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# Variety Membership Problem for Two Classes of Non-Finitely Based Semigroups

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**Abstract:** The variety membership problem for two classes of non-finitely based semigroups is considered. It is shown that a finite semigroup *S* belongs to the variety generated by one of these non-finitely based semigroups if and only if *S* satisfies four certain equations that involve at most $2|S|+1$ distinct variables.

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## 0 Introduction

Recall that the *variety* generated by an algebra *A*, denoted by $\langle A\rangle$, is the smallest class of algebras of the same type containing *A* that is closed under the formation of homomorphic images, subalgebras, and arbitrary direct products. By the celebrated theorem of Birkhoff [1], the variety $\langle A\rangle$ consists precisely of algebras that satisfy all equations of *A*. The *variety membership problem* for a finite algebra *A*, abbreviated by $\mathrm{VMP}(A)$, is the problem of deciding if a finite algebra belongs to the variety $\langle A\rangle$. In general, the problem $\mathrm{VMP}(A)$ for a finite algebra *A* is 2-EXPTIME-complete [2,3]. For a finite semigroup *S*, the precise complexity of $\mathrm{VMP}(S)$ has not been determined, but it is known to be NP-hard [4,5]. In any case, since an algebra *A* satisfies the same equations as the variety $\langle A\rangle$, the problem $\mathrm{VMP}(A)$ is solvable in polynomial time whenever *A* is *finitely based* in the sense that its equations are finitely axiomatizable. Hence the variety membership problem is nontrivial only for non-finitely based algebras. Finite algebras that are minimal with respect to being non-finitely based, or *minimal non-finitely based*, are naturally of interest.

In general, a minimal non-finitely based algebra has at least three elements [6]. As for semigroups—the main algebras of the present article—minimal non-finitely based members are of order six; up to isomorphism, there are precisely four such semigroups:

- The monoid $B_2^1$ obtained by adjoining an identity element 1 to the Brandt semigroup

$$B_2=\langle a,b\mid a^2=b^2=0,\ aba=a,\ bab=b\rangle,$$

- the monoid $A_2^1$ obtained by adjoining an identity element 1 to the orthodox 0-simple semigroup

$$A_2 = \langle a,b \mid a^2 = aba = a, bab = b, b^2 = 0\rangle$$

- the semigroup $A_2^g$ obtained by adjoining an external element $g$ to $A_2$ with multiplication involving $g$ given by $gA_2 = A_2 g = \{g\}$ and $g^2 = 0$, and
- the $J$-trivial semigroup

$$L_3 = \langle e,f \mid e^2 = e, f^2 = f, efe = 0\rangle$$

The absence of any other non-finitely based semigroups of order six or less follows from the solution to the finite basis problem for semigroups of order up to six[7-10]. Refer to Lee *et al* [11] for more information.

The semigroup $A_2^g$, due to Volkov *et al* [12], is the first published example of a non-finitely based finite semigroup whose variety membership problem is solvable in polynomial time; one important step in checking if a finite semigroup $S$ belongs to the variety $\langle A_2^g\rangle$ is the computation of the *core* of $S$, that is, the subsemigroup of $S$ generated by its idempotents. Now since the equality $\langle A_2^g\rangle = \langle A_2 \times \mathbb{Z}_2\rangle$ holds [13], where $\mathbb{Z}_n$ denotes the cyclic group of order $n$, it is instinctive to examine the problem $\mathrm{VMP}(A_2 \times \mathbb{Z}_n)$ for general $n$. For $n$=1, this problem is solvable in polynomial time because the semigroup $A_2 \times \mathbb{Z}_1 \cong A_2$ is finitely based [14]. If $n \geqslant 3$, then even though the semigroup $A_2 \times \mathbb{Z}_n$ is non-finitely based [15], arguments from Volkov *et al* [12] can be repeated to solve the problem $\mathrm{VMP}(A_2 \times \mathbb{Z}_n)$ in polynomial time. Consequently, for all $n \geqslant 1$, the problem $\mathrm{VMP}(A_2 \times \mathbb{Z}_n)$ is solvable in polynomial time.

In contrast, Jackson [4] proved that the problem $\mathrm{VMP}(B_2^1)$ is NP-hard. The complexity of the problems $\mathrm{VMP}(A_2^1)$ and $\mathrm{VMP}(L_3)$, however, is currently unknown, and Jackson [4] questioned if they are solvable in polynomial time.

The main objective of the present article is to exhibit, in Sections 1 and 2, easily verifiable solutions to the problems $\mathrm{VMP}(A_2 \times \mathbb{Z}_n)$ and $\mathrm{VMP}(L_3 \times \mathbb{Z}_n)$, respectively. In each section, it is shown that a finite semigroup of order $r \geqslant 2$ belongs to the variety of the section if and only if it satisfies four equations that involve at most $2r+1$ distinct variables. Solutions with complexity co-NP are therefore available for the problems $\mathrm{VMP}(A_2 \times \mathbb{Z}_n)$ and $\mathrm{VMP}(L_3 \times \mathbb{Z}_n)$. Although the solution for $\mathrm{VMP}(A_2 \times \mathbb{Z}_n)$ is less efficient than Volkov *et al* [12], it does not require the computation of cores of semigroups. On the other hand, the solution for the problem $\mathrm{VMP}(L_3 \times \mathbb{Z}_n)$ demonstrates the plausibility of it being solvable in polynomial time.

At the end of each section, it is also deduced that a finite semigroup generates the variety of the section if and only if it satisfies and violates certain equations from some finite list.

Acquaintance with rudiments of universal algebra is assumed of the reader. Refer to Burris and Sankappanavar [16] for more information.

# 1 The Varieties $\langle A_2 \times \mathbb{Z}_n\rangle$

**Theorem 1** Suppose that $S$ is any finite semigroup of order $r \geqslant 2$. Then the inclusion $S \in \langle A_2 \times \mathbb{Z}_n\rangle$ holds if and only if $S$ satisfies the equations

$$x^{n+2} \approx x^2 \tag{1}$$

$$x(yx)^{n+1} \approx xyx \tag{2}$$

$$xyxzx \approx xzxyx \tag{3}$$

$$\varPsi_r : \left(\prod_{i=1}^{r} x_i^n\right)^3 \approx \left(\prod_{i=1}^{r} x_i^n\right)^2$$

**Lemma 1** For each integer $n \geqslant 1$, the equations $\{(1),(2),(3),\varPsi_r \mid r = 2,3,4,\cdots\}$ constitute an equational basis for the variety $\langle A_2 \times \mathbb{Z}_n\rangle$.

**Proof** This follows from Proposition 1.1(i) in Lee and Volkov [13].

**Remark 1** If $n$ = 1, then the equation (1) implies the equation $\varPsi_r$ for all $r \geqslant 2$.

**Lemma 2** The equations $\{(1),\varPsi_{r+1}\}$ imply the equation $\varPsi_r$.

**Proof** By Remark 1, it suffices to assume that $n \geqslant 2$. Then by making the substitution $x_{r+1} \mapsto x_r$ in the equation $\varPsi_{r+1}$ and applying the equation (1) to eliminate excessive copies of the variable $x_r$ from both sides, the equation $\varPsi_r$ is obtained.

**Proof of Theorem 1** Necessity follows from Lemma 1. Conversely, suppose that the semigroup $S$ satisfies the equations $\{(1),(2),(3),\varPsi_r\}$. If $n=1$, then $S \in \langle A_2 \times \mathbb{Z}_n\rangle$ by Lemma 1 and Remark 1. Therefore it suffices to assume that $n \geqslant 2$. Then by Lemma 2, the semigroup $S$ satisfies the equations

$$\varPsi_2, \varPsi_3, \cdots, \varPsi_r \tag{4}$$

In the following, it is shown that $S$ satisfies the equation $\varPsi_t$ for all $t > r$, so that $S \in \langle A_2 \times \mathbb{Z}_n\rangle$ by Lemma 1.

Let $\varphi$ be any substitution into $S$. For notational brevity, write $\varphi(x_i) = a_i$. Since $|S| = r \geqslant 2$, the list $a_1, a_2, \cdots, a_{r+1}$ of elements from $S$ contains some repetition. Let $\ell$ be the least integer such that $a_\ell = a_{\ell+k}$ for some $k \geqslant 1$. Then there are two cases to consider.

Case 1: $a_\ell = a_{\ell+1}$. Let $\Theta_c^d = \prod_{i=c}^{d} a_i^n$. Then

$$\varphi\left(\left(\prod_{i=1}^{r+1} x_i^n\right)^2\right) = \left(\Theta_1^{\ell-1} \cdot a_\ell^n a_\ell^n \cdot \Theta_{\ell+2}^{r+1}\right)^2$$

$$\overset{(1)}{=} \left(\Theta_1^{\ell-1} \cdot a_\ell^n \cdot \Theta_{\ell+2}^{r+1}\right)^2 \overset{(4)}{=} \left(\Theta_1^{\ell-1} \cdot a_\ell^n \cdot \Theta_{\ell+2}^{r+1}\right)^3$$

$$\overset{(1)}{=} \left(\Theta_1^{\ell-1} \cdot a_\ell^n a_\ell^n \cdot \Theta_{\ell+2}^{r+1}\right)^3 = \varphi\left(\left(\prod_{i=1}^{r+1} x_i^n\right)^3\right).$$

Case 2: $a_\ell = a_{\ell+k}$ for some $k \geqslant 2$. Then

$$\varphi\left(\left(\prod_{i=1}^{r+1} x_i^n\right)^2\right) = \left(\Theta_1^{\ell-1} \cdot a_\ell^n \cdot \Theta_{\ell+1}^{\ell+k-1} \cdot a_\ell^n \cdot \Theta_{\ell+k+1}^{r+1}\right)^2$$

$$\overset{(2)}{=} \left(\Theta_1^{\ell-1} \cdot a_\ell^n \cdot \left(\Theta_{\ell+1}^{\ell+k-1} \cdot a_\ell^n\right)^{n+1} \cdot \Theta_{\ell+k+1}^{r+1}\right)^2$$

$$\overset{(4)}{=} \left(\Theta_1^{\ell-1} \cdot a_\ell^n \cdot \left(\Theta_{\ell+1}^{\ell+k-1} \cdot a_\ell^n\right)^{n} \cdot \Theta_{\ell+k+1}^{r+1}\right)^2$$

$$\overset{(4)}{=} \left(\Theta_1^{\ell-1} \cdot a_\ell^n \cdot \left(\Theta_{\ell+1}^{\ell+k-1} \cdot a_\ell^n\right)^{n} \cdot \Theta_{\ell+k+1}^{r+1}\right)^3$$

$$\overset{(4)}{=} \left(\Theta_1^{\ell-1} \cdot a_\ell^n \cdot \left(\Theta_{\ell+1}^{\ell+k-1} \cdot a_\ell^n\right)^{n+1} \cdot \Theta_{\ell+k+1}^{r+1}\right)^3$$

$$\overset{(2)}{=} \left(\Theta_1^{\ell-1} \cdot a_\ell^n \cdot \Theta_{\ell+1}^{\ell+k-1} \cdot a_\ell^n \cdot \Theta_{\ell+k+1}^{r+1}\right)^3$$

$$= \varphi\left(\left(\prod_{i=1}^{r+1} x_i^n\right)^3\right)$$

Therefore in both cases, the semigroup $S$ satisfies the equation $\Psi_{r+1}$.

The preceding procedure can be repeated to show that the semigroup $S$ satisfies the equation $\Psi_t$ for each subsequent $t > r+1$.

**Lemma 3** Let $S$ be any semigroup that satisfies the equation $x^{n+2} \approx x^2$. Then the inclusion $\mathbb{Z}_n \in \langle S \rangle$ holds if and only if $S$ violates every equation in

$$\{x^{d+2} \approx x^2 \mid d \text{ is a maximal proper divisor of } n\} \quad (5)$$

**Proof** This follows from Lemma .6.14 in Petrich and Reilly [17].

**Lemma 4** [18] Let $S$ be any semigroup that satisfies the equation (1). Then the inclusion $A_2 \in \langle S \rangle$ holds if and only if $S$ violates the equation

$$\left((x^n y)^n (yx^n)^n\right)^n \approx (x^n yx^n)^n \quad (6)$$

**Corollary 1** Let $S$ be any semigroup of order $r \geqslant 2$ that satisfies the equations $\{(1),(2),(3),\Psi_r\}$ and violates every equation in $\{(5),(6)\}$. Then the equality $\langle S \rangle = \langle A_2 \times \mathbb{Z}_n \rangle$ holds.

**Proof** The inclusion $\langle S \rangle \subseteq \langle A_2 \times \mathbb{Z}_n \rangle$ holds by Theorem 1, while the reverse inclusion $\langle S \rangle \supseteq \langle A_2 \times \mathbb{Z}_n \rangle$ holds by Lemmas 3 and 4.

## 2 The Varieties $\langle L_3 \times \mathbb{Z}_n \rangle$

**Theorem 2** Suppose that $S$ is any finite semigroup of order $r \geqslant 2$. Then the inclusion $S \in \langle L_3 \times \mathbb{Z}_n \rangle$ holds if and only if $S$ satisfies the equations

$$x^{n+2} \approx x^2 \quad (7)$$

$$x^{n+1} yx^{n+1} \approx xyx \quad (8)$$

$$xhykxty \approx yhxkytx \quad (9)$$

$$\Omega_r : \ w_r \approx w_r^\dagger$$

where

$$w_r = x\left(\prod_{i=1}^{r} (y_i h_i y_i)\right)x \quad \text{and} \quad w_r^\dagger = x\left(\prod_{i=r}^{1} (y_i h_i y_i)\right)x$$

**Lemma 5** For each integer $n \geqslant 1$, the equations $\{(7),(8),(9),\Omega_r \mid r = 2,3,4,\cdots\}$ constitute an equational basis for the variety $\langle L_3 \times \mathbb{Z}_n \rangle$.

**Proof** See Corollary 3.5 in Lee [19].

**Lemma 6** The equations $\{(7),(8),\Omega_{r+1}\}$ imply the equation $\Omega_r$.

**Proof** This implication holds because

$$w_r \overset{(8)}{\approx} xx^n\left(\prod_{i=1}^{r} (y_i z_i y_i)\right)x^n x$$

$$\overset{(7)}{\approx} xx^n\left(\prod_{i=1}^{r} (y_i z_i y_i)\right)\left(x^n x^n x^n\right)x^n x$$

$$\overset{\Omega_{r+1}}{\approx} xx^n\left(x^n x^n x^n\right)\left(\prod_{i=r}^{1} (y_i z_i y_i)\right)x^n x$$

$$\overset{(7)}{\approx} xx^n\left(\prod_{i=r}^{1} (y_i z_i y_i)\right)x^n x \overset{(8)}{\approx} w_r^\dagger .$$

**Lemma 7** The equations $\{(7),(8),(9),\Omega_2\}$ imply the equations

$$xyzxy \approx xzyxy \quad (10)$$

$$xyzxy \approx xyxzy \quad (11)$$

**Proof** Since

$$x^{n+1} yx \overset{(8)}{\approx} x^{2n+1} yx^{n+1} \overset{(7)}{\approx} x^{n+1} yx^{n+1} \overset{(8)}{\approx} xyx$$

$$xyx^{n+1} \overset{(8)}{\approx} x^{n+1} yx^{2n+1} \overset{(7)}{\approx} x^{n+1} yx^{n+1} \overset{(8)}{\approx} xyx$$

$$x^2 yx \overset{(8)}{\approx} x^n (x^{2n} x)(xyx)x^{3n} \overset{\Omega_2}{\approx} x^n (xyx)(x^{2n} x)x^{3n}$$

$$\overset{(8)}{\approx} xyx^{4n+2} \overset{(7)}{\approx} xyx^2$$

$$xyzxy \overset{(8)}{\approx} x(x^n)y(z)x(x^n)y \overset{(9)}{\approx} yx^n xzyx^n x \overset{(8)}{\approx} yxzyx$$

$$xzyxy \overset{(8)}{\approx} x(z)y(y^n)x(y^n)y \overset{(9)}{\approx} yzxy^{2n+1}x$$

$$\overset{(8)}{\approx} y^{n+1}zxy^{3n+1}x \overset{(7)}{\approx} y^{n+1}zxy^{n+1}x \overset{(8)}{\approx} yzxyx$$

the equations $\{(7),(8),(9),\Omega_2\}$ imply the equations

$$x^{n+1}yx \approx xyx \tag{12}$$

$$xyx^{n+1} \approx xyx \tag{13}$$

$$x^2yx \approx xyx^2 \tag{14}$$

$$xyzxy \approx yxzyx \tag{15}$$

$$xzyxy \approx yzxyx \tag{16}$$

Hence the equations $\{(7),(8),(9),\Omega_2\}$ imply the equation (10) because

$$xzyxy \overset{(16)}{\approx} yzxyx \overset{(12)}{\approx} y(y^n z)xyx \overset{(16)}{\approx} xy^n zyxy$$

$$\overset{(14)}{\approx} xy(zy)xyy^{n-1} \overset{(15)}{\approx} yxzy^2xy^{n-1}$$

$$\overset{(14)}{\approx} yxzyxy^n \overset{(15)}{\approx} xyzxy^{n+1} \overset{(13)}{\approx} xyzxy$$

By symmetry, the equations $\{(7),(8),(9),\Omega_2\}$ also imply the equation (11).

**Proof of Theorem 2** Necessity follows from Lemma 5. Conversely, suppose that the semigroup $S$ satisfies the equations $\{(7),(8),(9),\Omega_r\}$. Then by Lemmas 6 and 7, the semigroup $S$ also satisfies the equations $\{\Omega_2,\Omega_3,\cdots,\Omega_r,(10),(11)\}$. In the following, it is shown that $S$ satisfies the equation $\Omega_t$ for all $t > r$, so that $S \in \langle L_3 \times \mathbb{Z}_n \rangle$ by Lemma 5.

Let $\varphi$ be any substitution into $S$. For notational brevity, write $\varphi(x) = a$, $\varphi(y_i) = b_i$, and $\varphi(z_i) = c_i$. Since $|S| = r \geqslant 2$, the list $c_1, c_2, \cdots, c_{r+1}$ of elements from $S$ contains some repetition. Therefore there exist integers $\ell$ and $m$ with $1 \leqslant \ell < m \leqslant r+1$ such that $c_\ell = c_m$. Hence

$$\varphi(w_{r+1}) = aP \cdot b_\ell c_\ell b_\ell \cdot Q \cdot b_m c_\ell b_m \cdot Ra$$

where the products $P = \prod_{i=1}^{\ell-1}(b_i c_i b_i)$, $Q = \prod_{i=\ell+1}^{m-1}(b_i c_i b_i)$, and $R = \prod_{i=m+1}^{r+1}(b_i c_i b_i)$ are empty if $\ell = 1$, $\ell = m-1$, and $m = r+1$, respectively. Thus

$$\varphi(w_{r+1}) \overset{(11)}{=} aP \cdot b_\ell c_\ell \cdot Q \cdot b_m b_\ell c_\ell b_m \cdot Ra$$

$$\overset{(10)}{=} aP \cdot b_\ell c_\ell b_m \cdot Q \cdot b_\ell c_\ell b_m \cdot Ra$$

$$= a\left(\prod_{i=1}^{\ell-1}(b_i c_i b_i)\right) TQT \left(\prod_{i=m+1}^{r+1}(b_i c_i b_i)\right) a \tag{17}$$

where $T = b_\ell c_\ell b_m$. Note that the number $p$ of terms in (17) of the form *XYX* that are sandwiched between the two occurrences of $a$ is

$$p = |\{1,\cdots,\ell-1\}| + 1 + |\{m+1,\cdots,r+1\}|$$

$$= r + 1 + \ell - m$$

where $1 \leqslant p \leqslant r$. If $p \geqslant 2$, then the equation $\Omega_p$ can be used to reverse the order of the product in (17), giving

$$\varphi(w_{r+1}) = a\left(\prod_{i=r+1}^{m+1}(b_i c_i b_i)\right) TQT \left(\prod_{i=\ell-1}^{1}(b_i c_i b_i)\right) a \tag{18}$$

If $p = 1$, so that $P$ and $R$ are empty, then (18) holds vacuously. Therefore (18) holds in any case.

Now the number $q$ of terms in $Q$ of the form *XYX* is $q = m - \ell - 1$, so that $0 \leqslant q \leqslant r$. If $q \geqslant 2$, then the equation $\Omega_q$ can be used to reverse the order of product in $TQT = T\left(\prod_{i=\ell+1}^{m-1}(b_i c_i b_i)\right)T$, giving

$$TQT = T\left(\prod_{i=m-1}^{\ell+1}(b_i c_i b_i)\right)T \tag{19}$$

If $q \leqslant 1$, so that $Q$ is either $b_{\ell+1}c_{\ell+1}b_{\ell+1}$ or empty, then (19) holds vacuously. Therefore (19) holds in any case, whence

$$TQT \overset{(19)}{=} b_\ell c_\ell b_m \left(\prod_{i=m-1}^{\ell+1}(b_i c_i b_i)\right) b_\ell c_\ell b_m$$

$$\overset{(9)}{=} b_m c_\ell b_\ell \left(\prod_{i=m-1}^{\ell+1}(b_i c_i b_i)\right) b_m c_\ell b_\ell$$

$$\overset{(10)}{=} b_m c_\ell \left(\prod_{i=m-1}^{\ell+1}(b_i c_i b_i)\right) b_\ell b_m c_\ell b_\ell$$

$$\overset{(11)}{=} b_m c_\ell b_m \left(\prod_{i=m-1}^{\ell+1}(b_i c_i b_i)\right) b_\ell c_\ell b_\ell$$

$$= \prod_{i=m}^{\ell}(b_i c_i b_i) \text{ because } c_\ell = c_m$$

Since

$$\varphi(w_{r+1}) \overset{(18)}{=} a\left(\prod_{i=r+1}^{m+1}(b_i c_i b_i)\right) TQT \left(\prod_{i=\ell-1}^{1}(b_i c_i b_i)\right) a$$

$$= a\left(\prod_{i=r+1}^{m+1}(b_i c_i b_i)\right)\left(\prod_{i=m}^{\ell}(b_i c_i b_i)\right)\left(\prod_{i=\ell-1}^{1}(b_i c_i b_i)\right) a$$

$$= a\left(\prod_{i=r+1}^{1}(b_i c_i b_i)\right) a = \varphi(w_{r+1}^{\dagger}),$$

the semigroup $S$ satisfies the equation $\Omega_{r+1}$.

The preceding procedure can be repeated to show that the semigroup $S$ satisfies the equation $\Omega_t$ for each subsequent $t > r+1$.

**Corollary 2** The problem $\mathrm{VMP}(L_3 \times \mathbb{Z}_n)$ is co-NP.

**Proof** Suppose that $S$ is any finite semigroup of order $r \geqslant 2$. Then it follows from Theorem 2 that the exclusion $S \notin \langle L_3 \times \mathbb{Z}_n \rangle$ holds if and only if $S$ violates some equation from $\{(7),(8),(9)\}$ or $S$ violates the equation $\Omega_r$. The former task is solvable in polynomial time, while the latter task has complexity NP.

**Remark 2** The non-finitely based semigroup $L_3 \times \mathbb{Z}_n$, when endowed with a certain unary operation, is an involution semigroup that is finitely based [20]. Therefore the variety membership problem for this involution semigroup is solvable in polynomial time.

**Lemma 8** Let $S$ be any semigroup that satisfies the equation (7). Then the inclusion $L_3 \in \langle S \rangle$ holds if and only if $S$ violates the equation

$$x^n (y^n x^n)^{n+1} \approx x^n y^n x^n \tag{20}$$

**Proof** See Proposition 3.3 in Ref.[21].

**Corollary 3** Let $S$ be any semigroup of order $r \geqslant 2$ that satisfies the equations $\{(7),(8),(9),\Omega_r\}$ and violates every equation in $\{(5),(20)\}$. Then the equality $\langle S \rangle = \langle L_3 \times \mathbb{Z}_n \rangle$ holds.

**Proof** The inclusion $\langle S \rangle \subseteq \langle L_3 \times \mathbb{Z}_n \rangle$ holds by Theorem 2, while the reverse inclusion $\langle S \rangle \supseteq \langle L_3 \times \mathbb{Z}_n \rangle$ holds by Lemmas 3 and 8.

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