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Hamiltonian Theory for the DNLS Equation with a Squared Spectral Parameter

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Abstract: With a special gauge transformation, the Lax pair of the derivative nonlinear Shcrödinger (DNLS) equation turns to depend on the squared parameter $\lambda = k^2$ instead of the usual spectral parameter *k*. By introducing a new direct product of Jost solutions, the complete Hamiltonian theory of the DNLS equation is constructed on the basis of the squared spectral parameter, which shows that the integrability completeness is still preserved. This result will be beneficial to the further study of the DNLS equation, such as the direct perturbation method.

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0 Introduction

The derivative nonlinear Schrödinger (DNLS) equation has been recognized as a model for the change of form of weakly nonlinear and dispersive parallel magnetohydrodynamics (MHD) waves[1], which can well describe waves whose propagation is strictly parallel to the ambient magnetic field. It has been found that many physical phenomena, such as Alfven waves in space plasma[2-8], subpicosecond, or femtosecond pulses in single-mold optical fibers, and the weak nonlinear electromagnetic waves in ferromagnetic, antiferromagnetic, or dielectric systems under external magnetic fields, can be described with the DNLS equation[9]. In 1978, Kaup *et al*[10] first solved the DNLS equation with vanishing boundary condition (VBC) through the inverse scattering transform (IST) method constructed on complex *k*-plane[11], where *k* is the usual spectral parameter. In later years, a lot of literature was published in succession, and most of them tried to give 1-soliton and *N*-soliton solutions, as well as the related properties[12-18]. However, all the authors selected the usual spectral parameter *k* as the basic parameter and neglected to choose the squared spectral parameter $\lambda(\lambda = k^2)$ instead. Through learning the properties of the DNLS equation, we use the squared spectral parameter λ as the basic parameter to construct the Hamiltonian system of the DNLS equation.

If we choose the squared spectral parameter λ to replace the usual spectral parameter k[12, 19], the poles in the IST should no longer be found in both the 1st and 3rd quadrants but should appear in the upper half complex plane only. As a result, those ambiguities coming from the analytical properties of Jost solutions can be avoided, and the whole computation can be simplified. Specifically, in the case of solving the perturbed DNLS equation[18, 20-22], we find that using the $\lambda = k^2$ as spectrum parameter instead of k itself has some advantages. Compared with the NLS equation, the DNLS equation has several squared items that imply that choosing λ as the basic parameter is necessary. To verify the reasonableness of using λ as the basic parameter, we develop the Hamiltonian theory because this system is of complete integrability.

Therefore, it is appropriate to choose the squared spectral parameter λ as the basic parameter. The single soliton solution for the DNLS equation with VBC has been obtained through the Marchenko equation[23], which can be verified finally by direct substitution into the nonlinear equations. In Section 1, we give the basic formulas and through the gauge transformation rewrite those formulas first. From Section 2 to Section 4, we develop the Hamiltonian theory of the DNLS with λ in the cases of continuous and discrete spectrum, respectively, and give the conclusion.

1 Some Basic Formulas and Gauge Transformation

In 1978, Kaup *et al*[10] developed IST with some revisions for solving the DNLS equation with vanishing boundary conditions:

$$\mathrm{i}u_t + u_{xx} + \mathrm{i}(|u|^2 u)_x = 0 \tag{1}$$

with $u(x,k) \to 0$ as $x \to \infty$. The associated eigenvalue problem is given by the modified Zakharov-Shabat conditions of compatibility[23].

For the Lax pairs of DNLS equation, we have

$$\boldsymbol{L} = -\mathrm{i}k^2\boldsymbol{\sigma}_3 + k\boldsymbol{U},\ \boldsymbol{U} = \begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix} \tag{2a}$$

$$\boldsymbol{M} = -\mathrm{i}2k^4\boldsymbol{\sigma}_3 + 2k^3\boldsymbol{U} - \mathrm{i}k^2\boldsymbol{U}^2\boldsymbol{\sigma}_3 - k(-\boldsymbol{U}^3 + \mathrm{i}\boldsymbol{U}_x\boldsymbol{\sigma}_3) \tag{2b}$$

where k is the usual spectral parameter. According to the definition of the Jost solutions and their asymptotic behavior, we have the expression of the free Jost solution:

$$\boldsymbol{E}(x,k) = \mathrm{e}^{-\mathrm{i}k^2 x\sigma_3} \tag{3}$$

where $|x| \to \pm\infty,\ u \to 0,\ \boldsymbol{L} \to -\mathrm{i}k^2\boldsymbol{\sigma}_3$. Through (3), it is obvious that the basic parameter of the free Jost solution is k^2. Therefore, we suppose $\lambda = k^2$ as the basic parameter to replace k. Another problem is that k cannot be instead of λ directly, so we can make the gauge transformation as follows:

Because of the gauge transformation,

$$\boldsymbol{A}^{-1}(k)\boldsymbol{L}(k)\boldsymbol{A}(k) = \boldsymbol{L}(\lambda),\ \ \boldsymbol{A} = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \tag{4}$$

the Lax pairs (2a) can be rewritten as

$$\boldsymbol{L}(x,\lambda) = -\mathrm{i}\lambda\boldsymbol{\sigma}_3 + \boldsymbol{\Lambda U} = \begin{pmatrix} -\mathrm{i}\lambda & u \\ -\lambda\bar{u} & \mathrm{i}\lambda \end{pmatrix},\ \boldsymbol{\Lambda} = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \tag{2a$'$}$$

Be similar to (2a′), the expression of $\boldsymbol{M}$ can be obtained easily:

$$\boldsymbol{M} = -\mathrm{i}2\lambda^2\boldsymbol{\sigma}_3 + 2\lambda\boldsymbol{\Lambda U} - \mathrm{i}\lambda\boldsymbol{U}^2\boldsymbol{\sigma}_3 - \boldsymbol{\Lambda}(-\boldsymbol{U}^3 + \mathrm{i}\boldsymbol{U}_x\boldsymbol{\sigma}_c) \tag{2b$'$}$$

As $|x| \to \pm\infty,\ u \to 0$ and $\boldsymbol{L} \to -\mathrm{i}\lambda\boldsymbol{\sigma}_3$, the free Jost solution is expressed as

$$\boldsymbol{E}(x,\lambda) = \mathrm{e}^{-\mathrm{i}\lambda x\sigma_3} \tag{3$'$}$$

Moreover, it can be expressed as two independent solutions:

$$\boldsymbol{E}_{\cdot 1}(x,\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathrm{e}^{-\mathrm{i}\lambda x},\ \ \boldsymbol{E}_{\cdot 2}(x,\lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathrm{e}^{\mathrm{i}\lambda x} \tag{5}$$

Therefore, the Jost solutions are defined as

$$\begin{cases} \boldsymbol{\Psi}(x,\lambda) \to \boldsymbol{E}(x,\lambda),\ x \to +\infty \\ \boldsymbol{\Phi}(x,\lambda) \to \boldsymbol{E}(x,\lambda),\ x \to -\infty \end{cases} \tag{6}$$

Moreover, through the gauge transformation, the 2×2 matrix solution changes into

$$\boldsymbol{\Psi}(x,\lambda) = \begin{pmatrix} \tilde{\psi}_1(x,\lambda) & \lambda^{-1}\psi_1(x,\lambda) \\ \tilde{\psi}_2(x,\lambda) & \psi_2(x,\lambda) \end{pmatrix},$$

$$\boldsymbol{\Phi}(x,\lambda) = \begin{pmatrix} \phi_1(x,\lambda) & \lambda^{-1}\tilde{\phi}_1(x,\lambda) \\ \phi_2(x,\lambda) & \tilde{\phi}_2(x,\lambda) \end{pmatrix} \tag{7}$$

Obviously, because of the different asymptotic behavior, we can separate (7) into two column matrixes as $\boldsymbol{\Psi}(x,\lambda)=(\tilde{\boldsymbol{\psi}}(x,\lambda),\boldsymbol{\psi}(x,\lambda))$ and $\boldsymbol{\Phi}(x,\lambda)=(\boldsymbol{\phi}(x,\lambda),\tilde{\boldsymbol{\phi}}(x,\lambda))$. Since there are only two independent Jost solutions when λ is real, the unitary matrix $\boldsymbol{T}(\lambda)$ is introduced as

$$\boldsymbol{\Phi}(x,\lambda) = \boldsymbol{\Psi}(x,\lambda)\boldsymbol{T}(\lambda) \tag{8}$$

where $\boldsymbol{T}(\lambda) = \begin{pmatrix} a(\lambda) & -\lambda^{-1}\tilde{b}(\lambda) \\ b(\lambda) & \tilde{a}(\lambda) \end{pmatrix}$.

2 Poisson Brackets

2.1 Variation Operation $\delta u(x)$

According to equation (1), u is the complex field; therefore, a particular form of Poisson bracket is introduced:

$$\{u(x),\overline{u(y)}\} = -\frac{1}{2}\{\partial_x - \partial_y\}\delta(x-y) \tag{9}$$

which is extension of that for two real field densities at two points in the case of Korteweg-de Vries (KdV). Moreover, the Poisson bracket for the quantities Q and R

can be expressed as

$$\{Q,R\}=\frac{1}{2}\int \mathrm{d}x(\{\partial_x \frac{\delta Q}{\delta u(x)}\}\frac{\delta R}{\delta\overline{u(x)}}-\frac{\delta Q}{\delta u(x)}\{\partial_x \frac{\delta R}{\delta\overline{u(x)}}\}$$
$$+\{\partial_x \frac{\delta Q}{\delta\overline{u(x)}}\}\frac{\delta R}{\delta u(x)}-\frac{\delta Q}{\delta\overline{u(x)}}\{\partial_x \frac{\delta R}{\delta u(x)}\}) \qquad (10)$$

To be specific, the variation operation $\delta u(x)$ satisfy

$$\frac{\delta \boldsymbol{L}(z,\lambda)}{\delta u(x)}=\boldsymbol{\sigma}_+\delta(x-z)\,,\ \frac{\delta \boldsymbol{L}(z,\lambda)}{\delta\overline{u(x)}}=-\lambda\boldsymbol{\sigma}_-\delta(x-z)\,,$$

$$\frac{\delta \boldsymbol{T}(\lambda)}{\delta u(z)}=\boldsymbol{\Psi}^{-1}(z,\lambda)\boldsymbol{\sigma}_+\boldsymbol{\Phi}(z,\lambda),$$

$$\partial_z\frac{\delta \boldsymbol{T}(\lambda)}{\delta u(z)}=\boldsymbol{\Psi}^{-1}(z,\lambda)(\mathrm{i}2\boldsymbol{\sigma}_+-\lambda\overline{u}\boldsymbol{\sigma}_3)\boldsymbol{\Phi}(z,\lambda) \qquad (11)$$

By the same method, we can obtain the expressions of $\boldsymbol{T}(\lambda)$ with respect to $\overline{u(z)}$ and those of $\boldsymbol{T}^{-1}(\lambda)$ with respect to $u(z)$ and $\overline{u(z)}$.

2.2 Variations of Monodramy Matrix $\boldsymbol{T}(\lambda)$

According to the basic Poisson bracket and the general definition of direct product for matrices, the following expression can be given[24]:

$$\{\boldsymbol{T}(\lambda)\overset{\otimes}{,}\boldsymbol{T}(\lambda')\}$$
$$=\begin{pmatrix} \{a,\tilde{a}'\} & \{a,\lambda'^{-1}\tilde{b}'\} & -\{\lambda^{-1}\tilde{b},\tilde{a}'\} & -\{\lambda^{-1}\tilde{b},\lambda'^{-1}\tilde{b}'\} \\ -\{a,b'\} & \{a,a'\} & \{\lambda^{-1}\tilde{b},b'\} & -\{\lambda^{-1}\tilde{b},a'\} \\ \{b,\tilde{a}'\} & \{b,\lambda'^{-1}\tilde{b}'\} & \{\tilde{a},\tilde{a}'\} & \{\tilde{a},\lambda'^{-1}\tilde{b}'\} \\ -\{b,b'\} & \{b,a'\} & -\{\tilde{a},b'\} & \{\tilde{a},a'\} \end{pmatrix}$$

and the integrable equation can be expressed as

$$\{\boldsymbol{T}(\lambda)\overset{\otimes}{,}\boldsymbol{T}(\lambda')\}$$
$$=\frac{1}{2}\int \mathrm{d}x[\boldsymbol{\Psi}^{-1}(x,\lambda)\boldsymbol{\Phi}^{-1}(x,\lambda')\boldsymbol{R}\boldsymbol{\Phi}(x,\lambda)\boldsymbol{\Psi}(x,\lambda')] \qquad (12)$$

where

$$\boldsymbol{R}=(\mathrm{i}2\lambda\boldsymbol{\sigma}_+-\lambda\overline{u}\boldsymbol{\sigma}_3)\otimes(\lambda'\boldsymbol{\sigma}_-)+\boldsymbol{\sigma}_+\otimes(\mathrm{i}2\lambda'^2\boldsymbol{\sigma}_-+\lambda'u\boldsymbol{\sigma}_3)$$
$$-(\mathrm{i}2\lambda^2\boldsymbol{\sigma}_-+\lambda u\boldsymbol{\sigma}_3)\otimes\boldsymbol{\sigma}_+-(\lambda\boldsymbol{\sigma}_-)\otimes(\mathrm{i}2\lambda'\boldsymbol{\sigma}_+-\lambda'u\boldsymbol{\sigma}_3) \qquad (13)$$

According to the usual manner of making up the Hamiltonian theory, we can consider[24]

$$\partial_x\left(\boldsymbol{\Psi}^{-1}(x,\lambda)\boldsymbol{\Psi}(x,\lambda')\otimes'\boldsymbol{\Phi}^{-1}(x,\lambda')\boldsymbol{\Phi}(x,\lambda)\right)$$
$$=\boldsymbol{\Psi}^{-1}(x,\lambda)\boldsymbol{\Phi}^{-1}(x,\lambda')\boldsymbol{W}\boldsymbol{\Phi}(x,\lambda)\boldsymbol{\Psi}(x,\lambda') \qquad (14)$$

In formula (14), we use the symbol of another direct product "⊗′" for matrix[25] to simplify the integral calculation in (12). The usual spectral parameter k has been instead of the squared spectral parameter λ, therefore, the symmetry of the original matrix is broken. To construct the Hamiltonian theory by the method that has been offered in Ref.[26], we must consider the other factors. First, we definite two special matrices:

$$\boldsymbol{\Gamma}_1=\begin{pmatrix}\lambda & 0\\ 0 & \lambda'\end{pmatrix},\ \boldsymbol{\Gamma}_2=\begin{pmatrix}\lambda' & 0\\ 0 & \lambda\end{pmatrix} \qquad (15)$$

then,

$$\partial_x(\boldsymbol{\Psi}^{-1}(x,\lambda)\boldsymbol{\Gamma}_1\boldsymbol{\Psi}(x,\lambda')\otimes'\boldsymbol{\Phi}^{-1}(x,\lambda')\boldsymbol{\Gamma}_2\boldsymbol{\Phi}(x,\lambda))$$
$$=\boldsymbol{\Psi}^{-1}(x,\lambda)\boldsymbol{\Phi}^{-1}(x,\lambda')\boldsymbol{W}_+\boldsymbol{\Phi}(x,\lambda)\boldsymbol{\Psi}(x,\lambda') \qquad (16a)$$

$$\partial_x(\boldsymbol{\Psi}^{-1}(x,\lambda)\boldsymbol{\Gamma}_2\boldsymbol{\Psi}(x,\lambda')\otimes'\boldsymbol{\Phi}^{-1}(x,\lambda')\boldsymbol{\Gamma}_1\boldsymbol{\Phi}(x,\lambda))$$
$$=\boldsymbol{\Psi}^{-1}(x,\lambda)\boldsymbol{\Phi}^{-1}(x,\lambda')\boldsymbol{W}_-\boldsymbol{\Phi}(x,\lambda)\boldsymbol{\Psi}(x,\lambda') \qquad (16b)$$

$$\partial_x(\boldsymbol{\Psi}^{-1}(x,\lambda)\sigma_3\boldsymbol{\Gamma}_1\boldsymbol{\Psi}(x,\lambda')\otimes'\boldsymbol{\Phi}^{-1}(x,\lambda')\sigma_3\boldsymbol{\Gamma}_2\boldsymbol{\Phi}(x,\lambda))$$
$$=\boldsymbol{\Psi}^{-1}(x,\lambda)\boldsymbol{\Phi}^{-1}(x,\lambda')\boldsymbol{W}_3\boldsymbol{\Phi}(x,\lambda)\boldsymbol{\Psi}(x,\lambda') \qquad (16c)$$

where $\boldsymbol{W}_+=(\lambda-\lambda')$

$$\cdot\begin{pmatrix} 0 & \lambda'u & -\lambda u & 0 \\ -(\lambda+\lambda')\lambda\overline{u} & 0 & \mathrm{i}2\lambda^2 & \lambda u \\ (\lambda+\lambda')\lambda'\overline{u} & -\mathrm{i}2\lambda'^2 & 0 & -\lambda'u \\ 0 & -(\lambda+\lambda')\lambda'\overline{u} & (\lambda+\lambda')\lambda\overline{u} & 0 \end{pmatrix}$$

$$\boldsymbol{W}_-=(\lambda-\lambda')\begin{pmatrix} 0 & -\lambda u & \lambda'u & 0 \\ 0 & 0 & \mathrm{i}2\lambda'^2 & -\lambda'u \\ 0 & -\mathrm{i}2\lambda^2 & 0 & \lambda u \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{W}_3=$$
$$\begin{pmatrix} 0 & (\lambda+\lambda')\lambda'u & (\lambda+\lambda')\lambda u & 0 \\ (\lambda^2+\lambda'^2)\lambda\overline{u} & 0 & -\mathrm{i}2(\lambda-\lambda')\lambda^2 & -(\lambda+\lambda')\lambda u \\ (\lambda^2+\lambda'^2)\lambda'\overline{u} & \mathrm{i}2(\lambda-\lambda')\lambda'^2 & 0 & -(\lambda+\lambda')\lambda'u \\ 0 & -(\lambda^2+\lambda'^2)\lambda'\overline{u} & -(\lambda^2+\lambda'^2)\lambda\overline{u} & 0 \end{pmatrix}$$

Finally, we find

$$\boldsymbol{R}=f_+\boldsymbol{W}_-+f_-\boldsymbol{W}_-+f_3\boldsymbol{W}_3 \qquad (17)$$

and we can also obtain

$$f_+=\frac{1}{2}\frac{1}{\lambda-\lambda'},\ f_-=\frac{\lambda^2+\lambda\lambda'+\lambda'^2}{(\lambda-\lambda')(\lambda^2+\lambda'^2)},\ f_3=\frac{1}{2}\frac{\lambda-\lambda'}{\lambda^2+\lambda'^2} \qquad (18)$$

Substituting (17), (18), and (14) into (12), we obtain

$$\{\boldsymbol{T}(\lambda)\overset{\otimes}{,}\boldsymbol{T}(\lambda')\}=\{f_+\boldsymbol{\Psi}^{-1}(x,\lambda)\boldsymbol{\Gamma}_1\boldsymbol{\Psi}(x,\lambda')\otimes'\boldsymbol{\Phi}^{-1}(x,\lambda')\boldsymbol{\Gamma}_2\boldsymbol{\Phi}(x,\lambda)$$
$$+f_-\boldsymbol{\Psi}^{-1}(x,\lambda)\boldsymbol{\Gamma}_2\boldsymbol{\Psi}(x,\lambda')\otimes'\boldsymbol{\Phi}^{-1}(x,\lambda')\boldsymbol{\Gamma}_1\boldsymbol{\Phi}(x,\lambda)$$
$$+f_3\boldsymbol{\Psi}^{-1}(x,\lambda)\boldsymbol{\sigma}_3\boldsymbol{\Gamma}_1\boldsymbol{\Psi}(x,\lambda')\otimes'\boldsymbol{\Phi}^{-1}(x,\lambda')\boldsymbol{\sigma}_3\boldsymbol{\Gamma}_2\boldsymbol{\Phi}(x,\lambda)\} \qquad (19)$$

Comparing the expression of matrix (12) with (19), we obtain the result as

$$\{a(\lambda),b(\lambda')\}=\frac{1}{2}\frac{\lambda(\lambda^2+3\lambda'^2)}{\lambda'^2}\frac{\lambda+\lambda'}{\lambda-\lambda'+\mathrm{i}0}a(\lambda)b(\lambda') \qquad (20a)$$

$$\{\tilde{a}(\lambda),b(\lambda')\}=-\frac{1}{2}\frac{\lambda'(\lambda'^2+3\lambda^2)}{\lambda^2+\lambda'^2}\frac{\lambda+\lambda'}{\lambda-\lambda'-\mathrm{i}0}\tilde{a}(\lambda)b(\lambda') \qquad (20b)$$

$$\{|a(\lambda)|^2,b(\lambda')\}=-\mathrm{i}4\lambda^2\pi\delta(\lambda-\lambda')|a(\lambda)|^2\,b(\lambda) \qquad (20c)$$

3 Hamiltonian Theory Based on Action-Angle Variables

3.1 Continuous Spectrum

Among the Hamiltonian scheme, $a(\lambda)$ is the action variable that is not dependent on time t, while $b(\lambda)$ is the angle variable and $\arg b(\lambda)$ depended on t, namely,

$$b(t,\lambda) = b(0,\lambda)\mathrm{e}^{\mathrm{i}4\lambda^2 t} \tag{21}$$

The action variable $P(\lambda)$ is a function of $a(\lambda)$, and the angle variable $Q(\lambda)$ is proportional to the phase of $b(\lambda)$,

$$P(\lambda) = F(|a(\lambda)|^2),$$

$$Q(\lambda) = \arg b(\lambda) = \frac{1}{2\mathrm{i}}\ln\frac{b(\lambda)}{\tilde{b}(\lambda)} \tag{22}$$

Since

$$\{Q(\lambda), Q(\lambda')\} = \{P(\lambda), P(\lambda')\} = 0,$$

$$\{P(\lambda), Q(\lambda')\} = -\delta(\lambda - \lambda') \tag{23}$$

according to (20c), (22), and (23), we have

$$F'(|a(\lambda)|^2)4\pi\lambda^2 |a(\lambda)|^2 = 1 \tag{24}$$

where F' is derivative of F. Therefore, it leads to

$$P(\lambda) = F(|a(\lambda)|^2) = \frac{1}{4\pi\lambda^2}\ln|a(\lambda)|^2 \tag{25}$$

The Hamiltonian canonical equation demands

$$\{Q(\lambda), H\} = \partial_t Q(\lambda) = 4\lambda^2 \tag{26}$$

We assume the continuous Hamiltonian H_{c} has the form

$$H_{\mathrm{c}} = \int_{-\infty}^{\infty} \mathrm{d}\lambda f(\lambda)P(\lambda) \tag{27}$$

then

$$\{Q(\lambda), H\} = \int_{-\infty}^{\infty} \mathrm{d}\lambda' f(\lambda')\{Q(\lambda), P(\lambda')\} = f(\lambda) = 4\lambda^2 \tag{28}$$

Substituting (25) and (28) into (27), we obtain the expression of Hamiltonian in continuous spectrum as

$$H_{\mathrm{c}} = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda \ln|a(\lambda)|^2 \tag{29}$$

3.2 Discrete Spectrum

At first, we introduce the discrete action-angle variables P_n, Q_n, and the basic Poisson brackets are

$$\{P_n, P_m\} = 0, \quad \{Q_n, Q_m\} = 0, \quad \{P_n, Q_m\} = -\delta_{mn} \tag{30}$$

then, similar to the continuous spectrum, we obtain

$$b_n(t,\lambda_n) = b_n(0,\lambda_n)\mathrm{e}^{\mathrm{i}4\lambda_n^2 t},\ Q_n = \ln b_n = \ln|b_n|,$$

$$\partial_t Q_n = \mathrm{i}4\lambda_n^2 \tag{31}$$

The action variable is a function of λ_n, denoting as $P_n = G(\lambda_n)$, we have

$$\{P_n, Q_m\} = G'(\lambda_n)\frac{1}{b_m}\{\lambda_n, b_m\} \tag{32}$$

Assuming the discrete Hamiltonian has the form $H_{\mathrm{d}} = \sum_m f(P_m)$, and just as (26), we have

$$\{Q_n, H_{\mathrm{d}}\} = \sum_m f'(P_m)\{Q_n, P_m\} = f'(P_m) = -4P_m^{-2} \tag{33}$$

Like the continuous spectrum, we have

$$H(P_n) = f(P_n) = -4P_n^{-1} = -4\lambda_n \tag{34}$$

namely,

$$H_{\mathrm{d}} = \sum_m \frac{1}{3}(P_m^{\ 3} - \tilde{P}_m^{\ 3}) = \sum_m \frac{1}{3}(\lambda_m^{\ -3} - \tilde{\lambda}_m^{\ -3}) \tag{35}$$

Until now, we have derived the equations of the Hamilton system expressed by the action-angle variables.

In this section, we formulate the complete Hamiltonian theory for the DNLS equation with vanishing boundary conditions, in which (29) and (35) are expressed in continuous spectrum and discrete spectrum respectively. So far, through the construction of the Hamiltonian system, it shows that using the squared spectral parameter λ as an elementary parameter is reasonable.

4 Conclusion

In this paper, through the gauge transformation, we first introduce a new spectral parameter, the squared spectral parameter λ, to replace the usual spectral parameter k, and then develop the DNLS equation's Hamiltonian theory to justify that using λ as the basic parameter is reasonable. Taking the squared spectral parameter λ as the basic parameter is the key point of this paper. Comparing it with the simple spectral parameter, we derive all the formulas of Hamiltonian theory for the DNLS equation, which have the same modality as those derived by the simple spectral parameter. The success in applying the squared parameter here shows that a good choice of spectrum parameter is greatly helpful for IST method. Moreover, it is also beneficial to the perturbation computation of DNLS equation[20].

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