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SURVEY PAPER

Making university mathematics matter for secondary teacher preparation

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Abstract
Internationally, questions about the perceived utility of university mathematics for teaching school mathematics pose an ongoing challenge for secondary mathematics teacher education. This special issue is dedicated to exploring this topic and related issues in the preparation of secondary mathematics teachers—by which we mean teachers of students with ages, approximately, of 12–18 years. This article introduces this theme and provides a semi-systematic survey of recent related literature, which we use to elaborate and situate important theoretical distinctions around the problems, challenges, and solutions of university mathematics in relation to teacher education. As part of the special issue, we have gathered articles from different countries that elaborate theoretical and empirical approaches, which, collectively, describe different ways to strengthen university mathematics with respect to the aims of secondary teacher education. This survey paper serves to lay out the theoretical groundwork for the collection of articles in the issue.

Keywords Secondary teacher education · University mathematics · Mathematical preparation of teachers

1 Introduction

The focus of this special issue is on exploring and strengthening university mathematics for secondary teacher preparation—by which we mean teachers of students with ages, approximately, of 12–18 years. University mathematics plays an important role in secondary teacher preparation programs (e.g., CBMS, 2012), yet it is not without significant problems and challenges. Felix Klein (2016) was one of the first to point out some of the specific challenges in his formulation of the so-called "double discontinuity"—the gap prospective teachers experience in their transition to and from university mathematics studies. As educational research has progressed over the past century, other scholars' work has continued to shape and refine our recognition and understanding of the unique role and challenges of university mathematics in terms of teacher education; such as Shulman's (1986) notion of pedagogical content knowledge, Chevellard's (1989) notion of didactic transposition, the recognition of the importance of non-cognitive, affective domains that influence teachers (e.g., Thompson, 1992), as well as recent emphases on practice-based orientations to knowledge and teacher education (e.g., Lampert, 2010; Ball & Forzani, 2009). These ideas—which we elaborate on later—inform teacher education aims around the world, in that prospective teachers not only develop mathematical knowledge alone, but also pedagogical content knowledge, appropriate beliefs about teaching and learning, and practical teaching skills. Yet, in a university context, it can be difficult to adhere to these recommendations, hindering the potential of university mathematics in service to prospective teachers' professional formation.

To belabor the point, empirical studies repeatedly point to the minimal value secondary teachers report about their university mathematics coursework in relation to their classroom teaching—that is, the expectation that university mathematics will "trickle-down" to inform teaching practice, in the terms of Wu (2011), is not supported by empirical evidence (e.g., Cooney & Wiegel, 2003; Darling-Hammond,

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2000; Gool, 2013; Goulding et al., 2003; Hefendehl-Hebeker, 2013; Hoth et al., 2020; Wasserman et al., 2018; Zazkis & Leikin, 2010). Prospective secondary teachers have commonly criticized the lack of practical applicability of their university studies in mathematics to their future profession (Cooney & Wiegel, 2003; Hefendehl-Hebeker, 2013). The challenges posed by the high academic demands of the mathematical coursework, combined with the perceived low relevance of such courses to their future profession, may even contribute to high dropout rates of prospective teachers from teacher preparation programs (Clark & Lovric, 2009; Gueudet, 2008).

The purpose of this survey paper is to introduce the theme of the special issue and to situate it within a relevant literature base. Although the predominant discourse in the field, broadly-speaking, is around differences between (as well as opportunities to connect) university and school mathematics, practically-speaking, both innovations and challenges are realized in particular contexts—specifically, in particular mathematics courses. Hence, much of this special issue addresses a variety of initiatives that university instructors have developed to strengthen connections between university mathematics and school mathematics and to provide pre- or in-service teachers with a range of mathematical and didactical experiences that relate more to their professional practice. In addition to introducing the theme, we furthermore use this survey to help lay out the general theoretical groundwork for the collection of articles in this special issue in order to situate this work that is happening at the local level of courses. We start by briefly situating the importance of university mathematics within secondary teacher education, and outlining some of the historical developments in the field of mathematics education that have informed the field's sense of what university mathematics might focus on to accomplish teacher education goals. Then, we survey the recent literature pertaining to this theme, organizing it by different aspects that these papers have focused on in terms of the problems, challenges, and solutions for university mathematics. The important theoretical distinctions with respect to making university mathematics matter for secondary teacher preparation, which come out from the literature review, are then used to frame and introduce the articles in this special issue.

2 Situating university mathematics in secondary teacher education

Teacher education, globally, is a contextual enterprise. Ponte and Chapman (2008) point out that prospective teachers' experiences in teacher education are influenced by program elements, the characteristics of program instructors and other stakeholders, as well as their own characteristics, socio-cultural features of the society, the organization of the educational system, and the state of research. The key point is that teacher education is a complex system; one inherently linked to particular contexts. And yet, internationally, there also seem to be some common aspects across the contextual diversity.

In particular, there seems to be a reasonably common *structure* within university mathematics teacher education programs. Potari and da Ponte (2017) identify content knowledge, pedagogical knowledge, and didactical knowledge as three strands around which teacher education programs organize coursework; they also point out practicum experiences, or fieldwork, as another common component. Indeed, outside of other general requirements, Leung et al. (2015) report secondary mathematics teacher education courses in Asia according to these same four categories: mathematics; mathematics education; general pedagogy; and teaching practicum.[1] For our purposes, we will use the term pedagogy to refer to general aspects of teaching, and mathematics didactics or mathematics pedagogy to refer to mathematics-education-specific aspects of teaching (although we recognize some may make even further distinctions between mathematics didactics and mathematics pedagogy, e.g., Scheiner & Buchholtz, 2022). This structure also mirrors departmental distinctions frequently made at universities; e.g., mathematics departments typically offer mathematical coursework, and education departments offer pedagogical coursework. The mathematics didactical coursework can be situated in either the mathematics or education department, depending on the university or teacher education program. We note this structure has been criticized for its fragmentation of teacher education and flagged as an area in need of reform (Flores, 2016; Hudson & Zgaga, 2017). This is in part due to the fact that as a result of the fragmentation, pre-service teachers experience mathematical content and mathematics didactics in different university courses. Moreover, since pre-service teachers often ascribe greater importance to practice-oriented components, experiencing mathematics as detached from school requirements may lead to motivation problems in university mathematics courses (Hanke et al., 2021).[2]

[1] We also acknowledge there are exceptions to this common structure; programs with some components missing or scarcely incorporated.

[2] Recent mathematics education research in this area suggests that in order to overcome the structural fragmentation problem, a solution could be to consider specific course design principles of dovetailing and interlinking subject matter knowledge and subject matter didactical knowledge through boundary-crossing in order to shape a more coherent program of study for pre-service teachers (Hanke et al., 2021).

The Teacher Education and Development Study in Mathematics (TEDS-M) was a large international study of mathematics teacher education programs (Tatto et al., 2008), which provided some empirical support for convergent trends internationally, while still acknowledging the diversity. Aside from the common university "structure" described previously, the TEDS-M framework, which was grounded in extensive theoretical and empirical research worldwide, pointed to a general consensus in terms of the aims and desired *outcomes* of preparing future mathematics teachers. These outcomes can be delineated along cognitive, affective and conative domains. In the cognitive domain, teacher candidates should develop: mathematics content knowledge of various content domains; mathematics pedagogy knowledge (curricular knowledge, planning instruction, and enacting instruction); and general pedagogical knowledge (knowledge of students, classroom environment, instructional design, and diagnostics and assessment). In terms of the affective domain, teacher candidates are expected to develop productive beliefs about the nature of mathematics; beliefs about the nature of teaching mathematics; beliefs about the nature of learning mathematics; and beliefs about preparedness for teaching.[3] The conative domain, which is a domain of practice, involves situation-specific skills and, according to Blömeke et al.'s (2015) model of teacher competence, may act as a bridge between teachers' knowledge and beliefs and their observable classroom actions.

We point out there is a reasonable alignment between the typical structure of university coursework and the desired outcomes of teacher education. Figure 1 illustrates this alignment, with Blömeke et al.'s (2015) model overlaid (vertically) to help situate the TEDS-M learning outcomes that represent desirable general traits for teacher-candidates. The situation-specific teaching skills that Blömeke et al. position as a bridge to observable classroom practices are mainly aligned with, and developed through, practicum experiences—although these may also be supported in other types of coursework through practice-based teaching experiences, such as developing noticing skills (e.g., Dindyal et al., 2021). As mentioned previously, these structures and outcomes are common across contexts, but they themselves are, of course, framed by various social, cultural, institutional, and political contexts (Blömeke & Kaiser, 2017).

With regard to the positioning and the relevance of university mathematics—as situated within university mathematics courses—in teacher education, we make two comments with respect to Fig. 1. First, we point out that despite contextual differences, there seems to be a common approach for how the structure helps accomplish the outcomes. Namely,

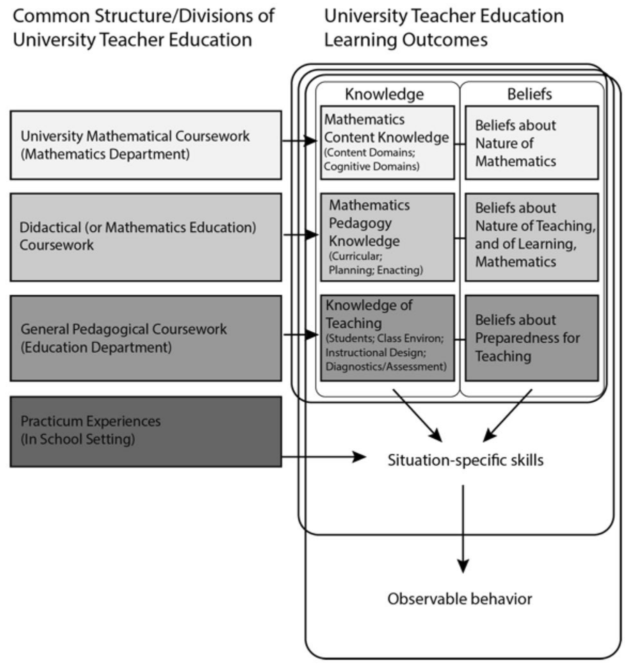

Fig. 1 An alignment between the common structure and outcomes of university teacher education

university mathematics coursework is tasked to develop content knowledge and beliefs about mathematics; mathematics didactics coursework to develop mathematics-specific pedagogical knowledge and beliefs about the teaching and learning of mathematics; general education coursework to develop general pedagogical knowledge of teaching and beliefs about teaching; and, lastly, practicum experiences to bring these general traits to bear in specific situations, developing situation-specific skills needed to accomplish the work of teaching. This division of labor is what we refer to as a "divide and conquer" approach; that each structure is addressing its portion of the desired outcomes. Yet, as this special issue captures, this need not be the case; many articles in this special issue describe how university mathematics coursework might be intentionally designed to contribute to a broader set of the learning outcomes. Second, while mathematical ideas might be discussed in other contexts, this special issue is particularly focused on the instantiation of university mathematics as it happens in courses typically offered by a mathematics department, such as calculus, differential equations, real analysis, linear algebra, abstract algebra, and so forth. Yet, the nature both of the audience, and of the course design, may have some subtleties. In terms of the audience, it may be the case that a mathematics course is offered to any mathematics student (e.g. Hanke & Schäfer, 2018) or is offered only to secondary mathematics education students (e.g., as a special section of a course); in terms of the course content, it may be that a mathematics course was

[3] TEDS-M also measured beliefs about program effectiveness, but this was not named as a learning outcome.

designed to cover typical mathematics content or specifically to cover mathematical content with secondary mathematics teachers in mind (e.g., a capstone course—cf., Murray & Star, 2013; Winsløw & Grønbæk, 2014). Determining which of each of these dichotomous distinctions is the case is pertinent for recognizing the nuances of a course context as well as how it might fit and relate to the general aims of teacher education; yet, for the purposes of this special issue, all these variants are considered under the umbrella of university mathematics coursework.

3 Developments in mathematics education that bear on university mathematics

Here, we briefly outline some of the historical developments in mathematics education scholarship that have impacted university mathematics—and university mathematics courses—in terms of secondary teacher education goals. These developments have informed the progressively changing expectations of teacher education, and also help frame some of the specific problems and challenges for university mathematics.

Felix Klein (2016) was perhaps one of the first to make prominent the challenges of university mathematics in secondary teacher education, and the gap perceived and experienced by teacher candidates between university level mathematics and the mathematics they are expected to teach in schools. Klein described the challenges as a "double discontinuity." By this, he meant that in the transition from school mathematics to university mathematics there is a (first) disconnect for students—what was studied at university had little resemblance to what students knew about mathematics from school. A (second) disconnect happens again for secondary teachers in the transition from having studied university mathematics and then returning back to the school mathematics they would be teaching. The key point is that there tends to be a large gap between school and university mathematics. Algebra in school mathematics, for example, might mean solving equations such as $2x + 1 = 12$; algebra in university mathematics, however, might refer to the study of algebraic structures (e.g., groups), where equation solving happens on abstract elements of sets, such as solving $a * b = c$ for a by operating by an inverse element to maintain equality, $(a * b) * b^{-1} = c * b^{-1}$, and leveraging algebraic properties to simplify the left side to $(a * b) * b^{-1} = a * \left(b * b^{-1}\right) = a * e = a$. Notably, the distinctions between university level mathematics and school mathematics cannot be simply reduced to a list of topics. The difference is often one of abstraction: university level mathematics primarily considers more abstract and general mathematical concepts, whereas school mathematics studies specific instances of them. The university level mathematics also tends to emphasize the systematic nature of the results, and the logical rigor of the justifications (cf., Dreher et al., 2018). Klein's framing of the double discontinuity between these two mathematical levels—along with his notion of "elementary mathematics from a higher standpoint"—has been a primary influence for conceptualizing the orientation of university mathematics toward school mathematics as it relates to secondary teacher education.

Discussing the origins of school mathematical content and its relationship to the body of disciplinary knowledge in mathematics, Chevallard (1989) introduced the notion of didactic transposition. Didactic transposition refers to the process by which a disciplinary body of scholarly knowledge, generally produced in universities and other scholarly institutions, is transposed into teachable knowledge. Thus, both the university level mathematics and school mathematics are explicitly selected, constructed and modified from the scholarly body of knowledge to create mathematical content, procedures, and practices that are teachable at the respective levels (Chevallard & Bosh, 2014). This didactic transposition, and didactics more generally, is given systematic study within the Anthropological Theory of the Didactic (ATD) (Chevallard & Sensevy, 2014), which shapes our understanding of transitions between these spaces and informs how praxeologies—consisting of task, technique, technology, and theory—can address the problem of how we might smooth out learning processes in university mathematics courses (Gueudet et al., 2016).

Another significant development in mathematics education was the conceptualization of teachers' professional knowledge. It was Shulman (1986, 1987) who advocated for a discipline-specific notion of teaching—one where teachers' pedagogical knowledge was in part shaped by their disciplinary knowledge. In addition to content knowledge (CK), and general pedagogical knowledge (PK), Shulman posited pedagogical content knowledge (PCK) as another domain of knowledge—an amalgam of content and pedagogical knowledge that mediates the work of teaching. The concept of PCK has been influential for theoretical and practical approaches to teacher education since its inception, as can be seen in Depaepe et al's (2013) review of several decades worth of research articles on this topic. This idea that teachers should develop a mathematics-specific pedagogical knowledge also resembles older traditions of mathematics didactics spread across many European countries (Blum et al., 2019). This tradition assumes pre-service teachers need to develop a specific body of knowledge of teaching mathematics—didactical knowledge—during teacher education, which is reconstructed from mathematics and structured under didactic criteria for learning and understanding (Scheiner & Buchholtz, 2022). This knowledge is relevant to practice, among other reasons, because it is shaped by pre-service teachers' own experiences with learning mathematics across multiple contexts—and their image of those experiences—which feeds back into their own teaching practice (Even, 2011; Kaur, 2017; Ponte, 1994). Today, the terms mathematical pedagogical content knowledge and mathematics didactical knowledge are widely used internationally as synonyms. Indeed, we point out that the structure of, and learning outcomes for, teacher education (see Fig. 1) mirror these mathematical, didactical, and pedagogical distinctions. In terms of the challenges for university mathematics, the close relation between mathematics and didactics suggests that the mathematical ideas developed in university mathematics courses should somehow be related to this professional knowledge base for teachers.

Since mathematics teaching is not solely a cognitive activity (Depaepe et al., 2020), in addition to aspects such as knowledge, the field increasingly has also incorporated other psychological ideas such as beliefs. Thompson (1992) provided an important overview of how affective notions like beliefs came to be studied in relation to mathematics education—and the important role they play in understanding mathematics teaching. Indeed, we see beliefs formalized as learning outcomes of teacher education (see Fig. 1), where we might have beliefs about different things, for instance, mathematical, didactical, and pedagogical beliefs. Theories about teachers' actions—such as Schoenfeld's (1998) theory of teaching-in-context, Rowland's (2014) knowledge quartet framework or Blömeke et al.'s (2015) model of teacher competence—increasingly have incorporated affective components. In relation to university mathematics, it suggests that, in addition to mathematical ideas, ideas about the discipline of mathematics should be taken into account when teaching university mathematics; the prospective teachers should be exposed to disciplinary beliefs and values in order to help form their view of mathematics (Eichler & Isaev, 2022).

Lastly, as a continuation of Shulman's legacy, which conceptualized professional knowledge as that which is used and drawn on while in the act of teaching, further work, like that of Grossman et al. (2009), expanded on these practice-based approaches to knowledge in terms of how they subsequently influence teacher education. Ball et al. (2008) studied the practice of mathematics teaching in elementary school contexts to further refine sub-domains of CK and PCK, leading to their Mathematical Knowledge for Teaching (MKT) framework. Others have extended their work to conceptualize MKT and its practices at the secondary and postsecondary levels (e.g., Howell et al., 2016; Martinovic et al., 2017; Speer et al., 2015; Wasserman, 2015). Also, other models of teacher knowledge exist, such as Carrillo et al.'s (2018) Mathematics Teachers Specialized Knowledge (MTSK), or Zazkis and Leikin's (2010) Advanced Mathematical Knowledge (AMK). And with these practice-based conceptions of knowledge, the approaches to developing knowledge—including within teacher education—were realigned accordingly. Ball and Forzani (2009), for example, describe utilizing practice-based approaches in teacher education. This emphasis on teaching practice is also seen within Blömeke et al.'s (2015) incorporation of situation-specific skills, such as noticing (cf., Philipp et al., 2014), into the teacher competence continuum. But the emphasis on practice in relation to teacher knowledge also has posed some challenges for university mathematics in that the mathematical ideas need to be developed in relation to their bearing on practice.

4 Literature survey

According to these pertinent developments in the mathematics education literature, including their bearing on some of the challenges, problems, and goals for university mathematics, we conducted a semi-systematic review of the relevant research literature addressing these issues and providing solution approaches. Semi-systematic reviews are particularly suitable for emergent and developing research areas where empirical evidence may be sporadic and limited. Such areas do not lend themselves naturally to systematic examination and evaluation of abundant scientific evidence in the way that well-established research fields do. Semi-systematic surveys are therefore more likely to refer to research literature on a scientific topic characterized by limited surveyability, which is the case for the current topic because many publications refer to university courses that are published in less accessible scholarly sources. The advantage of semi-systematic reviews is that they can help to determine how a research topic has developed over time or across research traditions in order to synthesize and detect themes and theoretical perspectives in a particular research field (Synder, 2019). In particular, we reviewed articles and papers that addressed the problems and challenges of university mathematics with the help of theoretical, practical or empirical approaches, intending to draw out themes in the literature related to the topic of this special issue: exploring and strengthening university mathematics for secondary teacher preparation. Specifically, we sought to answer the following question in our literature survey:

- What are the specific challenges of, and how can the aims of secondary mathematics teacher education be realized through, university level mathematics coursework?

To conduct our survey, we adhered to the following literature search procedure. In addition to our own knowledge of related literature in the field, we identified relevant publications from the past two decades, which included international handbooks in mathematics education; top international journals in mathematics education including

ZDM-Mathematics Education, *Educational Studies in Mathematics*, *Journal for Research in Mathematics Education*, and *Journal of Mathematics Teacher Education*; topical surveys and proceedings from recent international mathematics education conferences such as International Congress on Mathematics Education (ICME), Psychology of Mathematics Education (PME), Congress of the European Society for Research in Mathematics Education (CERME), International Network for Didactic Research in University Mathematics (INDRUM), Research in Undergraduate Mathematics Education (RUME) and related international seminar series like From University Mathematics to Mathematics Education (FUMME) and the Seoul National University (SNU) Mathematics Education Webinar. It is evident that there is a certain degree of subjectivity involved in the choice of literature in this process; however, to limit the scope of the articles under consideration we chose to begin with these sources because they are open to a large international authorship and because they ensure a high quality of scientific findings through standards such as peer-review. As a result, we did not necessarily take into account a large corpus of book chapters, monographs, or dissertations. Within these resources, we defined specific inclusion criteria to help us narrow down the scope of the examined literature; specifically, we searched for titles or abstracts that made clear a focus on all three of the following criteria: (i) teacher education, (ii) secondary teachers, and (iii) university mathematics. Meaning we excluded papers, for example, that discussed post-secondary, non-university based professional development of secondary teachers, or articles that dealt with the preparation of pre-service secondary mathematics teachers in areas that are exclusively or mainly pedagogical. We then considered only contributions in which the focus was explicitly related to secondary pre-service teachers' university mathematical coursework as part of their professional preparation for the teaching profession. To allow the incorporation of some other literature, we also used snowball sampling (Parker et al., 2019) from the lists of references to identify other articles or papers of interest to the topic.

5 Results

In what follows, we summarize the reviewed literature in sections that elaborate on three different aspects within the literature in terms of the problems, challenges, and solutions emphasized with respect to university mathematics and teacher preparation.

5.1 Bridging school and university mathematics

Since Klein (2016) pointed to the issue of the "double discontinuity" between school mathematics and university level mathematics, there has been an ongoing debate on how to support secondary teachers in developing a well-connected and strong knowledge base in mathematics while at the same time conveying university mathematics knowledge as applicable to their later professional practice as mathematics teachers (Gueudet et al., 2016; Winsløw & Grønbæk, 2014; Wood, 2001). Winsløw and Grønbæk (2014) point to Klein's own proposed solution to the problem: that university instruction must consider the needs of future teachers by exposing them to elementary mathematics from a higher standpoint (e.g., Kilpatrick, 2008). The critical aspect of this sort of approach is bridging the *mathematical* gap by identifying points of connection between school and university mathematics and helping prospective teachers see their fundamental coherence rather than their disconnectedness.

As Schubring (2019) described, Klein devised a series of lectures to help prospective secondary teachers see the mutual connection between problems in the various mathematical fields, and to emphasize the relation of these problems to those of school mathematics. Kilpatrick (2019) summarized Klein's mathematical approach as demonstrating how different branches of school and university mathematics might be unified, providing a unified treatment of geometry from school through university, and emphasizing the link between mathematics and its applications. In one of his lectures, for example, Klein elaborated approaches to solving equations with one, two, and three parameters, with real quantities, as well as equations with complex quantities, which are certainly part of university mathematics. Klein then related solving one-parameter equations $f(x) = k$—which harkens back to school mathematics—by reimagining them as a multivariable function $f(x, y) = 0$ drawn as a curve in the xy-plane and looking for intersections with the line $y = k$. Continuing Klein's tradition, Weigand et al. (2019) have collected recent works and theoretical descriptions of Klein's approach in an edited volume. For example, there is work describing the principle of intuition of mathematical content that Klein drew upon in his lectures (Buchholtz & Behrens, 2014) and a collection of specific examples from his lectures (Allmendinger, 2019).

Klein's discontinuity is sometimes tackled in two pieces: a first discontinuity, the secondary-tertiary transition, and a second discontinuity, the tertiary-secondary teaching transition. Gueudet (2008) elaborate on uses of ATD theory (Chevallard & Sensevy, 2014), amongst others, as influencing scholars' efforts to understand and ameliorate the first discontinuity. Some research approaches have used these ideas to try to smooth out this transition by developing curricular materials or courses that make connections to fill in these gaps (e.g., Biehler et al., 2011; Clark & Lovric, 2009; Derouet et al., 2018). DiMartino et al. (2022) provide a more systematic overview of literature on secondary to tertiary transitions in mathematics. Others have leveraged these theoretical perspectives and Klein's approach to think about the second discontinuity (e.g., Planchon, 2019). Dreher et al. (2018), for example, elaborate on "top-down" and "bottom-up" connections between university and school mathematical ideas; "top-town" connections focus on specific university mathematical ideas and how they might be treated in school mathematics classrooms, whereas "bottom-up" connections focus on specific school mathematics content and how university ideas might inform them.

The approach of bridging school and university mathematics up to today has inspired both theoretical and empirical work aimed at analyzing the professional knowledge base of future secondary mathematics teachers and efforts towards conceptualizing appropriate mathematics coursework to address the double discontinuity problem (Ableitinger et al., 2013; Ball & Bass, 2000; Buchholtz et al., 2013; Gueudet et al., 2016; Göller et al., 2017; Winsløw & Grønbæk, 2014; Wood, 2001). It has been influential in a variety of contexts—such as developing capstone courses (e.g., Buchbinder & McCrone, 2020; Cox et al., 2013; CBMS, 2012; Hoffmann & Biehler, 2020; Murray & Star, 2013), as well as designing materials or modules to be used within existing mathematical courses such as abstract algebra or complex analysis (e.g., Hanke & Schäfer, 2018; Suominem, 2018). Shiqi et al. (2008), for example, describe efforts in China to offer courses that integrate typically separated content areas to highlight connections (e.g., a single course, Higher Algebra and Analytic Geometry); Bauer (2013), as another, described what he calls "interface tasks" to address these challenges. Explicating connections between university and school mathematics has been utilized in textbooks designated specifically for secondary mathematics teachers (e.g., Bremigan et al., 2011; Shin, 2007; Sultan & Artzt, 2011; Usiskin et al., 2003), or textbooks for more "typical" undergraduate courses with chapters explicitly labeled as "connections" (e.g., Cuoco & Rotman, 2013).

However, the efficiency of these approaches, which rely solely on explicating connections between university level and school level mathematical content, has been questioned (e.g., Álvarez & White, 2018; Murray et al., 2017). Specifically, it has been suggested that making teachers aware of existing mathematical connections does not automatically translate to use, or improvement, in teaching practices in secondary classrooms (Wasserman, 2018a; Zazkis & Leikin, 2010).

5.2 Distinguishing mathematical concepts from mathematical practices

Another important distinction within the mathematical realm can be made between the content domain and the domain of *mathematical practices*, which refer to process activities like problem solving, representing, proving, defining, modeling, generalizing, etc. The distinction being made is between particular mathematical ideas and particularly mathematical ways of engaging with those ideas, sometimes called mathematical "habits of mind" (Cuoco et al., 1996; Heid et al., 2015; Mason et al., 2010; Rasmussen et al., 2005). Having students engage in such mathematical activities is the premise of many inquiry-based learning approaches in mathematics education (e.g., Laursen & Rasmussen, 2019). The importance of mathematical practices and their distinction from content are stressed across various documents and recommendations for mathematics education and teacher preparation internationally (e.g., MOE, 2012; NCTM, 2000; Ponte, 1994). These mathematical practices, or habits of mind, might be viewed as a "foundation" of all mathematical content—grounding how we develop and engage with mathematical ideas. We note that these activities, and this foundation, also encompasses various beliefs about mathematics and what it means to do mathematics.

In elaborating on different ways prospective teachers should understand mathematics—including connections across K-16 mathematics—Ball and Bass (2009) incorporated ideas such as "key mathematical practices" and "core mathematical values and sensibilities." That is, the lines of connection between university-level mathematics and school-level mathematics teaching may have to do with developing prospective teachers' notions of mathematical practice. Doing so certainly aligns with the TEDS-M aims of teacher education and with CBMS (2012) recommendations that teacher education programs provide rich opportunities for teachers to engage with mathematical practices, and develop habits of mind underlying various mathematical domains, such as algebra, geometry, analysis, modeling, and statistics. Stacey (2008) suggests that experiencing mathematics in action and knowing about mathematics are both critical components of disciplinary knowledge for secondary teachers. Shiqi et al. (2008) elaborate the Korean goal that university mathematics courses, in addition to foundational mathematical ideas, might induce one's ability to self-investigate and engage in personal mathematical activity. Indeed, Even (2011) reports on mathematicians sharing their belief that having teachers develop knowledge *about* mathematics, with its particular disciplinary culture, was more important than for teachers to develop knowledge *of* any specific mathematical concept or rule; meaning, foundational disciplinary practices, beliefs, values, sensibilities, and so forth, are deemed especially important with respect to teacher preparation.

University level mathematics courses typify mathematical ways of doing, being, and developing concepts and ideas, and so they might serve as especially productive opportunities for teachers to learn and develop their own sense of the discipline. Regardless of the structure of the course, the nature of the instructional activities it provides should be flexible enough to allow students to experience mathematical practices, such as problem solving, conjecturing, generalizing, and proving. Cho and Kwon (2017), for example, specify the ways that rigorously understanding the processes for making theorems and using and extending definitions is productive. This direction inspired research approaches which developed and examined course structures and activities which exposed students to different mathematical practices (e.g., Bauer & Kuennen, 2016; Christy & Sparks, 2015; Kempen & Biehler, 2019).

However, exposure to mathematical practices does not occur automatically by virtue of a course dealing with university level content. These practices are often left implicit. Therefore, engaging future teachers in particular disciplinary ways of "doing" mathematics requires that instructors consciously create such learning opportunities within their courses. Often, this involves an instructor modeling disciplinary practices at the board with varied levels of explicitness, or promoting instructional approaches that engage students in mathematical activity and call attention to that activity. There seems to be some empirical support for how university instructors' knowledge of mathematical practices may influence their pedagogical approach to teaching (e.g., Delgado-Rebolledo & Zakaryan, 2020), as well as connections between teachers' experiences as learners to their teaching practices (e.g., Cobb & Bowers, 1999; Shriki, 2010).

In some sense, the hope is that teachers own experiences with mathematical practices as learners will translate to their ways of teaching (e.g., CBMS, 2012). Nevertheless, the logic behind this seems to be reminiscent of the "trickle-down" effect of exposure to content connections. Just because teacher candidates experienced doing mathematics themselves in the context of their university-based program, and even developed positive dispositions toward such experiences, does not necessarily mean that they will be able to create similar learning opportunities for their students in the context of school mathematics. Weber et al. (2020) problematize the simplistic link often made between mathematicians' practice and how we conceive of mathematics instruction. Moreover, mathematical practices at the university level may indeed be somewhat different than the kind of practices useful in school mathematics—the transfer of one's own experiences (including in more advanced mathematical courses) into teaching expertise is an open question.

5.3 Recognizing the didactical realm and connecting to teaching

So far, we have identified two conditions that seem to be necessary for having university mathematics courses benefit the aims in teacher education. Both are situated in the mathematical realm: (i) ensuring that prospective teachers develop well-connected mathematical knowledge across university and school mathematics; and (ii) ensuring that prospective teachers have first-hand experience of mathematical practices and develop mathematical habits of mind, as well as productive beliefs about mathematics. Yet, these may not be sufficient conditions. Within the context of university mathematics courses, various scholars have introduced the need to consider not just mathematical aspects but (iii) didactical ones as well. Shiqi et al. (2008) explicitly describe the incorporation of didactical goals through mathematical coursework: "Korean innovative [university mathematics] courses attempt to adopt an integrated approach to connect subject knowledge and [didactics]" (p. 83). These connections would embody mathematics education specific notions, i.e., didactics (Biehler et al., 1994; Blum et al., 2019; Göller et al., 2017; Straesser, 2007), and not general pedagogical ones. In the literature, didactical connections can be established with respect to two aspects: connections between university mathematics courses and didactical ideas for teaching school mathematics, and the didactical approaches used by instructors of university mathematics courses themselves. In their CERME contribution, Hanke and Bikner-Ahsbahs (2019) describe such connection-making as the specific design principle of boundary-crossing, which can be taken into account in the development of university teaching for secondary mathematics pre-service teachers.

As a first consideration, some scholars have pointed to the distinction between understanding a mathematical connection, and having that mathematical connection inform an approach to teaching (e.g., Wasserman, 2018a). For example, picking up on the prior example from Klein, recognition that $f(x) = k$ can be solved as a multivariable function in the xy-plane does not necessarily mean that this knowledge would influence how one would teach school students how to solve $f(x) = k$. Similarly, even with mathematical practices, having prospective teachers, for example, make and prove conjectures within the context of university courses may not suggest to a prospective teacher how to design a proof-oriented classroom activity at the secondary school level (Buchbinder & McCrone, 2019). The key point is that, in addition to a mathematically powerful understanding, it is also important to consider the degree to which they have implications on teaching—an effect in the didactical realm (cf., Wasserman, 2018a). Approaches from various countries have deliberately created such learning opportunities for prospective secondary teachers (Allmendinger, 2019; Allmendinger et al., 2013; Beutelspacher et al., 2011; Deiser & Reiss, 2013; Hanke & Bikner-Ahsbahs, 2019; Lai & Donsig, 2018; Shamash et al., 2018; Wasserman, 2018c; Wasserman et al., 2022).

The literature suggests two ways in which this type of didactical connection has been incorporated in university mathematics courses. One approach may be characterized as "bottom-up," which takes school mathematics teaching as a starting point. For example, Heid et al. (2015) use specific teaching situations in school mathematics to motivate the study of more advanced mathematics. The underlying rationale is that doing so helps prospective teachers connect the advanced content not just to the school mathematics, but to situations in which it might arise in teaching school mathematics. Starting from the school teaching situation, these mathematical explorations may then delve into any related notions—including advanced university-level mathematics. Alternatively, we also find "top-down" approaches which take university mathematics as a starting point. For instance, Stylianides and Stylianides (2014) conceptualized mathematics for teaching as a form of mathematical application, in a way analogous to how mathematics might get applied to other professions, like business or engineering. That is, one can apply mathematical ideas from university mathematics to inform one's approach to teaching. For example, Christy and Sparks (2015) developed a project which had prospective teachers develop a lesson plan for teaching systems of linear equations which builds on and incorporates ideas about mathematical structures developed in a prior course in abstract algebra. On a slightly larger scale, the Mathematical Education of Teachers as an Application of Undergraduate Math (META-Math) project in the United States has taken this approach to develop ideas for curriculum in university mathematics courses (e.g., Álvarez et al., 2020). Others have combined both "bottom-up" and "top-down" approaches, like in Wasserman et al.'s, (2019). Upgrading Learning for Teachers in Real Analysis (ULTRA) project which developed a real analysis course using a building-up (from school practice) and stepping-down (to school practice) model to make the learning of real analysis relevant to students (see also, Wasserman & McGuffey, 2021). Such didactical connections might be used within the mathematical coursework itself, or as part of an interlinked didactic shadow course; Hanke and Bikner-Ahsbahs (2019), for example, describe the design of a complex analysis course that was accompanied by a didactic course to address curricularly appropriate school task design as a didactic activity of secondary teachers.

As a second consideration, the literature indicates that the didactical realm might also be connected to via the mathematical instruction occurring in the university courses themselves. At some level, this is connected to what Lortie (1975) described as prospective teachers' learning about teaching through the "apprenticeship of observation"—which has been considered with respect to university mathematics courses in secondary teacher education. Much has been written about the pedagogical challenges associated with the normative "lecture" paradigm in university mathematics instruction (e.g., Davis & Hersch, 1981; Dreyfus, 1991; Rosenthal, 1995; Rowland, 2002; Thurston, 1994), where prospective teachers may not be engaged in meaningful justification and reasoning (e.g., Presmeg, 2011). Despite this, scholars have considered possibilities for leveraging instruction in university mathematics courses as opportunities for their students (who are prospective teachers) to learn more productive ideas about teaching (e.g., Leikin et al., 2018)—"to show a teaching programme (or model)" (Shiqi et al., 2008, p. 71). Stacey (2008), for example, asserted that having good instruction modeled in university level mathematics might help teacher-candidates develop productive pedagogical moves and practices for their future teaching by having experienced them as a mathematical learner. This is a common rationale underlying the incorporation of inquiry-based mathematics education learning paradigms (Laursen & Rasmussen, 2019) as they relate to teacher preparation. The literature has also looked more specifically at particular kinds of instructional moves used; Solórzano (2014), for example, looked specifically at problem-posing in formal mathematics courses and its relation to accomplishing teacher preparation goals. Interestingly, there may be cultural differences with respect to emphasizing these kinds of approaches in teacher education. Yang and Leung (2011), for example, point to an emphasis in Eastern (as opposed to Western) countries on the importance of observing exemplary teaching in the teacher education development process. At the university level, these opportunities would be happening within teacher candidates' experiences as students in university mathematics courses—which Liang et al. (2022) report as being a resource for positively shaping their future mathematics teaching.

5.4 Theoretical distinctions

We organized and structured the survey results around some particular aspects of the research question that were emphasized in the literature in terms of the problems, challenges, and solutions with respect to university mathematics—and university mathematics courses—in relation to teacher education. The theoretical distinctions we identified in the literature revolved around: (i) school versus university mathematics (5.1); (ii) mathematical concepts versus mathematical practices (5.2); and (iii) the mathematical versus didactical realm (5.3). In identifying these theoretical components from the literature, we have

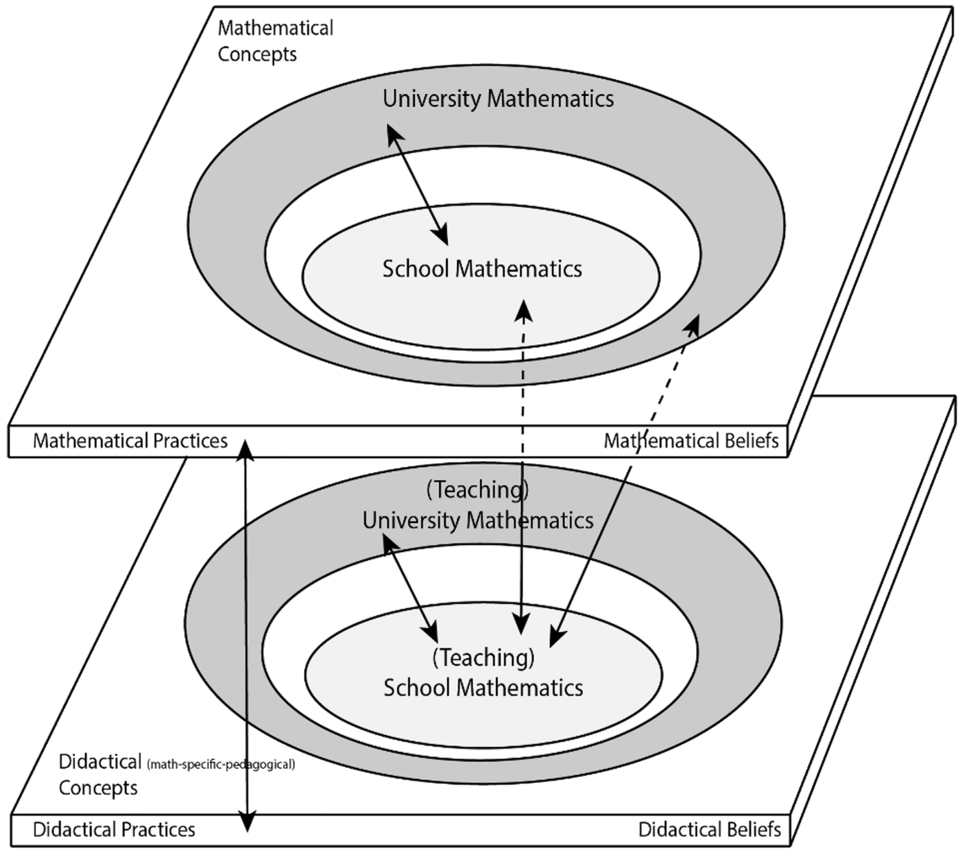

Fig. 2 Theoretical distinctions and relations for university mathematics in secondary teacher education

organized them into a figure to capture the multitude of distinctions.

Figure 2 depicts two "planes"—a mathematical plane and a didactical plane. The key premise is that didactical notions, e.g., mathematics-specific pedagogical ideas, are sufficiently distinct from mathematical ones. The two planes are similarly structured. On the "surface" of each plane is a landscape of concepts: for the mathematical plane, this would be the collection of all mathematical ideas and concepts (e.g., integers, addition, groups); for the didactical plane, this would be all the different ideas and considerations that shape didactical decision-making (e.g., project-based learning, multiple representations, "Bianshi teaching[4]"). On each surface, we have included two separated regions but depicted in concentric circles, which represent associated collections of ideas and how they relate to each other, in our conceptualization. Specifically, in the mathematical plane we differentiate between the mathematical concepts that are part of school mathematics, and those part of university mathematics. We similarly acknowledge that didactical concepts for teaching university and school mathematics are, at some level, sufficiently distinct.[5] We also note that the disciplinary body of mathematical concepts is constantly (albeit, slowly) evolving and reforming, and some of these changes find their way into university or school mathematics curricula. Lastly, the "foundation" of each plane is depicted as the underlying practices and beliefs that inform all the concepts located on the surface. That is, the disciplinary activities and practices of mathematics (e.g., proving, defining) are those that inform how we engage with those concepts in mathematical doing; similarly, didactical practices and beliefs are those that inform how we engage in doing teaching. The figure captures and relates the three kinds of theoretical distinctions from the survey; it also indicates bidirectional arrows across the gaps between various regions and planes in order to convey the different kinds of problems, challenges, and solutions identified across the literature.

[4] *Bianshi* is a teaching approach popular in Chinese culture; it is rooted in Confucian heritage and based on variation theory (Huang & Li, 2017).

[5] To reiterate, the surface of each plane represents collections of ideas and concepts which are sufficiently distinct yet intrinsically related. For example, while university and school mathematics teaching are sufficiently distinct from each other, they may share certain underlying didactical ideas and concepts. The same holds in the mathematical plane.

6 Summary of articles

University mathematics—and university mathematics coursework—is a cornerstone of the preparation of secondary mathematics teachers. As such, this special issue is focused on exploring and extending the prevailing issues around secondary teacher preparation within such coursework. As a whole, the special issue deals with how we might modify and reframe mathematical ideas or instructional approaches within such courses to improve the professional preparation for secondary teachers; this might be regarded as a practical approach—trying to improve and expand the developed teacher education aims within the existing structure of university mathematics coursework. The articles comprising this special issue build on the prevailing advancements and promote them in several important ways. The following questions broadly frame the articles in the special issue:

- What type of connections between higher mathematics and school mathematics can be established and how do they provide preservice teachers with learning opportunities for developing advanced mathematical understanding?
- How can university teacher education address the needs of secondary teachers by ensuring that learning advanced mathematics in mathematics courses is made relevant to, and incorporates practices (mathematical or pedagogical) for, secondary teachers?
- How might university mathematics courses integrate mathematical and mathematics educational content to advance future teachers' professional competence?

On one hand, these questions are cross-cutting in that each article addresses each of them in some capacity. On the other hand, each article tends to have a particular emphasis in which they address one question or another more explicitly. We introduce and summarize the articles in relation to these questions and briefly describe how each article connects to, and informs, larger themes in this area.

In particular, the theoretical framework developed in this survey paper provided some key concepts and relationships with respect to university mathematics and secondary teacher preparation. The articles themselves provide more specific details about *how* these things might be accomplished in university mathematics courses. Figure 3i–iv depicts the four categories of articles we describe in relation to the theoretical framework presented earlier—differentiating them by which components of the framework are highlighted. (The four categories are similar to those described in Wasserman, 2018b.) Specifically, one can understand each category as being the primary means by which certain studies attempt to strengthen university mathematics courses with respect to the aims of secondary teacher education. Although papers sometimes overlap across categories, the introduction here summarizes at least one of its key aspects.

6.1 Sharpening mathematical connections

Several articles in the special issue look at how we might strengthen the mathematical connections between school and university mathematics content (Fig. 3i). That is, their attempts to improve university mathematics courses primarily focus on initiatives—often via developing and studying curricular ideas and resources—that point out connections to school mathematics ideas. These might be connections in either direction, or both. Although these might be resources that could shape in-class instruction in a university mathematics course, they also might be particular ideas or exercises that could be assigned and explored outside of class.

Hoffmann and Biehler shape Klein's double discontinuity differently in their contribution, addressing pre-service teachers' view of the discontinuity. They look at how profession orientation as a design principle of university mathematics courses helps strengthening notions of congruence and symmetry via exploring various axiomatic systems in geometry courses with the help of so-called interface e-portfolios. Based on PSTs' written reflections, they investigate the benefits and effects in terms of PSTs' professionalization processes.

Cook, Lockwood, and Reed look at the use of conceptual analysis to strengthen mathematical connections between university mathematics and school mathematics in relation to the concepts of inverse and equivalence. They show how a conceptual analysis can lead to new insights on different conceptual understandings of mathematics and how productive engagement with school mathematics content can be supported with respective tasks used in university teacher education.

Scheiner and Bosch provide a theoretical comparison of how various research traditions look at the relationship between school mathematics and university mathematics. They position Klein, Shulman, and Chevellard as operating along different levels, and stress that cultural, ethical, and socio-political ideas—not just logical and epistemological ones—need to be considered in conceptualizing the relationships between university and school mathematics.

6.2 Highlighting disciplinary practices

Other articles in the special issue look at how we might highlight disciplinary mathematical practices in university coursework to help prepare secondary teachers to engage their own students in doing mathematics. In this sense, the mathematical practices are intended to shape—in some

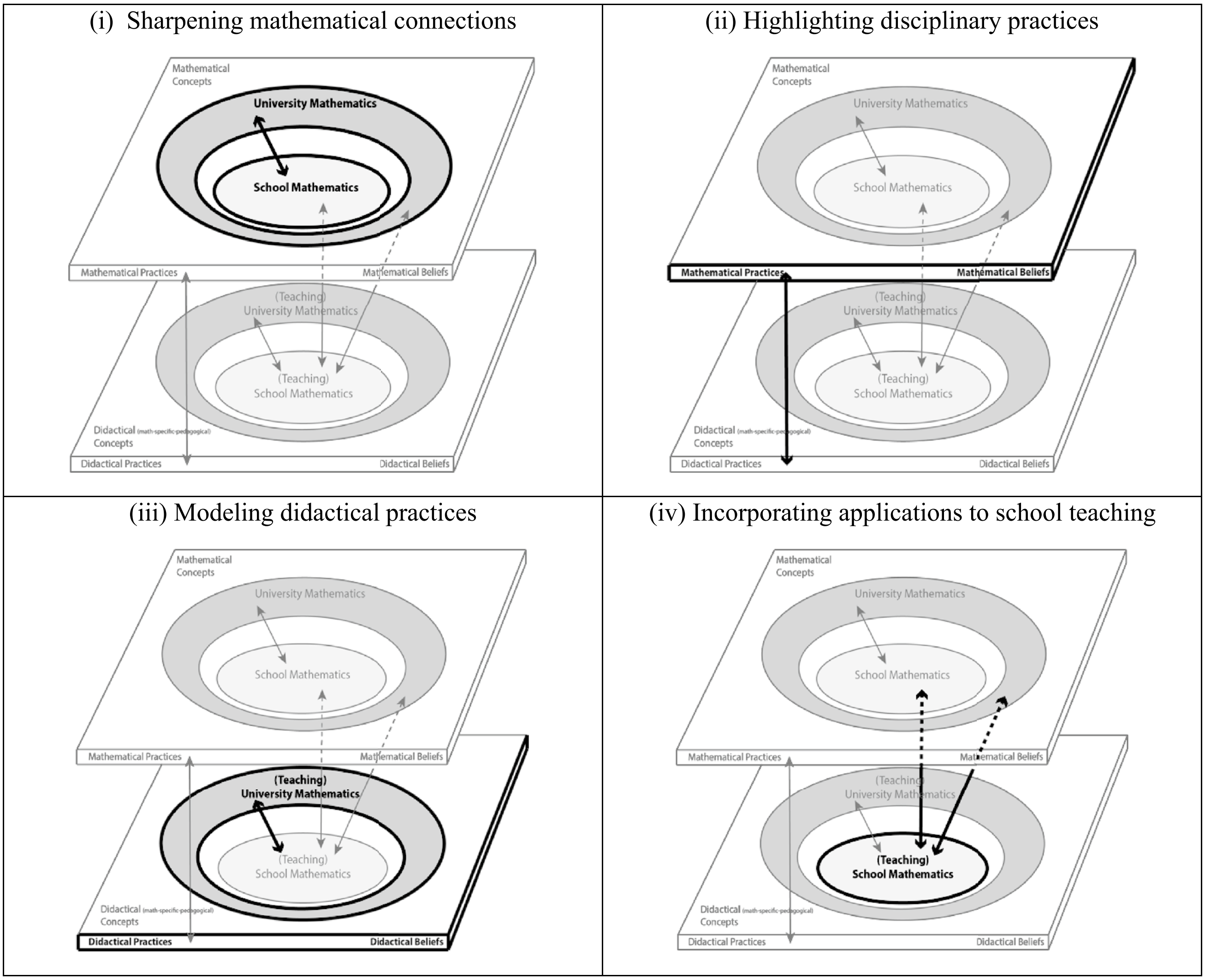

Fig. 3 Four categories (**i**–**iv**) of efforts to strengthen university mathematics with respect to secondary teacher education, highlighting different model aspects

manner—didactical practices for teaching secondary students. Just the mathematical practices were highlighted in Fig. 3ii because the focus in these articles is primarily on the mathematical aspects of the practice; the extent to which the connection to teaching practice is elaborated is varied. In sum, the attempts to improve university instruction hinge on engaging, developing, and making explicit certain mathematical practices—or certain aspects of mathematical practice.

Buchbinder and McCrone focus on bridging between the disciplinary practice of reasoning and proving and the teaching of secondary mathematics. Through a specially designed capstone course, which involved a practicum component, the article examines how the disciplinary practice of proof plays out in mathematics in order to support future teachers' competence for integrating reasoning and proof in secondary classrooms.

Mamolo and Glynn-Adey explore the creation and use of a tactile model—the dihedral calculator—in an abstract algebra course. The article considers how emphasizing visualization, making, and mathematical structure fosters mathematical awareness of connections and disciplinary ways of being that are essential for teaching.

Wasserman develops the construct of pedagogical mathematical practices, specifically studying the kinds of mathematical practices that teachers also find valuable from a pedagogical perspective. His article contributes four disciplinary practices that might serve as a productive starting point for discussion around course design in university mathematics courses because of their dual pedagogical nature.

6.3 Modeling didactical practices

In contrast to approaches that look to modify the kinds of mathematical ideas, problems, or connections being explored in a university mathematics course, some articles in the special issue look more closely at how courses might be structured for, and experienced by, prospective teachers. In this sense, the kinds of changes being discussed in these articles are less about particular curricular resources and more about the ways in which instruction occurs. That is, these articles suggest particularly productive ways that prospective teachers might learn about, and be apprenticed, in terms of teaching through observing and experiencing university level mathematics instruction. Some of these articles speak to instructors' actions; others speak to how future teachers might orient themselves to their own learning in such coursework. Regardless, the presumption here is that the didactical practices being modeled at the university level are general enough that they make sense for teaching secondary students as well—hence the emphasis on the foundational layer of the didactical plane in Fig. 3iii.

Apkarian, LeTona-Tequide, Habre, and Rasmussen consider inquiry-oriented approaches to the instruction of a differential equations class. They describe how inquiry orientated coursework can shape prospective teachers' perceptions of various conceptions of derivative—and the notion of rate of change more generally—and simultaneously shape their views of the importance of class communication, argumentation, and inquiry in relation their own instruction.

Kirwan, Winsor and Barker study the relationship between an instructor's actions and the opportunities for students' knowledge integration. Although not quite the same as modeling the kinds of instructional moves one would like students to adopt in their future teaching, the authors identify three types of instructional actions that instructors could incorporate into content courses that provide opportunities for prospective and practicing teachers in those courses to integrate their knowledge of mathematics, learners, and pedagogy.

Allmendinger, Aslaksen, and Buchholtz consider the teacher-student perspective. Specifically, they capture prospective teachers' mathematical orientation when learning university mathematics as an analytic category. By analyzing PSTs' reflections, they explore and describe connections between university mathematics and school mathematics that university mathematics courses can provide when aimed at student teachers.

6.4 Incorporating applications to school teaching

To some extent, every category aims toward a more integrated approach for accomplishing the aims of both mathematics and mathematics education; yet, some articles in the special issue focus more so than others on trying to establish explicit connections between the mathematical and the didactical planes (Fig. 3iv). For some, this might look like applying mathematical ideas to resolve pedagogical situations in secondary mathematics; for others, it might involve studying the particular ways that university mathematics ideas are used, or found within, secondary teaching situations, which might be connections in either direction, or both. In any case, these articles point toward ways that mathematical ideas and didactical concepts might be meaningfully entwined within the context of a university mathematics course. Similar to some of the other categories, although these might be resources that could shape in-class instruction in a university mathematics course, they also might be particular ideas or exercises that could be assigned and explored outside of class.

Lai and colleagues examined how courses designed to engage future teachers with mathematically intensive core teaching practices, such as learning about student understanding and promoting student conjecturing and justifying, helped to develop PSTs' value of these practices, their content knowledge, as well as their motivation and expectancy to enact these practices in their future classrooms.

Pinto and Cooper study the interplay between tertiary and secondary probability in an experimental tertiary course, in which secondary teachers and practicing mathematicians jointly analyzed dilemmas arising in videos of classrooms working on probability problems.

Burroughs and colleagues look at the use of analyzing fictitious student work as a genre of mathematical tasks in a variety of university mathematics courses; the ways in which these tasks seem to deepen student mathematical knowledge and beliefs about mathematics as human activity.

Hoffmann and Even consider the role of learning about Applied Mathematics as it relates to, and contributes to, teachers' classroom work in terms of the process of mathematical modeling and a repertoire of interesting mathematical applications. They are specifically interested in how applications as part of university mathematics can provide an adequate image of the discipline.

7 Closing remarks

It is clear from the many research approaches united in this special issue that the problem of discontinuity already described by Felix Klein is a problem that unifies and concerns mathematics educators internationally around the world. Prospective teachers should get opportunities in their university mathematics courses to develop knowledge, practices, skills and beliefs that are useful and applicable to their future professional practice. The articles in this special issue highlight some of the different ways that are currently being pursued to overcome this discontinuity—which include sharpening mathematical connections, highlighting disciplinary practices, modeling didactical practices, and incorporating applications to school teaching. Moreover, this collection of articles establishes an important empirical basis with respect to university mathematics on which the international research community can continue to build—providing opportunities to learn from each other, and preserve and extend the legacy of Felix Klein in order to improve secondary mathematics teacher education.

We note here that the perspectives we have adopted herein—the theoretical framework and the categories of articles presented above—are primarily in relation to individuals, meaning undergraduate students, prospective teachers, course instructors, and so forth. This suggests that the approaches to improving university mathematics with respect to secondary teacher education described in this special issue can, largely, be endeavored on at the individual scale. This perspective was in part motivated by the theoretical frameworks that informed our work, those of Klein (2016), Shulman (1986), Chevellard (1989), Grossman et al. (2009), and others mentioned above, and in part by the fundamental belief in the power of individuals to transform themselves and the world around them. The studies included in this special issue amply illustrate this idea. At the same time, we are keenly aware of the critical influence of larger social, cultural, institutional, and political contexts that shape and inform teacher education in different countries, as well as various methodological traditions that influence teacher education research, and we recognize that individuals cannot work in isolation from these larger contexts. Indeed, the different research traditions, methodological approaches, and contextual perspectives are likely significant factors in the problems, challenges, and solutions identified with respect to university mathematics courses. The analysis presented in this survey paper, and the studies reported in this special issue, should be interpreted within these larger contexts.

The collection of studies in this special issue represents just a sample of the contemporary research that is re-envisioning university mathematics—and university mathematics courses—for the benefit of prospective secondary mathematics teachers. Many of the articles collected in the issue not only describe how appropriate learning opportunities for future teachers might be structured and designed, but also support the effectiveness of the measures described with empirical findings. In sum, the variety of ideas, interventions, theoretical frameworks, and methodological approaches and measures represent a collective advancement in this field of study. Building on the scholarship represented in this special issue, future studies may examine such directions as scaling up successful interventions and exploring opportunities to study long-terms effects of these interventions on future secondary teachers' professional knowledge, beliefs, and practices. They might also extend the scope of theoretical lenses to incorporate elements of the broader social, cultural, institutional, and political environments within which individuals operate, learn, and develop. Regardless, further research and development should increasingly, and systematically, inform and help improve how university mathematics courses better accomplish the aims of teacher education.

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