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ORIGINAL PAPER

Using conceptual analyses to resolve the tension between advanced and secondary mathematics: the cases of equivalence and inverse

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Abstract
Advanced mathematics is seen as an integral component of secondary teacher preparation, and thus most secondary teacher preparation programs require their students to complete an array of advanced mathematics courses. In recent years, though, researchers have questioned the utility of proposed connections between advanced and secondary mathematics. It is simply not clear in many cases—to researchers, teacher educators, and teachers themselves—exactly how advanced mathematics content is related to secondary content. In this paper, we propose using a *conceptual analysis*—a form of theory in which one explicitly describes ways of reasoning about a particular mathematical idea—to address this issue. Specifically, we use conceptual analyses for the foundational notions of equivalence and inverse to illustrate how the ways of reasoning needed to support productive engagement with tasks in advanced mathematics can mirror and reinforce those that are similarly productive in school mathematics. To do so, we propose conceptual analyses for the key concepts of equivalence and inverse and show how researchers can use these conceptual analyses to identify connections to school mathematics in advanced mathematical tasks that might otherwise be obscured and overlooked. We conclude by suggesting ways in which conceptual analyses might be productively used by both teacher educators and future teachers.

Keywords Advanced mathematics · School mathematics · Teacher preparation · Abstract algebra · Conceptual analysis

1 Introduction

Developing knowledge of advanced mathematics is seen as an integral component of the preparation of future secondary mathematics teachers. This idea relies on the premise that future mathematics teachers benefit from viewing the secondary content they will soon be teaching through the lens of advanced mathematics. However, while this premise might seem imminently reasonable in theory, it has proved considerably difficult to implement in practice (e.g., Bukova-Guzel et al., 2010; Even, 2011; Kondratieva & Winsløw, 2018). As Wasserman (2017) noted, "a school teacher's knowledge of advanced mathematics, such as abstract algebra, [should] translate to their instructional practice in some way. And yet school mathematics teachers should not, in fact, end up teaching their students abstract algebra. This is a *difficult tension* to resolve" (p. 81, emphasis added). Similarly, some researchers have called into question the very utility of certain proposed connections between the two domains. Larsen et al. (2018), for example, argued that:

> The CBMS (2012) recommendations for the mathematical preparation of teachers [include] statements like, "it would be quite useful for prospective teachers to see how **C** can be built as a quotient of **R**[x]" (CBMS, 2012, p. 59). A very reasonable question to ask in response to such a statement is "Why?" Abstract algebra certainly provides a highly sophisticated perspective on a variety of secondary mathematics topics, but it simply does not follow that a teacher's pedagogical practice would (or even could) benefit from studying abstract algebra. Or perhaps, rather, we should say it does not follow simply (p. 74)

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As it pertains to this debate, we recognize that the current state of affairs is perhaps not ideal, but we also recognize that it is unlikely to change: future secondary teachers will continue to be required to take advanced mathematics courses. Operating within these constraints, we believe that researchers and teacher educators should focus less on the question of whether such courses are the *most useful* way to prepare teachers and focus more on how these courses might be made *as useful as possible*. Zazkis and Marmur (2018) succinctly characterized this pragmatic view as follows: "basic knowledge of group theory is in fact neither necessary nor obligatory for addressing the (more elementary) mathematics. Nevertheless, […] it can be helpful" (p. 379). But currently, research that specifically addresses *how* advanced mathematics might be made more helpful and useful remains relatively scarce. That is, it is still "a central question as to how the gap between these two kinds of mathematics can be bridged" (Dreher et al., 2016, p. 220).

Our central argument here is that one possible way advanced mathematics can be made more useful and helpful for school mathematics lies not in the surface-level differences in content but in the common ways of reasoning that underlie these differences. We center our argument on an empirical analysis of the ways in which students in advanced mathematics reason about the foundational, cross-domain ideas of *equivalence* and *inverse*. We use this empirical analysis to propose that the ways of reasoning that support successful completion of tasks in advanced mathematics mirror those needed to productively engage with school mathematics. (These objectives are reflected in the research questions we present in Sect. 2.4). Essential to these efforts was our use of *conceptual analyses* (Thompson, 2002) for the topics of equivalence and inverse. Broadly, a conceptual analysis is a theoretical tool that provides explicit descriptions of the ways in which one might reason about a particular mathematical idea. Specifically, we use conceptual analyses for equivalence and inverse to analyze task-based clinical interviews in abstract algebra; we then use the results of this empirical analysis to highlight commonalities in the ways of reasoning that are productive in both advanced mathematics and school mathematics. To conclude the paper, we argue that the empirically-grounded analyses we present here support a more general, theoretical hypothesis: that a conceptual analysis for a key topic can serve as a tool by which researchers and teacher educators can potentially make advanced mathematical study more useful for future teachers and, in doing so, address the 'difficult tension' between the two.

2 Literature and theory

2.1 Types of connections between advanced mathematics and school mathematics

Wasserman (2018) identified several ways in which advanced mathematics might be relevant for school mathematics, three of which we focus on here:

- *Content-based connections* are connections between the content of advanced mathematics and school mathematics.
- *Classroom teaching connections* involve content-based connections that are related in some way to a specific classroom situation.
- *Modeled instruction connections* focus on the idea that instructors of advanced mathematics can demonstrate effective instructional practices.

We agree that if connections of any kind are to be realized, then we must be explicit about them. We therefore see a need for theoretical tools that researchers and instructional designers can use to explicate connections (as well as illustrations of how these tools might be productively used). The primary connections we explore in this paper are *content-based connections*. Specifically, we aim to showcase how a conceptual analysis is a potentially valuable theoretical tool because it can "establish meaningful connections between seemingly disjoint areas of mathematical study" (Wasserman, 2018, p. 8). Kaiser et al. (2017) indicated that researchers have typically approached these kinds of issues in teacher education from two perspectives: *cognitive* and *situated*. Our work here aligns with the cognitive perspective. Cognitive approaches to content-based connections can be categorized as follows: (1) identifying ways of reasoning that are essential to both advanced mathematics and school mathematics (and thus are potentially valuable connections that future teachers might make), and (2) identifying fundamental ideas in school mathematics that establish a basis for learning more advanced mathematics. Our efforts in this paper fall into the first category. These kinds of connections are viewed as beneficial because they have the potential to reinforce pre-service teachers' understandings of mathematical ideas and to increase the coherence in their mathematical reasoning. In Sect. 5, we discuss the potential for extending the *content-based connections* in this paper to support the realization of *classroom teaching connections* and *modeled instruction connections*.

2.2 Using a conceptual analysis to identify potentially valuable connections

Consistent with our cognitive perspective, a *conceptual analysis* is form of theory that explicitly describes students' ways of reasoning about a particular mathematical idea (Thompson, 2002). Conceptual analyses, which are underpinned by a radical constructivist epistemology (von Glasersfeld, 1995), play an important theoretical role in cognitive studies because they are frameworks that researchers can use to generate explanations about what students' reasoning about a particular mathematical idea might entail. Thompson (2002) argued that conceptual analyses are essential in this respect because these kinds of explanations "[come] from *theories specific to what is being explained or described*" (p. 193, emphasis added). We interpret this to mean that investigating students' reasoning about an idea necessarily involves a theory that is specific to the idea being reasoned about. Put another way, examining content-based connections involving equivalence and inverse require conceptual analyses that are specific to equivalence and inverse.

Thompson (2002) explicated several different uses for a conceptual analysis. We use the conceptual analyses that we set forth for equivalence (in Sect. 2.3.1) and inverse (in Sect. 2.3.2) in two ways. First, a conceptual analysis can be used to describe (build models of) students' ways of reasoning. An implicit aspect of this use is that a conceptual analysis is necessarily grounded in *students'* reasoning (instead of, for example, being grounded solely in the reasoning of experts). We see this as a key point that might help explain why many proposed connections between advanced and school mathematics remain unrealized by future teachers: such connections might account only for the experiences and reasoning of *experts* without also account for those of *students*. Second, a conceptual analysis frames issues of *coherence* in terms of consistency amongst the ways of reasoning that are relevant across curricula. From this perspective, then, the question of content-based connections (i.e., coherence between advanced mathematics and school mathematics) becomes: *What ways of reasoning in advanced mathematics are also relevant in school mathematics?* We are therefore using conceptual analyses of equivalence and inverse to shed light on the issue of connections between advanced mathematics and school mathematics because (a) as content-specific theories, they are precisely the right grain-size for identifying content-based connections, (b) they involve clear articulations of specific ways of reasoning (thus addressing researchers' calls for connections to be made more explicitly), and (c) they center on *students*' conceptual experiences (helping to prevent the positing of spurious connections more likely to be made only by experts).

There are analogous notions to this approach stemming from other theoretical perspectives, including didactic analysis (e.g., Breda et al., 2017) and genetic decomposition (e.g., Dubinsky, 2002)—see also Sfard's (1991) analysis of concepts via the operational-structural duality. Generally, we believe that these other approaches can be used to achieve similar ends, and therefore our uses of the conceptual analyses in this paper are intended to showcase more broadly the utility of these kinds of theoretical tools (that focus on the common reasoning that might underlie surface-level differences) for teacher education. In fact, two key insights from this larger body of research inform our efforts in this paper: (1) explicating mathematical meanings and ways of reasoning is essential for the mathematical education of teachers, and (2) having teachers develop a conscious awareness of these meanings and ways of reasoning enables them to "identify and organize the multiple meanings of the concept they wish to teach and […] to select those meanings to be studied in the instruction processes" (Breda et al., 2017, p. 1897). Here we are primarily concerned with the first of these themes, though we do consider the second to be of commensurate importance (and therefore return to it in Sect. 5.3).

Jeschke et al. (2019) observed that "research on teachers' subject-specific professional knowledge usually has been conducted *within one subject*" (p. 4, emphasis added). Accordingly, we observe that conceptual analyses and their theoretical correlates have typically been used to investigate or strengthen pre- and in-service teachers' understandings of topics in *either* school mathematics *or* advanced mathematics. Research utilizing such approaches across *both* of these domains is less common (there has, however, been some recent progress in this regard—see, for example, Wasserman, 2018). Though we provide two content-specific conceptual analyses for illustrative purposes, our overarching goal in this paper is to illustrate a much more general (and far less emphasized) point: conceptual analysis is a theoretical tool that can help researchers and teacher educators to resolve the tension between advanced mathematics and school mathematics by identifying ways of reasoning that are productive in *both* domains.

2.3 Conceptual analyses for the topics of equivalence and inverse

2.3.1 A conceptual analysis of equivalence

Equivalence is one of the most fundamental notions in mathematics and, as such, is pervasive across the K-16 curriculum (e.g., Asghari & Tall, 2005; Baiduri, 2015; Berman et al., 2013; Godfrey & Thomas, 2008). Much of the research on equivalence has concentrated on notions of equality and the equal sign in K-12 arithmetic and algebra, a key theme of which is that it is advantageous for students to interpret the equal sign as an expression of sameness (e.g., Knuth et al., 2006; Molina et al., 2009; Zwetzschler & Prediger, 2013). In our previous work (Cook et al., 2022a), we built upon this idea by asking: *In what ways can two objects be considered 'the same'?* Our answer to this question included explicating two ways of reasoning.

Table 1 Ways of reasoning about equivalence in school mathematics

Way of reasoning	Prevalent examples in school mathematics	
	Algebraic expressions	Algebraic equations
Common characteristic	Expressions "f and g are equivalent [because] $f(x) = g(x)$ for all x in the common domain" (Solares & Kieran, 2013, p. 122)	Equations are equivalent when they share the same solution set (e.g., Alibali et al., 2007)
Transformational	Expressions are equivalent if one can be transformed into the other by using certain algebraic rules (e.g., Zwetzschler & Prediger, 2013)	Equations are considered equivalent when one can be manipulated into the other according to certain algebraic rules (e.g., Baiduri, 2015)

A *common characteristic* way of reasoning[1] involves interpreting or determining the sameness of objects in terms of a feature that the objects share. Consider, for example, the algebraic expressions[2] $2(x + 2) + 1$ and $2x + 5$. Referring to the former as f and the latter as g, we can say that "f and g are equivalent [because] $f(x) = g(x)$ for all x in the common domain" (Solares & Kieran, 2013, p. 122). This is an example of a *common characteristic* way of reasoning because it attributes sameness to the shared numerical value of the two expressions. Consistent with the premise that ways of reasoning are based in *students*' conceptual experiences, though, we note that a 'common characteristic' is not necessarily fixed and is instead based upon a characteristic that the student attends to or infers. The key aspects of this way of reasoning are:

- *Characteristic E1:* uses descriptors like *same*, *common*, *similar*, *invariant*, *identical*, *duplicate*, or *shared* (or a reasonable synonym), and
- *Characteristic E2*: explains the sameness of the objects in question by identifying an attribute that the objects themselves share.

A *transformational* way of reasoning involves interpreting or determining equivalence on the basis that one object can be manipulated into the other pursuant to an established procedure or set of actions, rules, or properties. For instance, one might interpret that $2(x + 2) + 1$ and $2x + 5$ are equivalent because "one expression can be transformed into the other following certain syntactic rules" (Solares & Kieran, 2013, p. 122). This demonstrates a *transformational* way of reasoning because the equivalence of $2(x + 2) + 1$ and $2x + 5$ is framed in terms of manipulating the former into the latter using, for example, distributivity and associativity. We characterize this way of reasoning as follows:

- *Characteristic E1*: uses descriptors like *same*, *common*, *similar*, *invariant*, *identical*, *duplicate*, or *shared* (or a reasonable synonym), and
- *Characteristic E3*: explains the sameness of the objects in question by enacting or describing a sequence of actions by which one object might be changed into another.

We observe that the *common characteristic* and *transformational* ways of reasoning are complementary and essential in school mathematics—see Table 1. For example, in the domain of algebraic expressions in the real numbers, a *transformational* way of reasoning is an essential complement to *common characteristic* on account of "[t]he impossibility of testing all possible numerical replacements [for a variable] in order to determine equivalence" (Kieran & Saldanha, 2005, p. 196). Though an overemphasis on transformational activity has been rightly identified as a source of students' difficulties (e.g., Pomerantsev & Korosteleva, 2003), it is nevertheless essential because it enables students to generate additional, perhaps more desirable representations of a given object. The key is that students should know that transformations preserve the equivalence relation in question (Knuth et al., 2006)—in the language of our conceptual analysis, it is advantageous for students in school mathematics to know that one's transformations preserve the common characteristic.

2.3.2 A conceptual analysis of inverse

Similar to equivalence, inverse is a significant idea in mathematics and is ubiquitous in the K-16 curriculum. Students first encounter it at the primary level as a way to reason about the relationships between addition and subtraction (e.g., Baroody & Lai, 2007), as well as multiplication and division (e.g., Vergnaud, 2012); the focus then shifts to notions of inverse elements, first in the integers and real numbers, and then to more advanced algebraic

[1] We first encountered the term *common characteristic* in a study by Hamdan (2006) about the nature of elements that have been grouped together in an equivalence class.

[2] For simplicity, in this paper the algebraic expressions we refer to are polynomial expressions in one variable over the real numbers.

contexts like functions and complex numbers (e.g., McGowen & Tall, 2013). The fact that inverse appears in so many different algebraic contexts spurred us to identify and describe three ways of reasoning that can support productive engagement with inverses *across* algebraic contexts (Cook et al., 2022b).

Inverse as an undoing involves viewing inverse in terms of "sequences of commands which undo the action of other sequences of commands" (Pinto & Schubring, 2018, p. 898). As this characterization suggests, the focus of *inverse as an undoing* is on the interplay between operations. For example, many students first encounter inverse in the form of *inversion*, which involves "viewing addition and subtraction as interrelated operations (e.g., for $3 + 1 - 1$, immediately recognizing that adding 1 is undone by subtracting 1" (Baroody & Lai, 2007, p. 133). We operationalize this way of reasoning via the following characteristics:

- *Characteristic U1*: inverse is viewed as a relationship between operations.
- *Characteristic U2*: the purpose of the operation (or sequence of operations) in question is to undo the effect of the original operation(s).

Inverse as a manipulated element involves viewing inverse in terms of a procedure by which an element is changed into its inverse element. *Inverse as a manipulated element* is immediately distinct from *inverse as an undoing* because its focus is on inverse *elements*. The procedure by which inverse elements are obtained can be viewed as a unary operation (Vlassis, 2008) that is applied to a single element. In the real numbers, for example, one can find the additive inverse by multiplying the original element by -1 and the multiplicative inverse (of a nonzero number) by taking the reciprocal. *Inverse as a manipulated element* has two definitive characteristics:

- *Characteristic M1*: inverse is viewed as an element.
- *Characteristic M2*: the inverse element is associated with a procedure by which a given element is manipulated into its inverse element.

Inverse as a coordination of the binary operation, identity, and set involves conceiving of inverse as a relationship between two elements such that the combination of those two elements via the relevant binary operation yields the identity element. There are three core characteristics of *inverse as a coordination*:

- *Characteristic C1*: inverse is viewed as a relationship between a pair of elements and their interaction via the relevant binary operation.
- *Characteristic C2*: involves an awareness that the two elements in question combine via the binary operation to produce the relevant identity element.
- *Characteristic C3*: attends in some way to the fact that an element an its inverse must both be elements of the set in question.

Wasserman (2016) argued that "[u]nderstanding the general notion of inverse, where additive, multiplicative, functional, etc., inverse become examples of the same concept, unified within some algebraic structure, can help provide a sense of consistency for teachers in developing and discussing these ideas" (p. 36). We propose that the 'general notion' of inverse involves being able to move flexibly between each of the aforementioned ways of reasoning as needed. We caution, however, that an overreliance on any particular way of reasoning can be problematic. For example, an overreliance on *inverse as an undoing* can lead to the belief that inverses always exist, and an overreliance on *inverse as a manipulated element* (and procedures such as 'switch-and-solve') can lead to compartmentalized and incoherent ways of reasoning about inverse (e.g., Kontorovich & Zazkis, 2017). It is the flexible interchange between all three that supports productive reasoning with inverses in school algebra contexts (Table 2).

2.4 Research questions

The conceptual analyses of equivalence and inverse set forth above provide a theoretical framing for the empirical analysis on which we base our arguments in this paper. We advance our argument by answering the following research questions:

- *RQ1*: What ways of reasoning about the foundational topics of equivalence and inverse support the successful completion of tasks in advanced mathematics?
- *RQ2*: How do these ways of reasoning about equivalence and inverse in advanced mathematics relate to those that support productive engagement with the same topics in school mathematics?

3 Methods

The episodes in this paper occur in the context of task-based clinical interviews (Clement, 2000), a methodology that, in accordance with one of the uses of a conceptual analysis, aims to develop models of students' intuitive mathematical reasoning. Researchers construct these models by observing students' mathematical behaviors and then proposing descriptions of students' ways of reasoning that might plausibly underlie and explain these behaviors. We observe that

Table 2 Ways of reasoning about inverse in school mathematics

Way of reasoning	Examples from school mathematics	
	Multiplicative inverses in the real numbers	Compositional inverse of a function
Inverse as an undoing	An inverse "will return you to the starting point. Let's say I pushed the wrong button on the calculator and multiplied by 5. For correcting this, I need to divide by 5" (Kontorovich & Zazkis, 2017, p. 31)	An inverse function is "the operation needed to go in the reverse direction, from the final state to the initial state" (Vergnaud, 2012, p. 441)
Inverse as a manipulated element	The multiplicative inverse of any nonzero real number can be found by taking its reciprocal (e.g., Clay et al., 2012)	An inverse function can also be viewed in terms of "switching the x and y variables and solving for y" (Pinto & Schubring, 2018, p. 900)
Inverse as a coordination	"We remember multiplication if we take a number and multiply it by its multiplicative inverse you will get the multiplicative identity 1" (Clay et al., 2012, p. 769)	The composition of a function with its inverse function yields the identity function (e.g., Vidakovic, 1996)

Table 3 Tasks used to elicit reasoning about equivalence and inverse

Episode 1	Episode 2
Task 1.1: Is $\phi : \mathbb{Q} \to \mathbb{Z}$ given by $\phi\left(\frac{a}{b}\right) = a + b$ a function? Explain	*Task 2.1:* Prove: for all $a, b \in \mathbb{Z}_3[i]$, all equations of the form $a + x = b$ have a unique solution in $\mathbb{Z}_3[i]$
Task 1.2: Is $g : \mathbb{Q} \to \mathbb{Q}$ given by $g\left(\frac{a}{b}\right) = \frac{a+b}{b}$ a function? Explain	*Task 2.2:* Prove: for all $a \in \mathbb{Z}_3[i]\backslash\{0\}$ and $b \in \mathbb{Z}_3[i]$, all equations of the form $ax = b$ have a unique solution in $\mathbb{Z}_3[i]$
Task 1.3: Is $f : \mathbb{Z}_4 \to \mathbb{Z}$ given by $f\left([a]_4\right) = a$ a function? Explain[a]	

[a]Here, $[a]_4$ represents the congruence class (modulo 4) that contains the integer a. It can be represented in multiple ways (e.g., $[0]_4$, $[4]_4$, and $[8]_4$ are all different representations of the same congruence class)

the uses of a conceptual analysis (as a theoretical framework) and task-based clinical interviews (as a methodology) align well with each other because they provide complementary tools (one theoretical and one empirical) by which researchers can develop clearer images (models) of students' ways of reasoning. For example, the conceptual analyses for equivalence and inverse that we proposed above—in addition to describing key ways of reasoning in school mathematics—provided the primary analytical framework that guided the development of these models.

3.1 Data collection and task design

The participants in these episodes—referred to by the pseudonyms Isaac (Episode 1) and Meagan and Josh (Episode 2)—were enrolled at a large research university in the United States. At this university, abstract algebra is a required course for pre-service mathematics teachers. All were selected for participation because (a) their mathematical experience indicated to us that they would be able to engage productively with tasks in abstract algebra, and (b) we anticipated that they would be able to articulate their thinking clearly and without reservation as they engaged with potentially challenging mathematical tasks. A typical session lasted approximately 90 min and involved a researcher (the second author for Episode 1, the first author for Episode 2) administering a series of tasks (here we focus on the tasks listed in Table 3). Upon the students' completion of a task, the researcher would (1) ask the student to explain and justify their general approach, and then (2) ask follow-up questions to clarify some aspect of the students' mathematical activity or test an emerging conjecture the researcher had developed about the students' underlying ways of reasoning. Isaac's activity in Episode 1 spans one session; Episode 2 includes excerpts from 3 sessions. Each session was recorded with an iPad application that created videos of students' writing with synchronized audio.

We designed the tasks featured in these episodes—which involve attending to multiple, equivalent representations in the domain of a proposed correspondence (Episode 1) and proving results about the structure of a finite field (Episode 2)—to reflect fundamental considerations in abstract algebra and elicit reasoning about the topics of equivalence and inverse. The tasks in Episode 1 were drawn from an analysis of examples and non-examples of functions given in abstract algebra textbooks (Uscanga & Cook, 2022). The tasks in Episode 2 were informed by researchers' observations that inverses can emerge in students' activity as they prove basic conjectures about a finite algebraic structure (Larsen, 2013). These tasks are particularly useful for our purposes here because they involve notions—multiple representations in the domain and finite fields—that have little (if any) relevance in school algebra, thus embodying the 'difficult tension' between abstract algebra and school algebra.

3.2 Data analysis

Our primary goal of data analysis was consistent with the first use of conceptual analyses given in Sect. 2.2: to create viable models of students' reasoning; we considered a model 'viable' insofar as it offered a plausible frame of reference for our observations of students' mathematical behaviors. To prepare for data analysis, all sessions were transcribed in full. We then created enhanced transcripts that incorporated images of students' written work. We then used our conceptual analyses in conjunction with Clement's (2000) stages for developing models of students' reasoning—see Table 4 for an illustration of the prominent role that our conceptual analyses played in this process. This process typically resulted in modifying the emerging model of the students' reasoning (e.g., by revising a hypothesis); it was iterated until stable, viable models of the students' ways of reasoning emerged.

4 Results

In this section, we analyze students' reasoning in response to abstract algebra tasks, which included, for example, examining a proposed correspondence with domain $\mathbb{Z}_n$ (Episode 1), and proving results about the structure of $\mathbb{Z}_3[i]$, the finite field of order 9 (Episode 2). Our objective is to identify the students' ways of reasoning about equivalence and inverse (informing RQ1) in order to examine their potential relevance to the ways of reasoning that are productive in school mathematics, such as those in Table 1 (informing RQ2).

4.1 Episode 1—reasoning about correspondences in abstract algebra

In this episode, we focus on the mathematical activity of Isaac as he engaged with function-based tasks in abstract algebra (Tasks 1.1–1.3). In response to Task 1.1, Isaac argued at first that ϕ is a function, explaining that "there's not an element when we input it into the function that maps to two different outputs." He based this claim on specific input–output pairs he computed. For example, he stated that "1/3 would map to 4" and "ϕ of 2/3 would go to $2+3$, and you have 5. […] That works out, you have one element and it goes to another element." Eventually the interviewer, noting that all of the rational numbers Isaac was using were in reduced form, prompted him to consider unreduced rational numbers (for which $\gcd(a,b) > 1$). He noted that "2/3, well, that's the same thing as 4/6, but they would map to a different element. […] And so now you no longer have a function […] because 2/3 and 4/6 are *equivalent*." In response to Task 1.2, Isaac, who correctly identified this proposed correspondence as a function, described his approach as follows (see Fig. 1):

Isaac: I would maybe see if, like, some sort of element would map to two different elements, right? And in this case, I would probably pick an element that is, quote unquote, equivalent. So I could probably pick like 1/3 or something. I guess I would maybe do it like $g(1/3)$ […] which is 4/3. And then see if I can find something that's like kind of equivalent to 1/3. So maybe like $g(2/6)$ […] which would be 8/6. […] 4/3 is the same thing as 8/6. And you would ultimately get some sort of function.

We note that Isaac's (correct) identification of the proposed correspondences in Tasks 1.1 and 1.2 as a nonfunction and function (respectively) hinged on his attention to equivalence in the domain. Isaac even pointed this out himself: "in my first example, I didn't even consider 2/3 and 4/6, you know, and stuff like that. And so, and then the second example, I was a little bit more careful. I was like, OK, well, there's some elements that are *equivalent* to each other."

Isaac explained that he approached Task 1.3 in a similar way to Tasks 1.1–1.2: "in example 1 and example 2 how I was, you know, picking fractions that ultimately looked different but represented the same thing, or the same property or sameness, or whatever. I wanted to do the exact same thing in this case." That is, attending to the fact that elements in $\mathbb{Z}_4$ (the domain) could be represented in different, equivalent ways, he set out to identify such equivalent representations in the domain and see if their image in the codomain was the same. He observed, for example, that even though $[0]_4 = [4]_4$, the images of $[0]_4$ and $[4]_4$ with respect to the correspondence f are not the same (the integers 0 and 4, respectively—see Fig. 2):

Isaac	You have 0, 4, 8 or whatever. They're all the exact same thing in $\mathbb{Z}_4$. But in my outputs, I'm getting different values, you know, and that is a no–no, in this case.
Interviewer	OK.
Isaac	So that would be my reasoning to say like, oh, like, bam, no, not a function.

Accordingly, we propose that Isaac's activity with these functions tasks in abstract algebra was supported by two distinct (yet complementary) ways of reasoning about equivalence. When first asked what it meant for elements of $\mathbb{Q}$ to be equivalent, Isaac framed his response in terms of a question: "do they have some sort of property in common?" He elaborated that, with respect to "equivalence, we're just looking at this sort of the same property between

Table 4 Example of our data analysis procedures

Stages of analysis (Clement, 2000)	Data excerpt	Analysis
Stage 1: Making observations of the students' mathematical behaviors	Isaac: I would probably pick an element that that is, quote unquote, equivalent. So I could probably pick like 1/3 or something. I guess I would maybe do it like $g(1/3)$ would have me map to $\frac{1+3}{3}$, which is 4/3. And then see if I can find something that's like kind of equivalent to 1/3. So maybe like $g(2/6)$. […] So then I would get $\frac{2+6}{6}$, which would be 8/6. […] 4/3 is the same thing as 8/6. […] They represent the same numerical value	Observations: Isaac describes 1/3 and 2/6 as "quote unquote *equivalent*." Similarly, "four over three *is the same thing as* eight over six." Isaac explains that these elements are the same because they "represent the same numerical value."
Stage 2: Using the conceptual analysis to formulate hypotheses about the ways of reasoning that underlie these behaviors	(same as above)	Our two observations above align with characteristics E1 and E2 (respectively) in our conceptual analysis of equivalence. We therefore propose that Isaac's behaviors demonstrate a *common characteristic* way of reasoning
Stage 3: Returning to the transcripts to identify additional observations of students' reasoning that affirm or contradict our initial hypothesis	Isaac: If you divide whatever integer you're looking at, and the remainder is so on and so forth. With four, then they're all the same thing. So like 0, and 4, and 8, you know, they're all evenly divided by 4. […] We would just write that $[0]_4$ is the same thing as $[4]_4$	We again observe characteristics E1 and E2 in Isaac's activity in this excerpt: he describes the integers 0, 4, and 8 as "the same thing" (characteristic E1) in this context because "they're all evenly divided by 4"—that is, they all have the same remainder upon division by 4 (characteristic E2)

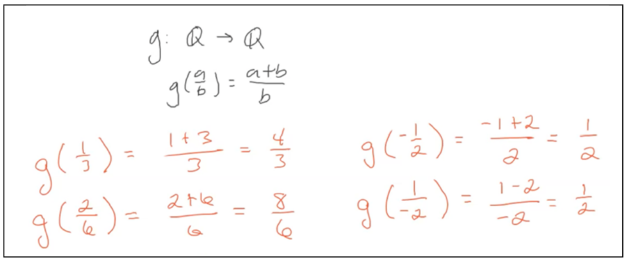

Fig. 1 Isaac uses equivalence to conclude that g is a function

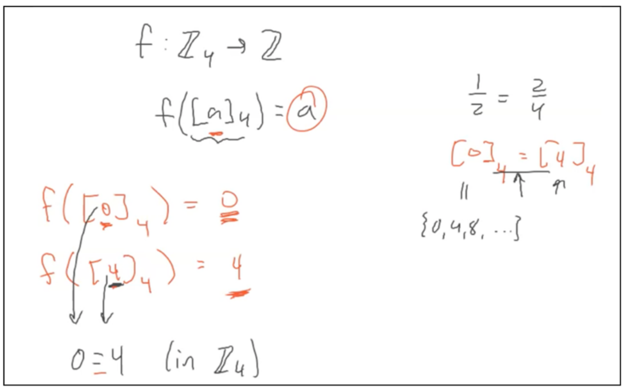

Fig. 2 Isaac attends to issues caused by multiple representations in $\mathbb{Z}_4$

whatever we're looking at, you know, they might not look the same but they have the same property". For Isaac, the "same property" in Tasks 1–2 was the numerical value of the quotient obtained by dividing the numerator by denominator. In Task 3, he framed equivalence in a similar way, suggesting a coherent, cross-cutting view of equivalence: "I'm just seeing if there's one property that they share in common. And if they have that property in common, then I would say that they're equivalent." For instance, referring to his written work (see Fig. 2), Isaac explained: "so like 0, and 4, and 8, they're all evenly divided by four." Put another way, each of these integers has the *same remainder*—zero—upon division by 4. These excerpts of Isaac's activity indicate a *common characteristic* way of reasoning about equivalence because Isaac identified collections of elements as the same (rational numbers in Tasks 1.1–1.2 and integers in Task 1.3; characteristic E1) and attributed this sameness to a common property shared by the rational numbers in each of these collections (the quotient in Tasks 1.1–1.2 and the remainder upon division by 4 in Task 1.3; characteristic E2).

Isaac also demonstrated strong notions of equivalence in terms of "simplifying" or "reducing" elements. For example, when explaining how to identify other elements that are equivalent to a particular element (i.e., populate an equivalence class), he said, "take a fraction and you see if you can simplify all the way down." He noted, for example, that 4/3, 8/6, and 12/9 are equivalent because "they can all reduce to 4/3." Additionally, he explained that, with respect to 1/3 and 2/6, "ultimately I could reduce one or think of them as the same thing." Isaac reduced these fractions using the canonical procedure of multiplying both the numerator and denominator by the same nonzero factor. The fractions that resulted, he noted, "all have this sameness property that I can reduce all of them to one of those fractions." Isaac exhibited a similar strategy when engaging with integers that are equivalent modulo 4: 0 in $\mathbb{Z}_4$ is the same as 4 in $\mathbb{Z}_4$, right? […] You know, I'm just, it's kind of like those fractions. We kind of reduce them down." The reduction procedure that Isaac employed in $\mathbb{Z}_4$ involved repeated subtraction (or addition) of the modulus 4. We therefore claim that Isaac is also demonstrating a *transformational* way of reasoning about equivalence because he was interpreting the sameness of elements (rational numbers in Tasks 1.1–1.2 and integers in Task 1.3; characteristic E1) in terms of a procedure by which one element might be manipulated into another (dividing the numerator and denominator by the same factor in Tasks 1.1–1.2 and repeatedly subtracting/adding the modulus in Task 1.3; characteristic E3).

We therefore propose that the *common characteristic* and *transformational* ways of reasoning about equivalence were central to his successful completion of Tasks 1.1–1.3, thus informing RQ 1. Additionally, on the surface, function-related tasks that hinge on issues of multiple representations in the domain appear to have very little to do with school mathematics. Our analysis, however, highlights that the underlying ways of reasoning that support productive engagement with such tasks in abstract algebra (in this case, *common characteristic* and *transformational*) are exactly the ways of reasoning needed to reason productively about equivalence in school mathematics (see Table 1), thus informing RQ 2. Importantly, our use of the conceptual analysis of equivalence (in Sect. 2.3.1) is what enabled us to identify this relationship between advanced mathematics and school mathematics. We discuss this point in greater depth in Sect. 5.

4.2 Episode 2—proving conjectures about a finite field in abstract algebra

In this episode, we discuss the reasoning of Josh and Meagan as they explored the algebraic structure of $\mathbb{Z}_3[i]$, the finite field of order 9. We focus here[3] on their mathematical activity as they attempted to prove that, for all $a, b \in \mathbb{Z}_3[i]$, all equations of the form $x + a = b$ have a unique solution

[3] Other aspects of and episodes from these sessions are discussed in Cook and Uscanga (2017).

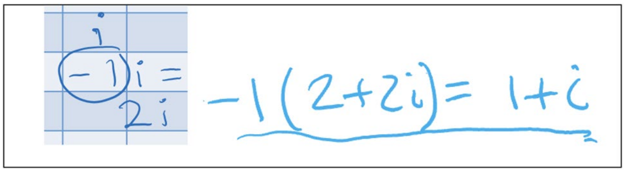

Fig. 3 Meagan (left) and Josh (right) demonstrate their procedure to manipulate an element of $\mathbb{Z}_3[i]$ into its additive inverse

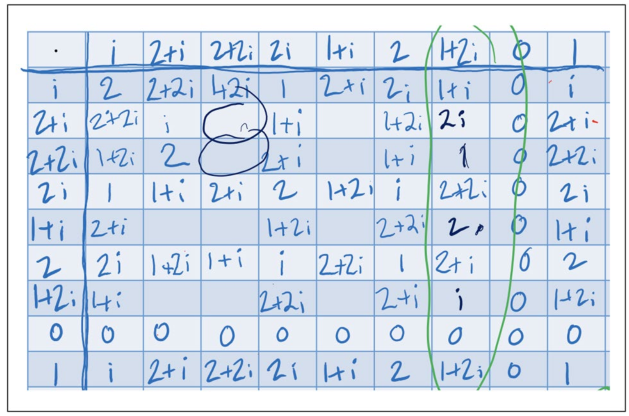

Fig. 4 Meagan used this multiplication table to identify a multiplicative inverse for each nonzero element in $\mathbb{Z}_3[i]$

in $\mathbb{Z}_3[i]$. As in Episode 1, we note that this task—exploring and proving conjectures about the algebraic structure of the finite field of order 9—is one whose connections to secondary algebra are initially neither obvious nor guaranteed.

When attempting to identify a solution candidate (the 'existence' part of the proof), Josh manipulated $x + (c + di) = (a + bi)$ to obtain $x = (a - c) + (bi - di)$. Meagan explained that they subtracted $c + di$ from both sides of the equation because "we want to get x by itself." We interpret that the students were demonstrating *inverse as an undoing* because they used operations (subtraction, characteristic U1) to undo the effects of addition (as evidenced by their desire to "get x by itself," characteristic U2). Hoping to encourage Josh and Meagan to focus on the binary operation of addition, the researcher asked how they might reformulate their use of subtraction in terms of addition. Josh amended his initial solution $x = (a + bi) - (c + di)$ to $x = (a + bi) + (-c - di)$. When asked how they could be sure that such an element $-c - di$ existed for each element $c + di$ in $\mathbb{Z}_3[i]$, Josh responded that "you multiply it by negative one" and then "simplify it from there." For example, Meagan used her knowledge of modular arithmetic (specifically that -2 and 1 are congruent modulo 3) to reason that the additive inverse "of $2i$ is $-2i$, which is just i." They were able to use this procedure to identify an additive inverse element for each of the 9 elements in $\mathbb{Z}_3[i]$ (see Fig. 3 for two additional examples). We interpret that Josh and Meagan were demonstrating an *inverse as a manipulated element* way of reasoning because they were viewing the inverse relationships in this task in terms of inverse *elements* (characteristic M1) that were obtained by manipulating the original element via a procedure (multiplying by -1 and then using modular congruence, characteristic M2).

Later, after attempting to prove the analogous result for multiplicative linear equations in $\mathbb{Z}_3[i]$, they experienced some difficulties justifying that each nonzero element has a multiplicative inverse. They employed a reciprocal-based procedure (which we interpreted as another demonstration of *inverse as a manipulated element*), but they were unable to adapt it so that it clearly identified which element of $\mathbb{Z}_3[i]$ was the multiplicative inverse of the given element. For example, when attempting to find the multiplicative inverse of 2, Josh and Meagan took the reciprocal to obtain 1/2 but were initially unable to identify an element of $\mathbb{Z}_3[i]$ to which this corresponded. Eventually, however, Meagan, using the multiplication table for $\mathbb{Z}_3[i]$ as a guide, realized that multiplying an element by its multiplicative inverse yields 1, enabling her to resolve the issue of the multiplicative inverse of 2, concluding that 2 is its own inverse because "2 and 2 equal 1." She went on to identify several more inverse pairs: "i times $2i$, and then, um, $2i$ times i again, obviously, and then, like, $2 + i$ and $1 + i$ also equal 1." Josh, also referencing the multiplication table, reasoning similarly, referred to $2 + i$ and $1 + i$ as "inverse pairs" and explicitly identified in the multiplication table that their product is 1 (see Fig. 4).

Josh and Meagan were able to reason in this way to identify a multiplicative inverse for all 8 nonzero elements of $\mathbb{Z}_3[i]$. We claim that they were demonstrating an *inverse as a coordination of the binary operation, identity, and set* way of reasoning here. We first observe that their attention has shifted from manipulating one element into another (as they did when enacting *inverse as a manipulated element*) to focusing simultaneously on *pairs* of elements and their image under the binary operation of multiplication. As evidence, consider that Josh and Meagan make repeated reference to (1) pairs of elements—both implicitly (e.g., Meagan's references to "2 and 2," "i times $2i$," and "$2 + i$ and $1 + i$") and explicitly (e.g., Josh's reference to "inverse pairs")—and (2) their image under multiplication (e.g., several mentions of "times" and "multiplying"); this satisfied characteristic C1. For characteristic C2, we note that Josh and Meagan identify that the image of these pairs of elements under multiplication is 1, the multiplicative identity (e.g., "2 and 2 *equal 1*"). Lastly, we observe that Josh and Meagan have attended to the fact that an element and its inverse must be in the same set (characteristic C3) both explicitly (in their work above with additive inverses, they adapted their manipulation procedure so that they could identify which element of $\mathbb{Z}_3[i]$—the relevant set from which the original element was taken—was the additive inverse) and implicitly (when identifying inverse elements with a multiplication table, the only elements in question are the elements of $\mathbb{Z}_3[i]$ that are arranged across the top row and leftmost column) (Fig. 4).

Our analysis here highlights that Meagan and Josh's efforts were supported by their demonstration of three ways of reasoning about inverse: *inverse as an undoing*, *inverse as a manipulated element*, and *inverse as a coordination* (informing RQ1). Furthermore, even though the relevance of this abstract algebra task—which required reasoning about the algebraic structure of a modular ring with a complex component—to school algebra is initially unclear, the underlying ways of reasoning mirror those needed to engage productively with inverse in school algebra contexts (see Table 2), affording insight into RQ2. As in Episode 1, the empirical analysis in this episode supports our more general, theoretical argument: that a conceptual analysis is a tool that can be used to highlight potentially valuable connections for future teachers to make between advanced mathematics and school mathematics.

5 Discussion

5.1 Revisiting the research questions

In this paper, we have considered the persistent and well-documented tension that pervades discussions of productive connections between advanced mathematics and secondary mathematics instruction; this tension is compounded by proposed connections that are vague and underspecified. In response, we adopted the pragmatic stance that advanced mathematics should be made *as useful as possible* for future teachers, and posed two research questions that aimed to illustrate a particular way in which advanced mathematics can be made useful for future teachers. Answering our first research question involved an empirical analysis of the ways of reasoning demonstrated by abstract algebra students in task-based interviews. In the case of equivalence, these ways of reasoning included *common characteristic* and *transformational*; in the case of inverse, *inverse as an undoing*, *inverse as a manipulated element*, and *inverse as a coordination*. In response to our second research question, we have also illustrated that these ways of reasoning about equivalence and inverse in advanced mathematics mirror those that support productive reasoning in secondary mathematics (observe, for example, how the ways of reasoning that emerged in Sects. 4.1 and 4.2 mirror those in Tables 1 and 2, respectively). Our capacity for answering these questions hinged on our use of two conceptual analyses (Thompson, 2002), which focused our attention on the ways of reasoning that underlie the surface-level differences in content that are in large part responsible for the 'difficult tension' between advanced mathematics and school mathematics. In this way, the specific ways of reasoning that form the foundation of our answers to our research questions illustrate a more general, theoretical point: that researchers and teacher educators can potentially use conceptual analyses to overcome the difficult tension between advanced mathematics and school mathematics and highlight coherent ways of reasoning that might otherwise be obscured.

5.2 Contributions

This paper's primary contribution to the literature stems from addressing an important need: though many researchers agree that advanced mathematics should be made as useful as possible for pre-service teachers, there are relatively few explicit theoretical tools and illustrations available to assist in achieving this goal. In this respect, the episodes featured here highlight a new insight: a conceptual analysis can help identify coherence that might otherwise be obscured by some of the obvious differences between advanced and secondary mathematics. This contribution addresses two notable gaps in the literature. First, as Larsen et al. (2018) quotation in the introduction underscores, many attempts to identify connections between advanced and secondary mathematics have focused on *researchers*' and *educators*' views of connections (in our view, a necessary but insufficient initial step). Our work here extends one step further by grounding the associated ways of reasoning in the conceptual experiences of *students*. Second, we observe that, while this general approach is not altogether novel, such studies have typically constrained their focus to *either* advanced mathematics *or* secondary mathematics. Our efforts here contribute to a small (but growing) body of literature that has used such approaches—which, at their core, examine the ways of reasoning that underlie topics and tasks—across *both* domains.

5.3 Limitations and future directions

We have focused most of our efforts on illustrating potential benefits of this approach for researchers and teacher educators, as researchers and teacher educators must first develop clear images of connections between advanced mathematics and school mathematics for themselves before they can support future teachers in doing so. We view this as a critical initial—though by no means final—step toward our pragmatic objective to make advanced mathematics as useful for teachers as possible. A notable limitation of this work, in fact, is that it addresses only mathematical—and not pedagogical—knowledge. Thus, it remains to be seen how the ideas we have discussed here might influence teachers' practice in various classroom situations and environments

Table 5 Stages for identifying potential connections to school mathematics

Stage	Specific example
Identify a key idea in secondary mathematics	The notion of identity appears in school algebra in many different forms, including the real numbers (addition and multiplication), functions (composition), and matrices (addition and multiplication)
Develop/use a conceptual analysis to describe ways in which students might reason about this idea	Reasoning about the identity as the element that 'does nothing' can support attention to a more coherent concept of identity across various contexts (e.g., Clay et al., 2012)
Identify various situations in advanced mathematics in which these ways of reasoning are useful	The notion of 'doing nothing' is also productive in advanced mathematical settings in which the connections to school mathematics are not immediately clear, including (a) the dihedral groups D_n in abstract algebra (e.g., Larsen, 2013) and (b) the invertibility of matrix transformations in linear algebra (Bagley et al., 2015)
Promote reflection on the similarity of the ways of reasoning across domains	Reflecting upon the similarities highlighted by notions of the 'do nothing' function across matrix transformations in linear algebra and polynomial functions in school algebra, a student commented, "essentially this [the vector (x, y)] is the vector **x**, so essentially I did end up with […] **x** as in the, whatever I had here. Yeah, it is identical [to the function case]. That's cool! I'm glad I did that, that's interesting" (Bagley et al., 2015, p. 44)

(Hoth et al., 2018). Another limitation is that this general insight emerged retrospectively, after our analysis of specific empirical episodes. Ideally, we would like for instructional designers at the postsecondary level to use this general insight prescriptively as a means of instructional design. In this section, we address how these points inform and might be realized in future research and in teacher education more extensively.

Broadly, we believe that conceptual analyses can address the tension between advanced and secondary mathematics in (at least) three ways, which we relate to the types of connections emphasized by Wasserman (2018) and discussed in Sect. 2.1. The first is to use conceptual analyses retrospectively (as we have here) to identify *content-based connections* that otherwise might have been obscured within advanced content. The second use involves the potential for supporting the development of *classroom teaching connections*. This could involve using a conceptual analysis *prescriptively* as a means of instructional design. That is, researchers and teacher educators might identify and support *classroom teaching connections* by starting with an idea in school mathematics and using a conceptual analysis of that idea to identify potential ideas in advanced mathematics that might reinforce ways of reasoning that are productive in that classroom situation (indeed, this is a limitation of our retrospective, empirically-driven approach in this paper). In order to partially address this limitation and position future research efforts that align with this objective, we provide an example of how we envision such a process unfolding for the key topic of identity—see Table 5. We further note that our analysis here is entirely *cognitive* (and not *situated*), and—as "both perspectives provide powerful insights into teacher professional knowledge" (Kaiser et al., 2017, p. 165)—there is still much work to be done to identify how the tools we have presented here might influence teachers' pedagogical knowledge and the factors that condition its implementation in various classroom situations and environments.

The third use involves a *modeled instruction connection*. Related to the point we raised in the previous paragraph, another way for the kinds of connections we propose in this paper to positively influence teachers' instruction is to make them explicit to teachers. Though we have primarily focused on how researchers and teacher educators can use conceptual analyses, we note that others have suggested that it might also be beneficial for teachers themselves to become explicitly aware of their own ways of reasoning so that they can draw upon more intentionally and strategically in their instruction (e.g., Breda et al., 2017). As a conceptual analysis is an explicit description of ways of reasoning, we suggest that teacher educators can make progress toward accomplishing this goal by (1) modeling their own use of conceptual analyses in their advanced mathematics instruction, and (2) promoting conversations in which future teachers reflect on and attend to a conceptual analysis and how it was (or might be) used. We thus also call for researchers to examine ways in which pre-service teachers might be encouraged to reflect upon their own mathematical reasoning in order to develop their capacity for identifying and leveraging such connections in the future.

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