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SURVEY PAPER

The multi-dimensionality of early algebraic thinking: background, overarching dimensions, and new directions

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Abstract

Early algebraic thinking is the reasoning engaged in by 5- to 12-year-olds as they build meaning for the objects and ways of thinking to be encountered within the later study of secondary school algebra. Ever since the 1990s when interest in developing algebraic thinking in the earlier grades began to emerge, there has been a steady growth in the research devoted to exploring ways of fostering this thinking. While in its early days this research had to grapple with the question of what kinds of algebraic thinking might be feasible for the younger student, the evolution of the field over the past 30 years has led to an ever-increasing range of activity that is truly multi-dimensional. In this survey paper, I have framed the multi-dimensionality of early algebraic thinking according to three overarching types, namely, that of analytic thinking, structural thinking, and functional thinking, with generalizing being the scarlet thread that runs through all three. The first part of the paper looks back to the history of the notion of early algebra and the initial research efforts aimed at characterizing early algebraic thinking. The second part delineates the three overarching theoretical dimensions of early algebraic thinking, presents a sampling of past empirical findings, and points to some of the more recent work in the field, including the contributions to this Special Issue. The paper concludes by highlighting the new directions of this domain of research and offering suggestions for further research.

Keywords Algebraic thinking · Analytic thinking · Structural thinking · Functional thinking · Generalizing · Primary level students

1 Introduction

When the notion of early algebra arose in the 1990s, there was no clear idea of what algebra for primary school students (i.e., 5- to 12-year-olds) might look like. Early algebra was not going to be algebra early (Carraher et al., 2008), but what was it going to be? The concept of algebraic thinking had already emerged, but even that concept was not clearly defined. So, the first part (Sect. 2) of this survey paper takes us on a brief historical tour of the context within which early algebra and the genesis of the idea of early algebraic thinking arose and evolved—a tour that also points to the unfolding of its multi-dimensionality. But the heart of the paper is its second part (Sects. 3, 4, and 5) in which I have framed the major theoretical dimensions of early algebraic thinking according to three overarching types, namely, analytic, structural, and functional—with generalizing being the scarlet thread that runs through all three. Each of Sects. 3, 4, and 5 presents not only the theoretical underpinnings of the given dimension, but also a sampling of past and more recent work in the field, including the contributions to this Special Issue. The paper concludes (Sect. 6) by highlighting the new directions of this domain of research and offers some suggestions of topics where further research is recommended. The approach taken in this paper, which involves going to some length in distinguishing the three main dimensions of the research on early algebraic thinking, as well as describing in some detail the most recent work, has necessitated abbreviating what would otherwise be a broader selection from a rather large pool of past empirical studies. For further information on research that is complementary to the synthesis presented in this paper, readers are urged to consult additional resources that include, for example, the ICME-13 monograph on algebraic thinking (Kieran, 2018a), the NCTM compendium chapter on algebraic thinking (Stephens et al., 2017a), the ICME-13

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topical survey on early algebra (Kieran et al., 2016), the volume on early algebraization (Cai & Knuth, 2011), the chapter on early algebra in the Second Handbook of Research on Mathematics Teaching and Learning (Carraher & Schliemann, 2007), and of course the pioneering publication on algebra in the early grades (Kaput et al., 2008).

2 Background

2.1 School algebra and its evolution throughout the years

We need to turn the calendar back to the year 825 for the first real text on algebra. Al-Khwarizmi's work, with its emphasis on equality transformations (al-jabr) and balancing (al-muqābala), put equation solving at the forefront of algebra. Equations containing letters that represented unknown numbers were solved with syntactic methods inspired by Al-Khwarizmi and later formalized by Viète and Descartes. While the roots of algebra can be traced back to even earlier than 825, it was only in the sixteenth century that algebra joined arithmetic and geometry as the third subfield of mathematics.

A century later, polynomial expressions consisting of indeterminates with no fixed value, coefficients, and non-negative integer exponents, and whose generalized terms could be manipulated according to the basic operations of addition, subtraction, and multiplication, entered explicitly into the subfield of algebra. School algebra in its early days of the eighteenth century mirrored the contents of this historical period with its overall perspective on algebra as that of generalized arithmetic—the generalization of ways of operating with numbers, in accordance with the basic properties of arithmetic and equality (Kilpatrick & Izsák, 2008).

Monument to Al-Khwarizmi in Khiva, Uzbekistan

The notion of the mathematical function, which developed later and within the context of the calculus, was viewed as quite distinct from algebra. Its basic concept encompassed the idea of how one quantity depended on another quantity that varied—requiring an interpretation of the letters used in the function relation that was quite unlike that of the unknowns and generalized numbers of generalized-arithmetic algebra. Functions also brought with them new representations, such as graphs and tables of values. When functions entered school mathematics curricula, they were initially viewed as a minor part of school algebra. With their gain in prominence in the 1960s—in the aftermath of the Bourbaki and "new math" movements, and heightened further in the 1980s with advances in graphing-capable digital tools (see, e.g., Fey, 1984; Heid, 1996)—functions came to be a full-fledged component of school algebra. The result was that there were now two rather different perspectives on the objects and techniques of school algebra: a generalized-arithmetic perspective and a functional perspective. No single definition of school algebra is widely adhered to today.

2.2 Algebraic thinking, the genesis of the idea of Early Algebra, and early algebraic thinking

When the teaching and learning of school algebra became an important topic in research in the 1980s (see, e.g., Kieran, 1992), the term *algebraic thinking*, though hardly defined, emerged as well. In fact, one of the recommendations emphasized by the international participants at the Research Agenda Conference on Algebra held in 1987 at the University of Georgia (Wagner & Kieran, 1989) was that the area of algebraic thinking was sorely in need of research attention. Despite contributions to this area during the decade that followed (e.g., Filloy & Rojano, 1989; Kieran, 1989, 1996; Lins, 1992; Radford, 2001), it was the idea that some form of early algebra needed to be engaged in before secondary school that underscored the importance of characterizing more explicitly algebraic thinking. However, the growing interest in algebraic thinking was not restricted to the young learner; the term algebraic thinking was indeed becoming central to the whole range of algebra research involving the older learner as well (e.g., Zazkis & Liljedahl, 2002). In the early 2000s, Radford (2006) posed the question: "What is it that makes algebraic thinking distinctive?" By way of answer, Mason and his colleagues (Mason et al., 1985) had proposed a decade or so earlier that "expressing generality" was at the heart of mathematical thinking, and that algebra was the language in which generality is expressed. However, as will be seen, algebraic thinking would come to be viewed more broadly in the years to come.

The research conducted during the latter decades of the twentieth century had documented the challenges faced by

13- to 15-year-olds when they began their first course of algebra at the secondary school level (e.g., Booth, 1984; Kieran, 1992; Küchemann, 1981; Sutherland et al., 2001; Wagner & Kieran, 1989). Students' arithmetic ways of thinking that had served them well in primary school had not prepared them for encountering the alphanumeric notation and ways of operating with unknowns and variables in equations and expressions that were the standard fare of secondary school algebra. While arithmetic thinking can be quite sophisticated, researchers during those years were stressing the consequences of an arithmetic thinking that was primarily computational in nature and were beginning to question whether a modified type of mathematical activity might better prepare primary level students for making the eventual transition to the more formal study of algebra. But the question that researchers in algebra education were asking was what form this might take.

Davis (1985, 1995) had been one of the first to consider the notion of whether the study of algebra should be spread throughout the primary and secondary curricula. Other influences were coming from Europe (e.g., Bolea et al., 1998; Steinweg, 2001), as well as from Russian experimental schools and from Chinese primary education; however, descriptions of these two latter bodies of work were not available in English research publications until somewhat later. Despite these early influences, the key figure in the beginning of the Early Algebra discussions was without doubt James Kaput (1998). His leadership was pivotal to conceptualizing alternative ways of construing algebra for the early grades.

However, it was the year 2001 that was noteworthy for the advancing of the idea of Early Algebra. That year witnessed for the first time not only a PME Research Forum dedicated to the theme of Early Algebra (Ainley, 2001), but also the designation of one of the thematic working groups at the 12th ICMI Study conference on algebra as the Early Algebra working group (Lins & Kaput, 2004).

In their chapter describing the activity of that working group, Lins and Kaput (2004) emphasized that the idea of Early Algebra was framed on understanding the early development of algebraic reasoning (as early as 5 or 6 years of age) and the larger views of algebra education in which this could occur. While no consensual definition of algebra was arrived at, working group participants could agree on two key characteristics of *algebraic thinking*:

- It involves acts of deliberate generalization and expression of generality.
- It involves, usually as a separate endeavor, reasoning based on the forms of syntactically-structured generalizations, including syntactically and semantically guided actions. (Lins & Kaput, 2004, p. 48)

These two facets are also reflected in the core aspects of Kaput's (2008, p. 11) characterization of algebraic reasoning within elementary school mathematics: (i) Algebra as systematically symbolizing generalizations of regularities and constraints, and (ii) Algebra as syntactically guided reasoning and actions on generalizations expressed in conventional symbol systems.

However, wider definitions of early algebraic thinking were being advanced by some researchers. Kieran adapted her prior model of algebraic activity (Kieran, 1996) and suggested the following definition of early algebraic thinking (Kieran, 2004, p. 149):

> Algebraic thinking in the early grades involves the development of ways of thinking within activities for which the letter-symbolic could be used as a tool, or alternatively within activities that could be engaged in without using the letter-symbolic at all, for example, analyzing relationships among quantities, noticing structure, studying change, generalizing, problem solving, justifying, proving, and predicting.

Carpenter and Franke (2001, p. 156) argued at the 2001 ICMI Study Conference on algebra that one of the hallmarks of algebraic thinking is a "shift from a procedural view to a relational view of equality, and that developing a relational understanding of the meaning of the equal sign underlies the ability to make and represent generalizations." Other aspects of early algebraic thinking that were promoted during the years that followed the Conference include, for example, Blanton et al.'s (2011) emphasizing that mathematical structure and relationships are central to the practice of developing early algebraic thinking, in particular the practices of generalizing, representing, justifying, and reasoning with mathematical relationships. For Britt and Irwin (2011), early algebraic thinking involves coming to use numbers and words to express arithmetic transformations in general terms—this emphasis in line with Malara and Navarra's (2003) stressing the role of natural language within generalizing. And, Carraher and Schliemann (2007) characterize early algebraic thinking in terms of basic forms of reasoning that express relations among number or quantities, in particular, functional relations. Clearly, early algebraic thinking is multi-dimensional.

Structuring the multi-dimensionality of early algebraic thinking for this paper into a form that both reflects the diverse ways in which the research in early algebra has evolved over the years and, at the same time, draws out a set of overarching dimensions that could connect with the two central perspectives on secondary school algebra was indeed a daunting task—made even more challenging by the various intersections that characterize early algebra research, as well as the presence of differing definitions of widely-used terms. While several structurings of the main lines of research are indeed possible, the one that was finally opted for is a threefold framing of early algebraic thinking that consists of the following dimensions: Analytic Thinking, Structural Thinking, and Functional Thinking. In that both analytic and structural thinking are rooted in the generalized-arithmetic perspective, where number is the basic mathematical object, there is considerable overlap and interaction between these two dimensions. While both could be collapsed into the single, more general dimension of relational thinking, it was considered that separating the two would allow for distinguishing between, on the one hand, the kind of thinking that underpins the transformations and equivalence aspects of equations and equation-solving, and on the other hand, the thinking that is more aligned with seeing and expressing structure and properties within numbers, operations, and expressions. And the third dimension of early algebraic thinking, namely, the functional, where function is the basic mathematical object, reflects its undeniable status in current school algebra curricula.

At this point, one would be justified in asking about generalizing, and where this kind of thinking fits into the above framing. In that generalizing is a scarlet thread that runs through the three dimensions, treating it separately proved unfeasible. Thus, it will be seen to permeate all three. As mentioned above, in the 1980s Mason and his colleagues (Mason et al., 1985) proposed that generalizing was at the heart of algebraic thinking. In their book titled *Routes to/Roots of Algebra*, Mason et al. used a variety of activities (e.g., patterns, functional situations, and think-of-a-number games) to exemplify how 11- to 13-year-olds might be introduced to those facets of algebraic thinking that they referred to as expressing, recording, and manipulating generality. In a later publication, Mason (1996, p. 65) argued further that "there is no single program for learning algebra through the expression of generality; it is a matter of awakening and sharpening sensitivity to the presence and potential for algebraic thinking." The notion that the recognition and articulation of generality is central to algebraic thinking has struck a positive chord with early algebra researchers and is an aspect of most studies, no matter which other dimension might predominate the researcher's theoretical framework. Thus, the expression of generality is what stitches together the three dimensions, albeit with its own particular flavor in each of them separately.

3 Analytic thinking

3.1 Theorizing related to analytic thinking

For Radford (2014), analyticity is central to algebraic thinking—the aspect that distinguishes it from arithmetic thinking. In fact, Viète referred to algebra as the Analytic Art. Radford characterizes thinking as algebraic when it deals with indeterminate quantities in an analytic way, that is, the indeterminate quantities (unknowns or variables) are considered as if they were known and calculations are carried out with them in the same way as would be done with known numbers, with emphasis on the deductive nature of these calculations. Note, however, that for Radford, the use of alphanumeric symbolism is neither necessary nor sufficient for thinking algebraically. The denotation of indeterminate quantities can be signified through recourse to a host of various semiotic resources, including natural language, gestures, and unconventional signs.

In that Radford (2018) also applies his notion of analyticity to the activity of patterning, analyticity is herein deemed to be only a part of the analytic-thinking dimension of early algebraic thinking. A return to Al-Khwarizmi, where the origins of algebra were based on equality transformations and balancing within equation solving, and extended to include the later introduction of the equal sign and adoption of the term equivalence, suggests that these additional notions be included within our characterization of analytic thinking. While notions of equality and equivalence could also be said to interweave with aspects of the structural-thinking dimension of early algebraic thinking, they are nevertheless central to activity involving equations and equation solving and to the kind of thinking related to transforming equations and their numerical counterparts. Furthermore, and in line with Radford (2014), the dimension of analytic thinking does not require alphanumeric signs; such thinking can be expressed with natural language, concrete materials, unconventional symbols, and even with numbers used in a generic manner.

Equivalence, a key component of our analytic-thinking dimension of early algebraic thinking, was only defined in the twentieth century; but as Asghari (2018) recounts, its history as a lived experience goes back to much earlier times. Its modern definition is that of a binary relation that is reflexive, symmetric, and transitive—the relation "is equal to" being its canonical example. Within equation solving, the properties of equality include not just reflexivity, symmetry, and transitivity, but also a property that was first formulated by Euclid: "if equals be added to equals, the wholes are equal" (and extended to include the other basic operations)—properties that serve to maintain the top-down equivalence between one equation and the next in the equation-solving chain. As Freudenthal (1983) reminds us, it is not the numerical value of left and right sides that remains the same throughout the equation-solving process, but rather the truth-value. Additionally, other properties are also involved in maintaining the balance of both sides within equation solving—the basic field properties applied to the individual expressions of the equations when they too are transformed.

In that equality is defined as a relationship between two quantities, or more generally two mathematical expressions—asserting either that the quantities have the same value or that the expressions represent the same mathematical object—the equality relation straddles two facets of equivalence: the computational and the structural. And, so it is that equation solving has both computational and structural aspects in its dealing with both indeterminates and determinates. However, it is noted that expressions, their properties, and their structurings are treated primarily in the section on structural thinking—equations and equalities being the main focus of this section on analytic thinking.

In sum, handling unknowns as if they were known, along with the thinking related to equivalence transformations and the notions of balance and truth-value, serve to characterize the overarching dimension of analytic thinking within early algebraic thinking. In the following sampling of empirical research related to this dimension, the sub-dimensions to be treated include: (i) thinking about the equal sign and equivalence (Sect. 3.2), (ii) distinguishing sameness from substitution (Sect. 3.3), (iii) solving equations in non-letter-symbolic form (Sect. 3.4), and (iv) solving equations in letter-symbolic form (Sect. 3.5).

3.2 Thinking about the equal sign and equivalence

From the 1970s, when research on the ways in which students view the equal sign began to emerge (e.g., Behr et al., 1976; Denmark et al., 1976; Kieran, 1981), and on up to the present day, studies have been carried out on the understandings that students come to have regarding the equal sign (e.g., Knuth et al., 2006; Lee & Pang, 2021; Rittle-Johnson et al., 2011), as well as on the ways in which instruction can affect those understandings (e.g., Baroody & Ginsburg, 1983; Blanton et al., 2018; Carpenter et al., 2003; McNeil, 2008; Molina & Ambrose, 2006; Sáenz-Ludlow & Walgamuth, 1998).

Briefly, young students' interpretations of the equal sign have been characterized in terms of varying degrees of "operational" and "relational" thinking. The operational interpretation is one where students view the equal sign computationally as a "do something signal" rather than as a relational sign that involves comparing expressions and equations (Jacobs et al., 2007). Stephens et al. (2013) further distinguish the relational view of the equal sign by separating it into the relational-computational and the relational-structural. These nuances brought forward by Stephens et al. suggest an interaction between students' arithmetic thinking and the beginnings of their algebraic thinking. Related to this aspect, Kieran and Martínez-Hernández (2022a, 2022b) report that computational underpinnings were central to young students' algebraic approaches to transforming numerical equalities. In fact, the relation between arithmetic and algebraic thinking, and attempts to more finely distinguish between the two, were a theme of discussion at the recent CERME12 conference with participants at the Working Group on Algebraic Thinking arguing that this is indeed an important area for further research (Hewitt et al., 2022).

Research has shown that the way in which primary school students are typically introduced to the equal sign, as reflected in the textbooks and teacher guides being used, is instrumental to the way in which they view the equal sign (e.g., Li et al., 2008). However, with appropriate instruction, students evolve in their thinking about and manner of dealing with the equal sign. For example, Carpenter et al. (2003) describe how students learn to establish the truth-value of numerical equalities such as $56+47=54+49$ by transforming them as follows: $56+47=(54+2)+47=54+(2+47)=54+49$ (these transformations expressed by the young students in their own words).

The kinds of tasks that have generally been used in the studies gauging students' views of the equal sign have tended to include (i) true–false equalities where students are asked to state their truth-value, (ii) open sentences requiring them to determine the value of the unknown, and (iii) the request to provide a definition of the equal sign. By means of such tasks, Rittle-Johnson et al. (2011), for example, were able to assess how U.S. students' knowledge of mathematical equivalence develops through the 7- to 12-year-old age range. Their results indicated that students evolved in their view of equivalence throughout the primary grades and that four levels of equivalence knowledge served to capture the transitional nature of their growing understandings: rigid operational, flexible operational, basic relational, and comparative relational. They noted in particular that generating a non-computational definition of the equal sign was much harder than solving or evaluating equations with operations on both sides. They also commented that compensation items were easier than items requiring explicit thinking about the addition/subtraction property of equality, that is, "judging $89+44=87+46$ to be true without computing" was easier than "recognizing and justifying that if $56+85=141$ is true, $56+85-7=141-7$ is also true" (Rittle-Johnson et al., 2011, p. 97). Lastly, their results suggested that by about the 5th grade most students have begun to compare both sides of an equation (i.e., they can accept and solve equations with one occurrence of the unknown and with operations on both sides).

The majority of the above research studies on students' thinking about numerical equalities and equations-with-unknowns have focused on left–right equivalence, that is, equivalence of the left and right sides of the equality/equation. In contrast, a focus on top-down equivalence was studied by Kieran and Martínez-Hernández (2022a, 2022b). They investigated students' thinking and ways of transforming numerical equalities in a manner that foreshadows the kind of thinking that would be called upon later in dealing with alphanumeric equations and identities. Sixth graders were presented with numerical equalities of the form $a+b=c+d$ and were asked whether the equality was true or false. Initially, they computed the total of both sides in order to answer the question. When then asked if they could determine the truth-value without calculating, the students wondered aloud: "What other way is there?" After some instructional intervention, and their ensuing attempt to respond to the question by rewriting the initial equality with unrelated decompositions on each side, they eventually came to generate a third common form for each side by means of top-down equivalence. When asked in a general way how they would show if an equality was true, without calculating, and regardless of the numbers involved in the equality (Kieran & Martínez-Hernández, 2022b), they responded (with the aid of generic examples) that they would "convert the numbers in a different way [from the initial equality], but that they should be the same [on both sides of the transformed equality]."

3.3 Distinguishing sameness from substitution

Jones (2009, p. 175) argues for a distinction between sameness and substitution in students' thinking about the equal sign, stating that "the sameness view promotes distinguishing statements by truthfulness; the exchanging view promotes distinguishing statements by form in terms of the notation they transform," and that one does not necessarily imply the other. Jones et al. (2012) carried out a study with English and Chinese 11- and 12-year-olds and found that the notion of *substitution* (i.e., "swapping", "changing sides", "exchanging") is an important part of a sophisticated understanding of mathematical equivalence—one that they claim Rittle-Johnson et al. (2011) did not take into account. Jones et al. (2013) drew further evidence for their argument from their research involving the *Sum Puzzles* digital environment: the students were asked to transform $31+40$ into a single result by selecting from among the various tiles that were offered, tiles that allowed the students to substitute as follows: $31+40$, $30+1+40$, $1+30+40$, $1+70$—which led to the desired single result of 71. They found that some students made substitutions without considering sameness/truth-value—thereby adding more support for their claim of a distinction between sameness and substitution. However, Kieran and Martínez-Hernández (2022b) suggest that the justification for $30+1$ in the *Sum Puzzles* being "substituted" for 31 would seem to be precisely because $30+1$ is the "same" as 31. In the Kieran and Martínez-Hernández (2022b) study on sameness, students used the language of sameness—both visible and invisible—to describe both the truth-value of their transformed equalities and the validity of the substituted expressions.

Further research on the substitution versus sameness perspective on equality was carried out by Donovan et al. (2022). These researchers compared the impact of a lesson focused on the sameness conception alone with a lesson focused on a dual sameness and substitutive conception. They found that the 4th and 5th grade participants of the Sameness condition were more likely to notice numerical relationships across the equal sign than were those in the Sameness + Substitutive condition. They suggest further that substitution follows logically from the sameness conception and that holding a substitutive view without a sameness view is potentially problematic.

3.4 Solving equations in non-letter-symbolic form

As noted above, indeterminates can be denoted without conventional symbols. In a teaching study by Molina and Castro (2021) with 3rd graders, where the indeterminates were represented by square brackets, the equations were designed in such a way (e.g., $12+7=7+[]$, $[]+4=5+7$) that the approach used to solve them, and the way in which students justified the truth-value of the solution obtained, could indicate whether they had used computational or non-computational approaches. As the study progressed, the intervention provided feedback that emphasized those approaches that were non-computational in nature, which led to an evolution in both non-computational solving strategies and ways of justifying true equalities.

Material objects—marbles, colored boxes, images of two children to indicate each side of an "equation"—were used in a study by Lenz (2022) to represent indeterminates and determinates (see also the use of varied materials in Cooper & Warren, 2011, and in Stephens et al., 2022). In one of the tasks of the Lenz study (adapted from Carraher et al., 2008), Kindergarten, Grade 2, and Grade 4 students were asked how they would determine the number of marbles hidden in the green box (see Fig. 1). The students, who had not had any prior experience with such tasks, were told that boxes of the same color always contain the same number of marbles and that each child has the same total number of marbles.

The students at all three grade levels found this a difficult task (success rates of 10%, 30%, and 50% respectively for the Kindergartners, 2nd graders, and 4th graders). Lenz points out that the presence of unknown quantities that did not have to be determined stimulated those who were successful to approach the "equation" in a structural manner and thereby solve it. In contrast, the tasks that initially had a single box on one side and loose marbles on the other side could be approached arithmetically by comparing and were considerably easier for these young students to solve.

Few studies in early algebra have tackled the issue of representing and solving equations of the type $ax+b=cx+d$ with two occurrences of the unknown, one on each side—a kind of equation that requires operating on the unknown as if it were known (see Filloy & Rojano, 1989). Radford (2022) introduced 3rd graders to such equations by means of the following story problem:

> Simon and Françoise have some hockey cards. Françoise has 6 cards and Simon has one card. Their mother puts some cards into three envelopes and makes sure to put the same number of cards in each envelope. She gives one envelope to Françoise and two to Simon. Now the two children have the same number of hockey cards. How many hockey cards are inside each envelope?

In Radford's study, this story is first represented with a Concrete Semiotic System (CSS) comprised of material objects, namely, paper envelopes that each contain the same unknown number of cards, loose cardboard cards, and the equal sign. The Iconic Semiotic System (ISS), which is used next, replaces the concrete objects with iconic drawings. The 3rd graders seen in Fig. 2 were faced for the first time with such an "equation" (i.e., $2x+1=6+x$). They knew from their previous experience with equations having just one occurrence of the unknown that they had to isolate one envelope; but the present situation with envelopes on both sides puzzled them. As Radford points out, they had to move from $2x=x+5$ to x equals something.

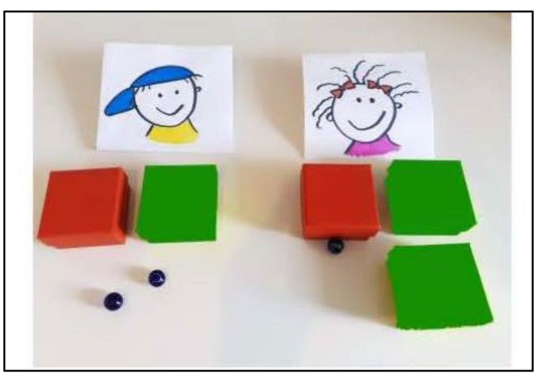

Fig. 1 Task involving unknowns in Lenz (2022)

The teacher's intervention was crucial to their learning that not only could an equal number of loose hockey cards be removed from both sides, but so too could an equal number of envelopes with their unknown number of cards within. The transformation that the students had applied to the knowns had now been generalized so as to apply to the unknowns.

Perhaps just as important from the report of this research is Radford's (2022) observation of how difficult it was for the students, when subsequently inventing their own story problem and solving it, to put the equality situation and the equal sign into words. This same phenomenon on the difficulty experienced by students in expressing with natural language their ideas about relational situations is found in Blanton et al. (2015a); in fact, students were more adept in the Blanton et al. study at using symbols to represent their thinking than they were with natural language. Recognizing this difficulty has led researchers such as Malara and her colleagues (e.g., Cusi et al., 2011; Malara & Navarra, 2003, 2018) to devote considerable effort to developing young students' language use in algebraic situations involving indeterminates and the equal sign.

3.5 Solving equations in letter-symbolic form

In China, first graders are introduced to indeterminates and to their representation by means of brackets "()" within the activity of equation solving, for example, $1+()=3$ (Xie & Cai, 2022). They are taught to solve such equations by means of inversing operations; that is, the subtraction $3-1=2$ is used for finding the number in the brackets. This method continues through to the beginning of the 5th grade, when the formal method of equation solving is introduced. The indeterminate is now referred to as an unknown, which is represented by a letter. The balance-scale model and the addition and multiplication properties of equality for solving the equations take center stage. By means of a post-test involving both the use of brackets and

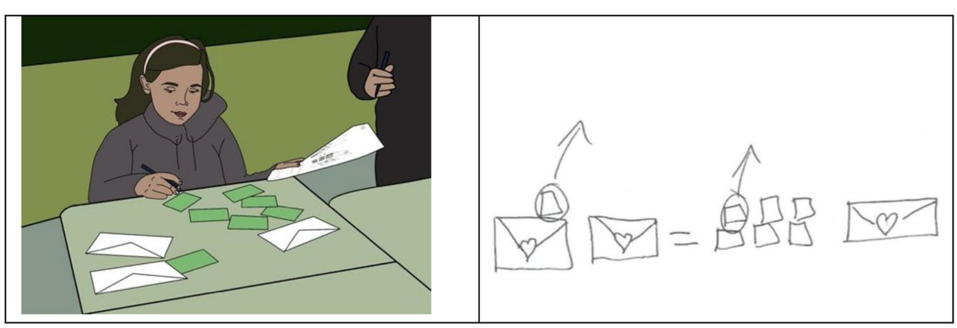

Fig. 2 Solving the equation $2x+1=6+x$ in the CSS and the ISS (Radford, 2022)

letters to represent the indeterminates in the equations, Xie and Cai found that half the students in their study of 5th graders attempted to apply their inversing strategy to the equations with letters, not always successfully due in some cases to an intermingling of aspects from both solving approaches. The inversing strategy clearly interfered with most students' learning of the formal method.

The above research finding from the Xie and Cai study raises questions about the transition to the letter-symbolic representation of the unknown within the balance-oriented approach to equation solving. Many studies with young students (e.g., Blanton et al., 2015a; Britt & Irwin, 2011; Molina & Ambrose, 2006) have involved representing the single occurrence of the unknown with various non-traditional notations (e.g., empty boxes, blank lines, question marks, ink blots, etc.) and using non-formal methods of equation solving (e.g., number facts, trial substitutions, cover-up method). And, in a number of Asian studies with primary level students, unknowns are represented by rectangular bars within a pictorial linear model that depicts the relations within a problem situation—sometimes referred to as the Singapore model method (e.g., Cai et al., 2011; Ng & Lee, 2009). While younger students initially solve problems arithmetically by means of this model, Cai et al. (2011) argue that by the 5th grade the rectangular unit of the pictorial equation is replaced by the letter x (yielding equations such as $x+x+100=410$), which Cai et al. claim allows students to treat unknowns as if they were knowns. However, the obstacles encountered by students in the Xie and Cai (2022) study suggest the need for further intervention-related research on the transitional challenges faced by young students as they move from the non-letter-symbolic to the letter-symbolic form of equations, especially equations of the type $ax+b=cx+d$ with an occurrence of the unknown on each side of the equation.

4 Structural thinking

4.1 Theorizing related to structural thinking

Mason et al. (2009, p. 10) have characterized structural thinking as follows:

> Recognising a relationship amongst two or more objects is not in itself structural or relational thinking, which, for us, involves making use of relationships as instantiations of properties. Awareness of the use of properties lies at the core of structural thinking. We define structural thinking as a disposition to use, explicate and connect these properties in one's mathematical thinking.

Similar to this is Carpenter et al.'s (2005, p. 54) definition of relational thinking:

> Relational thinking involves using fundamental properties of number and operations to transform mathematical expressions rather than simply calculating an answer following a prescribed sequence of procedures. This implies some level of awareness of the properties.

Kaput (2008) too has described an aspect of the generalized-arithmetic strand of algebra as including the generalizing of arithmetic operations and their properties, and reasoning about more general relationships from the structure of arithmetic. Clearly, relationships and properties are at the heart of structural thinking. While these characterizations of structural thinking within early algebraic thinking all focus on the numerical, another perspective on this dimension is one that was developed by Davydov and his colleagues (Davydov et al., 1999). The Davydovian view tends to be non-numerical in the early stages of learning and is based rather on relationships among quantities—a view that also involves the use of literal symbols right from the first grade (see also Dougherty, 2003; Schmittau & Morris, 2004).

Kieran (2018b) argues that the emphasis on using properties to transform the structure of numerical and algebraic expressions is one that would benefit from taking a wider perspective on structure and properties. In line with Freudenthal (1983, 1991), who refers to the order structure, addition structure, multiplicative structure, as well as structure according to divisors, structure according to multiples, and so on, there are many different but related properties associated with these structures—not simply the basic properties of arithmetic. Freudenthal also uses the phrasing, *means of structuring*, which puts forward the notion of alternative structurings that can be deduced from the basic structures. These structurings have properties, such as the basic properties of arithmetic, but also a multitude of other properties, such as the successor property, the sum of consecutive odd numbers property, the sum of even and odd numbers property, and so on.

In sum, the overarching dimension of structural thinking is characterized as encompassing the broader, Freudenthal-related perspective on structure and properties within activity aimed at seeing relations, properties, and structure within numbers, operations, and expressions. Such early algebraic thinking is considered to underpin the kind of thinking involved in transforming and generating equivalent algebraic expressions. In the following sampling of empirical studies related to the structural dimension of early algebraic thinking, the sub-dimensions that are treated include: (i) seeing structure within number and numerical expressions (Sect. 4.2), (ii) representing the structure of numerical operations (Sect. 4.3), (iii) extending the structural dimension beyond the realm of natural numbers (Sect. 4.4), and (iv) extending the structural dimension beyond number (Sect. 4.5).

4.2 Seeing structure within number and numerical expressions

Linchevski and Livneh (1999) propose that early-algebra instruction be designed to foster the development of structure sense by providing experience with equivalent structures of expressions and with their decomposition and recomposition. Subramaniam and Banerjee (2011, p. 91) suggest that "numerical expressions must be viewed not merely as encoding instructions to carry out a sequence of binary operations, but as revealing a particular operational composition of a number." Molina and Ambrose (2008) stress the importance of young students' learning to use both their number and operation sense so as to come to view arithmetic expressions from a structural rather than a procedural perspective. Malara and Navarra (2018) point to the value of expressing structural aspects of number in transparent, non-canonical ways, as illustrated by one student's representing of the sum of 5 and its successor as "5 + 6," whereupon one of her classmates suggested that "5 + 5 + 1" was better because it more clearly represented the successor property. Kieran (2018b) reports how 12-year-olds became aware of various structural properties related to multiples and divisors in their decomposing of numbers within the "Five Steps to Zero" activity—properties such as, "within every interval of n numbers, there is exactly one number divisible by n."

Carpenter et al. (2003) elaborate on classroom activity that fosters young students' conjecturing and justifying of properties such as, "when you multiply a number by another number that is not zero and then divide by the same number, you get the number you started with." Britt and Irwin (2011) describe how students noticed and used compensation properties in their structural transformations of numerical expressions. Similarly, Schifter (2018) illustrates the ways in which 3rd graders examined a sequence of numerical expressions and came to generalize the property that when 1 is subtracted from one addend and added to the other, the sum is unchanged. In another study with 3rd graders, Isler et al. (2013) report on conjecturing and justifying property-based rules such as "anytime you add three odd numbers, you always get an odd number." An often-referred-to example of this kind of structural activity with properties involves the use of numbers as quasi-variables (Fujii, 2003; Fujii & Stephens, 2001). Fujii introduced young Japanese students to algebraic thinking through generalizable numerical expressions, using numbers as if they were variables.

A somewhat more complex example of structural thinking leading to generalization is reported in a recent study by Coles and Ahn (2022). These researchers describe the journal entries of 11- and 12-year-olds on the "1089" task:

> Take a 3-digit number (whose first digit is larger than the last), reverse the digits and subtract the second number from the first. This gives you a 2- or 3-digit answer. You reverse the digits of the answer and add it to that answer. The result will be 198 or 1089, if you have done the subtraction and addition correctly.

After working for a couple of lessons on this 3-digit challenge, students went on to trying out 4- and 5-digit numbers. Coles and Ahn report that, after the students performed the algorithm on many numbers, some wrote about relations involving operations on unspecified numbers, classifying and generalizing their work. The way in which the students were able to structure and generalize the underlying relations between unspecified numbers of the same constrained set with both deduction and certainty led Coles and Ahn to suggest that there was an element of analyticity in the students' conjecturing and justifying activity that is related to Radford's (2014) theorizing on analyticity. The study by Coles and Ahn also points to an area of research that is presently underdeveloped in early algebra—an area that includes the use of numerical "puzzles" that always yield the same surprising result (e.g., see also the "think of a number" problems in Mason et al., 1985). Such puzzles can serve to motivate the introduction of alphanumeric symbols—symbols that allow for justifying those surprising outcomes, and for proving in early algebra more generally.

4.3 Representing the structure of numerical operations

The use of student-generated representations for expressing and proving generalizations about numerical operations underpins the research program of Schifter and Russell (2022). They delineate three criteria for characterizing a representation that supports students' arguments for a generalization about an operation: (i) the meaning of the operation is represented in diagrams, manipulatives, or story contexts; (ii) the representation shows how the conclusion of the generalization follows from the premise; and (iii) the representation can accommodate a class of instances, for instance, all whole numbers. The researchers illustrate with several examples drawn from the work of students in grades 1–5 the nature of the representations that support the students' arguments. For instance, with "Brian's blob," a drawing that a 4th grade student used to show the relation between 145–100 and 145–98, classmates explored Brian's general claim that "the less you subtract, the more you end up with and, in fact, the thing you end up with is exactly as much larger as the amount less that you subtracted."

Other kinds of representations that have been featured in various studies on developing students' structural understanding of operations include number lines for exemplifying the inverse relationship of addition and subtraction (Warren & Cooper, 2009) and story problems to represent, for example, the distributive property (Ding & Li, 2014)—the use of such representations usually being complemented by the students' expressing their thinking in natural language (e.g., Cooper & Warren, 2011; Empson et al., 2011).

4.4 Extending the structural dimension beyond the realm of natural numbers

Recent research extends the structural-thinking dimension of early algebraic thinking by introducing the notion of subtractive number (Vlassis & Demonty, 2022) and that of reverse fractions (Pearn et al., 2022). Vlassis and Demonty investigated the nature of students' structural thinking about negative numbers in a study with 6th graders. The researchers aimed to determine the extent to which the students who answered the compensation questions correctly also performed the operations with integers better than those who answered them incorrectly. They found that students' ability to see a number that was preceded by a subtraction operation as a transformation involving a unary use of the minus sign, which they referred to as a subtractive number, was a decisive factor in their success in operating with negative numbers. This study by Vlassis and Demonty illustrates how the structural dimension of early algebraic thinking can be appropriately extended to encompass a numerical domain that has not up to now been considered within its purview—that of negative number.

Another numerical domain that has received scant attention within the study of early algebraic thinking is that of fractions. An earlier venture into this area was carried out by Empson et al. (2011), who provided evidence for the notion that the kind of structural thinking that is based on the fundamental properties of operations also underpins student work with fractions (see also the related research with older students carried out by Hackenberg & Lee, 2015). More recent research by Pearn et al. (2022) focuses on reverse-fraction tasks, that is, finding an unknown whole when presented with a quantity representing a fraction of that whole. Interviews conducted with 10- to 12-year-olds who had just worked with reverse-fraction tasks involving specific numbers culminated with the general question: "What if I gave you *any* number of counters, and they represented *any* fraction of the number of counters I started with, how would you work out the number of counters I started with? Can you tell me what you would do? Please write it in your own words." The researchers found that the students who were reliant on diagrams or additive strategies struggled to solve the more generalized task. The successful students recognized the multiplicative structure that related the given fraction to its unknown whole.

4.5 Extending the structural dimension beyond number

A further extension to the structural-thinking dimension is embodied in the work of Tondorf and Prediger (2022), whose study focused on the interplay between numeric and geometric transformations within equivalence activity by 5th graders. The researchers explored the manner in which the design of successively refined instructional environments could enhance students' bridging of the gap between description equivalence and transformation equivalence. They first defined *result equivalence* (also referred to in the literature as the computational aspect of equivalence); then *transformation equivalence* as the transformation of one expression into another (also referred to in the literature as the structural aspect of equivalence); and *description equivalence* as the relating of two symbolic expressions to the same situation or geometric figure.

The geometric figure in their study was a drawing of two floor plans for the same room, which was accompanied by the numerical expressions $8\times12+2\times4$ and 26×4—these two expressions corresponding to the manner in which the two floor plans had been structured (see Fig. 3).

Tondorf and Prediger found that the gap between description and transformation equivalence was bridged by the students' direct modification of the structured figure. The researchers suggest that this *restructuring equivalence* replaces the more usual approach of comparing each of two numerical expressions to possible sub-structurings of a given figure so as to ensure that both expressions describe the same figure and are thereby equivalent. While existing research tends to focus rather on connecting result equivalence with transformation equivalence (e.g., Banerjee & Subramaniam, 2012; Kieran & Martínez-Hernández, 2022a; Schwarzkopf et al., 2018), Tondorf and Prediger's study adds another facet to the structural component of equivalence and thereby

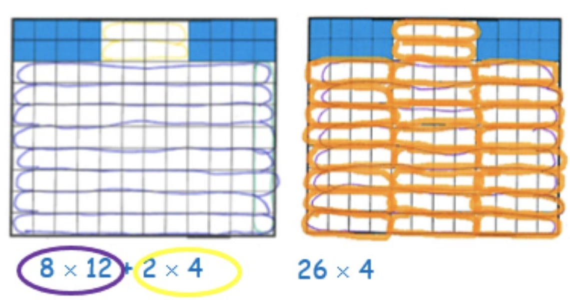

Fig. 3 Geometric floor plans used in the Tondorf and Prediger (2022) study

contributes to expanding the structural-thinking dimension of early algebraic thinking.

5 Functional thinking

5.1 Theorizing related to functional thinking

Kaput (2008, p.13) states that the functional strand of algebraic reasoning can be thought of as "describing systematic variation of instances across some domain." Blanton et al. (2011, p. 13) propose that functional thinking "entails generalizing relationships between co-varying quantities, expressing those relationships in words, symbols, tables, or graphs, and reasoning with these various representations to analyze function behavior."

More broadly, Blanton and Kaput (2011, p. 8) have specified that the framework underpinning their research on the functional thinking found in classroom data is based on the three modes of analyzing patterns and relationships outlined by Smith (2008): (i) recursive thinking, which involves finding variation within a single sequence of values; (ii) covariational thinking, which is based on analyzing how two quantities vary simultaneously (e.g., "as x increases by one, y increases by three"; and (iii) correspondence thinking, which is based on identifying a correlation between variables (e.g., "y is 3 times x plus 2").

During the years that followed Blanton and Kaput's (2011) use of the above framework for analyzing functional thinking, further theorizing related to functional thinking occurred, namely, the development of two more-elaborated frameworks that have served as the basis for additional research in this area. The first of these frameworks—in fact, a learning trajectory to describe first graders' (6-year-olds) thinking about generalizing functional relationships—was the outcome of a study carried out by Blanton et al. (2015b). During an 8-week sequence of classroom teaching sessions and student interviews, with functional task situations such as, "the relationship between Keisha's and Janice's age if Janice is 2 years younger than Keisha" (adapted from Carraher et al., 2006), the students were taught how to organize their problem data into tables of values. The analysis of the levels of sophistication in students' thinking arising from their exploration of functional relationships led to the following learning trajectory:

- Level 1: Prestructural;
- Level 2: Recursive-Particular;
- Level 3: Recursive-General;
- Level 4: Functional-Particular;
- Level 5: Primitive Functional-General;
- Level 6: Emergent Functional-General;
- Level 7: Condensed Functional-General; and
- Level 8: Function as Object.

The Blanton et al. (2015b) framework was subsequently refined by Stephens et al. (2017b) in the light of the latter's study on grades 3–5 students' abilities to generalize and represent functional relationships. During the instructional interventions, the teacher, who followed the students as they moved up through the grades, prompted them to reason about the recursive, covariational, and correspondence relationships of the patterns being investigated and to represent them in multiple ways (i.e., pictures, words, variables, tables, and graphs). The trajectory of students' levels of thinking that was generated was based on assessment data obtained at the beginning of grade 3 and at the end of grades 3, 4, and 5—one of the two written assessment tasks being the "Brady birthday party at school" task: a pattern sequence involving desks being joined together with persons sitting on opposite sides of each desk. The trajectory developed by Stephens et al. (2017b, p. 153) consists of the following levels:

- L0: No Evidence of Functional Thinking;
- L1: Variational Thinking (i.e., recursive pattern-particular);
- L2: Variational Thinking (i.e., recursive pattern-general);
- L3: Covariation Thinking (i.e., covariation relationship);
- L4: Correspondence Thinking (i.e., single instantiation);
- L5: Correspondence Thinking (i.e., functional-particular);
- L6: Correspondence Thinking (i.e., functional-basic);
- L7/8: Correspondence Thinking (i.e., functional-emergent)—student identifies incomplete function rule in variables (L7) or words (L8);
- L9/10: Correspondence Thinking (i.e., functional-condensed)—student identifies function rule in variables (L9) or words (L10).

It is noted that, in contrast to the Blanton et al. (2015b) trajectory, the Stephens et al. (2017b) trajectory makes explicit the distinctions among variation, covariation, and correspondence. As well, the researchers draw attention to the separation within the functional-emergent (L7/8) and the functional-condensed L9/10) levels of two sub-levels that allowed for distinguishing generalizations expressed with variables (L7 and L9), which were easier for the students, from generalizations using words (L8 and L10).

In sum, the overarching dimension of functional thinking within early algebraic thinking is characterized as seeking, expressing, and reasoning about relationships between co-varying quantities (whose initial presentation might be in any of several possible formats) in a manner that makes explicit in a general way the underlying relation of the given function, by means of some type of conventional or unconventional representation. For the following sample of empirical studies related to the functional-thinking dimension of early algebraic thinking, two sub-dimensions are featured: (i) generalizing within various representations of functional situations (Sect. 5.2), and (ii) generalizing with respect to recursive, covariation, and correspondence modes of functional thinking (Sect. 5.3).

5.2 Generalizing within various representations of functional situations

Functional situations can be presented to students in a variety of ways. Many of the function-related studies with primary school students favor the use of figural or non-figural patterns—presented with a brief storyline and represented by an accompanying visual of the first three or so items of the sequence—and aim at investigating the manner in which students come to generalize the pattern in the form of a rule, as well as the influence of the particular type of task presentation on students' thinking.

Based on his extensive work in this area, Rivera (2010) argues that visual growing patterns support students' attempts to generalize relationships between two quantities in ways that non-figural function tasks do not. However, instructional effort is needed to assist students in detecting the salient features of such patterns. Warren (2005a, 2005b) reports on a study that involved 9- and 10-year-olds working with growing patterns, but without accompanying tables of values. Students were required to find the general rule for these growing patterns directly from the visual geometry of the objects. According to Warren, it was very difficult for the students to detect a commonality that they could extend to a general rule. Ultimately, they benefitted from learning how to do a visual analysis of growing patterns (i.e., learning how to separate a figural/geometric pattern into its components).

In fact, distinguishing between what is invariant and what it is that is varying constitutes a crucial first stage in the activity of patterning (Kieran, 2006; Mason, 2005). According to Rivera (2013), and in line with Mason et al. (2009), the development of attention to structural aspects within patterning activity involves the recognition of relationships of similarity and difference within a structure, followed by the perceiving of properties that characterize the objects being analyzed, and then by reasoning on the basis of the identified properties. And for the case of figural patterns, Radford (2011, p. 19) argues that:

> Generally speaking, to extend a figural sequence, the students need to grasp a regularity that involves the linkage of two different structures; one *spatial* and the other *numerical*; from the spatial structure emerges a sense of the figures' *spatial position*, whereas their numerosity emerges from a numerical structure. (italics in the original)

Radford (2011) describes the nature of the specific teaching interventions that assisted the 7- and 8-year-olds of his study to make the links between the numerical and spatial structures of the various growing patterns. Papic and Mulligan (2007) too emphasize the importance of teacher intervention in developing within preschool children the ability to focus on the structure of repeating patterns—intervention that enabled the children to spontaneously go on to extending the growing patterns they were presented with. In a study by Moss and London McNab (2011), a combination of function-machine tasks involving representation with tables of values and tile-based growing patterns, with movement back and forth between the two types of functional tasks, helped to bring out the salience of the geometric and numerical structural integration required to find commonalities and generate functional rules. Wilkie and Clarke (2016) report how upper level primary students' visualization of the structure of a geometric pattern in different ways and the use of this variety promoted the generalization of the functional relationship.

Function machines of the sort used by Moss and London McNab (2011)—a device that favors an input–output notion of functions (see Doorman et al.'s, 2012, related research with older students)—also served as a prelude for a programming activity with 6th graders' involving the visual programming language of Scratch (Kilhamn et al., 2022). After playing with the cardboard function machines and guessing the "secret" rules, the students were given the programming task of examining a block of code in Scratch that featured the rule $y = 2x + 1$. They were tasked with finding the rule within the programming code and with figuring out how the program worked. Further activity involved inspecting a table of values that was produced by a program and trying to deduce the rule—a task that the students found to be challenging. According to the researchers, learning to coordinate the way in which variables are treated in programming code (where running a code requires assigning starting values to the variables) with the way in which the same variables are treated in standard functional usage requires specific teacher intervention. In this regard, Benton et al. (2017) have noted that the integration of programming within mathematics classes does not automatically enhance mathematics learning.

In a study involving only the table-of-values representation (and without an accompanying storyline or pattern) that was carried out by Xolocotzin et al. (2022) with 3rd and 5th graders, the students were found to experience considerable difficulty in reorganizing their arithmetic knowledge in such a way that would allow them to reconceptualize numbers as instances of an indeterminate quantity and thereby extract the general rules for functions and express them in symbolic-equation format (see Ellis, 2011, for related results involving middle schoolers). Nevertheless, a large number of studies have relied on the use of tables of values as a central tool to assist in the development of students' generalizing of functional relations and have pointed to the positive role that this kind of representation can play, when used in conjunction with stories and other representations in an instructional context (e.g., Blanton & Kaput, 2011; Blanton et al., 2015b; Brizuela et al., 2015; Carraher et al., 2008).

The use of specific cultural contexts and representations was at the heart of a study by Hunter and Miller (2022), who investigated the ways in which 6-year-old New Zealand students of Pacific descent identified functional structures in growing patterns related to contexts that were part of their cultural heritage—arranging tables at family feasts, and the design of quilts (see Fig. 4).

Interestingly, the use of these familiar contexts led to a large variety of ways of structuring the patterns—a variety that called upon specific teaching interventions to assist the students in reaching the desired higher levels of functional thinking. One of the pivotal interventions was pressing the students to find far terms—an aspect of the generalizing process emphasized in other studies (e.g., Pinto & Cañadas, 2021; Radford, 2011; Twohill, 2018).

The use of familiar situations was also an important component of a study carried out by Goñi-Cervera et al. (2022), involving 26 students (aged 6–12 years) who were all diagnosed with autism spectrum disorder. The students were presented with a story situation and visual display of people seated around tables in a restaurant, which was based on the function $y = 2x + 2$, and adapted from Carraher et al. (2008). Having been supplied with paper and building blocks for exploring the functional situation, the students were found to rely strongly on the use of drawings within their generalizing attempts, especially for the near items of the pattern. Those who were successful at generalizing used a combination of drawing and numerical approaches.

The way in which students justify their function-related generalizations has also been an area of research interest. In fact, Lannin (2005) argues that generalization cannot be separated from justification. In his study of 6th graders, one of the tasks involved the Theater Seats pattern for which students were asked to *explain the formula* they had generated. Lannin found that they tended to test their formulas

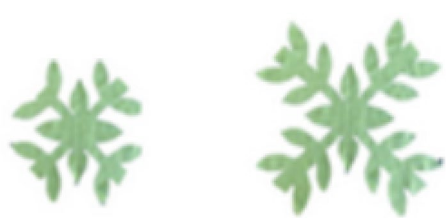

Fig. 4 Part of a growing leaf pattern from the design for a Cook Island quilt (Hunter & Miller, 2022)

by empirical justification, even though they considered this an inadequate type of proof. In Blanton et al. (2015b), one 6-year-old was able to justify her functional rule by relating its parts to the specific items of the story context. Rivera and Becker (2011) found that the 6th graders of their study justified their generalized linear pattern rules in one of four ways: (i) using more examples to verify the correctness of a formula, (ii) describing a structural similarity in an imagined general instance, (iii) demonstrating the validity of a formula on various figure items, and (iv) using a numerical explanation that fits the formula to the corresponding generated table of values—this last being the most common method of justification. Rivera and Becker add that the students were in fact constructing and validating their formulas at the same time. Ellis (2007) suggests that justifying be introduced early on in the generalizing process and not be left until after students have arrived at a final generalization.

5.3 Generalizing with respect to recursive, covariation, and correspondence modes of functional thinking

Blanton et al. (2015b) emphasize that the levels of sophistication that are developed within young students' functional thinking are tightly connected to the instructional sequence that gives rise to them. Thus, their finding that the 1st graders' thinking about recursive patterns did not become as sophisticated as their thinking about the functional correspondence relationships is due to the fact that their instructional interventions did not focus on eliminating recursive thinking, but rather on developing the more advanced forms of functional thinking. They did notice, however, that some students might use a recursive form of thinking with sequential values of a table. While several researchers have noted that students tend to gravitate initially toward recursive approaches (e.g., Lannin et al., 2006), others have found that this is not always so (e.g., Pinto & Cañadas, 2021). It is widely considered that generalizing recursively (e.g., to find the elements in Term 100: "you keep adding 2, and 2, and 2 to Term 1 until you get to Term 100") is arithmetic in nature (e.g., Radford, 2014) and does not support the development of algebraic thinking.

This particular point of view has led some researchers, for example, Moss and London McNab (2011), to experiment with the use of non-consecutive values in a table. These researchers describe how the 7- and 8-year-olds of their study of function-machine activity were encouraged to generate non-sequential pairs of input and output numbers. Assisting students to steer away from recursive approaches and to focus on correspondence relationships has involved various types of interventions, such as decomposing a repeating pattern into its repeats and recognizing the synergy between the visual growing pattern and its table of values

(Warren & Cooper, 2008). Despite the apparent effectiveness of such interventions, Amit and Neria (2008) document the limited success encountered by students who begin with a recursive approach and attempt to switch over to a correspondence approach; a similar challenge has been noted in students' efforts to coordinate covariation and correspondence approaches (Ellis, 2011). Complementary to the role of instruction in developing student thinking regarding the correspondence relationship is the positive role that can be played by collaborative student discussion of their various approaches (Moss & Beatty, 2006).

With respect to the covariation mode of thinking, Stephens et al. (2017b) comment that this mode is not often emphasized in the primary levels of schooling. An exception is a study by Levin and Walkoe (2022) who explored the influence of covariational thinking across a range of activities using their *Seeds of Algebraic Thinking* frame (Walkoe & Levin, 2020). Among the examples they provide of the use of covariation is that of 4th graders programming a Sphero robot to complete an obstacle course created by their teacher. Covariation also underpinned a study carried out by Ramirez et al. (2022), who presented a group of high-achieving 6th graders with a set of tasks that involved quadratic functions with continuous variables in the geometric setting of an area problem. It is reported that while the successful students identified regularity in particular cases and extended it to include the set of numbers in a particular interval, they nevertheless relied primarily on the use of natural language to express their generalizations in the face of their difficulties with using symbols correctly.

Despite the challenges reported in students' moving from a recursive to a correspondence mode of thinking, or from a covariational to a correspondence mode, Stephens et al. (2017b) found that, after three years of early algebra lessons, most of the students evolved in their ability to express correspondence rules in both words and variables. The researchers observed that the students operated at different levels depending on the context. It was also noted that they were more apt to express function rules in variables than in words as the complexity of the function increased. Brizuela et al. (2015) and Molina et al. (2018) report similar results.

In contrast, Pinto and Cañadas (2021) found that all the 5th graders in their study, which involved the function $y=2x+6$, generalized the relationship, primarily using natural language. This same reliance on natural language to express a generalization was also reported in the study by Goñi-Cervera et al. (2022). The role played by language was further addressed, albeit somewhat differently, in a study by Ayala-Altamirano et al. (2022) where functional word problems with questions on the general case were presented using natural language, drawn figures, non-algebraic symbols, and letters. The keyword "many" within the natural language mode of questioning was found to induce the expression of generalization more successfully than questions using other modes of representation for the indeterminates. In view of the disparity in the findings of various studies regarding the role of natural language (in contrast to the use of symbols) in the expression of generality for functional situations, it would seem appropriate for more research to address this particular facet of functional thinking.

Reflecting the importance accorded by most studies to the role of instruction in the development of functional thinking, Pang and Sunwoo (2022) designed a pattern and correspondence unit for a revised Korean textbook for primary school mathematics—a unit that was underpinned by key instructional elements. A review of the literature on functional thinking, in particular the frameworks developed by Blanton et al. (2015b) and Stephens et al. (2017b), led Pang and Sunwoo to integrate certain specific instructional elements into their curriculum design. The students who were taught with the new draft were found to be more successful than their counterparts in exploring and representing the relationship between two quantities, thereby illustrating the role that can be played by fostering functional thinking through changes to curricular materials that draw on existing research findings. The final version of the unit that was developed is now in use in all schools of the country.

6 Summing up and new directions

In this survey paper, I have framed the overarching dimensions of early algebraic thinking according to the analytic, the structural, and the functional. Within the treatment of each dimension, I first described its theoretical underpinnings and then offered a sampling of the related empirical research. The first and second dimensions—the analytic and the structural—were related to the generalized-arithmetic perspective on algebra and the third, quite obviously, to the mathematical-functional perspective on algebra. The focus of the analytic-thinking dimension was primarily on the notion of dealing with unknowns as if they were knowns, along with the thinking related to the balance-maintaining equivalence transformations of equations and equation solving in both non-symbolic and symbolic form—including the thinking related to the equal sign in various equation formats. To complement the analytic-thinking dimension, the focus of the structural-thinking dimension of early algebraic thinking was primarily on seeing and expressing structure and properties within numbers, operations, and expressions. The focus of the functional-thinking dimension was that of generalizing within different representations of functional situations, as well as generalizing related to the recursive, covariation, and correspondence modes of functional thinking. However, generalizing was not restricted to the functional dimension; it was a salient thread that ran through all three dimensions. In presenting examples of research exemplifying the three overarching dimensions, I also signalled some of the newer developments and directions, together with areas in need of further research. I return to these latter aspects now.

With respect to new directions, we are reminded that the title of this Special Issue refers not only to the multi-dimensionality of early algebraic thinking, but also to its ever-expanding vision. This ever-expanding vision comprises both the theoretical and the empirical. In the theoretical area, the new developments that were highlighted in this survey paper are of two types: the first involving novel theoretical ideas and the second, the act of building upon the existing frameworks developed previously by researchers in this same field of early algebra. Some of the novel theoretical ideas include, as a first example, the *Seeds of Algebraic Thinking* frame initiated and further developed in this Special Issue by Levin and Walkoe (2022). A second example is the theorizing by Tondorf and Prediger (2022) related to equivalence: that of *restructuring equivalence*, which some students used as a means to bridge transformation and description equivalence. A third example is that offered by Vlassis and Demonty (2022) whose work on the structural notion of the *subtractive number* brought the domain of negative numbers into the theorizing related to early algebraic thinking.

The new developments related to drawing upon theoretical frames developed in and for the field of early algebra is a striking aspect of recent research, one that illustrates the evolution in the field. In this regard, but related more broadly to the story of mankind, the explorer Salopek (2021) reminds us that it was only when human populations grew large and stable enough to retain and build upon past breakthroughs that we could truly advance. Within our own field, the theoretical frames developed earlier by Blanton et al. (2015b) and refined by Stephens et al. (2017b) provided the foundation for Pang and Sunwoo's (2022) development of a curricular unit aimed at fostering student functional thinking related to pattern and correspondence. The same frameworks were used by Hunter and Miller (2022) to analyze 2nd graders' levels of generalization with growing patterns, as well as by Goñi-Cervera et al. (2022) in their study with autistic students. Another example of building on theory previously developed by researchers in our own field is the use by Coles and Ahn (2022) of Radford's (2014) notion of analyticity in their analyzing of students' journal writing on the relations in the "1089" problem and for their subsequent proposal that this notion be extended to include conjecturing and justifying of a generalized nature.

In the empirical area, the ever-expanding vision relates not only to the individual contributions of each of the papers in this Special Issue, but also and especially to work in three novel areas: curriculum development, research with special education students, and research in programming environments. Pang and Sunwoo's (2022) work in developing a curricular unit, which was based on extracting key instructional elements from the review of the literature on functional thinking, serves as a model of how curriculum development aimed at fostering early algebraic thinking could be carried out. Research with special education students, which was realized by Goñi-Cervera et al. (2022) with autistic students, sheds light on the design of teaching methods that can help these students develop functional thinking. Research by Kilhamn et al. (2022) in the Scratch programming environment emphasizes the specific didactic interventions that are required in order to link the non-standard usage of variables within programming environments to their standard usage in algebra.

This survey paper, with its sampling of past and more recent research on early algebraic thinking, also pointed to some areas where further research would be in order. These areas include, but are not limited to, the following:

- There is a need for detailed intervention studies of the transitional challenges faced by young students as they move from the non-letter-symbolic to the letter-symbolic form of equations, especially equations of the type $ax+b=cx+d$ with an occurrence of the unknown on each side of the equation, and with the solving of such equations by performing the same operation on both sides.
- The contrasting findings regarding the use of natural language versus the use of symbols to express generalized functional rules suggests that more research is needed so as to better characterize the nature of the circumstances that lead to one finding or the other.
- More studies are needed that research the ways in which students can be assisted in analyzing the visual structure of growing patterns, and in generating related diagrams, so as to better equip them to develop in their functional thinking.
- Further research is needed to explore student thinking about sameness and substitution with respect to equivalence and to the roles that these two notions play in supporting students' thinking about the equal sign, equation solving, and equation-solving transformations.
- The tendency in many countries towards integrating computational thinking and programming into the mathematics curriculum right from the earliest grades of primary school suggests the need for research programs that focus on ways in which early algebraic thinking can be fostered in learning environments involving digital resources.
- Long-term research is needed regarding the impact of early algebraic thinking on students' later study of algebra.

- A presently underdeveloped area of research includes a focus on number puzzles with their surprising, always-the-same results (e.g., think-of-a-number puzzles), which can serve to motivate the introduction of alphanumeric symbols as a tool for justifying the surprising results of such puzzles, and for proving in early algebra more generally.
- More research is needed that describes explicitly the nature of the instructional interventions that are found to be pivotal to developing the different dimensions of students' early algebraic thinking.
- The relationship between arithmetic thinking and algebraic thinking remains an area where further research that better differentiates the two is warranted.
- While an abundance of research exists on seeing relations, properties, and structure within the numerical domain, there is comparatively little at the primary school level on the development of structural thinking related to the transformation of alphanumeric expressions within algebraic equation solving.

It is hoped that the story told by this synthesis of research, with its particular theoretical framing and descriptions of related empirical work on the multi-dimensionality of early algebraic thinking, both reflects the dynamism of the research being carried out in this exciting field and proves useful to members of the early algebra community and beyond.

Acknowledgments Photo credit: Photo of the Al-Khwarizmi monument by Nick Stooke.

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