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Critical Chords of Convex Bodies of Constant Width

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Abstract: In this paper, we show that when Minkowski measure of asymmetry of convex body K of constant width is bigger than $\alpha(n-1)$, K has at least n+1 critical chords, where $\alpha(n)=\frac{n+\sqrt{2n(n+1)}}{n+2}$.

Key words: measure of asymmetry; critical chord; affine diameter; Grünbaum conjecture; constant width

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0 Introduction

Let C be a convex body (compact convex subset of $\mathbf{R}^n$). Affine diameter of C is a chord of C such that through the endpoints of this chord there exist two parallel support hyperplanes of C. For convex body of constant width K, each affine diameter is precisely a diameter.

In Ref.[1], Grünbaum conjectured that for each convex body C, there exists at least one point such that it is the intersection of at least $n+1$ affine diameters. It is called Grünbaum's conjecture for affine diameter (see Refs. [2, 3]).

In Ref. [4], Klee proved that if the Minkowski measure of asymmetry of C, $\mathrm{as}_\infty(C) > n-1$, then C has unique critical point x_0, and $x_0 \in \operatorname{int}\operatorname{conv} C^\ell$ (see below for definitions), which implies that C has at least $n+1$ critical chords. Since each critical chord is an affine diameter. Therefore, when $\mathrm{as}_\infty(C) > n-1$, Grünbaum's conjecture for affine diameters is correct.

In Ref.[5], Jin and Guo showed that each convex body of constant width has unique critical point. In this paper, we prove that if K is a convex body of constant width, and $\mathrm{as}_\infty(K) > \alpha(n-1)$, then the unique critical point o of K satisfies that $o \in \operatorname{int}\operatorname{conv} K^\ell$, which implies that K has at least $n+1$ critical chords, where $\alpha(n)=\frac{n+\sqrt{2n(n+1)}}{n+2}$. So, when $\mathrm{as}_\infty(K) > \alpha(n-1)$, Grünbaum's conjecture for affine diameters is correct.

1 Prelimilary

$\mathbf{R}^n$ denotes the usual n-dimensional Euclidean space with the canonical inner product $\langle\cdot,\cdot\rangle$. A bounded closed convex subset C of $\mathbf{R}^n$ is called a convex body (convex domain for $n=2$) if it has non-empty interior (int for brief). The family of all convex bodies in $\mathbf{R}^n$ is denoted by $\mathcal{K}^n$.

A convex body K is said to be of constant width ω if its projection on any straight line is a segment of universal length $\omega > 0$, which is equivalent to the geometrical fact that any two parallel support hyperplanes of K are always separated by a distance ω. Notice that the width ω of a constant width body is clearly its diameter. The convex bodies of constant width in $\mathbf{R}^2$ and $\mathbf{R}^3$ are also called *orbiforms* and *spheroforms*, respectively. Euclidean balls are obviously convex bodies of constant width, however, there are many others (see Refs. [6, 7]). We denote by W^n the set of all n-dimensional convex bodies of constant width. We refer readers to Ref. [8] for other notation and terms.

Given a convex body $C \in \mathcal{K}^n$ and $x \in \mathrm{int}(C)$. For a chord l of C through x, let $\gamma(l,x)$ be the ratio, not less than 1, in which x divides the length of l, and denote $\gamma(C,x) = \max\{\gamma(l,x) \mid x \in l\}$, then the Minkowski measure of asymmetry of C is defined by

$$\mathrm{as}_\infty(C) = \min_{x \in \mathrm{int}(C)} \gamma(C,x)$$

A point $x \in \mathrm{int}(C)$ satisfying $\gamma(C,x) = \mathrm{as}_\infty(C)$ is called a *critical point* of C. The set of all critical points of C is denoted by C. It is known that C is a non-empty convex set [1]. A chord l satisfying $\gamma(C,x) = \mathrm{as}_\infty(C)$ is called a critical chord of C. Recent results on Minkowski measure of asymmetry see Refs.[5, 9-11].

Denote for any $x \in \mathrm{int}(C)$,

$$S_C(x) = \left\{ p \in \mathrm{bd}(C) \mid x \in pq(\text{chord}) \text{ and } \frac{xp}{xq} = \gamma(C,x) \right\}$$

where bd denotes the boundary, and pq denotes the segment with endpoints p, q or its length alternatively if no confusing is caused. It is proved that $S_C(x) \neq \varnothing$ (see Ref.[4]).

The following is a list of some properties of the Minkowski measure of asymmetry (see Ref.[4] for proofs).

Property 1 If $C \in \mathcal{K}^n$, then $1 \quad \mathrm{as}_\infty(C) \quad n$. Equality holds on the left-hand side if and only if C is central, and on the right-hand side if and only if C is a simplex.

Property 2 For $C \in \mathcal{K}^n$, $\mathrm{as}_\infty(C) + \dim C^* \quad n$.

Property 3 Given $x \in \mathrm{ri}(C^*)$, the relative interior of C, then for any $y \in S_C(x)$,

$$y + \frac{\mathrm{as}_\infty(C)+1}{\mathrm{as}_\infty(C)}(C^* - y) \subset \mathrm{bd}(C)$$

and

$$y \in S_C(x') \text{ for } \forall x' \in C^*.$$

This property shows that the set $S_C(x)$ does not vary as x ranges over $\mathrm{ri}(C^*)$. We denote this set by C^ℓ.

Property 4 C^ℓ contains at least $\mathrm{as}_\infty(C)+1$ points.

It is well-known that each convex body K of constant width $\omega(K)$, the circum-sphere $S_O K$ and the insphere $S_i(K)$ of K are concentric and their radii, denoted by $R(K)$ and $r(K)$, respectively, satisfy $R(K) + r(K) = \omega(K)$ (see Ref.[6]). In Ref.[5], Jin and Guo got the following results.

Property 5 For $K \in W^n$, $\mathrm{as}_\infty(C) = \dfrac{R(K)}{r(K)}$, and K has unique critical point which is the center of circumsphere of K.

2 Contact Points of Convex Body of Constant Width and Its Circumsphere

For convex body K of constant width $\omega(K)$, let o be the unique critical point and $K^\#$ be the set of contact points of K and its circumsphere. Now we give the following proposition.

Proposition 1 If $K \in W^n$, then $S_K(o) = K^\#$. Recall that

$$S_K(o) = \left\{ p \in \mathrm{bd}(K) \mid q \in \mathrm{bd}(K), o \in pq, \frac{op}{oq} = \gamma(K,o) = \mathrm{as}_\infty(K) \right\}$$

In order to prove Proposition 1, we need the following lemma.

Lemma 1[4] If $C \in \mathcal{K}^n$, $x \in \mathrm{int}(C)$ and pq is a chord through x with $p \in S_C(x)$, (i.e., $\dfrac{px}{xq} = \gamma(C,x)$), then there exist two parallel hyperplane H_1, H_2 supporting C at p, q, respectively.

Remark 1 In Lemma 1, pq is an affine diameter of C. Particularly, each critical chord is an affine diameter.

Proof of Proposition 1

1) If $p \in S_K(o)$, then there exists $q \in \mathrm{bd}(K)$ such

that $\frac{op}{oq}=\gamma(K,o)=\text{as}_\infty(K)$. By Lemma 1, there exist two parallel hyperplanes H_1,H_2 supporting K at p,q, respectively. Since $K\in W^n$, have $pq \quad \omega(K)$. By Property 5, we have $op=R(K), oq=r(K)$, which implies that $p\in K^\#$.

2) If $p\in K^\#$, then there exists $q\in bd(K)$ such that $o\in pq$. Since $pq=op+oq=R(K)+oq \quad \omega(K)$, we have $oq \quad r(K)$, which implies that $\frac{op}{oq} \quad \frac{R(K)}{r(K)}= \gamma(K,o)$. By the definition of $\gamma(K,o)$, we have $\frac{op}{oq} \quad \frac{R(K)}{r(K)}=\gamma(K,o)$. So, $\frac{op}{oq}=\frac{R(K)}{r(K)}$, which implies that $p\in S_K(o)$.

3 Main Theorem

For general convex body $C\in\mathcal{K}^n$, we have

$$1 \quad \text{as}_\infty(C) \quad n$$

Equality holds on the left-hand side if and only if *C* is central, and on the right-hand side if and only if *C* is a simplex. In Ref.[4], Klee proved the following theorem.

Theorem 1 For $C\in\mathcal{K}^n$, if $\text{as}_\infty(C)>n-1$, then *C* has unique critical point *o*, and $o\in\text{int conv}\, C^\ell$.

Remark 2 Theorem 1 implies that *C* has at least $n+1$ critical chords. Since each critical chord is an affine diameter. Therefore, when $\text{as}_\infty(C)>n-1$, Grünbaum conjecture for affine diameters is correct.

In Ref.[12], we gain the following results:

Lemma 2 If $K\in W^n$, then

$$1 \quad \text{as}_\infty(K) \quad \alpha(n)$$

where $\alpha(n)=\frac{n+\sqrt{2n(n+1)}}{n+2}$. The equality holds on the left-hand side if and only if *K* is an Euclidean ball and on the right-hand if and only if *K* is a completion of a regular simplex.

Now we give the main theorem.

Theorem 2 For $K\in W^n$, if $\text{as}_\infty(K)>\alpha(n-1)$, then the unique critical point *o* of *K* satisfies that $o\in\text{int conv}\, K^\ell$, which implies that *K* has at least $n+1$ critical chords, where $\alpha(n)=\frac{n+\sqrt{2n(n+1)}}{n+2}$.

Remark 3 Theorem 2 shows that for $K\in W^n$, if $\text{as}_\infty(K)>\alpha(n-1)$, then Grünbaum's conjecture for affine diameters is correct.

Proof of Theorem 2

Since *o* is the center of the circumsphere of *K* (see Property 5), we have $o\in\text{conv}\, K^\#$. By Proposition 1, we have $o\in\text{conv}\, K^\ell$.

Now we prove that $d:=\dim(\text{conv}\, K^\ell)=n$. In fact, if $d \quad n-1$, let $H_o=\text{aff}(\text{conv}\, K^\ell)$, the affine hull of $\text{conv}\, K^\ell$, then $K^\ell=H_o\cap S_o(K)\cap\text{bd}(K)$. Le $P_{H_o}K$ be the projection of *K* onto H_o. Then $P_{H_o}K$ is *d*-dimensional convex body of constant width $\omega(K)$. So, the *d* dimensional ball $B_o:=H_o\cap B_o(K)\supset P_{H_o}K$ and $B_i:=H_o\cap B_i(K)\subset P_{H_o}K$, where $B_o(K)=\text{conv}(S_o(K))$ and $B_i(K)=\text{conv}(S_i(K))$. Let $S_o:=\text{relbd}(B_o)$, the relative boundary of B_o, and $S_i:=\text{relbd}(B_i)$. Therefore, $S_o\cap P_{H_o}K=K^\ell=K^\#$. Since $o\in\text{conv}\, K^\#$, we have S_o is the circumsphere of $P_{H_o}K$ in the *d*-dimensional affine subspace H_o. In H_o, we have

$$\text{as}_\infty(P_{H_o}K)=\frac{R(K)}{r(K)}=\text{as}_\infty(K) \quad \alpha(d) \quad \alpha(n-1)$$

it is a contradiction.

Then we prove that $o\in\text{int conv}\, K^\ell$. If not, then $o\in\text{bd conv}\, K^\ell$. So, there exists $L\subset K^\ell$ such that $o\in\text{conv}\, L$ and $\dim(\text{conv}\, L) \quad n-1$. Let $H_o:=\text{aff}(\text{conv}\, L)$. By the same argument as above, we can get a contradiction.

By Property 4, it is easy to get that for $C\in\mathcal{K}^n$, any point $x\in C^*$ belongs to at least 3 affine diameters of *C*. In Ref.[13], Soltan gave an example to show that there exists a convex body *T*, which has only three affine diameters (critical chords) through the unique critical point of *T*. However, for convex bodies of constant width, we can still conjecture that each convex body of constant width in $\mathbf{R}^n$ has at least $n+1$ critical chords through the unique critical point, which means for convex bodies of constant width, Grünbaum conjecture for affine diameters is correct.

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