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A Type of Busemann-Petty Problems for General *Lp*-Intersection Bodies

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Abstract: Recently, the notion of general (containing symmetric and asymmetric) *Lp*-intersection bodies was given. In this article, by the *Lp*-dual mixed volumes and the general *Lp*-dual Blaschke bodies, we study the *Lp*-dual affine surface area forms of the Busemann-Petty problems for general *Lp*-intersection bodies. Our works belong to a new and rapidly evolving asymmetric *Lp*-Brunn-Minkowski theory.

Key words: Busemann-Petty problem; *Lp*-dual affine surface area; general *Lp*-intersection body

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0 Introduction

Let K^n denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space $\mathbf{R}^n$. For the set of convex bodies whose centroid lie at the origin in $\mathbf{R}^n$, we write K_c^n.

Let S_o^n, S_c^n, S_{os}^n denote the set of star bodies (about the origin), the set of star bodies whose centroid lie at the origin and the set of origin-symmetric star bodies in $\mathbf{R}^n$, respectively. Let S^{n-1} denote the unit sphere in $\mathbf{R}^n$, and $V(K)$ denote the n-dimensional volume of body K. For the standard unit ball B in $\mathbf{R}^n$, we use $\omega_n = V(B)$ to denote its volume.

If K is a compact star-shaped (about the origin) in $\mathbf{R}^n$, its radial function, $\rho_K = \rho(K,\cdot):\mathbf{R}^n\setminus\{0\}\to[0,+\infty)$ is defined by[1]

$$\rho(K,x)=\max\{\lambda\geqslant 0:\lambda x\in K\},\quad x\in\mathbf{R}^n\setminus\{0\}.$$

The notion of intersection bodies was introduced by Lutwak[2]: For $K\in S_o^n$, the intersection body, IK, of K is a star body whose radial function in the direction $u\in S^{n-1}$ is equal to the $(n-1)$-dimensional volume of the section of K by $u^{\perp}$, the hyperplane orthogonal to u, i.e. for all $u\in S^{n-1}$, $\rho(IK,u)=V_{n-1}(K\cap u^{\perp})$, where V_{n-1} denotes $(n-1)$-dimensional volume.

In 2006, Haberl and Ludwig[3] defined the Lp-intersection body as follows: For $K\in S_o^n$, $0<p<1$, the Lp-intersection body, I_pK, of K is the origin-symmetric star body whose radial function is defined by

$$\rho_{I_pK}^p(u)=\int_K|u\cdot x|^{-p}\mathrm{d}x$$

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$$=\frac{1}{n-p}\int_{S^{n-1}}\left|u\cdot v\right|^{-p}\rho_K^{n-p}(v)\mathrm{d}S(v) \qquad (1)$$

for all $u\in S^{n-1}$.

Meanwhile, they[3] defined the following asymmetric *Lp*-intersection bodies I_p^+K. For $K\in S_o^n$, $0<p<1$, define

$$\rho_{I_p^+K}^p(u)=\int_{K\cap u^+}\left|u\cdot x\right|^{-p}\mathrm{d}x \qquad (2)$$

for all $u\in S^{n-1}$, where $u\cdot x$ denotes the standard inner product of u and x, and $u^+=\{x:u\cdot x\geqslant 0,x\in\mathbf{R}^n\}$. They also defined that

$$I_p^-K=I_p^+(-K) \qquad (3)$$

From definitions (2) and (3), we can see that

$$\rho_{I_p^-K}^p(u)=\rho_{I_p^+(-K)}^p(u)=\int_{-K\cap u^+}\left|u\cdot x\right|^{-p}\mathrm{d}x$$

$$=\int_{K\cap(-u)^+}\left|u\cdot x\right|^{-p}\mathrm{d}x \qquad (4)$$

Recently, Wang and Li[4,5] gave the notion of general *Lp*-intersection body with a parameter τ as follows: For $K\in S_o^n$, $0<p<1$ and $\tau\in[-1,1]$, the general *Lp*-intersection body, $I_p^\tau K\in S_o^n$, of K is defined by

$$\rho_{I_p^\tau K}^p(u)=f_1(\tau)\rho_{I_p^+K}^p(u)+f_2(\tau)\rho_{I_p^-K}^p(u) \qquad (5)$$

for all $u\in S^{n-1}$, where

$$\begin{cases} f_1(\tau)=\dfrac{(1+\tau)^p}{(1+\tau)^p+(1-\tau)^p} \\ f_2(\tau)=\dfrac{(1-\tau)^p}{(1+\tau)^p+(1-\tau)^p} \end{cases} \qquad (6)$$

From (6), we easily know that

$$f_1(-\tau)=f_2(\tau),\quad f_2(-\tau)=f_1(\tau);$$

$$f_1(\tau)+f_2(\tau)=1 \qquad (7)$$

Obviously, if $\tau=0$, by (1), (2), (4), (5) and (6), we see $I_p^0K=I_pK$.

The general *Lp*-intersection bodies belong to a new and rapidly evolving asymmetric *Lp*-Brunn-Minkowski theory that has its own origin in the work of Ludwig, Haberl and Schuster[3-8]. For the further researches of asymmetric *Lp*-Brunn-Minkowski theory, also see Refs.[9-22].

Associated with the *Lp*-dual mixed volume $\tilde{V}_p(M,N)$, Wang *et al*[23] gave the notion of *Lp*-dual affine surface area as follows: For $K\in S_o^n$, and $0<p<n$, the *Lp*-dual affine surface area, $\tilde{\Omega}_p(K)$, of K is defined by

$$n^{-\frac{p}{n}}\tilde{\Omega}_p(K)^{\frac{n+p}{n}}=\sup\{n\tilde{V}_p(K,Q^*)V(Q)^{\frac{p}{n}}:Q\in K_c^n\} \qquad (8)$$

Here Q^* denotes the polar of Q which is defined by Ref.[1]

$$Q^*=\{x\cdot y \quad 1,y\in Q\},\qquad x\in\mathbf{R}^n.$$

In 2014, Wang *et al*[24] improved definition (8) from $K\in K_c^n$ to $Q\in S_c^n$: For $K\in S_o^n$, and $0<p<n$, the *Lp*-dual affine surface area, $\tilde{\Omega}_p(K)$, of K is defined by

$$n^{-\frac{p}{n}}\tilde{\Omega}_p(K)^{\frac{n+p}{n}}=\sup\{n\tilde{V}_p(K,Q^*)V(Q)^{\frac{p}{n}}:Q\in S_c^n\} \qquad (9)$$

Let Z_p^n denote the set of polar of all *Lp*-intersection bodies, then $Z_p^n\subseteq S_c^n$. If $Q\in Z_p^n$ in (9), $\tilde{\Omega}_p^\circ(K)$ is written by

$$n^{-\frac{p}{n}}\tilde{\Omega}_p^\circ(K)^{\frac{n+p}{n}}=\sup\{n\tilde{V}_p(K,Q^*)V(Q)^{\frac{p}{n}}:Q\in Z_p^n\} \qquad (10)$$

According to (9) and (10), Wang *et al* [24] studied the *Lp*-dual affine surface area forms of the Busemann-Petty problems for the *Lp*-intersection bodies.

Theorem 1 For $K,L\in S_o^n$, $0<p<1$, if $I_pK\subseteq I_pL$, then $\tilde{\Omega}_p^\circ(K) \quad \tilde{\Omega}_p^\circ(L)$, with the equality if and only if $I_pK=I_pL$.

Theorem 2 For $K\in S_o^n$ and $0<p<1$, if K is not origin-symmetric, then there exists $L\in S_{os}^n$, such that $I_pK\subset I_pL$, but $\tilde{\Omega}_p^\circ(K)>\tilde{\Omega}_p^\circ(L)$.

In this paper, associated with *Lp*-dual affine surface area, we will investigate the Busemann-Petty problem for the general *Lp*-intersection bodies.

For the convenience of our work, we improve definition (10) as follows : Let $Z_p^{\tau,n}$ denote the set of polar of all general *Lp*-intersection bodies, for $K\in S_o^n$, $0<p<1$ and $\tau\in[-1,1]$, the *Lp*-dual affine surface area, $\tilde{\Omega}_p^\bullet(K)$, of K is given by

$$n^{-\frac{p}{n}}\tilde{\Omega}_p^\bullet(K)^{\frac{n+p}{n}}=\sup\{n\tilde{V}_p(K,Q^*)V(Q)^{\frac{p}{n}}:Q\in Z_c^{\tau,n}\} \qquad (11)$$

When $\tau=0$, definition (11) is just definition (10).

Especially, when $Q\in S_{os}^n$ in (9), we write $\tilde{\Omega}_p^*(K)$ by

$$n^{-\frac{p}{n}}\tilde{\Omega}_p^*(K)^{\frac{n+p}{n}}=\sup\{n\tilde{V}_p(K,Q^*)V(Q)^{\frac{p}{n}}:Q\in S_{os}^n\} \qquad (12)$$

According to definition (11), we first give an affirmative form of the Busemann-Petty problem for general *Lp*-intersection bodies, i.e., a general form of Theorem 1 is obtained.

Theorem 3 For $K,L\in S_o^n$, $\tau\in[-1,1]$ and $0<p$

<1, if $I_p^{\tau}K \subseteq I_p^{\tau}L$, then

$$\tilde{\Omega}_p^{\bullet}(K) \quad \tilde{\Omega}_p^{\bullet}(L) \tag{13}$$

with equality if and only if $I_p^{\tau}K = I_p^{\tau}L$.

Next, combining with definition (12), we get a negative form of the Busemann-Petty problem for the general *Lp*-intersection bodies.

Theorem 4 For $K \in S_{\text{o}}^n$, $\tau \in (-1,1)$ and $0 < p < 1$, if K is not origin-symmetric, then there exists $L \in S_{\text{o}}^n$, such that $I_p^{\tau}K \subset I_p^{\tau}L$, but $\tilde{\Omega}_p^*(K) > \tilde{\Omega}_p^*(L)$.

Finally, corresponding to Theorem 2, we extend a negative form of the Busemann-Petty problem for the *Lp*-intersection bodies from $L \in S_{\text{os}}^n$ to $L \in S_{\text{o}}^n$.

Theorem 5 For $K \in S_{\text{o}}^n$, $0 < p < 1$, if K is not origin-symmetric, then there exists $L \in S_{\text{o}}^n$, such that $I_pK \subset I_pL$, but $\tilde{\Omega}_p^*(K) > \tilde{\Omega}_p^*(L)$.

The proofs of Theorems 3-4 will be completed in Section 2 of this paper.

1 Preliminaries

1.1 *Lp*-Dual Mixed Volume

The notion[6,25] of *Lp*-dual mixed volume was introduced as follows: For $K, L \in S_{\text{o}}^n$ and real number $p > 0$, the *Lp*-dual mixed volume, $\tilde{V}_p(K,L)$, of K of L is defined by

$$\tilde{V}_p(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho_K(u)^{n-p}\rho_L(u)^p \,\mathrm{d}u \tag{14}$$

From (14), we easily know that

$$\tilde{V}_p(K,K) = V(K) = \frac{1}{n}\int_{S^{n-1}} \rho_K(u)^n \,\mathrm{d}u \tag{15}$$

1.2 General *Lp*-Dual Blaschke Body

For $K, L \in S_{\text{o}}^n$, $0 < p < n$ and $\lambda, \mu \geqslant 0$ (not both zero), the *Lp*-dual Blaschke combination, $\lambda \otimes K \check{+}_p \mu \otimes L$, of K and L is defined by

$$\rho(\lambda \otimes K \check{+}_p \mu \otimes L, \bullet)^{n-p} = \lambda\rho(K,\bullet)^{n-p} + \mu\rho(L,\bullet)^{n-p} \tag{16}$$

Here, $\lambda \otimes K = \lambda^{\frac{1}{n-p}}K$. Take $\lambda = \mu = \frac{1}{2}$, $L = -K$ in $\lambda \otimes K \check{+}_p \mu \otimes L$, then the *Lp*-dual Blaschke body, $\overline{\nabla}_p K$, of K is defined by

$$\overline{\nabla}_p K = \frac{1}{2} \otimes K \check{+} \frac{1}{2} \otimes (-K) \tag{17}$$

Obviously, the *Lp*-dual Blaschke body $\overline{\nabla}_p K$ is origin-symmetric.

Associated with (6), Wang and Li[4] gave the notion of general *Lp*-dual Blaschke body. For $K \in S_{\text{o}}^n$, $p > 0$ and $\tau \in [-1,1]$, the general *Lp*-dual Blaschke body, $\overline{\nabla}_p^{\tau} K \in S_{\text{o}}^n$, of K is defined by

$$\rho(\overline{\nabla}_p^{\tau} K, \bullet)^{n-p} = f_1(\tau)\rho(K,\bullet)^{n-p} + f_2(\tau)\rho(-K,\bullet)^{n-p} \text{ i.e.,}$$

$$\overline{\nabla}_p^{\tau} K = f_1(\tau) \otimes K \check{+}_p f_2 \otimes (-K) \tag{18}$$

Here $f_1(\tau)$ and $f_2(\tau)$ satisfy (6). Obviously, if $\tau = 0$, then $\overline{\nabla}_p^{\tau} K = \overline{\nabla}_p K$.

2 Proofs of Theorems

In this section, we will complete the proofs of Theorems 3-5. In order to prove Theorem 3, we require a lemma as follows:

Lemma 1 [4] If $K, L \in S_{\text{o}}^n$, $0 < p < 1$, and $\tau \in [-1,1]$, then $\tilde{V}_p(K, I_p^{\tau}L) = \tilde{V}_p(L, I_p^{\tau}K)$.

Proof of Theorem 3 From (14), since $I_p^{\tau}K \subseteq I_p^{\tau}L$, thus for any $Q \in S_{\text{o}}^n$,

$$\tilde{V}_p(Q, I_p^{\tau}K) \quad \tilde{V}_p(Q, I_p^{\tau}L) \tag{19}$$

with equality if and only if $I_p^{\tau}K = I_p^{\tau}L$.

Therefore, from Lemma 1, we have

$$\tilde{V}_p(K, I_p^{\tau}Q) \quad \tilde{V}_p(L, I_p^{\tau}Q) \tag{20}$$

Let $M^* = I_p^{\tau}Q$, then $M \in Z_p^{\tau,n}$. From (11) and (20), we get

$$\begin{aligned} & n^{-\frac{p}{n}}\tilde{\Omega}_p^{\bullet}(K)^{\frac{n+p}{n}} \\ & = \sup\{n\tilde{V}_p(K,M^*)V(M)^{\frac{p}{n}}, M \in Z_p^{\tau,n}\} \\ & \quad \sup\{n\tilde{V}_p(L,M^*)V(M)^{\frac{p}{n}}, M \in Z_p^{\tau,n}\} \\ & = n^{-\frac{p}{n}}\tilde{\Omega}_p^{\bullet}(L)^{\frac{n+p}{n}} \end{aligned}$$

i.e., (13) is obtained.

According to the equality of (19), we know that the equality holds in (13) if and only if $I_p^{\tau}K = I_p^{\tau}L$.

In order to prove the negative form for the Busemann-Petty type problem, we need the following two lemmas.

Lemma 2 If $K, L \in S_{\text{o}}^n$, $0 < p < 1$ and $\tau \in (-1,1)$, then

$$\tilde{\Omega}_p^*(f_1(\tau) \otimes K \check{+}_p f_2(\tau) \otimes L)^{\frac{n+p}{n}}$$

$$f_1(\tau)\tilde{\Omega}_p^*(K)^{\frac{n+p}{n}} + f_2(\tau)\tilde{\Omega}_p^*(L)^{\frac{n+p}{n}} \tag{21}$$

with equality if and only if K and L are dilates.

Proof From (12), (14) and (16), we have

$$n^{-\frac{p}{n}}\tilde{\Omega}_p^*(f_1(\tau)\otimes K \check{+}_p f_2(\tau)\otimes L)^{\frac{n+p}{n}}$$
$$=\sup\{n\tilde{V}_p(f_1(\tau)\otimes K \check{+}_p f_2(\tau)\otimes L, M)V(M^*)^{\frac{p}{n}}, M\in S_{\mathrm{os}}^n\}$$
$$=\sup\{nf_1(\tau)\tilde{V}_p(K,M)V(M^*)^{\frac{p}{n}}$$
$$+nf_2(\tau)\tilde{V}_p(L,M)V(M^*)^{\frac{p}{n}}: M\in S_{\mathrm{os}}^n\}$$
$$\sup\{nf_1(\tau)\tilde{V}_p(K,M)V(M^*)^{\frac{p}{n}}: M\in S_{\mathrm{os}}^n\}$$
$$+\sup\{nf_2(\tau)\tilde{V}_p(L,M)V(M^*)^{\frac{p}{n}}: M\in S_{\mathrm{os}}^n\}$$
$$=f_1(\tau)n^{-\frac{p}{n}}\tilde{\Omega}_p^*(K)^{\frac{n+p}{n}}+f_2(\tau)n^{-\frac{p}{n}}\tilde{\Omega}_p^*(L)^{\frac{n+p}{n}}$$

Thus

$$\tilde{\Omega}_p^*(f_1(\tau)\otimes K \check{+}_p f_2(\tau)\otimes L)^{\frac{n+p}{n}}$$
$$f_1(\tau)\tilde{\Omega}_p^*(K)^{\frac{n+p}{n}}+f_2(\tau)\otimes\tilde{\Omega}_p^*(L)^{\frac{n+p}{n}}.$$

The equality holds if and only if $f_1(\tau)\otimes K \check{+}_p f_2(\tau)\otimes L$ are dilates with K and L, respectively. This means that equality holds in (21) if and only if K and L are dilates.

Corollary 1 If $K\in S_{\mathrm{o}}^n$, $0<p<1$ and $\tau\in(-1,1)$, then

$$\tilde{\Omega}_p^*(\overline{\nabla}_p^\tau K)\quad \tilde{\Omega}_p^*(K) \tag{22}$$

with equality if and only if K is origin-symmetric.

Proof Taking $L=-K$ in (21), by (18) we get

$$\tilde{\Omega}_p^*(\overline{\nabla}_p^\tau K)^{\frac{n+p}{n}}\qquad f_1(\tau)\tilde{\Omega}_p^*(K)^{\frac{n+p}{n}}+f_2(\tau)\tilde{\Omega}_p^*(-K)^{\frac{n+p}{n}} \tag{23}$$

According to the equality condition of (21), we see that equality holds in (23) if and only if K and $-K$ are dilates, i.e., K is origin-symmetric.

Since $M\in S_{\mathrm{os}}^n$, thus M is origin-symmetric, i.e., $\rho_M(u)=\rho_{-M}(u)=\rho_M(-u)$ for all $u\in S^{n-1}$. From this, by (14) we get

$$\tilde{V}_p(-K,M)=\frac{1}{n}\int_{S^{n-1}}\rho_{-K}(u)^{n-p}\rho_M(u)^p\,\mathrm{d}u$$
$$=\frac{1}{n}\int_{S^{n-1}}\rho_K(-u)^{n-p}\rho_M(-u)^p\,\mathrm{d}u$$
$$=\tilde{V}_p(K,M) \tag{24}$$

Thus, associated with (12) and (24), we have

$$\tilde{\Omega}_p^*(-K)=\tilde{\Omega}_p^*(K) \tag{25}$$

Therefore, from (23) and (25), we know

$$\tilde{\Omega}_p^*(\overline{\nabla}_p^\tau K)\quad \tilde{\Omega}_p^*(K)$$

According to the equality condition of (23), we easily see that equality holds in (22) if and only if K is an origin-symmetric star body.

Lemma 3 [4] If $K\in S_{\mathrm{o}}^n$, $0<p<1$ and $\tau\in[-1,1]$, then

$$I_p^+(\overline{\nabla}_p^\tau K)=I_p^\tau K \tag{26}$$

and

$$I_p^-(\overline{\nabla}_p^\tau K)=I_p^{-\tau}K \tag{27}$$

Proof of Theorem 4 Since K is not origin-symmetric, so from Corollary 1, we know that for $\tau\in(-1,1)$, $\tilde{\Omega}_p^*(\overline{\nabla}_p^\tau K)<\tilde{\Omega}_p^*(K)$.

Choose $\varepsilon>0$, such that $\tilde{\Omega}_p^*((1+\varepsilon)\overline{\nabla}_p^\tau K)<\tilde{\Omega}_p^*(K)$. Therefore, let $L=(1+\varepsilon)\overline{\nabla}_p^\tau K\in S_{\mathrm{o}}^n$, then $\tilde{\Omega}_p^*(L)<\tilde{\Omega}_p^*(K)$.

But by (26) and notice $n>p$ ($0<p<1$), then

$$\rho(I_p^+L,\bullet)=\rho(I_p^+(1+\varepsilon)\overline{\nabla}_p^\tau K,\bullet)$$
$$=\rho((1+\varepsilon)^{\frac{n-p}{n}}I_p^+\overline{\nabla}_p^\tau K,\bullet)$$
$$=\rho((1+\varepsilon)^{\frac{n-p}{n}}I_p^\tau K,\bullet)>\rho(I_p^\tau K,\bullet) \tag{28}$$

Similarly, from (27), we obtain

$$\rho(I_p^-L,\bullet)>\rho(I_p^{-\tau}K,\bullet) \tag{29}$$

Notice that $\tau\in(-1,1)$ is equivalent to $-\tau\in(-1,1)$, then by (29) we can get

$$\rho(I_p^-L,\bullet)>\rho(I_p^\tau K,\bullet) \tag{30}$$

From (28) and (30), and combined with (5), we have that for $\tau\in(-1,1)$,

$$\rho(I_p^\tau K,\bullet)^p<\rho(I_p^\tau L,\bullet)^p,$$

i.e.,

$$I_p^\tau K\subset I_p^\tau L.$$

The proof of Theorem 5 requires the following a lemma.

Lemma 4 If $K\in S_{\mathrm{o}}^n$, $0<p<1$ and $\tau\in[-1,1]$, then

$$I_p\overline{\nabla}_p^\tau K=I_pK \tag{31}$$

Proof From (1), (18) and (7), and notice $I_p(-K)=I_pK$, we have that for all $u\in S^{n-1}$,

$$\rho_{I_p\overline{\nabla}_p^\tau K}^p(u)=\frac{1}{n-p}\int_{S^{n-1}}|u\cdot v|^{-p}\rho_{\overline{\nabla}_p^\tau K}^{n-p}(v)\mathrm{d}S(v)$$
$$=f_1(\tau)\rho_{I_pK}^p(u)+f_2(\tau)\rho_{I_p(-K)}^p(u)=\rho_{I_pK}^p(u)$$

This yields (31).

Proof of Theorem 5 Similar to proof of Theorem 4, by (31) and compare (28), we easily complete the proof of Theorem 5.

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