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Approximate Self-Similarity of a Class of Cookie-Cutter-Like Sets

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Abstract: According to the bounded variation theory and the bounded distortion property of cookie-cutter-like (CCL) sets, the approximate self-similarity of cookie-cutter-like sets satisfying certain conditions is studied. Based on the mean value theorem, it is proved that a class of special cookie-cutter-like sets is approximately self-similar. The results obtained in this paper extend the corresponding results that have already existed.

Key words: cookie-cutter-like set (CCL); strong separation condition; approximate self-similar set

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0 Introduction

Self-similar sets[1,2] are a class of important and typical fractals, the studies on which are the most rich and thorough[3-6]. While it is difficult in essence to research fractal sets without self-similarity, to which scholars have been addressed themselves, an important way to deal with such problems is to study them with the theory of self-similar. Bedford[7,8], Ruelle[9], and Falconer and Marsh[10] have respectively studied the dimensions of such approximate self-similar (or quasi-self-similar) sets as cookie-cutter sets, Julia sets, and quasi-cycles. By introducing Gibbs-like measures, Ma *et al*[11] calculated the Hausdorff dimension of cookie-cutter-like (CCL) sets by means of self-similar approximation. Based on Ref.[11], this paper studies the characters of any n-order-basic-intervals of a class of special cookie-cutter-like sets, and proves that these sets are approximately self-similar, using the bounded variation theory and the bounded distortion property of cookie-cutter-like sets. The preliminary results obtained can extend the corresponding results in Ref.[8]. In addition, more importantly, they can help studying whether there is invariant measure on cookie-cutter-like sets, and whether there is promoted form of ergodic theorem associated with such invariant measure. Besides, these results will do benefit to the studies on the lower and upper density of cookie-cutter-like sets.

1 Definitions and Notations

Definition 1 A mapping f is called a cookie-cutter (see Ref.[8]), if there exists a finite collection of disjoint closed intervals $J_1,\cdots,J_q \subset J=[0,1]$, such that

(a) f is defined in a neighborhood of each J_j, $1\leqslant j\leqslant q$, the restriction of f to each initial interval J_j is 1-1, and onto, the corresponding branch inverse is denoted by $\varphi_j=(f|_{J_j})^{-1}:J\to J_j$;

(b) f is differentiable with Holder continuous derivative Df, i.e., there exist constants $c_f>0$, and $\gamma_f\in(0,1]$ such that for $x,y\in J_j, 1\leqslant j\leqslant q$,

$$|Df(x)-Df(y)|\leqslant c_f|x-y|^{\gamma_f};$$

(c) f is boundedly expanding in the sense that:

$$1<b_f:=\inf_x\{|Df(x)|\}\leqslant\sup_x\{|Df(x)|\}:=B_f<+\infty.$$

$\left[\bigcup_{j=1}^{q}J_j;c_f,\gamma_f,b_f,B_f\right]$ is called the defining data of the cookie-cutter mapping f.

Now, consider a sequence of cookie-cutters $\{f_k\}_{k\geqslant1}$ with defining data $\left[\bigcup_{j=1}^{q_k}J_j^{(k)};c_k,\gamma_k,b_k,B_k\right]$. In this paper, we always assume that

$$c=\sup_{k\geqslant1}\{c_k\}\in(0,+\infty),\ \gamma=\inf_{k\geqslant1}\{\gamma_k\}\in(0,+\infty),$$

$$b=\inf_{k\geqslant1}\{b_k\}\in(1,+\infty),\ B=\sup_{k\geqslant1}\{B_k\}\in(1,+\infty).$$

Thus, $\forall k\geqslant1$, $x,y\in J_j^{(k)}(1\leqslant j\leqslant q_k)$, we have $|Df_k(x)-Df_k(y)|\leqslant c|x-y|^{\gamma}$, and

$$b\leqslant|Df_k(x)|\leqslant B \tag{1}$$

Let $1\leqslant m\leqslant n$, define the coding spaces by

$$\Omega_{m,n}=\prod_{k=m}^{n}\{1,2,\cdots,q_k\},\ \Omega_n=\Omega_{1,n},\ \Omega^*=\bigcup_{n=1}^{\infty}\Omega_n$$

An element $\sigma=(i_m,i_{m+1},\cdots,i_n)(i_k\in\{1,2,\cdots,q_k\},m\leqslant k\leqslant n)$ of the coding space $\Omega_{m,n}$ is called a code with length $|\sigma|=n-m+1$, and the concatenation of two codes $\sigma=(i_m,\cdots,i_n)$ and $\tau=(i_{n+1},\cdots,i_l)\in\Omega_{n+1,i}$ is a new code $\sigma*\tau=(i_m,\cdots,i_n,i_{n+1},i_l)$. $\forall\sigma\in\Omega^*$, σ^* is the code obtained by deleting the last letter of σ.

For $\sigma\in\Omega_n$, define the basic interval of order n corresponding to σ by

$$J_\sigma=J_{i_1,i_2,\cdots,i_n}=\varphi_{1,i_1}\circ\varphi_{2,i_2}\circ\cdots\circ\varphi_{n,i_n}(J)$$

where $1\leqslant i_k\leqslant q_k$, $1\leqslant k\leqslant n$, and φ_{k,i_k} is the corresponding branch inverse of f_k.

It is easy to verify that these basic intervals posses the following net properties:

① $J_{\sigma*j}\subset J_\sigma$, for each $\sigma\in\Omega_n$, $1\leqslant j\leqslant q_{n+1}$;

② $J_{\sigma_1}\cap J_{\sigma_2}=\varnothing$, if $\sigma_1,\sigma_2\in\Omega_n$, and $\sigma_1\neq\sigma_2$, $n\geqslant1$;

③ $|J_\sigma|\to0$ $(|\sigma|\to\infty)$, since $B^{-|\sigma|}\leqslant|J_\sigma|\leqslant b^{-|\sigma|}$.

For any $n\geqslant1$, $\sigma=(i_1,i_2,\cdots,i_n)\in\Omega_n$, where $1\leqslant i_n\leqslant q_n-1$, let

$$d_\sigma\triangleq\mathrm{dist}(J_\sigma,J_{\sigma^**(i_n+1)})$$

Then we shall use the notation $d^{(n)}=\max\{d_\sigma;\sigma=(i_1,\cdots,i_n)\in\Omega_n,1\leqslant i_n\leqslant q_n-1\}$ to denote the maximal gap between the basic intervals of order n.

Now let $E=\bigcap_{n=1}^{\infty}\bigcup_{\sigma\in\Omega_n}J_\sigma$, with the net properties above, we can conclude that E is a perfect, nowhere dense and totally disconnected subset of J, then set E is called a cookie-cutter-like(CCL) set satisfying the strong separation condition, which is generated by the cookie-cutter sequence $\{f_k\}_{k\geqslant1}$.

2 Main Results and Proofs

The following lemmas will be used in the proofs of the main results, which are called respectively the bounded variation theory and the bounded distortion property.

Lemma 1 (see Ref.[11]) There exists a constant $0<\xi<\infty$ such that for each $n\geqslant1$, $\sigma=(i_1,\cdots,i_n)\in\Omega_n$, and $x,y\in J_\sigma$, we have

$$\xi^{-1}\leqslant\frac{|DF_n(x)|}{|DF_n(y)|}\leqslant\xi$$

where $F_n(x)=f_n\circ f_{n-1}\circ\cdots\circ f_1(x)$.

Lemma 2 (see Ref.[11]) For any $n\geqslant1$, $\sigma\in\Omega_n$, $x\in J_\sigma$, we have

$$\xi^{-1}\leqslant|J_\sigma|\cdot|DF_n(x)|\leqslant\xi$$

Moreover, for each $1\leqslant j\leqslant q_{n+1}$, we get $|J_{\sigma*j}|\geqslant\xi^{-1}B^{-1}|J_\sigma|$, where ξ is the constant of Lemma 1.

Theorem 1 Let E be a CCL set satisfying strong separation condition, and $\{d^{(n)}\}_{n\geqslant1}$ are descending series. Then

(a) For any $n\geqslant1$, $\sigma\in\Omega_n$,

$$B^{-1}\xi^{-1}|J_\sigma|\leqslant\mathrm{dist}_{1\leqslant j\leqslant q_{n+1}-1}(J_{\sigma*j},J_{\sigma*(j+1)})\leqslant|J_\sigma|;$$

(b) Suppose that $\lambda=B^{-2}\xi^{-2}$, for any $n\geqslant1$, $\sigma\in\Omega_n$, if $x\in J_\sigma\cap E$, and $|J_\sigma|\leqslant r<|J_\sigma|\cdot B\xi$, then

$$B(x,\lambda r)\cap E\subset J_\sigma\cap E\subset B(x,r)$$

Proof (a) For $\forall n\geqslant1$, $\sigma\in\Omega_n$, on one hand, notice that

$$J_{\sigma*1}\cup J_{\sigma*2}\cup\cdots\cup J_{\sigma*q_{n+1}}\subset J_\sigma$$

which implies that

$$\mathrm{dist}_{1\leqslant j\leqslant q_{n+1}-1}(J_{\sigma*j},J_{\sigma*(j+1)})\leqslant|J_\sigma|$$

On the other hand, for any $1\leqslant j\leqslant q_{n+1}-1$, take

$y\in J_{\sigma * j}$ and $z\in J_{\sigma *(j+1)}$, such that

$$\operatorname{dist}(J_{\sigma * j},J_{\sigma *(j+1)})=|y-z|$$

Then $F_n(y)=f_n\circ f_{n-1}\circ\cdots\circ f_1(y)\in\varphi_{n+1,j}(J)$, $F_n(z)=f_n\circ f_{n-1}\circ\cdots\circ f_1(z)\in\varphi_{n+1,j+1}(J)$, and

$$\operatorname{dist}(\varphi_{n+1,j}(J),\varphi_{n+1,j+1}(J))\leqslant|F_n(y)-F_n(z)|$$

Moreover, according to the mean value theorem and (1), we get

$$\begin{aligned}\operatorname{dist}(\varphi_{n+1,j}(J),\varphi_{n+1,j+1}(J))&=|\varphi_{n+1,j}(1)-\varphi_{n+1,j+1}(0)|\\&=|f_{n+1}^{-1}(1)-f_{n+1}^{-1}(0)|\\&=|Df_{n+1}^{-1}(\omega)|=B^{-1},\qquad\omega\in J\end{aligned}$$

thus $B^{-1}\leqslant|F_n(y)-F_n(z)|$.

Now notice that $F_n:J_\sigma\to J$ is differentiable homeomorphism. By using the mean value theorem and Lemma 2, we have therefore

$$\begin{aligned}B^{-1}|J_\sigma|&\leqslant|J_\sigma|\cdot|F_n(y)-F_n(z)|=|J_\sigma|\cdot|DF_n(x)|\cdot|y-z|\\&\leqslant\xi|y-z|=\xi\operatorname{dist}(J_{\sigma * j},J_{\sigma *(j+1)})\end{aligned}$$

which leads to the theorem.

(b) For any $n\geqslant1$, $\sigma\in\Omega_n$, $x\in J_\sigma\cap E$, on one side, since $r\geqslant|J_\sigma|$, it is easy to see that $J_\sigma\cap E\subset B(x,r)$ holds.

On the other side, since $r<|J_\sigma|\cdot B\xi$, then by (a) and the property of $\{d^{(n)}\}_{n\geqslant1}$, we obtain

$$\begin{aligned}\lambda r&<B^{-2}\xi^{-2}|J_\sigma|\cdot B\xi=B^{-1}\xi^{-1}|J_\sigma|\\&\leqslant\operatorname{dist}_{1\leqslant j\leqslant q_{n+1}-1}(J_{\sigma * j},J_{\sigma *(j+1)})<\operatorname{dist}(J_\sigma,J_\omega)\end{aligned}$$

where $\forall\omega\in\Omega_n$, $\sigma\neq\omega$. Hence $B(x,\lambda r)\cap J_\omega=\varnothing$. Consequently $B(x,\lambda r)\cap E\subset J_\sigma\cap E$. This completes the proof of the theorem.

Remark 1 Theorem 1 demonstrates that for any $n\geqslant1$ and $\sigma\in\Omega_n$, subintervals of J_σ, which can be denoted by $J_{\sigma * 1},J_{\sigma * 2},\cdots,J_{\sigma * q_{n+1}}$, are rationally separated. Moreover, J_σ is comparable to the sphere (interval) in a consistent way.

Theorem 2 Let E be a CCL set satisfying strong separation condition, and $\{d^{(n)}\}_{n\geqslant1}$ are descending series. Then there are constants $c>0$ and $r_0>0$, such that for any sphere B, with center in E and radius $r<r_0$, we can find a mapping $h:E\cap B\to E$ satisfying

$$c^{-1}r^{-1}|y-z|\leqslant|h(y)-h(z)|\leqslant cr^{-1}|y-z|,\quad(y,z\in E\cap B)$$

Proof Firstly, by Lemma 2, for any $n\geqslant1$, $\sigma\in\Omega_n$, $1\leqslant j\leqslant q_{n+1}$, we have

$$\xi^{-1}B^{-1}|J_\sigma|\leqslant|J_{\sigma * j}|\leqslant b^{-1}|J_\sigma|$$

Take $r_0=B^{-1}\xi^{-1}|J|=B^{-1}\xi^{-1}$, then for any $r<r_0$ and $x\in E$, we can surely find a natural number n and $\sigma\in\Omega_n$, such that $x\in J_\sigma$ and

$$\xi^{-2}B^{-2}|J_\sigma|\leqslant r<B^{-1}\xi^{-1}|J_\sigma|,$$

i.e.
$$\xi^{-2}B^{-2}r^{-1}\leqslant|J_\sigma|^{-1}<B^{-1}\xi^{-1}r^{-1}\tag{2}$$

Thus, by Theorem 1(b), we get

$$B(x,r)\cap E\subset J_\sigma\cap E\subset J_\sigma\tag{3}$$

Note that $f_n\circ f_{n-1}\circ\cdots\circ f_1:J_\sigma\to J$ is differentiable homeomorphism, using the mean value theorem we can see that for any $y,z\in J_\sigma$, there exists $\omega\in J_\sigma$ such that

$$\begin{aligned}&|f_n\circ f_{n-1}\circ\cdots\circ f_1(y)-f_n\circ f_{n-1}\circ\cdots\circ f_1(z)|\\&=|D(f_n\circ f_{n-1}\circ\cdots\circ f_1)(\omega)|\cdot|y-z|\end{aligned}$$

Lemma 2 follows that

$$\begin{aligned}\xi^{-1}|y-z|&\leqslant|f_n\circ f_{n-1}\circ\cdots\circ f_1(y)-f_n\circ f_{n-1}\circ\cdots\circ f_1(z)|\cdot|J_\sigma|\\&\leqslant\xi|y-z|\end{aligned}$$

Then by (3), we get the mapping $f_n\circ f_{n-1}\circ\cdots\circ f_1: B(x,r)\cap E\to E$ satisfying

$$\begin{aligned}\xi^{-1}|J_\sigma|^{-1}\cdot|y-z|&\leqslant|f_n\circ f_{n-1}\circ\cdots\circ f_1(y)-f_n\circ f_{n-1}\circ\cdots\circ f_1(z)|\\&\leqslant\xi|J_\sigma|^{-1}\cdot|y-z|\end{aligned}$$

where $\forall y,z\in B(x,r)\cap E$. By (2), we have immediately

$$\begin{aligned}\xi^{-3}B^{-2}r^{-1}|y-z|&\leqslant|f_n\circ f_{n-1}\circ\cdots\circ f_1(y)-f_n\circ f_{n-1}\circ\cdots\circ f_1(z)|\\&\leqslant B^{-1}r^{-1}|y-z|\end{aligned}$$

Setting $h=f_n\circ f_{n-1}\circ\cdots\circ f_1$ and $c=B^2\xi^3$ gives that for any $y,z\in B(x,r)\cap E$,

$$c^{-1}r^{-1}|y-z|\leqslant|h(y)-h(z)|\leqslant cr^{-1}|y-z|$$

holds as claimed.

Remark 2 Theorem 2 implies that we can find a double Lipschitz mapping to map each sphere with center in E onto the larger part of E and the Lipschitz constant is comparable to the size of each sphere. It is to say that "under the condition of few distortion" the smaller part of E can be mapped onto the lager one. This suggests that the CCL set above mentioned is approximately self-similar.

Theorem 3 Let E be a CCL set satisfying strong separation condition, then there are constants $c>0$ and $r_0>0$, such that for any sphere B, with center in E and radius $r<r_0$, we can find a mapping $h:E\to E\cap B$ satisfying

$$c^{-1}r|y-z|\leqslant|h(y)-h(z)|\leqslant cr|y-z|\ (y,z\in E)$$

Proof Take $r_0=|J|=1$, then for any $x\in E$ and $r<r_0$, we can surely find $n\geqslant1$ and $\sigma\in\Omega_n$, such that $x\in J_\sigma$, and $\xi^{-1}B^{-1}r<|J_\sigma|\leqslant r$.

Hence $J_\sigma \subset B(x,r)$.

Because $f_n \circ f_{n-1} \circ \cdots \circ f_1 : J_\sigma \to J$ is differentiable homeomorphism, by the mean value theorem and Lemma 2, it follows that for any $y,z \in J$, there exist $\omega, \tau \in J_\sigma$, so that

$$\omega = \varphi_{1,i_1} \circ \varphi_{2,i_2} \circ \cdots \circ \varphi_{n,i_n}(y)$$

$$\tau = \varphi_{1,i_1} \circ \varphi_{2,i_2} \circ \cdots \circ \varphi_{n,i_n}(z)$$

Moreover

$$\xi^{-1}\left|\varphi_{1,i_1} \circ \varphi_{2,i_2} \circ \cdots \circ \varphi_{n,i_n}(y) - \varphi_{1,i_1} \circ \varphi_{2,i_2} \circ \cdots \circ \varphi_{n,i_n}(z)\right|$$
$$\leqslant |y-z| \cdot |J_\sigma|$$
$$\leqslant \xi\left|\varphi_{1,i_1} \circ \varphi_{2,i_2} \circ \cdots \circ \varphi_{n,i_n}(y) - \varphi_{1,i_1} \circ \varphi_{2,i_2} \circ \cdots \circ \varphi_{n,i_n}(z)\right|$$

This implies that

$$\xi^{-1}|J_\sigma| \cdot |y-z|$$
$$\leqslant \left|\varphi_{1,i_1} \circ \varphi_{2,i_2} \circ \cdots \circ \varphi_{n,i_n}(y) - \varphi_{1,i_1} \circ \varphi_{2,i_2} \circ \cdots \circ \varphi_{n,i_n}(z)\right|$$
$$\leqslant \xi |J_\sigma| \cdot |y-z|$$

Then it is easy to see that the mapping $\varphi_{1,i_1} \circ \varphi_{2,i_2} \circ \cdots \circ \varphi_{n,i_n} : J \to J_\sigma$ is homeomorphism, and

$$\xi^{-2} B^{-1} r |y-z|$$
$$\leqslant \left|\varphi_{1,i_1} \circ \varphi_{2,i_2} \circ \cdots \circ \varphi_{n,i_n}(y) - \varphi_{1,i_1} \circ \varphi_{2,i_2} \circ \cdots \circ \varphi_{n,i_n}(z)\right|$$
$$\leqslant \xi r |y-z|$$

where $\forall y,z \in J$. Let h be the restriction of $\varphi_{1,i_1} \circ \varphi_{2,i_2} \circ \cdots \circ \varphi_{n,i_n}$ on E, i.e.

$$h = \varphi_{1,i_1} \circ \varphi_{2,i_2} \circ \cdots \circ \varphi_{n,i_n}|_E : E \to E \cap B, \quad c = B\xi^2,$$

so, we get the conclusion of the Theorem.

Remark 3 Theorem 3 turns out that "under the condition of few distortion", the CCL set E can be mapped onto the smaller neighborhood of E. This indicates that the CCL set E is "relatively" approximately self-similar in the meantime.

3 Conclusion

This paper analyzes characters of any n-order-basic-intervals of a class of special cookie-cutter-like sets, and proves the approximate self-similarity of these sets. It is demonstrated that the CCL set E, which satisfies the strong separation condition and has a descending maximal gap between the basic intervals of order n, is approximately self-similar. And in the meantime it has been proved that the CCL set mentioned above can be mapped onto the smaller neighborhood of E, which suggests the set E is "relatively" approximately self-similar.

We know that the measures on the self-similar sets are invariant and ergodic with respect to the transformation defined by themselves, which is the same as cookie-cutter sets. As Gibbs-like measures are not invariant, it is therefore natural to ask whether there is invariant measure on CCL sets, and whether there is promoted form of ergodic theorem associated with such invariant measure. These topics will be our future work, which would be the basis of our study on upper, lower and average density. Obviously, the results obtained in this paper can give good preliminaries for these problems.

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