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On the Dual *p*-Measures of Asymmetry for Star Bodies

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Abstract: Recently, the connection between *p*-measures of asymmetry and the L_p-mixed volumes for convex bodies was found soon after the *p*-measure of asymmetry was proposed, and the Orlicz-measures of asymmetry was proposed inspired by such a kind of connection. In this paper, by a similar way the dual *p*-measures of asymmetry for star bodies (naturally for convex bodies) is introduced first. Then the connection between dual *p*-measures of asymmetry and L_p-dual mixed volumes is established. Finally, the best lower and upper bounds of dual *p*-measures and the corresponding extremal bodies are discussed.

Key words: convex body; dual *p*-measures of asymmetry; L_p-dual mixed volumes
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0 Introduction

As one of the most important geometric invariants, the measures of asymmetry (or symmetry) for convex bodies (i.e., compact convex sets in the Euclidean n-space with nonempty interior) initiated from the early Minkowski's work[1] and formulated in the well-known paper[2] have stayed stably as a quite popular topic in convex geometry. Various kinds of measures of asymmetry or their extensions have been proposed and studied (see Refs.[3-9] and the references therein). Recently some new measures of asymmetry were discovered [10-19].

Among these measures of asymmetry proposed recently, the *p*-measures of asymmetry defined by the third named author in Ref.[10], is of some significance in the sense. The method used in Ref.[10] to construct the *p*-measures of asymmetry provides a new way to construct new geometric invariants (see Ref.[9]), e.g., soon after Ref.[10], Jin, Leng and Guo in Ref.[13] revealed a connection between the *p*-measures of asymmetry and the L_p-mixed volumes and further established the so-called Orlicz-measures of asymmetry.

In this article, we introduce the dual *p*-measure of asymmetry for star bodies (and naturally for convex bodies), which is a dual concept of the *p*-measure of asymmetry in some sense, and discuss its properties.

$\mathbf{R}^n$ denotes the usual *n*-dimensional Euclidean space with the canonical inner product $\langle\cdot,\cdot\rangle$. The family of all convex bodies is denoted by K^n. For a convex set C, $\operatorname{int} C$, $\operatorname{ri} C$ denote the interior and relative interior of C, respectively. We refer to Ref. [20] for general notations.

For $C \in K^n$, its support function $h(\cdot) = h(C,\cdot)$: $S^{n-1} \to [0,+\infty)$ is defined as $h(C,u) = \max_{y\in C}\langle x,u\rangle$, $u \in S^{n-1}$, where S^{n-1} is the $(n-1)$-dimensional unit sphere. More generally, for $x \in \mathbf{R}^n$, we denote also

$$h_x(C,u) = \max_{y\in C}\langle y-x,u\rangle,\ u \in S^{n-1},$$

called the support function of C based on x. Clearly, $h_0(C,\cdot) = h(C,\cdot)$ and it is easy to see that

$$h_x(C,\cdot) = h(C,\cdot) - \langle x,\cdot\rangle$$

and

$$h_x(C,\cdot) = h(C_x,\cdot)$$

where $C_x := C - x$.

A subset $C \subset \mathbf{R}^n$ is called a star-shaped set if there is $x \in C$ such that $\lambda x + (1-\lambda)y \in C$ for all $y \in C$ and $0 \leqslant \lambda \leqslant 1$. For $C \subset \mathbf{R}^n$, we denote

$$\mathrm{cor}C := \{x \in C \mid \lambda x + (1-\lambda)y \in C\}$$

for all $y \in C$ and $0 \leqslant \lambda \leqslant 1$, called the core of C. Clearly C is star-shaped if and only if $\mathrm{cor}(C) \neq \varnothing$. We say C is a star-shaped about x whenever $x \in \mathrm{cor}(C)$. Observe that $\mathrm{cor}(C) = C$ for $C \in K^n$ and more generally, $\mathrm{cor}(C)$ is closed and convex for closed star-shaped set C.

Given a compact star-shaped set $C \subset \mathbf{R}^n$ and $x \in \mathrm{cor}(C)$, we define its radial function $\rho_x(\cdot) = \rho_x(C,\cdot)$, $S^{n-1} \to [0,+\infty)$, with respect to (w.r.t. for brevity) x by

$$\rho_x(u) = \rho_x(c,u) = \max\{\lambda \geqslant 0 \mid \lambda u + x \in C\}, u \in S^{n-1}$$

when $x = 0$, we often write ρ simply instead of ρ_0. It is easy to see that $\rho(C_x,u) = \rho_x(C,u)$ and $\rho(C_x,u) = \rho(C_x,-u) = \rho_x(C,-u)$, $u \in S^{n-1}$, for every compact star-shaped set C and $x \in \mathrm{cor}(C)$.

A compact star-shaped set K is called a star body about x if $x \in \mathrm{cor}K$ and $\rho_x(K,\cdot)$ is positive and continuous. Observe that if K is a star body about x, then $x \in \mathrm{int}K$. Denote φ^n the set of star bodies and φ_0^n the set of star bodies about 0.

Clearly, a convex set (respectively convex body) is a star-shaped set (respectively star body). In this article, we write customarily C, D for convex bodies or star bodies and K, L for star bodies about 0.

The Hausdorff metric $d_H(C,D)$ between $C, D \in K^n$ is defined as

$$d_H(C,D) := \max_{u\in S^{n-1}} |h(C,u) - h(D,u)|$$

By $C_k \to C$ we mean $d_H(C_k,C) \to 0$ as $k \to +\infty$.

1 *p*-Measures of Asymmetry and *Lp*-Mixed Volumes

In this section, we introduce some properties of the *p*-measures of asymmetry[10] and L_p-mixed volumes [21] for convex bodies.

Given $C \in K^n$, for a fixed $x \in \mathrm{int}(C)$, we define $m_x(C,\cdot)$, a probability measure on S^{n-1}, by

$$m_x(C,\omega) := \frac{\int_\omega h_x(C,u)\mathrm{d}S_{n-1}(C,u)}{nV_n(C)}$$

for any measurable $\omega \subset S^{n-1}$, where $S_{n-1}(C,\cdot)$ denotes the surface area measure of C on S^{n-1} and $V_n(C)$ denotes the n-dimensional volume of C. Then we write

$$\mu_p(C,x) := \begin{cases} \left(\int_{S^{n-1}} \alpha_x(C,u)^p \mathrm{d}m_x(C,u)\right)^{\frac{1}{p}}, & \text{if } 1 \leqslant p < +\infty \\ \sup_{u\in S^{n-1}} \alpha_x(C,u), & \text{if } p = +\infty \end{cases}$$

where $\alpha_x(C,u) = \dfrac{h(C_x,-u)}{h(C_x,u)}$.

Then, the *p*-measure of asymmetry $\mathrm{as}_p(C)$ is defined by

$$\mathrm{as}_p(C) := \inf_{x\in \mathrm{int}(C)} \mu_p(C,x)$$

A point $x \in \mathrm{int}(C)$ satisfying $\mu_p(C,x) = \mathrm{as}_p(C)$ is called a *p*-critical point of C.

Remark 1 1) It is easy to see that as_∞ is just the well-known Minkowski measure of asymmetry.

2) The following statements are confirmed in Ref.[10].

i) for $C \in K^n$ and $1 \leqslant p \leqslant +\infty$, $1 \leqslant \mathrm{as}_p(C) \leqslant n$, and $\mathrm{as}_p(C) = 1$ if and only if C is (centrally) symmetric; $\mathrm{as}_p(C) = n$ if and only if C is a simplex.

ii) for any $C \in K^n$ and $1 \leqslant p < +\infty$, its *p*-critical point is unique; the set of ∞-critical points is a non-empty closed convex set [2] while the set of 1-critical points is exactly $\mathrm{int}(C)$.

In Ref.[13], the authors observed a nice connection between the *p*-measures of asymmetry and the L_p-mixed volumes (see Ref.[21] for definitions): for $1 \leqslant p \leqslant +\infty$,

$$\mu_p^p(C,x) = \frac{V_p(C_x,-C_x)}{V_n(C)}$$

and in turn

$$\mathrm{as}_p(C) = \inf_{x\in \mathrm{int}C}\left(\frac{V_p(C_x,-C_x)}{V_n(C)}\right)^{\frac{1}{p}}$$

$V_p(\cdot,\cdot)$ denotes the L_p-mixed volume, and they thereby introduced the so-called Orlicz-measures of asymmetry for convex bodies which have similar properties to those of *p*-measures.

2 Dual *p*-measures of Asymmetry and L_p-dual Mixed Volumes

In this section, we define the dual *p*-measures of asymmetry for star bodies (automatically for convex bodies), starting with some necessary definitions and notations.

Definition 1[21] For $p \geqslant 1$ and $K, L \in \varphi_0^n$, the L_p-dual mixed volume $\tilde{V}_{-p}(K,L)$ is defined by

$$\tilde{V}_{-p}(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n+p}\rho(L,u)^{-p}\,\mathrm{d}S(u)$$

where $S(\cdot)$ denotes the spherical Lebesgue measure on S^{n-1}, the unit sphere of $\mathbf{R}^n$.

Especially, $\tilde{V}_{-p}(K,K) = \int_{S^{n-1}} \rho(K,u)^n \mathrm{d}S(u) = V_n(K)$ for $K \in \varphi_0^n$.

If $p \geqslant 1$ and $K, L \in \varphi_0^n$, it is easy to check that, for any $T_0 \in \mathrm{GL}(n)$, the general linear group,

$$\tilde{V}_{-p}(T_0K, T_0L) = |\det T_0|\,\tilde{V}_{-p}(K,L)$$

where "det" denotes the determinant. It is shown in Ref. [20] that the L_p-Minkowski dual mixed volumes inequality holds: for $K, L \in \varphi_0^n$ and $p \geqslant 1$,

$$\tilde{V}_{-p}(K,L)^n \geqslant V_n(K)^{n+p}V_n(L)^{-p} \tag{1}$$

and

$\tilde{V}_{-p}(K,L)^n = V_n(K)^{n+p}V_n(L)^{-p}$ if and only if $K = \lambda L$ for some $\lambda > 0$.

Now, we introduce the dual *p*-measures of asymmetry for star bodies (automatically for convex bodies). First, given $C \in \varphi^n$, $x \in \mathrm{ri}(\mathrm{cor}C)$, we define a probability measure $\tilde{m}_x(C,\cdot)$ on S^{n-1} by

$$\mathrm{d}\tilde{m}_x(C,\omega) = \frac{\rho_x(C,u)^n \mathrm{d}S(u)}{nV_n(C)}$$

for any measurable $\omega \subset S^{n-1}$.

Then we write

$$\mu_p^\circ(C,x) = \begin{cases} \left(\int_{S^{n-1}} \tilde{\alpha}_x(C,u)^p \mathrm{d}m_x(C,u)\right)^{\frac{1}{p}}, & \text{if } 1 \leqslant p < +\infty \\ \sup\limits_{u\in S^{n-1}} \alpha_x(C,u), & \text{if } p = +\infty \end{cases}$$

where $\tilde{\alpha}_x = \dfrac{\rho_x(C,u)}{\rho_x(C,-u)}$.

Definition 2 For $C \in \varphi^n$, $1 \leqslant p \leqslant +\infty$, we define dual *p*-measure of asymmetry $\mathrm{as}_p^\circ(C)$ of *C* by

$$\mathrm{as}_p^\circ(C) := \inf_{x\in \mathrm{ri}(\mathrm{cor}(C))} \mu_p^\circ(C,x)$$

A point $x^* \in \mathrm{ri}(\mathrm{cor}C)$ satisfying $\mu_p^\circ(C,x^*) = \mathrm{as}_p^\circ(C)$ is called a p°-critical point of *C*.

Remark 2 Clearly, $\mathrm{as}_\infty^\circ(C) = \mathrm{as}_\infty(C)$ for $C \in K^n$ (cf. Ref. [4] and the references therein). By the continuity of $\mu_p^\circ(C,x)$ w.r.t. *x* and $\lim\limits_{x\to \mathrm{bd}(C)} \mu_p^\circ(C,x) = +\infty$, we see that p°-critical points exist, i.e. $\mathrm{as}_p^\circ(C)$ is attainable.

As an analogue of the volume-normalized version of L_p-mixed volumes introduced by Lutwak, Yang and Zhang in Ref.[22], we introduce the following volume-normalized version $\bar{V}_{-p}(\cdot,\cdot)$ of L_p-dual mixed volumes.

Given $K, L \in \varphi_0^n$, we denote, for each $p \geqslant 1$,

$$\bar{V}_{-p}(K,L) = \begin{cases} \left(\dfrac{\tilde{V}_{-p}(K,L)}{V_n(K)}\right)^{\frac{1}{p}}, & \text{if } 1 \leqslant p < +\infty \\ \max\left\{\dfrac{\rho(K,u)}{\rho(L,u)} \,\middle|\, u \in S^{n-1}\right\}, & \text{if } p = +\infty \end{cases} \tag{2}$$

Here, we mention without proofs some properties of $\bar{V}_{-p}(\cdot,\cdot)$, all of which can be checked by the definition and/or Jensen's inequality:

$$\bar{V}_{-p}(T_0K, T_0L) = \bar{V}_{-p}(K,L) \tag{3}$$

for all $T_0 \in \mathrm{GL}(n)$, $p \in [1,+\infty]$;

$\bar{V}_{-p}(K,L) < \bar{V}_{-q}(K,L)$, $1 \leqslant p < q \leqslant +\infty$, unless $\dfrac{\rho_K}{\rho_L}$ is a constant on S^{n-1}; And

$$\lim_{p\to p_0} \bar{V}_{-p}(K,L) = \bar{V}_{-p_0}(K,L),\ p_0 \in [1,+\infty]$$

We are now in the position to show the connection between dual *p*-measures and L_p-dual mixed volumes.

Theorem 1 For $C \in \varphi^n$, $x \in \mathrm{ri}(\mathrm{cor}C)$ and $1 \leqslant p \leqslant +\infty$, we have

$$\mu_p^\circ(C,x) = \bar{V}_{-p}(C_x, -C_x)$$

and in turn $\mathrm{as}_p^\circ(C) = \inf\limits_{x\in \mathrm{ri}(\mathrm{cor}C)} \bar{V}_{-p}(C_x, -C_x)$.

Proof For $1 \leqslant p < +\infty$, by the definitions of $\mu_p^\circ(C,x)$ and $\tilde{V}_{-p}(C,D)$, we have

$$\mu_p^\circ(C,x)^p=\int_{S^{n-1}}\tilde{\alpha}_x(C,u)^p\mathrm{d}\tilde{m}_x(C,u)$$
$$=\int_{S^{n-1}}\frac{\rho_x(C,u)^p}{\rho_x(C,-u)^p}\frac{\rho_x(C,u)^n\mathrm{d}S(u)}{nV_n(C)}$$
$$=\frac{1}{nV_n(C)}\int_{S^{n-1}}\rho_x(C,u)^{n+p}\rho_x(C,-u)^{-p}\mathrm{d}S(u)$$
$$=\frac{1}{nV_n(C)}\int_{S^{n-1}}\rho(C_x,u)^{n+p}\rho(-C_x,u)^{-p}\mathrm{d}S(u)$$
$$=\frac{\tilde{V}_{-p}(C_x,-C_x)}{V_n(C)}=(\bar{V}_{-p}(C_x,-C_x))^p$$

Observing $V_n(C)=V_n(C_x)$ and in turn

$$\mathrm{as}_p^\circ(C)=\inf_{x\in\mathrm{ri}(\mathrm{cor}C)}\bar{V}_{-p}(C_x,-C_x)$$

$\mathrm{as}_p^\circ(C)=\inf_{x\in\mathrm{ri}(\mathrm{cor}C)}\bar{V}_{-\infty}(C_x,-C_x)$ follows directly from the definitions of $\mathrm{as}_\infty^\circ(\cdot)$ and $\bar{V}_{-p}(\cdot,\cdot)$.

The following theorem ensures that the dual p-measures of asymmetry are indeed measures of asymmetry.

Theorem 2 For any $1\leqslant p,q\leqslant+\infty$ and $C\in\varphi^n$, the following statements are true:

i) $\mathrm{as}_p^\circ(\cdot)$ is continuous w.r.t. the Hausdorff metric and affine invariant.

ii) $\mathrm{as}_p^\circ(C)\leqslant\mathrm{as}_q^\circ(C)$ for any $C\in\varphi^n$ and $1\leqslant p\leqslant q\leqslant+\infty$.

iii) $1\leqslant\mathrm{as}_p^\circ(C)\leqslant n$ and $\mathrm{as}_p^\circ(C)=1$ if and only if C is (centrally) symmetric.

Proof i) The continuity of as_p° follows from the continuity of L_p-dual mixed volumes (noticing Theorem 1). We show only the affine invariance of as_p°. Let T be an invertible affine transform on $\mathbf{R}^n$, then $T=T_1\circ T_0$, where $T_0\in\mathrm{GL}(n)$ and T_1 is a translation.

For $T_0\in\mathrm{GL}(n)$, since

$$(T_0C)_{T_0y}=T_0C-T_0y=T_0(C-y)=T_0C_y,$$
$$\mathrm{cor}(T_0C)=T_0(\mathrm{cor}C)$$

and so $\mathrm{ri}(\mathrm{cor}(T_0C))=T_0(\mathrm{ri}(\mathrm{cor}C))$, we have by Theorem 1 (writing $x=T_0(y)$)

$$\mathrm{as}_p^\circ(T_0C)=\inf_{x\in\mathrm{ri}(\mathrm{cor}(T_0C))}\bar{V}_{-p}((T_0C)_x,-(T_0C)_x)$$
$$=\inf_{y\in\mathrm{ri}(\mathrm{cor}C)}\bar{V}_{-p}((T_0C)_{T_0y},-(T_1C)_{T_0y})$$
$$=\inf_{y\in\mathrm{ri}(\mathrm{cor}C)}\bar{V}_{-p}(T_0(C_y),-T_0(-C_y))$$
$$=\inf_{y\in\mathrm{ri}(\mathrm{cor}C)}\bar{V}_{-p}(C_y,-C_y)$$
$$=\mathrm{as}_p^\circ(C)$$

For the translation T_1, since $(T_1C)_{T_1y}=C_y$, we have by Theorem 1 (writing $x=T_1(y)$)

$$\mathrm{as}_p^\circ(T_1C)=\inf_{x\in\mathrm{ri}(\mathrm{cor}(T_1C))}\bar{V}_{-p}((T_1C)_x,-(T_1C)_x)$$
$$=\inf_{y\in\mathrm{ri}(\mathrm{cor}C)}\bar{V}_{-p}((T_1C)_{T_1y},-(T_1C)_{T_1y})$$
$$=\inf_{y\in\mathrm{ri}(\mathrm{cor}C)}\bar{V}_{-p}(C_y,-C_y)$$
$$=\mathrm{as}_p^\circ(C)$$

Hence, $\mathrm{as}_p^\circ(TC)=\mathrm{as}_p^\circ(C)$.

ii) It follows from Theorem 1 and (3) .

iii) First, by ii) and 2) in Remark 1, we have

$$\mathrm{as}_p^\circ(C)\leqslant\mathrm{as}_\infty^\circ(C)=\mathrm{as}_\infty(C)\leqslant n$$

Then, we show the other conclusions hold for as_1°. Since $\mathrm{as}_1^\circ(C)=\inf_{x\in\mathrm{ri}(\mathrm{cor}C)}\frac{\tilde{V}_{-1}(C_x,-C_x)}{V_n(C)}$ and the volume is translation-invariant, by (1) we have

$$\tilde{V}_{-1}(C_x,-C_x)\geqslant V_n(C_x)^{\frac{n+1}{n}}V_n(-C_x)^{-\frac{1}{n}}$$
$$=V_n(C),x\in\mathrm{ri}(\mathrm{cor}C)$$

which implies $\mathrm{as}_1^\circ(C)\geqslant1$.

Now

$\mathrm{as}_1^\circ(C)=1\Leftrightarrow\mu_1^\circ(C,x^*)=1$ for some $x^*\in\mathrm{ri}$ $(\mathrm{cor}C)$ (by Remark 2)

$\Leftrightarrow\bar{V}_{-1}(C_{x^*},-C_{x^*})=1$ for some $x^*\in\mathrm{ri}(\mathrm{cor}C)$ (by Theorem 1)

$\Leftrightarrow\tilde{V}_{-1}(C_{x^*},-C_{x^*})=V_n(C)=V_n(C_{x^*})^{\frac{n+1}{n}}V_n(-C_{x^*})^{-\frac{1}{n}}$ (by the definition of $\bar{V}_{-1}(\cdot,\cdot)$)

$\Leftrightarrow C_{x^*},-C_{x^*}$ are dilates (by (1))

and the last statement is equivalent to that C is symmetric w.r.t. x^*.

Next, we show the other conclusions hold for as_∞°: by ii) and what just confirmed for as_1° we have $\mathrm{as}_\infty^\circ\geqslant\mathrm{as}_1^\circ\geqslant1$. If $\mathrm{as}_\infty^\circ(C)=1$, we have

$$1\leqslant\mathrm{as}_1^\circ(C)\leqslant\mathrm{as}_\infty^\circ(C)=1$$

by ii) which leads to $\mathrm{as}_1^\circ(C)=1$ and in turn that C is centrally symmetric by what just proved for as_1°. Conversely, if C is symmetric with the center x^*, then $\frac{\rho(C_{x^*},u)}{\rho(C_{x^*},-u)}=1$ for all $u\in S^{n-1}$ which implies clearly $\mathrm{as}_\infty^\circ(C)=\mu_\infty^\circ(C,x^*)=1$.

Finally, for $1<p<+\infty$, the conclusion can be deduced simply by $\mathrm{as}_1^\circ\leqslant\mathrm{as}_p^\circ\leqslant\mathrm{as}_\infty^\circ$ and what we proved for as_1° and as_∞°.

Remark 3 If $C \in K^n$, then $\mathrm{cor}(C) = C$, so Theorem 1 and 2 hold for $C \in K^n$.

3 The Best Upper Bound of Dual *p*-Measure of Asymmetry and the Extremal Body

A set $C \in \varphi^n$ (respectively K^n) is called an extremal body w.r.t. as_p° if

$$\mathrm{as}_p^\circ(C_0)=\min(\text{or}\max)\{\mathrm{as}_p^\circ(C) \mid C \in \varphi^n (\text{respectively } K^n)\}$$

In Section 2, we see the best lower bound 1 for as_p° and the corresponding extremal bodies: symmetric star bodies. However, no information was given for the best upper bound and the corresponding extremal bodies, in contrast to the case of p measures of asymmetry for convex bodies. This seems not an easy task, so in this section we consider only the best upper bound and the corresponding extremal bodies in K^n. Even so our answers are still not satisfactory.

First, we show that the best upper bound of as_p° exists and is attainable.

Proposition 1 There exists $C_0 \in K^n$, such that $\mathrm{as}_p^\circ(C_0) = \sup\{\mathrm{as}_p^\circ(C) \mid C \in K^n\} = M_p$.

Proof First, by

$$\mathrm{as}_p^\circ(C) \leqslant \mathrm{as}_\infty^\circ(C) = \mathrm{as}_\infty(C) \leqslant n$$

(by 2) in Remark 1), we have $M_p \leqslant n$. Thus, by the definition of supremum there is a sequence $\{C_k\}_{k=1}^\infty \subset K^n$ such that $\mathrm{as}_p^\circ(C_k) \to M_p$ as $k \to +\infty$. Since for each $C \in K^n$ the Banach-Mazur distance

$$d_{\mathrm{BM}}(C, B_2^n) := \inf\{\lambda \geqslant 1 \mid B_2^n \subset TC \subset \lambda(B_2^n - x) + x\} \leqslant n$$

where B_2^n is the Euclidean unit ball and the infimum is taken over all applicable invertible affine map T and $x \in \mathbf{R}^n$ (see, e.g. Ref.[23] or Ref.[3] and the references therein), and as_p° is affinely invariant, without loss of generality we may assume $B_2^n \subset C_k \subset nB_2^n$ for all k. Thus, by the well-known Blaschke's selection theorem, there are $\{C_{k_j}\} \subset \{C_k\}$ and $C_0 \in K^n$ such that $C_{k_j} \to C_0 \in K^n$ as $j \to \infty$. Thus

$$\mathrm{as}_p^\circ(C_0) = \lim_{j\to+\infty} \mathrm{as}_p^\circ(C_{k_j}) = M_p$$

by the continuity of as_p°.

Since $\mathrm{as}_\infty^\circ = \mathrm{as}_\infty$, by 2) in Remark 1 we know that $M_\infty = n$ and the extremal bodies for the best upper bound of as_∞° are simplices. However, we know neither the exact values of M_p nor their extremal bodies for $1 \leqslant p < +\infty$. What we know is just the following partial answer.

Theorem 3 $M_p < n$ for $1 \leqslant p < +\infty$.

Proof Suppose $M_p = n$, then there is $C_0 \in K^n$ such that $\mathrm{as}_p^\circ(C_0) = n$. Thus we have $\mathrm{as}_\infty(C_0) = n$ since $\mathrm{as}_p^\circ(C_0) \leqslant \mathrm{as}_\infty^\circ(C_0) = \mathrm{as}_\infty(C_0) \leqslant n$. So C_0 is a simplex by 2) in Remark 1.

Let $\upsilon_1, \upsilon_2, \cdots, \upsilon_{n+1}$ be the vertices of C_0 and x^* be the centroid (i.e. the ∞-critical point) of C_0. Then since the continuous function (w.r.t.u)

$$\tilde{\alpha}_{x^*} = \frac{\rho_{x^*}(C_0, u)}{\rho_{x^*}(C_0, -u)} \leqslant n$$

and the equality holds only at points $u_i := \dfrac{\upsilon_i - x^*}{|\upsilon_i - x^*|}, 1 \leqslant i \leqslant n+1$, and the measure

$$\mathrm{d}\tilde{m}_{x^*}(C_0, \omega) = \frac{\rho_{x^*}(C_0, u)^n \,\mathrm{d}S(u)}{nV_n(C_0)}$$

is not concentrated at $u_1, u_2, \cdots, u_{n+1}$, we have

$$\mu_p^\circ(C_0, x^*) = \left(\int_{S^{n-1}} \tilde{\alpha}_{x^*}(C_0, u)^p \,\mathrm{d}\tilde{m}_{x^*}(C_0, u)\right)^{\frac{1}{p}}$$

$$< \left(\int_{S^{n-1}} n^p \,\mathrm{d}\tilde{m}_{x^*}(C_0, u)\right)^{\frac{1}{p}} = n$$

which leads to $\mathrm{as}_p^\circ(C_0) \leqslant \mu_p^\circ(C_0, x^*) < n$, a contradiction.

Final Remark In this article, we introduce the so-called dual *p*-measures of asymmetry for star bodies (automatically for convex bodies) and study their basic properties. Some properties of the dual *p*-measure for convex bodies are exactly the same as those of the *p*-measures. However, unfortunately, the most important conclusion for the extremal bodies corresponding to best upper bounds is missing and even the values of best upper bounds are not known. So, a valuable (probably hard as well) problem left is to find the exact values of M_p and those mysterious extremal bodies which might have some interesting properties.

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