ *norm.*

Proof We may assume that $\|\chi\|_{L^2} = 1$. Let $\chi_1 = \chi \otimes \chi$, and let $\mathcal{F}_1 F$ denotes the partial Fourier transform of $F(x,y)$ with respect to the x variable. By Parseval's formula we get

$$\begin{aligned}
\|F\|^2_{M^2_{(\omega)}} &= \iiiint |(V_{\chi\otimes\chi}F)(x,y,\xi,\eta)\omega(x,\xi)|^2\, dxdyd\xi d\eta \\
&= \iint \left(\iint |(\mathcal{F}(F\,\chi_1(\,\cdot\, - (x,y)))\,(\xi,\eta)\omega(x,\xi)|^2\, dyd\eta \right) dxd\xi \\
&= \iint \left(\iint |(\mathcal{F}_1\,(F(\,\cdot\,,z)\,\chi(\,\cdot\, - x))\,(\xi)\chi(z-y)\omega(x,\xi)|^2\, dydz \right) dxd\xi \\
&= \iint \left(\int |(\mathcal{F}_1\,(F(\,\cdot\,,z)\,\chi(\,\cdot\, - x))\,(\xi)\omega(x,\xi)|^2\, dz \right) dxd\xi = \|F\|,
\end{aligned}$$

where the right-hand side is the same as $\|F\|$ in (5.3). The proof is complete. □

We omit the proof of the next lemma, since the result agrees with [38, Lemma 3.2].

Lemma 5.4 *Assume that* $s,t \in \mathbf{R}$ *satisfies* $(-1)^j s^{-2} + (-1)^k t^{-2} = 1$, *for some choice of* $j,k \in \{0,1\}$, *and that* $a,b \in \mathcal{S}(\mathbf{R}^{2d})$. *Also let* $T_{j,z}$ *for* $j \in \{0,1\}$ *and* $z \in \mathbf{R}^d$ *be the operator on* $\mathcal{S}(\mathbf{R}^{2d})$, *defined by the formula*

$$(T_{0,z}U)(x,y) = (T_{1,z}U)(y,x) = U(x-z, y+z), \quad U \in \mathcal{S}(\mathbf{R}^{2d}).$$

Then

$$A(a(s\,\cdot\,) * b(t\,\cdot\,)) = (2\pi)^{d/2}|st|^{-d} \int (T_{j,sz}(Aa))(s^{-1}\,\cdot\,)(T_{k,-tz}(Ab))(t^{-1}\,\cdot\,)\, dz. \tag{5.4}$$

We note that for the involved spaces in Theorems 0.3′ and 0.4′, and Propositions 5.1 and 5.2 we have

$$s^A_p(1/\omega,\vartheta) \subseteq s^A_p(\mathbf{R}^{2d}) \subseteq s^A_p(\omega, 1/\vartheta), \quad \text{when} \quad \omega,\vartheta \ge c, \tag{5.5}$$

for some constant $c > 0$, and similarly when s^A_p is replaced by s^w_p. This is an immediate consequence of Remark 4.8 (4) and the fact that the embeddings $M^{2,2}_{(\omega)} \subseteq M^{2,2} =$

$L^2 \subseteq M^{2,2}_{(1/\omega)}$ hold when ω is bounded from below. In particular, if $C_B(\mathbf{R}^{2d})$ is the set of continuous functions on $\mathbf{R}^{2d}$ vanishing at the infinity, then

$$s_1^A(1/\omega, \vartheta) \subseteq s_1^A(\mathbf{R}^{2d}) \subseteq C_B(\mathbf{R}^{2d}) \cap \mathcal{F}C_B(\mathbf{R}^{2d}) \cap L^2(\mathbf{R}^{2d}), \qquad \text{when} \quad \omega, \vartheta \ge c, \tag{5.6}$$

and similarily when s_1^A is replaced by s_1^w. Here the latter embedding follows from Propositions 1.5 and 1.9 in [39].

Proof of Proposition 5.1 in the case $N = 2$ We only consider the case $j_1 = 1$ and $j_2 = 0$, i.e. $t^{-2} - s^{-2} = 1$ when $t_1 = s$ and $t_2 = t$. The other cases follows by similar arguments and are left for the reader. We start to prove the theorem in the case $p = q = r = 1$. By Propositions 4.10, 4.11 and a simple argument of approximations, it follows that we may assume that $a_1 = u$ and $a_2 = v$ are rank one elements in $\mathcal{S}$ and satisfy

$$\|u\|_{s_1^A(1/\omega_1,\vartheta_1)} \le C \quad \text{and} \quad \|v\|_{s_1^A(1/\omega_2,\vartheta_2)} \le C,$$

for some constant C. Then $Au = f_1 \otimes \overline{f}_2$, $Av = g_1 \otimes \overline{g}_2$ and

$$\|f_1\|_{M^2_{(\vartheta_1)}} \|f_2\|_{M^2_{(\omega_1)}} \le C_1 \|u\|_{s_1^A(1/\omega_1,\vartheta_1)},$$
$$\|g_1\|_{M^2_{(\vartheta_2)}} \|g_2\|_{M^2_{(\omega_2)}} \le C_1 \|v\|_{s_1^A(1/\omega_2,\vartheta_2)},$$

where $f_1, f_2, g_1, g_2 \in \mathcal{S}$ are such that

$$\|f_1\|_{M^2_{(\vartheta_1)}} \le C_2, \quad \|f_2\|_{M^2_{(\omega_1)}} \le C_2, \quad \|g_1\|_{M^2_{(\vartheta_2)}} \le C_2, \quad \|g_2\|_{M^2_{(\omega_2)}} \le C_2,$$

for some constants C_1 and C_2.

Set

$$F(x,z) = f_2(x/s + sz) g_1(x/t + tz), \qquad G(y,z) = f_1(y/s - sz) g_2(y/t - tz).$$

It follows from (5.4) that

$$A(u_s * v_t)(x,y) = (2\pi)^{d/2} |st|^{-d} \int F(x,z) G(y,z)\, dz.$$

This implies that

$$\begin{aligned} \|u_s * v_t\|_{s_1^A(1/\omega,\vartheta)} &\le (2\pi)^{d/2} |st|^{-d} \int \|F(\,\cdot\,,z)\|_{M^2_{(\vartheta)}} \|G(\,\cdot\,,z)\|_{M^2_{(\omega)}}\, dz \\ &\le C|st|^{-d} I_1 \cdot I_2, \end{aligned} \tag{5.7}$$

where

$$I_1 = \left(\iiint |V_\chi (F(\,\cdot\,, z))(x,\xi)\vartheta(x,\xi)|^2\, dxdzd\xi \right)^{1/2},$$
$$I_2 = \left(\iiint |V_\chi (G(\,\cdot\,, z))(x,\xi)\omega(x,\xi)|^2\, dxdzd\xi \right)^{1/2}. \tag{5.8}$$

Hence, $I_1 \le C\|F\|_{M^2_{(\vartheta_0)}}$ and $I_2 \le C\|G\|_{M^2_{(\omega_0)}}$ by Lemma 5.3, when $\omega_0(x,y,\xi,\eta) = \omega(x,\xi)$ and $\vartheta_0(x,y,\xi,\eta) = \vartheta(x,\xi)$.

We need to estimate $\|F\|_{M^2_{(\vartheta_0)}}$ and $\|G\|_{M^2_{(\omega_0)}}$. In order to estimate $\|F\|_{M^2_{(\vartheta_0)}}$ we choose the window function $\chi \in \mathcal{S}(\mathbf{R}^{2d})$ as

$$\chi(x,z) = \chi_0(x/s + sz)\chi_0(x/t + tz),$$

for some real-valued $\chi_0 \in \mathcal{S}(\mathbf{R}^d)$. By taking $(x_1/s + sz_1, x_1/t + tz_1)$ as new variables when evaluating $V_\chi F$ we get

$$\begin{aligned}
&V_\chi F(x,z,\xi,\zeta)\\
&= (2\pi)^{-d} \iint F(x_1,z_1)\chi(x_1 - x, z_1 - z)e^{-i\langle x_1,\xi\rangle - i\langle z_1,\zeta\rangle}\, dx_1dz_1\\
&= (2\pi)^{-d}|st|^{-d} \iint \overline{f_2(x_1)}g_1(z_1)\chi_0(x_1 - (x/s + sz))\chi_0(z_1 - (x/t + tz))\\
&\quad \times e^{-i(\langle t^{-1}z_1 - s^{-1}x_1,\xi\rangle + (st)^{-1}\langle t^{-1}x_1 - s^{-1}z_1,\zeta\rangle}\, dx_1dz_1\\
&= |st|^{-d}\overline{V_{\chi_0}f_2(s^{-1}x + sz, s^{-1}\xi - (st^2)^{-1}\zeta)}V_{\chi_0}g_1(t^{-1}x + tz, t^{-1}\xi - (s^2t)\zeta).
\end{aligned}$$

Furthermore, by (0.12), (5.2) and the fact that $t^{-2} - s^{-2} = 1$, we obtain

$$\begin{aligned}
\vartheta(x,\xi) &= \vartheta\Big((t^{-2}x + z) - (s^{-2}x + z), (t^{-2}\xi - (st)^{-2}\zeta) - (s^{-2}\xi - (st)^{-2}\zeta)\Big)\\
&\le C\omega_1(s^{-1}x + sz, s^{-1}\xi - (st^2)^{-1}\zeta)\vartheta_2(t^{-1}x + tz, t^{-1}\xi - (s^2t)^{-1}\zeta)
\end{aligned}$$

A combination of these relations now gives

$$|V_\chi F(x,z,\xi,\zeta)\vartheta(x,\xi)| \le C|st|^{-d} J_1 \cdot J_2, \tag{5.9}$$

where

$$J_1 = |V_{\chi_0}f_2(s^{-1}x + sz, s^{-1}\xi - (st^2)^{-1}\zeta)\omega_1(s^{-1}x + sz, s^{-1}\xi - (st^2)^{-1}\zeta)|$$

and

$$J_2 = |V_{\chi_0}g_1(t^{-1}x + tz, t^{-1}\xi - (s^2t)\zeta)\vartheta_2(t^{-1}x + tz, t^{-1}\xi - (s^2t)^{-1}\zeta)|.$$

By applying the L^2 norm in (5.9) and taking

$$s^{-1}x + sz, \quad t^{-1}x + tz, \quad s^{-1}\xi - (st^2)^{-1}\zeta, \quad t^{-1}\xi - (s^2t)^{-1}\zeta$$

as new variables of integration we get

$$\|F\|_{M^2_{(\vartheta)}} \le C|st|^{-2d}\|f_2\|_{M^2_{(\omega_1)}}\|g_1\|_{M^2_{(\vartheta_2)}}. \tag{5.10}$$

By similar computations it also follows that

$$\|G\|_{M^2_{(\omega)}} \le C|st|^{-2d}\|f_1\|_{M^2_{(\vartheta_1)}}\|g_2\|_{M^2_{(\omega_2)}}. \tag{5.11}$$

Hence, a combination of Proposition 4.11, Lemma 5.3, (5.7), (5.8), (5.10) and (5.11) gives

$$\begin{aligned}\|u_s * v_t\|_{s_1^A(1/\omega,\vartheta)} &\le C_1|st|^{-d}\|f_1\|_{M^2_{(\vartheta_1)}}\|f_2\|_{M^2_{(\omega_1)}}\|g_1\|_{M^2_{(\vartheta_2)}}\|g_2\|_{M^2_{(\omega_2)}}\\ &\le C_2|st|^{-d}\|u\|_{s_1^A(1/\omega_1,\vartheta_1)}\|v\|_{s_1^A(1/\omega_2,\vartheta_2)}.\end{aligned}$$

This proves the result in the case $p = q = r = 1$.

Next we consider the case $p_1 = r = \infty$, which implies that $p_2 = 1$. Assume that $a \in s_\infty^A(1/\omega_1, \vartheta_1)$ and that $b, c \in \mathcal{S}(\mathbf{R}^{2d})$. Then

$$(a_s * b_t, c) = |s|^{-4d}(a, \widetilde{b}_{t_0} * c_{s_0}),$$

where $\widetilde{b}(X) = \overline{b(-X)}$, $s_0 = 1/s$ and $t_0 = t/s$. We claim that

$$\|\widetilde{b}_{t_0} * c_{s_0}\|_{s_1^A(\omega_1,1/\vartheta_1)} \le C|s^2/t|^{2d}\|b\|_{s_1^A(1/\omega_2,\vartheta_2)}\|c\|_{s_1^A(\omega,1/\vartheta)} \tag{5.12}$$

Admitting this for a while, it follows by duality, using Theorem 4.12 that

$$\|a_s * b_t\|_{s_\infty^A(1/\omega,\vartheta)} \le C|s^2/t|^{2d}s^{-4d}\|a\|_{s_\infty^A(1/\omega_1,\vartheta_1)}\|b\|_{s_1^A(1/\omega_2,\vartheta_2)},$$

which gives (0.14). The result now follows in the case $p_1 = r = \infty$ and $p_2 = 1$ from the fact that $\mathcal{S}$ is dense in $s_1^A(1/\omega_2, \vartheta_2)$. In the same way the result follows in the case $p_2 = r = \infty$ and $p_1 = 1$.

For general $p_1, p_2, r \in [1, \infty]$ the result follows by multi-linear interpolation, using Theorem 4.4.1 in [1] and Remark 4.9 (3).

It remains to prove (5.12) when $b, c \in \mathcal{S}(\mathbf{R}^{2d})$. The condition (0.10) is invariant under the transformation $(t, s) \mapsto (t_0, s_0) = (t/s, 1/s)$. Let

$$\begin{aligned}\widetilde{\omega} = 1/\omega_1, \quad \widetilde{\vartheta} = 1/\vartheta_1, \quad \widetilde{\omega}_1 = 1/\omega,\\ \widetilde{\vartheta}_1 = 1/\vartheta, \quad \widetilde{\omega}_2 = \vartheta_2 \quad \text{and} \quad \widetilde{\vartheta}_2 = \omega_2.\end{aligned}$$

If $X_1 = -(X + Y)/s$ and $X_2 = Y/s$, then it follows that

$$\omega(X_1 + X_2) \le C\vartheta_1(-sX_1)\omega_2(tX_2)$$

and

$$\vartheta(X_1+X_2)\le C\omega_1(-sX_1)\vartheta_2(tX_2),$$

is equivalent to

$$\widetilde{\omega}(X+Y)\le C\widetilde{\vartheta}_1(-s_0X)\widetilde{\omega}_2(t_0Y)$$

and

$$\widetilde{\vartheta}(X+Y)\le C\widetilde{\omega}_1(-s_0X)\widetilde{\vartheta}_2(t_0Y).$$

Hence, the first part of the proof gives

$$\begin{aligned}\|\widetilde{b}_{t_0}*c_{s_0}\|_{s_1^A(\omega_1,1/\vartheta_1)} &= \|\widetilde{b}_{t_0}*c_{s_0}\|_{s_1^A(1/\widetilde{\omega},\widetilde{\vartheta})}\\ &\le C|s_0t_0|^{-2d}\|\widetilde{b}\|_{s_1^A(1/\widetilde{\omega}_2,\widetilde{\vartheta}_2)}\|\widetilde{c}\|_{s_1^A(1/\widetilde{\omega}_1,\widetilde{\vartheta}_1)}\\ &= C|s_0t_0|^{-2d}\|\widetilde{b}\|_{s_1^A(1/\vartheta_2,\omega_2)}\|\widetilde{c}\|_{s_1^A(\omega,1/\vartheta)}\\ &= C|s_0t_0|^{-2d}\|b\|_{s_1^A(1/\omega_2,\vartheta_2)}\|\widetilde{c}\|_{s_1^A(\omega,1/\vartheta)},\end{aligned}$$

and (5.12) follows. The proof of the case $N=2$ is complete. □

Remark 5.5 A proof without any use of interpolation in the case of trivial weight is presented in Sect. 2.3 in [36].

We need the following lemma for the proof of Proposition 5.1 in the general case.

Lemma 5.6 *Let $\omega_{j,k}$ and $\vartheta_{j,k}$ be as in Proposition 5.1, and assume that $\rho, t_1,\dots,t_N\in \mathbf{R}\backslash 0$ fulfills (0.10)′ and $\rho^{-2}+(-1)^{j_N}t_N^{-2}=1$. For $t_j'=t_j/\rho$ set*

$$\begin{aligned}\widetilde{\omega}(X) &= \inf \omega_{j_1,1}(t_1'X_1)\cdots\omega_{j_{N-1},N-1}(t_{N-1}'X_{N-1}) \quad \text{and}\\ \widetilde{\vartheta}(X) &= \inf \vartheta_{j_1,1}(t_1'X_1)\dots\vartheta_{j_{N-1},N-1}(t_{N-1}'X_{N-1}),\end{aligned}$$

where the infima are taken over all $X_1,\dots,X_{N-1}$ such that $X=X_1+\cdots+X_{N-1}$. Then the following is true:

(1) $\widetilde{\omega},\widetilde{\vartheta}\in\mathcal{P}(\mathbf{R}^{2d})$;

(2) *for each $X_1,\dots,X_{N-1}\in\mathbf{R}^{2d}$ it holds*

$$\begin{aligned}\widetilde{\omega}(X_1+\cdots+X_{N-1}) &\le \omega_{j_1,1}(t_1'X_1)\cdots\omega_{j_{N-1},N-1}(t_{N-1}'X_{N-1}), \quad \text{and}\\ \widetilde{\vartheta}(X_1+\cdots+X_{N-1}) &\le \vartheta_{j_1,1}(t_1'X_1)\cdots\vartheta_{j_{N-1},N-1}(t_{N-1}'X_{N-1});\end{aligned}$$

(3) *if C is the same as in (0.12)′, then for each $X,Y\in\mathbf{R}^{2d}$ it holds*

$$\omega(X+Y)\le C\widetilde{\omega}(\rho X)\omega_N(t_NY) \quad \text{and} \quad \vartheta(X+Y)\le C\widetilde{\vartheta}(\rho X)\vartheta_N(t_NY).$$

Proof The assertion (2) follows immediately from the definitions of $\widetilde{\omega}$ and $\widetilde{\vartheta}$, and (3) is an immediate consequence of (0.12)$'$.

In order to prove (1) we assume that $X = X_1 + \cdots + X_{N-1}$. Since $\omega_{j_1,1} \in \mathscr{P}(\mathbf{R}^{2d})$, it follows that

$$\begin{aligned}\widetilde{\omega}(X+Y) &\le \omega_{j_1,1}(t_1'(X_1+Y)) \dots \omega_{j_{N-1},N-1}(t_{N-1}'X_{N-1})\\ &\le \omega_{j_1,1}(t_1'X_1) \dots \omega_{j_{N-1},N-1}(t_{N-1}'X_{N-1})v(Y),\end{aligned}$$

for some $v \in \mathscr{P}(\mathbf{R}^{2d})$. By taking the infimum over all representations $X = X_1 + \cdots + X_N$, the latter inequality becomes $\widetilde{\omega}(X+Y) \le \widetilde{\omega}(X)v(Y)$. This implies that $\widetilde{\omega} \in \mathscr{P}(\mathbf{R}^{2d})$, and in the same way it follows that $\widetilde{\vartheta} \in \mathscr{P}(\mathbf{R}^{2d})$. The proof is complete.□

Proof of Proposition 5.1 for general N We may assume that $N > 2$ and that the proposition is already proved for lower values on N. The condition on t_j is that $c_1 t_1^{-2} + \cdots + c_N t_N^{-2} = 1$, where $c_j \in \{\pm 1\}$. For symmetry reasons we may assume that $c_1 t_1^{-2} + \cdots + c_{N-1} t_{N-1}^{-2} = \rho^{-2}$, where $\rho > 0$. Let $t_j' = t_j/\rho$, $\widetilde{\omega}$ and $\widetilde{\vartheta}$ be the same as in Lemma 5.6, and let $r_1 \in [1,\infty]$ be such that $1/r_1 + 1/p_N = 1 + 1/r$. Also set $\omega_0 = \widetilde{\omega}$ and $\vartheta_0 = \widetilde{\vartheta}$. Then $c_1(t_1')^{-2} + \cdots + c_{N-1}(t_{N-1}')^{-2} = 1$, $r_1 \ge 1$ since $p_N \le r$, and

$$1/p_1 + \cdots + 1/p_{N-1} = N - 2 + 1/r_1.$$

By the induction hypothesis and Lemma 5.6 (2) it follows that

$$b = a_{1,t_1'} * \cdots * a_{N-1,t_{N-1}'} = \rho^{d(2N-4)}(a_{1,t_1} * \cdots * a_{N-1,t_{N-1}})(\cdot/\rho)$$

makes sense as an element in $s_{r_1}^A(1/\omega_0,\vartheta_0)$, and

$$\|b\|_{s_{r_1}^A(1/\omega_0,\vartheta_0)} \le C \prod_{j=1}^{N-1} |t_j'|^{-2d/p_j} \|a\|_{s_{p_j}^A(1/\omega_j,\vartheta_j)},$$

for some constant C. Since $1/r_1 + 1/p_N = 1 + 1/r$, it follows from Lemma 5.6 (3) that $b_\rho * a_{N,t_N}$ makes sense as an element in $s_r^A(1/\omega,\vartheta)$, and

$$\begin{aligned}\|(a_{1,t_1} * \cdots * a_{N-1,t_{N-1}}) * a_{N,t_N}\|_{s_r^A(1/\omega,\vartheta)} &= \rho^{-d(2N-4)}\|b_\rho * a_{N,t_N}\|_{s_r^A(1/\omega,\vartheta)}\\ &\le C_1 \|a_1\|_{s_{p_1}^A(1/\omega_1,\vartheta_1)} \cdots \|a_N\|_{s_{p_N}^A(1/\omega_N,\vartheta_N)},\end{aligned}$$

where

$$C_1 = C\rho^{d(4-2N-2/r_1)}|t_N|^{-2d/p_N} \prod_{j=1}^{N-1} |t_j'|^{-2d/p_j} = C \prod_{j=1}^{N} |t_j|^{-2d/p_j}.$$

This proves the extension assertions. The uniqueness as well as the symmetry assertions follow from the facts that $\mathcal{S}$ is dense in s_p^A when $p < \infty$ and dense in s_∞^A with respect to the weak* topology, and that at most one p_j is equal to infinity due to the Young condition. The proof is complete. □

Proof of Proposition 5.2 The continuity assertions follow by combining Proposition 4.7, Proposition 5.1 and (5.1).

When verifying the positivity statement we may argue by induction as in the proof of Proposition 5.1. This together with Proposition 1.2 and some simple arguments of approximation shows that it suffices to prove that $a_s b_t$ is positive semi-definite when $\pm s^2 \pm t^2 = 1$, $st \neq 0$, and $a, b \in \mathcal{S}(\mathbf{R}^{2d}) \cap \mathcal{S}'_+(\mathbf{R}^{2d})$ are rank-one element.

We write

$$a_s b_t = \pi^{-d} \mathcal{F}_\sigma(\mathcal{F}_\sigma a_s * \mathcal{F}_\sigma b_t) = \pi^{-d} |st|^{-2d} \mathcal{F}_\sigma((\mathcal{F}_\sigma a)_{1/s} * (\mathcal{F}_\sigma b)_{1/t}).$$

If we set for any $U \in \mathcal{S}(\mathbf{R}^{2d})$,

$$U_{0,z}(x, y) = U_{1,z}(-y, -x) = U(x + z, y + z),$$

then it follows from Lemmas 1.1 and 5.4 that

$$A(a_s b_t)(x, y) = (2/\pi)^{d/2} |st|^{-d} \int (Aa)_{j,z/s}(sx, sy)(Ab)_{k,-z/t}(tx, ty)\, dz,$$

for some choice of $j, k \in \{0, 1\}$. Since $a, b \in S'_+$ are rank-one elements, it follows that the integrand is of the form $\phi_z(x) \otimes \overline{\phi_z(y)}$ in all these cases. This proves that $A(a_s b_t)$ is a positive semi-definite operator. □

Remark 5.7 Theorem 0.3′ can also be generalized to involve $s_{t,p}$ spaces, for general $t \in \mathbf{R}$.

In fact, assume that p_j, r, t_j, ω, ω_j, ϑ and ϑ_j for $1 \le j \le N$ are the same as in Theorems 0.3 and 0.4. Also assume that $t \in \mathbf{R}$, and let $\tau_k = t$ when $j_k = 0$ and $\tau_k = 1 - t$ when $j_k = 1$ (the numbers j_k are the same as in (0.10)′).

Then the mapping $(a_1, \dots, a_N) \mapsto a_{1,t_1} * \cdots a_{N,t_N}$ on $\mathcal{S}(\mathbf{R}^{2d})$, extends uniquely to a continuous mapping from

$$s_{\tau_1,p_1}(1/\omega_1, \vartheta_1) \times \cdots \times s_{\tau_N,p_N}(1/\omega_N, \vartheta_N)$$

to $s_{t,r}(1/\omega, \vartheta)$. Furthermore it holds

$$\|a_{1,t_1} * \cdots * a_{N,t_N}\|_{s_r^A(1/\omega,\vartheta)} \le C^d \|a_1\|_{s_{\tau_1,p_1}(1/\omega_1,\vartheta_1)} \cdots \|a_N\|_{s_{\tau_N,p_N}(1/\omega_N,\vartheta_N)}. \tag{5.13}$$

where $C = C_0^N |t_1|^{-2a/p_1} \cdots |t_N|^{-2/p_N}$ for some constant C_0 which is independent of $N, t_1, \dots, t_N$ and d.

Moreover, $\mathrm{Op}_t(a_{1,t_1} * \cdots * a_{N,t_N}) \ge 0$ when $\mathrm{Op}_{\tau_j}(a_j) \ge 0$ for each $1 \le j \le N$.

When proving this we first assume that $a_1, \dots, a_N \in \mathcal{S}$. By Proposition 4.7 we get

$$\begin{aligned}\|a_{1,t_1} * \cdots * a_{N,t_N}\|_{s_{t,r}(1/\omega,\vartheta)} &= \|e^{-i(t-1/2)\langle D_x,D_\xi\rangle}(a_{1,t_1} * \cdots * a_{N,t_N})\|_{s^w_r(1/\omega,\vartheta)}\\ &= \|b_1 * \cdots * b_N\|_{s^w_r(1/\omega,\vartheta)},\end{aligned}$$

where

$$b_k = e^{-i(-1)^{j_k}(t-1/2)\langle D_x,D_\xi\rangle/t_k^2}(a_k(t_k\,\cdot)) = (e^{-i(-1)^{j_k}(t-1/2)\langle D_x,D_\xi\rangle}a_k)(t_k\,\cdot).$$

Hence by Theorem 0.3′ we get

$$\|a_{1,t_1} * \cdots * a_{N,t_N}\|_{s_{t,r}(1/\omega,\vartheta)} \le C I_1 \cdots I_N,$$

where

$$I_k = \|e^{-i(-1)^{j_k}(t-1/2)\langle D_x,D_\xi\rangle}a_k\|_{s^w_{p_k}(1/\omega_k,\vartheta_k)} = \|a_k\|_{s_{\tau_k,p_k}(1/\omega_k,\vartheta_k)}.$$

This gives (5.13).

The result now follows from (5.13) and the fact that $\mathcal{S}$ is dense in $s_{t,p}(\omega_1,\omega_2)$ when $p<\infty$, and dense in $s_{t,\infty}(\omega_1,\omega_2)$ with respect to the weak* topology.

6 Some applications and further extensions

In this section we apply the results in previous section. We use Proposition 5.2 to prove that if v is submultiplicativ, then $s_1^A(1/v,v)$ is stable under composition with odd entire analytic functions. Thereafter we use Theorem 0.3 to extend the definition of Toeplitz operators to include appropriate dilations of s_p^w as permitted Toeplitz symbols.

We start by considering compositions of elements in $s_1^A(1/v,v)$ with analytic functions. In these considerations we restrict ourself to the case when $v=\check v \in \mathcal{P}(\mathbf{R}^{2d})$ is submultiplicative. We note that each element in $s_1^A(1/v,v)$ is a continuous function which turns to zero at infinity, since (5.6) shows that $s_1^A(1/v,v) \subseteq C_B(\mathbf{R}^{2d})$.

Since our investigation also involve positivity, we recall from [38] that $a \in C(\mathbf{R}^{2d}) \cap \mathcal{S}'(\mathbf{R}^{2d})$ is called *σ-positive*, if $Aa \ge 0$. We let $C_+(\mathbf{R}^{2d})$ be the set of all σ-positive functions, i.e. $C_+(\mathbf{R}^{2d}) = C(\mathbf{R}^{2d}) \cap S'_+(\mathbf{R}^{2d})$.

It follows that any product of odd numbers of elements in $s_1^A(1/v,v)$ are again in $s_1^A(1/v,v)$. In fact, assume that $a_1,\dots,a_N \in s_1^A(1/v,v)$, $|\alpha|$ is odd, and that $t_j=1$. Then it follows from Theorem 5.2 that $a_1^{\alpha_1}\dots a_N^{\alpha_N} \in s_1^A(1/v,v)$, and

$$\|a_1^{\alpha_1}\cdots a_N^{\alpha_N}\|_{s_1^A(1/v,v)} \le C_0^{d|\alpha|}\prod \|a_j\|^{\alpha_j}_{s_1^A(1/v,v)}, \tag{6.1}$$

for some constant C_0 which is independent of α and d.

Furthermore, if in addition $a_1,\dots,a_N$ are σ-positive, then the same is true for $a_1^{\alpha_1}\dots a_N^{\alpha_N}$. The following result is an immediate consequence of these observations.

Proposition 6.1 *Assume that* $a_1, \dots, a_N \in s_1^A(1/v, v)$*, where* $v = \check{v} \in \mathcal{P}(\mathbf{R}^{2d})$ *is submultiplicative,* C_0 *is the same as in* (6.1)*, and assume that* $R_1, \dots, R_N > 0$*. Also assume that* f, g *are odd analytic functions from the polydisc*

$$\{\, z \in \mathbf{C}^N \,;\ |z_j| < C_0 R_j \,\}$$

to $\mathbf{C}$*, with expansions*

$$f(z) = \sum_\alpha c_\alpha z^\alpha \quad \text{and} \quad g(z) = \sum_\alpha |c_\alpha| z^\alpha .$$

Then $f(a) = f(a_1, \dots, a_N)$ *is well-defined and belongs to* $s_1^A(1/v, v)$*. One has the estimate*

$$\| f(a) \|_{s_1^A(1/v,v)} \le g(C_0 \|a_1\|_{s_1^A(1/v,v)}, \dots, C_0 \|a_N\|_{s_1^A(1/v,v)}).$$

If in addition $a_1, \dots, a_N \in C_+(\mathbf{R}^{2d})$*, then* $g(a) \in C_+(\mathbf{R}^{2d})$*.*

An open question for the author is whether the results on dilated convolutions and multiplications in the present section are true for other dilations. This might then lead to improvements of Proposition 6.1. In this context we note that $s_1^w(\mathbf{R}^{2d})$, and therefore $s_\infty^w(\mathbf{R}^{2d})$ by duality, are not stable under dilations (see Proposition 2.1.12 in [36] or Proposition 5.4 in [39]).

For rank one elements we also have the following generalization of [38, Proposition 4.10].

Proposition 6.2 *Assume that* $v, v_1 \in \mathcal{P}(\mathbf{R}^{2d})$ *are even, submultiplicative and fulfill* $v_1 = v(\,\cdot\,/\sqrt{2}\,)$*. Also assume that* $u \in s_\infty^w(1/\omega, \omega)$ *is an element of rank one, and let* $a(X) = |u(X/\sqrt{2})|^2$*. Then* $a \in s_1^w(1/v_1, v_1)$*, and* $\operatorname{Op}^w(a) \ge 0$*.*

Proof Since u is rank one, it follows from Proposition 4.7 that $u, \overline{u} \in s_1^w(1/v, v)$, which implies that $a \in s_1^w(1/v_1, v_1)$ in view of Theorem 0.3. The result now follows from this fact and Proposition 4.10 in [38]. □

We finish the section by applying our results on Toeplitz operators. The following result, parallel to Theorems 3.1 and 3.5 in [45], generalizes [40, Proposition 4.5].

Theorem 6.3 *Assume that* $p \in [1, \infty]$ *and* $\omega, \omega_0, \vartheta, \vartheta_j \in \mathcal{P}(\mathbf{R}^{2d})$ *for* $j = 0, 1, 2$ *satisfy*

$$\omega(X_1 - X_2) \le C \omega_0(\sqrt{2}\, X_1) \vartheta_2(X_2), \quad \text{and}$$
$$\vartheta(X_1 - X_2) \le C \vartheta_0(\sqrt{2}\, X_1) \vartheta_1(X_2)$$

Then the definition of $\operatorname{Tp}_{h_1,h_2}(a)$ *extends uniquely to each* $a \in \mathcal{S}'(\mathbf{R}^{2d})$ *and* $h_j \in M^2_{(\vartheta_j)}$ *for* $j = 1, 2$ *such that* $a(\sqrt{2}\,\cdot\,) \in s_p^w(1/\omega_0, \vartheta_0)$*, and for some constant* C *it holds*

$$\| \operatorname{Tp}_{h_1,h_2}(a) \|_{\mathcal{I}_p(M^2_{(1/\omega)}, M^2_{(\vartheta)})} \le C \| a(\sqrt{2}\,\cdot\,) \|_{s_p^w(1/\omega_0, \vartheta_0)} \|h_1\|_{M^2_{(\vartheta_1)}} \|h_2\|_{M^2_{(\vartheta_2)}} .$$

Furthermore, if $h_1 = h_2$ *and* $\operatorname{Op}^w(b) \geq 0$*, where* $b = a(\sqrt{2}\,\cdot)$*, then* $\operatorname{Tp}_{h_1,h_2}(a) \geq 0$.

Proof Since $W_{h_2,h_1} \in s_1^w(1/\vartheta_1, \vartheta_2)$, the result is an immediate consequence of (1.14) and Theorem 0.3. □

We finish the section by presenting some possibilities of further extensions to the case when the symbols belong to appropriate classes of ultra-distributions. In fact, it seems that the analysis in Sect. 4 works also for a larger family of Hilbert spaces, where $\mathcal{S}$ and $\mathcal{S}'$ in Definition 4.1 are replaced by appropriate classes of Gelfand–Shilov spaces and their duals. Furthermore, the conditions on the weights in the definition of weighted Lebesgue and modulation spaces in Sects. 2–5 can be relaxed in such way that it is only assumed that v in (1.15) should be subexponential. Such modulation spaces are then not necessary contained in $\mathcal{D}'$. We refer to [18,34] for an introduction to such spaces and we refer to [34,35] and the references therein for an introduction to pseudo-differential operators in context of modulation spaces with subexponentially moderated weights and Gelfand–Shilov spaces.

References

1. Bergh, J., Löfström, J.: Interpolation Spaces, An Introduction. Springer, Berlin (1976)
2. Birman, M.S., Solomyak, M.Z.: Estimates for the singular numbers of integral operators (Russian). Usbehi Mat. Nauk. **32**, 17–84 (1977)
3. Bony, J.M., Chemin, J.Y.: Espaces functionnels associés au calcul de Weyl-Hörmander. Bull. Soc. Math. France **122**, 77–118 (1994)
4. Boulkhemair, A.: L^2 estimates for pseudodifferential operators. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **22**(4), 155–183 (1995)
5. Boulkhemair, A.: L^2 estimates for Weyl quantization. J. Funct. Anal. **165**, 173–204 (1999)
6. Cordero, E., Gröchenig, K.: Time–frequency analysis of localization operators. J. Funct. Anal. **205**(1), 107–131 (2003)
7. Cordero, E., Gröchenig, K.: Symbolic calculus and Fredholm property for localization operators. J. Fourier Anal. Appl. **12**, 371–392 (2006)
8. Cordero, E., Okoudjou, K.: Multilinear localization operators. J. Math. Anal. Appl. **325**, 1103–1116 (2007)
9. Feichtinger, H.G.: Banach spaces of distributions of Wiener's type and interpolation. In: Butzer, P., Nagy, B.Sz., Görlich, E. (eds.) Proceedings of Conference Oberwolfach, Functional Analysis and Approximation, August 1980, vol. 69, pp. 153–165. Int. Ser. Num. Math. Birkhäuser Verlag, Basel (1981)
10. Feichtinger, H.G.: Banach convolution algebras of Wiener's type. In: Proceedings of Functions, Series, Operators in Budapest, Colloquia Math. Soc. J. Bolyai. North Holland Publication Co., Amsterdam (1980)
11. Feichtinger, H.G.: Modulation spaces on locally compact abelian groups. In: Krishna, M., Radha, R., Thangavelu, S. (eds.) Wavelets and their Applications, Technical Report. University of Vienna, Vienna, 1983, pp. 99–140. Allied Publishers Private Limited, New Dehli (2003)
12. Feichtinger, H.G.: Atomic characterizations of modulation spaces through Gabor-type representations. In: Proceedings of Conference on Constructive Function Theory, Rocky Mountain J. Math., vol. 19, pp. 113–126 (1989)
13. Feichtinger, H.G.: Modulation spaces: looking back and ahead. Sampl. Theory Signal Image Process. **5**, 109–140 (2006)
14. Feichtinger, H.G., Gröchenig, K.H.: Banach spaces related to integrable group representations and their atomic decompositions, I. J. Funct. Anal. **86**, 307–340 (1989)
15. Feichtinger, H.G., Gröchenig, K.H.: Banach spaces related to integrable group representations and their atomic decompositions, II. Monatsh. Math. **108**, 129–148 (1989)
16. Feichtinger, H.G., Gröchenig, K.H.: Gabor frames and time-frequency analysis of distributions. J. Functional Anal. **146**(2), 464–495 (1997)

17. Folland, G.B.: Harmonic analysis in phase space. Princeton University Press, Princeton (1989)
18. Gröchenig, K.H.: Foundations of Time–Frequency Analysis. Birkhäuser, Boston (2001)
19. Gröchenig, K.H.: Composition and spectral invariance of pseudodifferential operators on modulation spaces. J. Anal. Math. **98**, 65–82 (2006)
20. Gröchenig, K.H., Heil, C.: Modulation spaces and pseudo-differential operators. Integr. Equ. Operator Theory **34**(4), 439–457 (1999)
21. Gröchenig, K., Toft, J.: Localization operator representation of modulation spaces (preprint, 2009). arXiv:0905.4954
22. He, Z., Wong, M.W.: Localization operators associated to square integrable group representations. Panamer. Math. J. **6**(1), 93–104 (1996)
23. Holst, A., Toft, J., Wahlberg, P.: Weyl product algebras and modulation spaces. J. Funct. Anal. **251**, 463–491 (2007)
24. Hörmander, L.: The Analysis of Linear Partial Differential Operators, vol. I, III. Springer, Berlin (1983, 1985)
25. Labate, D.: Pseudodifferential operators on modulation spaces. J. Math. Anal. Appl. **262**, 242–255 (2001)
26. Lieb, E.H.: Integral bounds for radar ambiguity functions and Wigner distributions. J. Math. Phys. **31**, 594–599 (1990)
27. Muramatu, T.: Estimates for the norm of pseudodifferential operators by means of Besov spaces. In: Pseudodifferential Operators, Proceedings of the Conference Held in Oberwolfach, 2–8 February 1986, Lecture Notes in Mathematics, vol. 1256. Springer, Berlin (1987)
28. Schatten, R.: Norm ideals of completely continuous operators. Springer, Berlin (1960)
29. Schulze, B.W., Tarkhanov, N.N.: Pseudodifferential operators with operator-valued symbols. Israel Math. Conf. Proc. **16** (2003)
30. Shubin, M.A.: Pseudodifferential Operators and Spectral Theory. Springer, Berlin (1987)
31. Simon, B.: Trace ideals and their applications I, Lecture Note Series. London Mathematical Society, Cambridge University Press, Cambridge (1979)
32. Sjöstrand, J.: An algebra of pseudodifferential operators. Math. Res. L. **1**, 185–192 (1994)
33. Sugimoto, M.: L^p boundedness of pseudodifferential operators satisfying Besov estimates I. J. Math. Soc. Jpn. **40**, 105–122 (1988)
34. Teofanov, N.: Ultramodulation spaces and pseudodifferential operators. Endowment Andrejević, Beograd (2003)
35. Teofanov, N.: Modulation spaces, Gelfand–Shilov spaces and pseudodifferential operators. Sampl. Theory Signal Image Process **5**, 225–242 (2006)
36. Toft, J.: Continuity and Positivity Problems in Pseudo-Differential Calculus, Thesis. Department of Mathematics, University of Lund, Lund (1996)
37. Toft, J.: Subalgebras to a Wiener type algebra of pseudo-differential operators. Ann. Inst. Fourier **51**(5), 1347–1383 (2001)
38. Toft, J.: Continuity properties for non-commutative convolution algebras with applications in pseudo-differential calculus. Bull. Sci. Math. **126**(2), 115–142 (2002)
39. Toft, J.: Positivity properties for non-commutative convolution algebras with applications in pseudo-differential calculus. Bull. Sci. Math. **127**(2), 101–132 (2003)
40. Toft, J.: Continuity properties for modulation spaces with applications to pseudo-differential calculus, I. J. Funct. Anal. **207**(2), 399–429 (2004)
41. Toft, J.: Continuity properties for modulation spaces with applications to pseudo-differential calculus, II. Ann. Global Anal. Geom. **26**, 73–106 (2004)
42. Toft, J.: Convolution and embeddings for weighted modulation spaces. In: Boggiatto, P., Ashino, R., Wong, M.W., (eds.) Advances in Pseudo-Differential Operators, Operator Theory: Advances and Applications, vol. 155, pp. 165–186. Birkhäuser Verlag, Basel (2004)
43. Toft, J.: Continuity and Schatten properties for pseudo-differential operators on modulation spaces. In: Toft, J. Wong, M.W., Zhu, H. (eds.) Modern Trends in Pseudo-Differential Operators, Operator Theory: Advances and Applications, pp. 173–206. Birkhäuser Verlag, Basel (2007)
44. Toft, J.: Continuity and Schatten properties for Toeplitz operators on modulation spaces. In: Toft, J., Wong, M.W., Zhu, H. (eds). Modern Trends in Pseudo-Differential Operators, Operator Theory: Advances and Applications, pp. 313–328. Birkhäuser Verlag, Basel (2007)
45. Toft, J., Boggiatto, P.: Schatten classes for Toeplitz operators with Hilbert space windows on modulation spaces. Adv. Math. **217**, 305–333 (2008)