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Existence of Four Periodic Solutions of a Ratio-Dependent Predator-Prey Model with Exploited Terms

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Abstract: We study a non-autonomous ratio-dependent predator-prey model with exploited terms. This model is of periodic coefficients, which incorporates the periodicity of the varying environment. By means of the coincidence degree theory, we establish sufficient conditions for the existence of at least four positive periodic solutions of this model.

Key words: predator-prey model; ratio-dependent; exploited term; periodic solution; coincidence degree
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0 Introduction

In this paper, we consider a predator-prey model with exploited terms. Generally, the model with exploited terms is described as follows[1]:

$$x'(t) = xf(x,y) - h,\ \ y'(t) = yg(x,y) - k$$

where x and y are functions of time representing densities of prey and predator, respectively; h and k are exploited terms standing for the harvests. Particularly, a non-autonomous ratio-dependent predator-prey model with exploited terms is described by the following system of ordinary differential equations

$$\begin{cases} x'(t) = x(a - bx - \dfrac{cy}{my + x}) - h \\ y'(t) = y(-d + \dfrac{fy}{my + x}) - k \end{cases} \tag{1}$$

where a, c, d, f, m are the prey intrinsic growth rate, capture rate, death rate of predator, conversion rate, half saturation-parameter, respectively. Moreover, for the biological background of model (1), we always assume that all of the parameters are positive constants. For the detailed biological meanings, one can refer to Refs.[2-4] and references cited therein. Since realistic models require the inclusion of the effect of changing environment, it motivates us to consider the following model:

$$\begin{cases} x'(t) = x(t)(a(t) - b(t)x(t) - \dfrac{c(t)y(t)}{m(t)y(t) + x(t)}) - h(t) \\ y'(t) = y(t)(-d(t) + \dfrac{f(t)y(t)}{m(t)y(t) + x(t)}) - k(t) \end{cases} \tag{2}$$

Correspondingly, we assume the parameters in (2) are positive ω-periodic functions. The assumption of periodicity of the parameters is a way of incorporating periodicity of the environment (e.g. seasonal effects of weather, food supplies, mating habits etc).

In recent years, the existences of periodic solutions in population models are widely studied by applying the continuation theorem of coincidence degree theory, for example, see Refs.[3,5-9]. In Ref.[6], a ratio-dependent predator-prey model with prey's harvest ($k(t)\equiv 0$, $h(t)>0$) is analyzed. The aim of this paper is to establish sufficient conditions for the existence of at least four positive ω-periodic solutions of system (2).

1 Preliminary and Notation

For the readers' convenience, we first introduce a few concepts from the book by Gaines and Mawhin[10].

Let X, Z be normed vector spaces, $L:\mathrm{Dom}L\subset X\to Z$ a Fredholm operator of index zero and $P:X\to X,\ Q:Z\to Z$ continuous projects such that $\mathrm{Im}\,P=\mathrm{Ker}L$, $\mathrm{Ker}Q=\mathrm{Im}\,L$, $X=\mathrm{Ker}L\oplus\mathrm{Ker}P$ and $Z=\mathrm{Im}\,L\oplus\mathrm{Im}\,Q$. Denote the generalized inverse (of L) by $K_P:\mathrm{Im}\,L\to\mathrm{Ker}P\bigcap\mathrm{Dom}\,L$ and an isomorphism of $\mathrm{Im}Q$ onto $\mathrm{Ker}P$ by $J:\mathrm{Im}\,Q\to\mathrm{Ker}\,L$.

Lemma 1 Let L,P,Q and K_P be as above, and let $\Omega\subset X$ be an open bounded set and $N:X\to Z$ be a continuous mapping, which is L-compact on $\overline{\Omega}$ (i.e., $QN:\overline{\Omega}\to Z$ and $K_P(i-Q)N:\Omega\to X$ are compact). Assume

(i) for each $\lambda\in(0,\ 1)$, $x\in\partial\Omega\bigcap\mathrm{Dom}\,L$, $Lx\neq\lambda Nx$;

(ii) for each $x\in\partial\Omega\bigcap\mathrm{Ker}\,L$, $QNx\neq 0$;

(iii) $\deg\{JQN,\Omega\bigcap\ker L,0\}\neq 0$,

Then $Lx=Nx$ has at least one solution in $\overline{\Omega}$.

Now we give some notations employed in the following discussion.

$$\overline{g}=\frac{1}{\omega}\int_0^{\omega}g(t)\,\mathrm{d}t,\ g^L=\min_{t\in[0,\omega]}g(t),\ g^M=\max_{t\in[0,\omega]}g(t),$$

where g is a continuous ω-periodic function.

2 Main Results

In this section, we shall state and prove our main result of this paper.

Theorem 1 Assume the following

(i) $(a-\frac{c}{m})^L>2\sqrt{b^Mh^M}$;

(ii) $(\overline{h}(f-d)-\overline{a}\,\mathrm{e}^{2\omega\overline{a}}\,km)^L\geqslant 2\,\mathrm{e}^{\omega\overline{a}}\sqrt{(dm)^M k^M\overline{a}\overline{h}}$.

Then system (2) has at least four positive ω-periodic solutions.

Proof Consider the following system:

$$\begin{cases}u'(t)=a(t)-b(t)\mathrm{e}^{u(t)}-\dfrac{c(t)\mathrm{e}^{v(t)}}{m(t)\mathrm{e}^{v(t)}+\mathrm{e}^{u(t)}}-h(t)\mathrm{e}^{-u(t)}\\ v'(t)=-d(t)+\dfrac{f(t)\mathrm{e}^{u(t)}}{m(t)\mathrm{e}^{v(t)}+\mathrm{e}^{u(t)}}-k(t)\mathrm{e}^{-v(t)}\end{cases}\tag{3}$$

It is easy to see that if system (3) has an ω-periodic solution $(u^*(t),\ v^*(t))^{\mathrm{T}}$, then

$$(x^*(t),\ y^*(t))^{\mathrm{T}}=(\exp u^*(t),\ \exp v^*(t))^{\mathrm{T}}$$

is a positive ω-periodic solution of system (2). So, to prove Theorem 1, it suffices to show that system (3) has at least four ω-periodic solutions.

For $\lambda\in(0,\ 1)$, we take a system as follows

$$\begin{cases}u'(t)=\lambda(a(t)-b(t)\mathrm{e}^{u(t)}-\dfrac{c(t)\mathrm{e}^{v(t)}}{m(t)\mathrm{e}^{v(t)}+\mathrm{e}^{u(t)}}-h(t)\mathrm{e}^{-u(t)})\\ v'(t)=\lambda(-d(t)+\dfrac{f(t)\mathrm{e}^{u(t)}}{m(t)\mathrm{e}^{v(t)}+\mathrm{e}^{u(t)}}-k(t)\mathrm{e}^{-v(t)})\end{cases}\tag{4}$$

Suppose that $(u(t),\ v(t))^{\mathrm{T}}$ is an ω-periodic solution of system (4) for $\lambda\in(0,\ 1)$. Integrating (4) over $[0,\ \omega]$, we obtain

$$\omega\overline{a}=\int_0^{\omega}(b(t)\mathrm{e}^{u(t)}+\frac{c(t)\mathrm{e}^{v(t)}}{m(t)\mathrm{e}^{v(t)}+\mathrm{e}^{u(t)}}+h(t)\mathrm{e}^{-u(t)})\,\mathrm{d}t\tag{5}$$

and

$$\omega\overline{d}=\int_0^{\omega}(\frac{f(t)\mathrm{e}^{u(t)}}{m(t)\mathrm{e}^{v(t)}+\mathrm{e}^{u(t)}}-k(t)\mathrm{e}^{-v(t)})\,\mathrm{d}t\tag{6}$$

From (4)-(6), it follows that

$$\begin{aligned}\int_0^{\omega}|u'(t)|\,\mathrm{d}t&\leqslant\lambda\int_0^{\omega}(a(t)+b(t)\mathrm{e}^{u(t)}+\frac{c(t)\mathrm{e}^{v(t)}}{m(t)\mathrm{e}^{v(t)}+\mathrm{e}^{u(t)}}\\&\quad+h(t)\mathrm{e}^{-u(t)})\,\mathrm{d}t\\&<\omega\overline{a}+\int_0^{\omega}(b(t)\mathrm{e}^{u(t)}+\frac{c(t)\mathrm{e}^{v(t)}}{m(t)\mathrm{e}^{v(t)}+\mathrm{e}^{u(t)}}\\&\quad+h(t)\mathrm{e}^{-u(t)})\,\mathrm{d}t=2\omega\overline{a}\end{aligned}\tag{7}$$

and

$$\begin{aligned}\int_0^{\omega}|u'(t)|\,\mathrm{d}t&\leqslant\lambda\int_0^{\omega}[d(t)+(\frac{f(t)\mathrm{e}^{v(t)}}{m(t)\mathrm{e}^{v(t)}+\mathrm{e}^{u(t)}}-k(t)\mathrm{e}^{-v(t)})]\,\mathrm{d}t\\&=2\omega\overline{d}\end{aligned}\tag{8}$$

Choose $t_i,\tau_i\in[0,\ \omega],\ i=1,2$, such that

$$u(t_1)=\min_{t\in[0,\omega]}u(t),\ \ u(\tau_1)=\max_{t\in[0,\omega]}u(t)\tag{9}$$

$$v(t_2)=\min_{t\in[0,\omega]}v(t),\ \ u(\tau_2)=\max_{t\in[0,\omega]}v(t)\tag{10}$$

From (5) and (9), we obtain

$$\begin{aligned}\omega\overline{a}&=\int_0^{\omega}(b(t)\mathrm{e}^{u(t)}+\frac{c(t)\mathrm{e}^{v(t)}}{m(t)\mathrm{e}^{v(t)}+\mathrm{e}^{u(t)}}+h(t)\mathrm{e}^{-u(t)})\,\mathrm{d}t\\&>\int_0^{\omega}b(t)\,\mathrm{d}t\geqslant\omega\overline{b}\,\mathrm{e}^{u(t_1)}\end{aligned}$$

which reduces to $u(t_1) < \ln(\frac{\bar{a}}{\bar{b}})$. This, together with (7), gives that for any $t \in [0, \omega]$

$$u(t) \leqslant u(t_1) + \int_0^\omega |u'(t)|\,\mathrm{d}t < \ln(\frac{\bar{a}}{\bar{b}}) + 2\omega\bar{a} := \delta_1 \quad (11)$$

Multiplying the first equality of (4) by $\mathrm{e}^{u(t)}$, and integrating over $[0, \omega]$, we obtain

$$\int_0^\omega a(t)\mathrm{e}^{u(t)}\mathrm{d}t = \int_0^\omega b(t)\mathrm{e}^{2u(t)}\mathrm{d}t + \int_0^\omega \frac{c(t)\mathrm{e}^{u(t)+v(t)}}{m(t)\mathrm{e}^{v(t)} + \mathrm{e}^{u(t)}}\mathrm{d}t + \int_0^\omega h(t)\mathrm{d}t$$

Again from (9), this implies that

$$\mathrm{e}^{u(\tau_1)}\omega\bar{a} \geqslant \int_0^\omega a(t)\mathrm{e}^{u(t)}\mathrm{d}t > \int_0^\omega h(t)\mathrm{d}t \geqslant \omega\bar{h}$$

which reduces to $u(\tau_1) > \ln(\frac{\bar{h}}{\bar{a}})$. This, together with (7), gives that for any $t \in [0, \omega]$

$$u(t) \geqslant u(\tau_1) - \int_0^\omega |u'(t)|\,\mathrm{d}t > \ln(\frac{\bar{h}}{\bar{a}}) - 2\omega\bar{a} := \delta_2 \quad (12)$$

From (9) and the first equality of (4), we also have

$$a(\tau_1) - b(\tau_1)\mathrm{e}^{u(\tau_1)} - \frac{c(\tau_1)\mathrm{e}^{v(\tau_1)}}{m(\tau_1)\mathrm{e}^{v(\tau_1)} + \mathrm{e}^{u(\tau_1)}} - h(\tau_1)\mathrm{e}^{-u(\tau_1)} = 0$$

Because of $\frac{c(\tau_1)\mathrm{e}^{v(\tau_1)}}{m(\tau_1)\mathrm{e}^{v(\tau_1)} + \mathrm{e}^{u(\tau_1)}} < \frac{c(\tau_1)}{m(\tau_1)}$, this implies that

$$b(\tau_1)\mathrm{e}^{2u(\tau_1)} - [a(\tau_1) - \frac{c(\tau_1)}{m(\tau_1)}]\mathrm{e}^{u(\tau_1)} + h(\tau_1) > 0$$

Solving the inequality, we obtain

$$\mathrm{e}^{u(\tau_1)} < \frac{[a(\tau_1) - \frac{c(\tau_1)}{m(\tau_1)}] - \sqrt{[a(\tau_1) - \frac{c(\tau_1)}{m(\tau_1)}]^2 - 4b(\tau_1)h(\tau_1)}}{2b(\tau_1)} \leqslant l_-$$

or

$$\mathrm{e}^{u(\tau_1)} > \frac{[a(\tau_1) - \frac{c(\tau_1)}{m(\tau_1)}] + \sqrt{[a(\tau_1) - \frac{c(\tau_1)}{m(\tau_1)}]^2 - 4b(\tau_1)h(\tau_1)}}{2b(\tau_1)} \geqslant l_+$$

where $l_\pm = \frac{(a - \frac{c}{m})^L \pm \sqrt{[(a - \frac{c}{m})^L]^2 - 4b^M h^M}}{2b^M}$. Namely, $\mathrm{e}^{u(\tau_1)} < l_-$, or $\mathrm{e}^{u(\tau_1)} > l_+$. Similarly, from (9) and the first equality of (4), it follows that $\mathrm{e}^{u(t_1)} < l_-$, or $\mathrm{e}^{u(t_1)} > l_+$. These, combined with (11)-(12), give that for any $t \in [0, \omega]$

$$\delta_2 < u(t) < \ln l_-, \text{ or } \ln l_+ < u(t) < \delta_1 \quad (13)$$

From (6) and (10)-(11), we have

$$\omega\bar{d} = \int_0^\omega (\frac{f(t)\mathrm{e}^{u(t)}}{m(t)\mathrm{e}^{v(t)} + \mathrm{e}^{u(t)}} - k(t)\mathrm{e}^{-v(t)})\mathrm{d}t$$

$$< \int_0^\omega \frac{f(t)\mathrm{e}^{u(t)}\mathrm{d}t}{m(t)\mathrm{e}^{v(t)} + \mathrm{e}^{u(t)}} < \int_0^\omega \frac{f(t)\mathrm{e}^{u(t)}\mathrm{d}t}{m(t)\mathrm{e}^{v(t)}}$$

$$\leqslant \int_0^\omega \frac{f(t)\mathrm{e}^{\delta_1}\mathrm{d}t}{m(t)\mathrm{e}^{v(t_2)}} = \frac{1}{\mathrm{e}^{v(t_2)}}\frac{\bar{a}}{\bar{b}}(\overline{\frac{f}{m}})\exp(2\omega\bar{a})\omega$$

which reduces to $v(t_2) < \ln[\frac{\bar{a}}{\bar{b}\bar{d}}(\overline{\frac{f}{m}})] + 2\omega\bar{a}$. Therefore, this together with (8) gives that for any $t \in [0, \omega]$,

$$v(t) \leqslant v(t_2) + \int_0^\omega |v'(t)|\,\mathrm{d}t < \ln[\frac{\bar{a}}{\bar{b}\bar{d}}(\overline{\frac{f}{m}})] + 2\omega(\bar{a} + \bar{d}) := \delta_3 \quad (14)$$

From (6) and the condition (ii) of Theorem 1 that implies $\bar{f} > \bar{d}$, we have

$$\omega\bar{d} = \int_0^\omega \frac{f(t)\mathrm{d}t}{m(t)\mathrm{e}^{v(t)-u(t)} + 1} - \int_0^\omega k(t)\mathrm{e}^{-v(t)}\mathrm{d}t < \omega\bar{f} - \frac{\omega\bar{k}}{\mathrm{e}^{v(\tau_2)}}$$

which reduces to $v(\tau_2) > \ln(\frac{\bar{k}}{\bar{f} - \bar{d}})$. This, together with (8), gives that for any $t \in [0, \omega]$

$$v(t) \geqslant v(\tau_2) - \int_0^\omega |v'(t)|\,\mathrm{d}t < \ln(\frac{\bar{k}}{\bar{f} - \bar{d}}) - 2\omega\bar{d} := \delta_4 \quad (15)$$

From (6) and (10), we also obtain

$$-d(t_2) + \frac{f(t_2)\mathrm{e}^{u(t_2)}}{m(t_2)\mathrm{e}^{v(t_2)} + \mathrm{e}^{u(t_2)}} - \frac{k(t_2)}{\mathrm{e}^{v(t_2)}} = 0$$

Noticing that $\frac{f(t_2)\mathrm{e}^{u(t)}}{m(t_2)\mathrm{e}^{v(t_2)} + \mathrm{e}^{u(t)}}$ is increasing with $u(t)$, again from (12), this implies that

$$-d(t_2) + \frac{f(t_2)\mathrm{e}^{\delta_2}}{m(t_2)\mathrm{e}^{v(t_2)} + \mathrm{e}^{\delta_2}} - \frac{k(t_2)}{\mathrm{e}^{v(t_2)}} < 0$$

Because of $m(t_2)\mathrm{e}^{v(t_2)} + e^{\delta_2} > 0$, eliminating the denominator, we obtain

$$d(t_2)m(t_2)\mathrm{e}^{2v(t_2)} - [f(t_2)\mathrm{e}^{\delta_2} - d(t_2)\mathrm{e}^{\delta_2} - k(t_2)m(t_2)]\mathrm{e}^{v(t_2)} + k(t_2)\mathrm{e}^{\delta_2} > 0 \quad (16)$$

For the sake of convenience and simplicity, define

$$\varphi(t) = f(t)\mathrm{e}^{\delta_2} - d(t)\mathrm{e}^{\delta_2} - k(t)m(t)$$

Then, the inequality (16) becomes

$$d(t_2)m(t_2)\mathrm{e}^{2v(t_2)} - \varphi(t_2)\mathrm{e}^{v(t_2)} + k(t_2)\mathrm{e}^{\delta_2} > 0$$

Because of the condition (ii) of Theorem 1, we give by solving the inequality

$$\mathrm{e}^{v(t_2)} > \frac{\varphi(t_2) + \sqrt{[\varphi(t_2)]^2 - 4d(t_2)m(t_2)k(t_2)\mathrm{e}^{\delta_2}}}{2d(t_2)m(t_2)} \geqslant r_+$$

or

$$\mathrm{e}^{v(t_2)} < \frac{\varphi(t_2) + \sqrt{[\varphi(t_2)]^2 - 4d(t_2)m(t_2)k(t_2)\mathrm{e}^{\delta_2}}}{2d(t_2)m(t_2)} \leqslant r_-$$

where $r_{\pm}=\dfrac{\varphi^L+\sqrt{(\varphi^L)^2-4(dm)^M k^M \mathrm{e}^{\delta_2}}}{2(dm)^M}$. Namely, $\mathrm{e}^{v(t_2)}<r_-$, or $\mathrm{e}^{v(t_2)}>r_+$. Similarly, from (6) and (10), it follows that $\mathrm{e}^{v(\tau_2)}<r_-$, or $\mathrm{e}^{v(\tau_2)}>r_+$. These, combined with (14) and (15), give that for any $t\in[0,\ \omega]$

$$\delta_4<v(t)<\ln r_-,\ \text{ or } \ln r_+<v(t)<\delta_3 \tag{17}$$

Clearly, δ_1-$\delta_4, l_{\pm}, r_{\pm}$ are independent of λ.

Now consider the following set of two equations:

$$\begin{cases}\bar{a}-\bar{b}\mathrm{e}^u-\dfrac{1}{\omega}\displaystyle\int_0^{\omega}\dfrac{c(t)\mathrm{e}^v}{m(t)\mathrm{e}^v+\mathrm{e}^u}\mathrm{d}t-\bar{h}\mathrm{e}^{-u}=0\\ -\bar{d}+\dfrac{1}{\omega}\displaystyle\int_0^{\omega}\dfrac{f(t)\mathrm{e}^u}{m(t)\mathrm{e}^v+\mathrm{e}^u}\mathrm{d}t-\bar{k}\mathrm{e}^{-v}=0\end{cases} \tag{18}$$

where $(u,v)^{\mathrm{T}}$ is a constant vector. Let (u,v) be a set of solutions of (18). From the first equation of (18), it follows that

$$\bar{a}-\bar{b}\mathrm{e}^u-\frac{1}{\omega}\int_0^{\omega}\frac{c(t)\mathrm{e}^{v-u}}{m(t)\mathrm{e}^{v-u}+1}\mathrm{d}t-\bar{h}\mathrm{e}^{-u}=0$$

This implies that $\bar{a}-\bar{b}\mathrm{e}^u-(\overline{\frac{c}{m}})-\bar{h}\mathrm{e}^{-u}<0$. Solving the inequality, we have $\mathrm{e}^u<l_-$, or $\mathrm{e}^u>l_+$.

Further, by using the arguments of (5)-(13), we obtain

$$\delta_2<u<\ln l_-,\ \text{ or } \ln l+<u<\delta_1 \tag{19}$$

For the second equation of (18), i.e., $-\bar{d}+\frac{1}{\omega}\int_0^{\omega}\frac{f(t)\mathrm{e}^u}{m(t)\mathrm{e}^v+\mathrm{e}^u}\mathrm{d}t$, from the fact that $\frac{f(t)\mathrm{e}^u}{m(t)\mathrm{e}^v+\mathrm{e}^u}$ is increasing with e^u, and from (19), it follows that

$$-\bar{d}+\frac{1}{\omega}\int_0^{\omega}\frac{f(t)\mathrm{e}^{\delta_2}}{m(t)\mathrm{e}^v+\mathrm{e}^{\delta_2}}\mathrm{d}t-\frac{\bar{k}}{\mathrm{e}^v}<0$$

This is,

$$\frac{1}{\omega}\int_0^{\omega}\frac{-d(t)m(t)\mathrm{e}^{2v}+\varphi(t)\mathrm{e}^v-k(t)\mathrm{e}^{\delta_2}}{(m(t)\mathrm{e}^v+\mathrm{e}^{\delta_2})\mathrm{e}^v}\mathrm{d}t<0$$

Further, we obtain

$$\frac{1}{\omega}\int_0^{\omega}\frac{-(dm)^M\mathrm{e}^{2v}+\varphi^L\mathrm{e}^v-k^M\mathrm{e}^{\delta_2}}{(m(t)\mathrm{e}^v+\mathrm{e}^{\delta_2})\mathrm{e}^v}\mathrm{d}t<0$$

which reduces to

$$(dm)^M\mathrm{e}^{2v}-\varphi^L\mathrm{e}^v+k^M\mathrm{e}^{\delta_2}>0$$

Solving the inequality, we give

$$\mathrm{e}^v>\frac{\varphi^L+\sqrt{(\varphi^L)^2-4(dm)^M k^M\mathrm{e}^{\delta_2}}}{2(dm)^M}=r_+$$

or

$$\mathrm{e}^v<\frac{\varphi^L-\sqrt{(\varphi^L)^2-4(dm)^M k^M\mathrm{e}^{\delta_2}}}{2(dm)^M}=r_-$$

By using the arguments of (14)-(16), we obtain

$$\delta_4<v<\ln r_-,\ \text{ or } \ln r_+<v<\delta_3 \tag{20}$$

We take

$$X=Z=\{(u(t),v(t))^{\mathrm{T}}\in \mathrm{C}(\mathbf{R},\mathbf{R}^2)\mid u(t+\omega)=u(t),\ v(t+\omega)=v(t)\}$$

and equipped the norm $\|(u(t),v(t))^{\mathrm{T}}\|=\max\limits_{t\in[0,\omega]}|u(t)|+\max\limits_{t\in[0,\omega]}|v(t)|$. Then, X (or Z) is a Banach space. Let $L:\mathrm{Dom}\,L\subset X\to X$, $L(u(t),v(t))^{\mathrm{T}}=(u'(t),v'(t))^{\mathrm{T}}$, where $\mathrm{Dom}\,L=\{(u(t),v(t))^{\mathrm{T}}\in X\mid (u(t),v(t))^{\mathrm{T}}\in \mathrm{C}^1(\mathbf{R},\mathbf{R}^2)\}$. Again let $N:X\to X$,

$$N\begin{bmatrix}u(t)\\v(t)\end{bmatrix}=\begin{bmatrix}a(t)-b(t)\mathrm{e}^{u(t)}-\dfrac{c(t)\mathrm{e}^{v(t)}}{m(t)\mathrm{e}^{v(t)}+\mathrm{e}^{u(t)}}-h(t)\mathrm{e}^{-u(t)}\\ -d(t)+\dfrac{f(t)\mathrm{e}^{u(t)}}{m(t)\mathrm{e}^{v(t)}+\mathrm{e}^{u(t)}}-k(t)\mathrm{e}^{-v(t)}\end{bmatrix}$$

Define projectors P and Q by

$$P\begin{bmatrix}u(t)\\v(t)\end{bmatrix}=Q\begin{bmatrix}u(t)\\v(t)\end{bmatrix}=\begin{bmatrix}\bar{u}\\ \bar{v}\end{bmatrix},\quad \begin{bmatrix}u(t)\\v(t)\end{bmatrix}\in X$$

Obviously, $\mathrm{Ker}\,L=\mathrm{Im}\,P=\mathbf{R}^2$, $\mathrm{Im}\,L=\mathrm{Ker}\,Q=\{(u(t),v(t))^{\mathrm{T}}\in X\mid \bar{u}=\bar{v}=0\}$ is closed in X, and $\dim\mathrm{Ker}\,L=\dim(Z/\mathrm{Im}\,L)=2$. Thus, L is a Fredholm operator of index zero. Moreover, define the inverse K_P of L as follows:

$$K_P:\mathrm{Im}\,L\to \mathrm{Dom}L\cap \mathrm{Ker}P,$$

$$K_P\begin{bmatrix}u(t)\\v(t)\end{bmatrix}=\begin{bmatrix}\displaystyle\int_0^t u(s)\mathrm{d}s-\frac{1}{\omega}\int_0^{\omega}\int_0^s u(s)\mathrm{d}s\mathrm{d}t\\ \displaystyle\int_0^t v(s)\mathrm{d}s-\frac{1}{\omega}\int_0^{\omega}\int_0^s v(s)\mathrm{d}s\mathrm{d}t\end{bmatrix}$$

Define again

$$\Omega_1=\{(u(t),v(t))^{\mathrm{T}}\in X\mid \delta_2<u(t)<\ln l_-,\ \delta_4<v(t)<\ln r_-\},$$
$$\Omega_2=\{(u(t),v(t))^{\mathrm{T}}\in X\mid \ln l_+<u(t)<\delta_1,\ \delta_4<v(t)<\ln r_-\},$$
$$\Omega_3=\{(u(t),v(t))^{\mathrm{T}}\in X\mid \delta_2<u(t)<\ln l_-,\ \ln r_+<v(t)<\delta_3\},$$
$$\Omega_4=\{(u(t),v(t))^{\mathrm{T}}\in X\mid \ln l_+<u(t)<\delta_1,\ \ln r_+<v(t)<\delta_3\}.$$

All Ω_i's are open bounded subsets of X. Since $l_-<l_+$ and $r_-\leqslant r_+$, we have $\bigcup\limits_{i=1}^{4}\Omega_i=\varnothing$. From (19) and (20), we can see that (18) all has solutions is each of Ω_i. It is easy to show that QN and $K_P(i-Q)N$ are continuous by the Lebesgue convergence theorem, and by Arzela-Ascoli theorem $QN(\overline{\Omega_i})$ and $KP(i-Q)N(\overline{\Omega_i})$ are compact. Therefore, N is L-compact on each of $\overline{\Omega_i}$. Here $i=1,2,3,4$.

According to (15) and (17), we have proven that for each $\lambda\in(0,\ 1)$ and $(u(t),v(t))^{\mathrm{T}}\in\partial\Omega_i\cap\mathrm{Dom}\,L$, $L(u(t),$

$v(t))^{\mathrm{T}} \neq \lambda N(u(t),v(t))^{\mathrm{T}}$ for $i=1,2,3,4$. Namely, the condition (i) in Lemma l is satisfied.

Next we have to prove that the condition (ii) and (iii) in Lemma 1 are satisfied.

Assume $(u,v)^{\mathrm{T}} \in \partial\Omega_i \cap \mathrm{Ker}L = \partial\Omega_i \cap \mathbf{R}^2$, i.e., $(u,v)^{\mathrm{T}}$ is a constant vector in $\mathbf{R}^2$. Then, it follows from (19) and (20) that $QN(u,v)^{\mathrm{T}} \neq 0$. This shows that the condition (ii) in Lemma l is satisfied. Finally we will prove that the condition (iii) of Lemma l is satisfied. Define $\phi_{1,2}: \mathrm{Dom}\,L \times [0,\ 1] \to X$,

$$\phi_1(u,v,\mu) = \begin{bmatrix} \bar{a} - \bar{b}\mathrm{e}^u - \dfrac{\mu}{\omega}\displaystyle\int_0^\omega \dfrac{c(t)\mathrm{e}^v}{m(t)\mathrm{e}^v + \mathrm{e}^u}\mathrm{d}t - \bar{h}\mathrm{e}^{-u} \\ -\bar{d} + \dfrac{1}{\omega}\displaystyle\int_0^\omega \dfrac{f(t)\mathrm{e}^u}{\mu m(t)\mathrm{e}^v + \mathrm{e}^u}\mathrm{d}t - \bar{k}\mathrm{e}^{-v} \end{bmatrix}$$

$$\phi_2(u,v,\mu) = \begin{bmatrix} \bar{a} - \bar{b}\mathrm{e}^u - \dfrac{1}{\omega}\displaystyle\int_0^\omega \dfrac{c(t)\mathrm{e}^v}{m(t)\mathrm{e}^v + \mathrm{e}^u}\mathrm{d}t - \bar{h}\mathrm{e}^{-u} \\ -\bar{d} + \dfrac{1}{\omega}\displaystyle\int_0^\omega \dfrac{f(t)\mathrm{e}^u}{m(t)\mathrm{e}^v + \mathrm{e}^u}\mathrm{d}t - \mu\bar{k}\mathrm{e}^{-v} \end{bmatrix}$$

where $\mu \in [0,\ 1]$ is a parameter. From (19) and (20), we see that $\phi_{1,2}(u,v,1)=0$ have at least one set of solutions in each of $\Omega_i (i=1,2,3,4)$. From the first equation of $\phi_1(u,v,\mu)=0 (\mu \in [0,\ 1])$, by using the same arguments of (11)-(12), we have

$$\delta_2 < u < \delta_1 \tag{21}$$

Again from the fact that $\dfrac{\mu}{\omega}\displaystyle\int_0^\omega \dfrac{c(t)\mathrm{e}^v}{m(t)\mathrm{e}^v + \mathrm{e}^u}\mathrm{d}t \leqslant \dfrac{1}{\omega}\displaystyle\int_0^\omega \dfrac{c(t)\mathrm{e}^v}{m(t)\mathrm{e}^v + \mathrm{e}^u}\mathrm{d}t$, it follows that

$$u < \ln l_-, \text{ or } u > \ln l_+ \tag{22}$$

From the second equation of $\phi_1(u,v,\mu)=0 (\mu \in [0,\ 1])$, it follows that

$$\delta_4 < v < \ln r_-, \text{ or } v > \ln r_+ \tag{23}$$

Expressions (21)-(23) show that

$$(0,\ 0)^{\mathrm{T}} \notin \phi_1(\partial\Omega_i,\mu),\ \ i=1,2,\ \ \mu \in [0,\ 1]$$

Similarly, from $\phi_2(u,v,\mu)=0 (\mu \in [0,\ 1])$, it follows that $\delta_2 < u < \ln l_-$ or $\ln l_+ < u < \delta_1$ and $v < \ln r_-$ or $\ln r_+ < v < \delta_3$. These show that

$$(0,\ 0)^{\mathrm{T}} \notin \phi_2(\partial\Omega_i,\mu),\ \ i=3,4,\ \ \mu \in [0,\ 1].$$

From the first equation of $\phi_{1,2}(u,v,\mu)=0$, one can see that $\bar{b}\mathrm{e}^u - \dfrac{\bar{h}}{\mathrm{e}^u} \neq 0$, where $\delta_2 < u < \ln l_-$, or $\ln l_+ < u < \delta_1$. Further, by applying the property of topological degree and taking $J=I: \mathrm{Im}Q \to \mathrm{Ker}L$, $J(u,v)^{\mathrm{T}} = (u,v)^{\mathrm{T}}$, we give by straightforwardly calculating

$$\begin{aligned} &\deg\{JQN(u,v)^{\mathrm{T}}, \Omega_i \cap \mathrm{Ker}L, 0\} \\ &= \deg\{\phi_1(u,v,1), \Omega_i \cap \mathrm{Ker}L, 0\} \\ &= \deg\{\phi_1(u,v,0), \Omega_i \cap \mathrm{Ker}L, 0\} \\ &= \mathrm{sgn}\{(-\bar{b}\mathrm{e}^u + \frac{\bar{h}}{\mathrm{e}^u})(\bar{k}\mathrm{e}^{-u})\} \neq 0 \quad (i=1,2) \end{aligned}$$

$$\begin{aligned} &\deg\{JQN(u,v)^{\mathrm{T}}, \Omega_i \cap \mathrm{Ker}L, 0\} \\ &= \deg\{\phi_2(u,v,1), \Omega_i \cap \mathrm{Ker}L, 0\} \\ &= \deg\{\phi_2(u,v,0), \Omega_i \cap \mathrm{Ker}L, 0\} \\ &= \mathrm{sgn}\{(\bar{b}\mathrm{e}^u - \frac{\bar{h}}{\mathrm{e}^u})\int_0^\omega \frac{f(t)m(t)\mathrm{e}^v\mathrm{d}t}{(m(t)\mathrm{e}^v + \mathrm{e}^u)}\} \neq 0 \quad (i=3,4) \end{aligned}$$

Summarizing up the above discussion, we have proven that every $\Omega_i (i=1,2,3,4)$ satisfies all requirements in Lemma l. Hence, system (3) has at least one ω-periodic solution in each of $\Omega_i (i=1,2,3,4)$. Thus, the proof of Theorem l is completed.

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