ORIGINAL PAPER

Seeds of algebraic thinking: a Knowledge in Pieces perspective on the development of algebraic thinking

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Abstract

In this paper, we elaborate the *seeds of algebraic thinking* perspective, drawing upon Knowledge in Pieces as a heuristic epistemological framework. We argue that students' pre-instructional experiences in early childhood lay the foundation for algebraic thinking and are a largely untapped resource in developing students' algebraic thinking in the classroom. We theorize that seeds of algebraic thinking are cognitive resources abstracted over many interactions with the world in children's pre-instructional experience. Further, we provide examples to demonstrate how the same seeds of algebraic thinking present in early childhood can be invoked in reasoning across contexts, grade levels, and different levels of formality of algebraic instruction. The examples demonstrate how the seeds perspective differs from other accounts of the relationship between children's early activity and their engagement in algebraic reasoning processes. We anticipate this new theoretical direction for characterizing the nature and development of algebraic thinking will lay the foundation for a robust agenda that sheds light on the development of algebraic thinking and informs algebra instruction, particularly how teachers notice and respond to children's developing algebraic thinking.

1 Introduction

Children come to school already with years of experience living in the world. Many of the facets of algebraic thinking that we value—for example, recognizing patterns and expressing relationships generality—have plenty of recurring opportunities to develop in children's daily lives. Children's early play experiences and language development are intimately linked with seeking out patterns and fitting them into larger systems of understanding of the way the world works. These observations give way to the natural question: What implications do these insights around children's early experiences have for mathematics education, and algebra learning, in particular?

For decades, algebra has held a critical role in school mathematics. Not only is it a prerequisite for advanced math and science courses, succeeding in algebra is necessary for access to many STEM career opportunities, giving the subject an enormous amount of power and consequence

(Moses, 1995; Steele et al., 2016). Historically, formal algebra instruction has emphasized symbol manipulation with instruction commencing at adolescence once a child has fluency in arithmetic and number operation. Algebra courses have traditionally privileged the formal manipulation of symbols, focusing on "unknowns," "variables" and "solving equations." This has not been without issue, as children have struggled in traditional algebra classes for a number of reasons, including its abstract nature and the difficult transition from arithmetic to algebra.

An approach to increasing both access to and understanding of algebra has been to reconsider the assumptions underlying the instructional approaches described above. Do children need to wait until adolescence to begin to think algebraically? Do they need to master arithmetic before they can think algebraically? Is algebra inherently more "abstract" than students' prior ways of thinking about mathematics?

In the past few decades, the notion of "algebraic thinking" has been extended to include ways of thinking that are less reliant on formal symbols and symbol manipulation, including, for example, competences related to generalizing, formalizing, and modeling (Carraher et al., 2008; Ellis, 2011; Lannin, 2005; Strachota et al., 2018). A significant body of work documenting young children's productive

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engagement with this broader set of algebraic competences has convinced the field that "algebra" instruction need not be postponed until middle or secondary grades (Bodanskii, 1991). This work has led researchers to study the impact of early algebraic experiences in elementary grades (e.g., Ayala-Altamirano & Molina, 2021; Blanton & Kaput, 2005; Blanton et al., 2019a, 2019b; Cai & Knuth, 2011; Fonger et al., 2018; Kaput et al., 2017; Mulligan et al., 2020; Stephens et al., 2017, and many others), demonstrating that these experiences can support children's future learning of formal, symbolic algebra and thus ease the historically difficult transition from arithmetic to algebra.

While there is consensus among early algebra researchers that it is beneficial to introduce algebraic ways of thinking such as generalization and symbolization earlier, there are differing approaches to when and how algebraic thinking should be introduced. One approach is to use children's existing knowledge of arithmetic to introduce algebraic ideas. For example, we can think about generalizing arithmetic and using quasi variables to explore relational thinking (e.g., Fujii & Stephens, 2001). Another way researchers have thought about introducing children to algebra is to highlight patterns and principles in reasoning with arithmetic operations in order to support thinking in both arithmetic and algebra (e.g., Carpenter et al., 2003; Jacobs et al., 2007). Lastly, yet another school of thought is that algebra structures can serve as a foundational basis from which to learn arithmetic. That is, proponents of this position advocate introducing children to algebraic ways of thinking, grounded in experience with continuous quantities, before arithmetic (e.g., Schmittau, 2011). Our position is that the existing approaches that focus on the transition from one school topic to another (e.g., arithmetic to algebra), or even the interplay between school math topics more generally, background the important role that pre-instructional experiences may play in developing competent algebraic reasoning.

Outside of early algebra research, there is a growing interest in the role that children's pre-instructional experiences might play in their later learning of mathematics. Research on the mathematical potential of children's pre-school play activities (Ginsburg, 2006; Helenius, et al., 2016; Mulligan & Vergnaud, 2006; Parks, 2015; Wager & Parks, 2014) introduces a wealth of pre-instructional mathematical experiences that children have available to build upon in learning school mathematics related to number, operations, spatial relations and patterns. Coming from a different perspective, research on embodied cognition similarly demonstrates that even babies and young children are developing important mathematical resources through their embodied experience (Dehaene et al., 1999; Lakoff & Núñez, 2000; Smith & Gasser, 2005).

Thus, synthesizing these important findings across fields is necessary: (1) Early algebra researchers have demonstrated that it is both possible and productive to spread out the learning of school algebra, allowing access to cross-cutting ideas of algebra to all students, and at younger ages, and (2) Embodied cognition and early childhood researchers have demonstrated that there is a wealth of pre-instructional mathematical experience that children have available to draw upon and that some of the same ideas stemming from this early embodied experience play a role in later forms of mathematical thinking. To advance this line of drawing upon the work in embodied cognition and pre-instructional experiences to inform the way we think about the development of algebraic thinking, the field needs a better understanding of both the nature of pre-instructional resources that could be drawn upon in school algebra instruction and the ways these resources may manifest themselves in later forms of algebraic reasoning.

In this paper, we elaborate on the research agenda we described in Walkoe and Levin (2020), aimed at conceptualizing the nature of algebraic thinking, its development, and its relationship to children's pre-instructional algebraic ideas. The perspective we develop conceptualizes larger concepts of algebra as being built from sub-conceptual pieces that have their roots in the pre-instructional activity of children. The sub-conceptual knowledge elements may themselves be used across a wide variety of contexts, including, but not limited to, algebra. To illustrate this, we will provide examples of three seeds: covariation schemes, inbetweeness, and closing-in and the contexts that may elicit them. Then, to speak to the role of seeds in the development of algebraic reasoning, we provide examples of students' reasoning that invokes these seeds across a variety of contexts and levels of formalization. We will close with a discussion of how these ideas are being used in teacher education and how we envision the practical implications of the shift in perspective we offer.

2 Developing a Knowledge in Pieces perspective on algebraic thinking

Our objective is to elaborate our theoretical perspective on the nature and development of algebraic thinking that provides guidance for recognizing potentially productive cognitive resources for algebraic thinking in children's early experiences. We call these resources *seeds of algebraic thinking* (Walkoe & Levin, 2020). Before we turn to discussing seeds, we provide background on Knowledge in Pieces (diSessa, 1993; diSessa et al., 2016), the general perspective on knowledge and learning that undergirds our perspective and guides our theorizing.
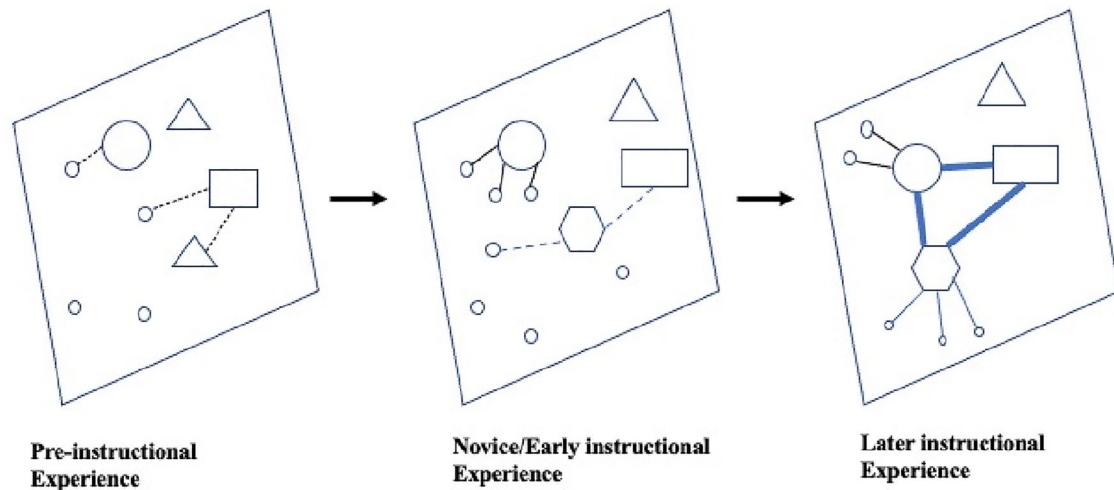
Pre-instructional Experience → Novice/Early instructional Experience → Later instructional Experience

Fig. 1 Tuning toward expertise

2.1 Knowledge in Pieces

Knowledge in Pieces (KiP) is a heuristic epistemological perspective that models knowledge use and development as a knowledge system of elements of diverse form and function. KiP grew out of Piagetian constructivism, taking from it the fundamental assumption that new knowledge is built out of old. It also has roots in cognitive modeling, sharing its commitment to building runnable models of real-time cognition. In this view, learning is a system-level phenomenon of improving how a knowledge system functions. That is, through experiences that provide feedback, such as interactions with peers, teachers, or computational designs, learners come to recognize resources as appropriate for a context or not. Figure 1 illustrates a knowledge system beginning in an initial organization. It includes many different kinds of knowledge elements (represented by the square, circle and triangle, as well as dots). In gaining experience and feedback, the system is re-organized. Some of the same elements are in the re-organized system, but they are connected to each other differently. Some of the linkages between elements become stronger in this system and even new elements are part of the system. Finally, with even more experiences and feedback, the system undergoes further reorganization. Again, several elements remain, but are linked differently, and in some cases more strongly. Importantly, there are some elements that are no longer activated as part of the system. This illustrates a core point that initial and more conceptually competent knowledge systems can share some of the very same elements, but the organization is key to distinguishing reasoning patterns.

Initial work developing the KiP perspective focused primarily on abstractions of experience interacting in the physical world, which led to the articulation of the construct of phenomenological primitive (p-prim) (diSessa, 1993). P-prims are sub-conceptual knowledge elements, the function of which is to account for individuals' expectations about how the physical world works. P-prims are evoked as a whole and are "self-explanatory," accounting for comfort in one situation and surprise in another. A large corpus of p-prims was empirically identified, including discussion of their source, developmental history, encoding, and local processes around their invocation and use. A prototypical example of a p-prim is Ohm's prim, that schematizes the phenomenological experience that exerting more effort results in more effect (and more resistance leads to less effect).

This formulation of the organization of knowledge has strong implications for instruction. For example, diSessa (1996) used the knowledge systems perspective to illustrate how physics subjects' multiple explanations of a ball tossed vertically—some normative and some exhibiting a classic "misconception"—could instead be modeled as being comprised of multiple productive p-prims used in concert and being differentially activated based upon what the participant was perceptually focusing on. Without the Knowledge in Pieces perspective, such reasoning was labeled a "misconception" and the instructional implication would be to replace it with correct reasoning about the context. Thus, instead of replacing "misconceptions" about how the world works, instruction in formal physics needed to engage individuals' intuitive knowledge and re-contextualize it more appropriately for given contexts (Smith et al., 1994).

Though KiP has its roots in intuitive physics, over the past 30 years the perspective has guided studies in an increasing number of domains across STEM disciplines and beyond. Within mathematics, KiP has been used in studies of statistical reasoning (Abu-Ghalyoun, 2021; Kapon et al., 2015; Wagner, 2006), calculus and linear algebra (Adiredja,

2021; Adiredja & Zandieh, 2020; Jones, 2013), modeling physical situations with variables (Izsák, 2000), students' understanding of multiplication (Izsák, 2005), pre-service teachers' developing understanding of proportional reasoning (Izsák et al., 2021), and developing knowledge of how to control linear variation to solve word problems (Levin, 2018). Across this research, common themes emerge that include: (1) intuitive knowledge based on experience in a domain provides rich resources for learning and (2) the activation and use of knowledge resources depends sensitively on context. More detail on the landscape of KiP studies, as well as principles and counter-principles for studying knowledge in a way that is consistent with Knowledge in Pieces is given in diSessa et al. (2016) and diSessa and Levin (2021).

2.2 Seeds of algebraic thinking

Guided by Knowledge in Pieces as a heuristic epistemological framework, we conjecture that algebra-relevant knowledge is a large knowledge system composed of cognitive resources gained through experience. We term these resources *seeds of algebraic thinking*.

Seeds of algebraic thinking are heterogeneous in form— some seeds of algebraic thinking are available for children to reason with simply because of their experience in human bodies—that is, embodied and image schemata developed through repeated sensorimotor experience are an example of what we consider a seed of algebraic thinking. However, we also consider seeds to be patterns of thinking that come from children's repeated experiences in the world, and that are not necessarily embodied in origin. In both cases, the function of these cognitive resources is to help children make sense of future experiences based on their prior experiences.

Seeds of algebraic thinking are resources that are involved in reasoning about algebraic structures, ideas, and concepts. In this view, what we observe and label as "algebraic thinking" or "proto-algebraic thinking" is conceptualized as an ensemble of more fine-grained cognitive resources, including seeds of algebraic thinking. Seeds of algebraic thinking are not inherently "correct" or "incorrect," but instead, their activation and use is appropriate or not in a given context. As one gains more experience their ideas of what is productive or not in a situation becomes strengthened. As described in the previous section in Fig. 1, this process is called *tuning towards expertise*. Development of algebraic thinking over time is a process of refining and reorganizing one's knowledge system, in order to more appropriately activate and use relevant cognitive resources (seeds).

To summarize, seeds of algebraic thinking have the following properties:

1. *Formed in early experience* Seeds are patterns of thinking that are abstracted over many interactions with the world and others in children's early experience, *prior to formal instruction.*
2. *Small in Grain-size* Seeds are "sub-conceptual" in grain size and thus a different form than earlier forms of school algebra ideas.
3. *Contextually Recognized* Seeds, as patterns of expectation based on experience, are neither right or wrong propositions in and of themselves. The appropriateness of their recognition and use in a given context is what confers validity.

A key aspect of our perspective on the development of algebraic thinking is that we consider cognitive resources of a sub-conceptual grain-size to be important, both empirically and theoretically, in conceptualizing the nature of algebraic thinking and its development. That is, the seeds that we identify are relevant for algebraic thinking (and thus they are called "seeds of algebraic thinking"), but these same cognitive resources are themselves not fundamentally algebraic in nature. The way resources are coordinated in larger chains of reasoning are where we see the "algebraic" character as visible. We illustrate this in the next section.

2.3 Seeds of algebraic thinking: co-variation schemes, in-betweeness, and closing-in

We introduce three specific seeds of algebraic thinking: covariation schemes (in particular *same direction change*: increase in X results in increase in Y), *inbetweeness*, and *closing-in* that are involved in the episodes of reasoning discussed in the next section.

A co-variation scheme (described in Levin, 2018) is a knowledge element that functions to help an individual predict the effect of controlling a "cause" (or an independent variable). Relevant features of this knowledge element are (1) input, (2) causal link from input to output, (3) resulting output, (4) hypothetical change in input, (5) resulting change in output given the hypothetical change in input. As an example of a real-life context which could be structured by a co-variation scheme, consider the situation of one asking oneself "How long should I pour in order to fill a glass up with water?" In this example, time is the input that one has control over, the relationship between time and the height of the liquid is given by a linear function (assuming one is pouring at a constant rate). The water level rises as one continues to pour. One can either use qualitative descriptions like "I need to pour a lot longer because the glass has been filling up slowly" or one could alternatively determine the exact amount the liquid level goes up per unit time.

Knowledge resources related to covariation can be traced to early childhood, in particular in situations in which children need to predict the effect of controlling a cause (Buchanan & Sobel, 2011). Children encounter many such

situations in their pre-school experience. Consider the everyday situation of a child watching a caregiver fill a tub with water for their bath. The longer the water runs, the higher the collected height of the bath. In another context, when playing with a stringed instrument, the child notices that in order to tune up the strings, the pegs or fine tuners need to be turned. As they are tightened, the pitch increases (and as they are loosened, the pitch decreases). Examples across a wide variety of experiences have been documented in investigations into phenomenological primitives (More effort begets more result; diSessa, 1993) and intuitive rules (More A—More B; Stavy & Tirosh, 1996). Notice that co-variation schemes, and same-direction change in particular, are neither correct nor incorrect as a knowledge resource. Right/wrong as an assessment depends on the appropriateness to the context of application. For example, the belief that turning a thermostat up higher will heat the room quicker (Kempton, 1987) would not be an appropriate situation to invoke same-direction change. Covariation and change are themes that extend well into a variety of later mathematical topics.

The second seed that we discuss is called "inbetweeness." Inbetweeness is a knowledge element that helps organize thinking about the location of a target value in between two other known values. Opportunities for experiencing situations that activate this pattern of thinking abound, going back even to young children's spatial structuring of experience via image schemata (Lakoff & Núñez, 2000). Inbetweeness is linked to the development of a basic sense of continuity of existence (e.g., If I am in my room right now and later I find myself in another bedroom, I know that at some point, I was in the hallway). A version of this seed includes the tale of Goldilocks and the three bears in which Goldilocks is searching for porridge that is the correct temperature to eat—she first finds daddy bear's porridge to be too hot, then mommy bear's to be too cold, and finally she finds the baby bear's porridge to be just right. An opportunity for the activation of a continuous version of this same seed in pre-school children's experience is the activity of measuring (e.g., growth of plants, children's height charts on their walls). If a child is one height at the beginning of the summer and then they are measured again at the end of the summer, you know there were points during the summer at which they took all of the intermediate height values. Eventually the idea of inbetweeness can be formally expressed as the Intermediate Value Theorem.

The third example of a seed that will be highlighted in the data we discuss is "closing in." Closing-in describes the strategy of purposefully creating bounds in order to narrow in on a goal. Closing-in is related to, and is often used in conjunction with the seed listed above, inbetweeness. Examples of closing-in in early childhood include playing "getting hotter—getting colder" type games, tuning an instrument

(iteratively tightening the pegs too much and then too little, until the sought after pitch is attained), and modulating effort when playing target-hitting games such as launching marshmallows into hot cocoa mugs with homemade catapults. Later out of school examples of this include trying to find a location (a park or friend's house), realizing you've gone past it and then correcting and going in the opposite direction. Doing this allows you to establish an interval on which you know the location must be. To find the location, you can make the interval narrower and narrower until you find it. This idea is related to many ideas and approaches in later mathematics, including Newton's algorithm for approximating roots of polynomials. We note that the difference between inbetweeness and closing-in, is that closing-in is a strategy that one uses and in doing so invokes the idea of inbetweeness. In-betweeness can be invoked without a particular goal of successively zeroing in on a target.

These examples allow us also to underline the heterogeneous nature of seeds—in characterizing algebraic thinking in terms of seeds, we are interested in identifying patterns of thinking that have a productive history of use and get invoked in later formal reasoning processes. There is no defining structural feature common to all seeds. In all cases, we emphasize that seeds are patterns of thought that emerge from repeated experiences. It is not necessary to have a given set of experiences to develop a particular seed. As we have described above, the normal experience of children provides an ample basis for developing covariation schemes, inbetweeness, and closing-in as patterns of thought.

3 Three examples illustrating seeds and their coordination into chains of reasoning

We now illustrate our perspective on the character and development of algebraic thinking with three examples. These examples come from our prior work with student interviews and classroom observations. The data from the first and second examples have been discussed in our prior work (Levin, 2018; Weintrop et al., 2020), but with a different focus. The first example comes from an elementary classroom where two children are programming a small robot to go a given distance. The second example comes from an interview with a middle school student about a formal algebra problem (though the student has not been taught formal algebraic methods of solving the problem yet). The third example comes from a corpus of student reasoning in middle and secondary algebra classrooms collected as part of a study of the second author on teacher noticing of algebraic thinking. In this vignette, students are analyzing a graph of a parabola and trying to find the location of the vertex. Across the three examples, we see three seeds of algebraic thinking arise in

different contexts, with slightly different organizations. We also see, in the second example, an extension that could be interpreted as bridging "arithmetic" and "algebraic" problem solving approaches. Our goal in sharing these examples is to highlight how analyzing student thinking in terms of smaller-grained knowledge resources (seeds), that have their roots in pre-instructional activities, can illustrate the continuity between contexts (formal and informal) as well as approaches that would be considered "arithmetic" and "algebraic." We argue that in order to observe the transformation to the "algebraic" approach it is necessary to show a finer-grained description of student knowledge and reasoning, making the point that learning can be productively modeled as a process of reorganization and refinement of knowledge systems.

3.1 Programming a Sphero robot

Our first example comes from a fourth grade curriculum, developed by the second author and colleagues, as part of an on-going project whose goal is to integrate computational thinking (CT) and mathematics in elementary mathematics classrooms (Weintrop et al., 2021). The children in this example, Evan and Blake (both aged 9), are working on a class activity that is part of a larger exploration. In this exploration, the class was tasked with programming a small robot (www.sphero.com; Fig. 2) using block programming on an iPad or iPhone, to complete an obstacle course created by their teacher. The objectives of the activity were for children to use CT competencies (e.g., decomposition, abstraction) and mathematical ideas (e.g., proportional reasoning) to program the Sphero to run an obstacle course using only a scaled down map (Fig. 3). The course was laid out on the classroom floor and made up of cardboard and tape along with other classroom materials (Fig. 4). The students were not allowed to "test" aspects of their program on the actual course before submitting their full program.

The obstacle course was created over carpet with a grid of rectangles that were 24 in by 21 in. The first task Evan and Blake worked on was to figure out how to program Sphero to move the length of one rectangle (24 inches) exactly, using the roll command (Fig. 5).

The roll command includes three variables. The first is the angle (direction) of the roll from a calibrated "forward" direction. The second is the speed, which is a non-standard unit that can be set from − 255 to 255, and the third is the time to roll, in seconds. Evan and Blake decided to keep both the angle (0°) and speed (50) constant and vary only the time traveled to see what time value corresponded with rolling exactly 24 inches. Blake was holding the iPad and adjusting the program while Evan was on the floor with the Sphero and the ruler. They started with a time value of 0.7 s. When the students ran this command, the Sphero rolled past the 24 inch mark on the yardstick by about 10 inches. Blake reacted with surprise at how far the Sphero rolled past their goal. Evan suggested they try 0.5 s, but Blake suggested maybe they try 0.3 s. Blake's reaction to the amount they were "over" may have prompted him to put a bid in for a shorter time. Blake plugged in 0.3 s and the Sphero rolled just 5
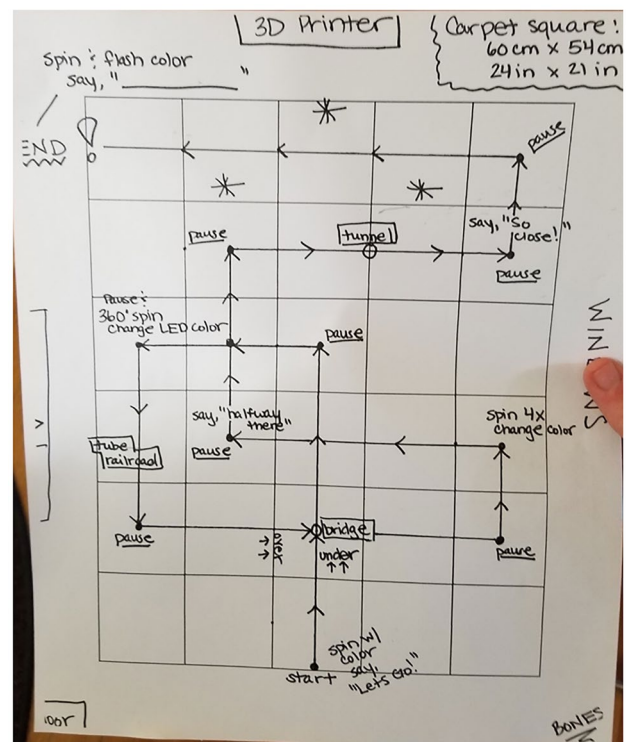

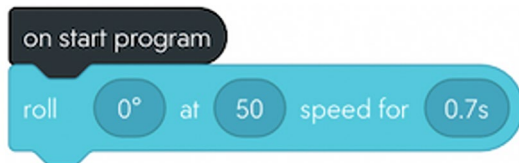**Fig. 3** The map

Fig. 4 The obstacle course

Fig. 5 The roll command

inches (about 19 inches short of their goal). Blake immediately adjusted and said "OK, 0.5," realizing that 0.3 s was way too short. Blake entered 0.5 and this time the Sphero rolled about 21 inches, 3 inches short of the 24 inch goal. Evan suggests 0.6 s for the next try, but reconsiders and both students agree to try 0.57 s. Evan comments, "Because I know 0.6 will go too far, probably." The students realize that 0.3 s fell *way* too short, 0.5 s fell a few inches short, and 0.7 s fell *way* too far. They use that information to arrive at the value 0.57 s. This value caused Sphero to roll just short of 24 inches (23.5 inches). Blake thought maybe that would be close enough, commenting, "Oh yeah, [the teacher] would give us that." But Evan pushed back, "No, that's like an inch less… because if we kept going like that, then… it would um…" Here, Evan recognized that "a little short" would add up over time, as they programmed Sphero to run the obstacle course. Blake agrees, and suggests 0.573 s. With this time

value, Sphero rolled exactly 24 inches, to which Evan commented, "Right on the dot… line, I mean."

Here we see the students reasoning in what at first might look like simply "guessing and checking." However, we argue they are engaged in more purposeful sense-making. First, they realize their first guess goes past the goal, by 10 inches, and choose their second guess based on that, ultimately choosing one that falls short by a good deal, almost as if they are compensating. Then they try numbers that slowly converge on the exact value they need, to the thousandths of a second. The students are using two seeds, *inbetweeness* ('If a guess for an input is too low and another is too high, then the true value should lie in between these two values.') and *closing-in* ('if the guess is a lot too high/ low, then I need to adjust the guess to lessen that distance). We also see the students using the "*same-direction-change*" covariation seed. That is, Blake and Evan understand that raising the time value causes Sphero to roll farther (same-direction-change). As they engage with Sphero, they gain a better sense of *how much* change in distance is determined by smaller and smaller changes in time. We see the students working to understand *how* the change in amount of time corresponds to the distance rolled. Eventually they come to realize that changing the time by just hundredths and even thousandths (their last guess) of a second will help them fine tune the distance Sphero will roll enough to arrive at the exact value they need. As they experiment, the same-direction-change covariation seed is becoming more quantified for them. While we wouldn't argue that the children are engaged in formal algebra, we see seeds of thinking that we also see in later, more formal algebraic reasoning, as our next example illustrates. In the next example, we see the same-direction-change covariation seed become even more quantified.

3.2 Algebra word problems

The empirical context for our second case is a study of the emergence of a novel mathematical strategy of a pre-algebra student, Liam (age 12, grade 7), who largely independently constructed a deterministic and essentially algebraic algorithm for solving algebra word problems prior to instruction in algebra (see Levin, 2018 for details on the full arc of invention; we summarize only the starting and ending points here). Liam's strategy gradually emerged over the course of his work on several problems in the context of semi-structured sessions with a tutor/researcher. The goal for the sessions was to investigate how introducing children to using a chart form might support transitioning from spontaneous informal approaches, such as guessing and checking, to modeling and solving problems with algebraic symbolism. What was striking about the case of Liam was that,

instead of moving in the direction of algebraic symbolism, the use of the chart form instead also supported him in making observations about quantitative relationships between his trial values that resulted in his inventing an algorithm that he recognized as general. This case offers the opportunity to see the same student activating a collection of seeds of algebraic thinking (same-direction-change, inbetweeness, and closing-in) and then with time and experience, coming to coordinate them in a more sophisticated way.

Liam's approach at the beginning of the study was based simply on "guessing and checking" trial values. Early in the sessions, Liam solved the following problem: "The base of a rectangle is three more than twice its height. If the perimeter is 60 inches, what are the base and height of the rectangle?" This is the first problem he solved after the tutor-researcher suggested organizing his approach of spontaneous guessing and checking (exhibited on a baseline problem) into a chart form. Liam's initial strategy involved choosing trial values that resulted in outputs that were closer and closer to the target value (closing-in). This assessment is supported by inbetweeness when Liam notices that his trial value for a guess of 10 inches for the height results in a perimeter that is too high and a guess of 8 inches results in a perimeter that is too low, so he concludes that it must be 9 inches (assuming the problem has an integer solution). In making his choices for next guesses, he noted whether a specific trial value was "too high" or "too low" relative to the target value of the perimeter of 60 inches (consistent with same-direction-change "If I increase the input, the output increases").

In Liam's later work, he had constructed a linear interpolation algorithm, using the outputs of two trial values to explicitly determine how much variation from one trial input would be needed to achieve the solution to the problem. That is, when Liam encountered the task "Anne is twice as old as Paul. Bill is five less than Anne's age. Together, Anne's and Bill's ages sum to 147. How old are Paul, Bill, and Anne?" His later strategy (Fig. 6) involved quantifying how much too high/low each trial value was relative to the target value and concluding that a change in input of 5 resulted in a change of output of 20, so each change in input

by one was worth a change in output of four. He determined that this implied he would need to increase the input by 3 in order to achieve a change in output of 12 (increasing output from 135 to 147, thus solving the problem). Liam was aware of the generality of his approach and spontaneously used it on other problems.

While indistinguishable in written form, and relying upon similar seeds of algebraic thinking, the thinking processes behind Liam's initial and later approaches have a very different character with respect to algebraic thinking and require attending to and extracting different kinds of quantitative information. For instance, the first approach, while purposeful, is responsive to the specific numbers in the given problem. The second approach is a general algorithm that works for an entire class of problems (those that have an underlying linear structure).

3.3 Locating "the bounce" of a parabola

Our final example illustrates an additional context in which the seeds of same-direction-change, inbetweeness, and closing-in are activated. This example comes from a ninth-grade formal algebra classroom (students aged 14–15), where the students are engaged in reasoning with standard symbolic and graphical forms of quadratic functions. The students in this vignette were discussing the graph of the function, given the following: the equation $f(x) = x^2 + x$ and the points (3, 12), (− 3, 6), (5, 30), and (9, 90). One student presents her graph, obtained by connecting the given points (Fig. 7).

The student presenting, Jesse, reports that she thinks the graph is supposed to be a parabola, based on the equation, but that she wasn't sure how to make the graph look like a parabola. The teacher poses the question to the class, and another student, Cathy, explains that Jesse connected the
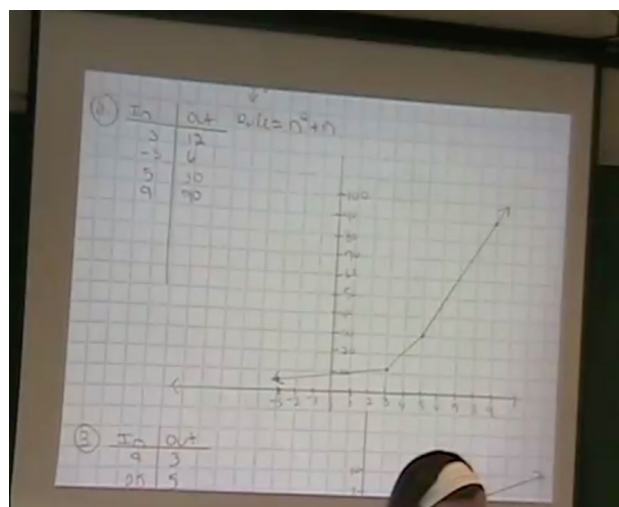

Fig. 6 Liam's linear interpolation strategy
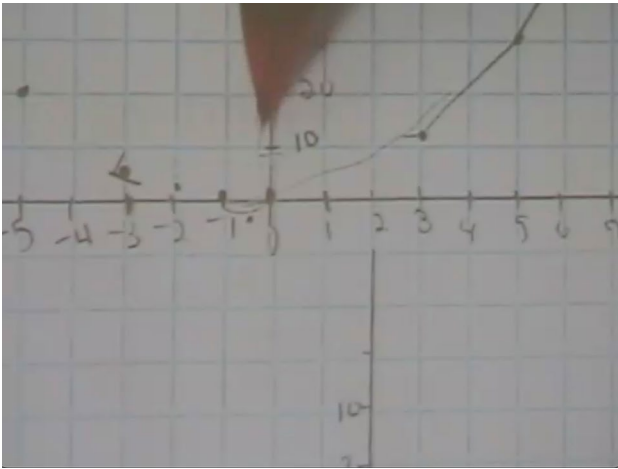

Fig. 7 Student's initial (incorrect) graph

Fig. 8 Corrected graph with approximation of vertex

points (3, 12) and (− 3, 6) but that the graph would need to "dip" down somewhere between those points. Students in the class discuss where the dip, or, what they rename "bounce" [the vertex of the parabola] might be. One student, Maggie, comments that the bounce wouldn't be at the origin. She suggests that the bounce would be "lower" around $x = -0.2$ or $x = -0.3$. Another student, Hannah, agrees that the bounce would be between integers and have a decimal value. The teacher asks Jesse how she could "enhance her graph." Jesse suggests trying more values. She computes the output value of the function for an input of $x = 0$, and plots a point at the origin. She then tries other negative x values and sees that the graph keeps going up to the left. Another student, Alexis, suggests Jesse try "[the x coordinates] − 1 or − 2… so it [the y values on the graph] goes lower." Jesse plots (− 1, 0) and (− 2, 2). Maggie offers, "Well, since [the graph] crosses the x axis at both [x coordinates] − 1 and 0, that means that dip is somewhere in between (*holds up her first finger and thumb to indicate an interval*). It goes below zero, on the y-axis at like, − 0.5 (*draws a U shape with her finger*)." The class continues to discuss and eventually Maggie comes to the overhead with Jesse and sketches the parabola, with the vertex around $x = -0.5$ (Fig. 8).

In this example, students are using inbetweenness to see where the x value of the vertex might be located. Maggie uses in-betweenness when she realizes that because the y values of $x = -3$ and 3 are not the same, that the vertex must not be at $x = 0$. She goes on to conclude that, based on the corresponding y values, the vertex must be at an x value below $x = 0$. In-betweenness is activated again for Hannah, who agrees that the vertex must be between two integers. When Alexis sees the trend of the graph as the x values get lower, she uses closing-in to suggest Jesse try the x coordinates of − 1 and − 2. Based on the y values at $x = 0$ and $x = -1$, Maggie coordinates in-betweenness and closing-in

to conclude that the vertex must be at an x value between 0 and − 1.

3.4 Discussion across the three cases

For the purposes of developing our perspective on early algebra and the development of algebraic reasoning, our goal in selecting the three examples in the previous sections was to demonstrate how the same set of knowledge resources—that have their roots in early childhood experience—can be reused and repurposed across contexts and levels of experience with school algebra. We observed the same seeds: same-direction-change, inbetweeness, and closing-in, elicited across all three contexts we explored (programming a Sphero robot, solving algebra word problems, and coordinating information about a quadratic function to find the vertex of its graph). This contrasting activation of the same knowledge resources underlines one of the main points across our three cases: knowledge resources from early experiences get incorporated into later reasoning processes. As in Fig. 1 (that describes the process of "tuning towards expertise"), the same knowledge resources can be present both in initial and more expert reasoning processes. One of the most significant kinds of changes in increasing sophistication is learning what to attend to in a problem situation. Shifts in attention can spur the creation of new conceptual categories and relations, which can lead to novel approaches that use the same set of knowledge resources in new ways.

4 Discussion

In this paper, we expanded on our work on developing the seeds of algebraic thinking perspective. We articulated the origin of our framework, informed by the Knowledge in Pieces epistemological framework, conceptualizing algebraic thinking as a complex system that consists of sub-conceptual pieces (seeds) gained through life experiences that get refined and reorganized as children gain formal algebra expertise. We underlined that important knowledge elements involved in algebraic reasoning processes are not subject specific, and appear across algebraic contexts. Additionally, they are not experience-level specific, meaning the same knowledge elements can show up in both novice and expert level algebraic thinking. That is, the "seeds" that we observe in children's activity are *also* used in more competent algebraic reasoning processes, even by adults.

It is important to note the boundaries of what we theorize. We do not expect that all reasoning processes can be decomposed into *only* seeds of algebraic thinking. There are many other knowledge structures relevant in describing the moment-by-moment algebraic reasoning processes of individuals, many of which certainly come from students'

experiences with school mathematics and conventions. Instead, our claim is that understanding shifts towards increasingly algebraic reasoning processes does not necessarily need to involve knowledge of a different form than prior forms of reasoning.

In further developing this line of work, there is a need for both additional work to identify seeds of algebraic thinking and work to trace how seeds become incorporated into later reasoning processes. We project the need to work from both directions to elaborate the seeds perspective further including (1) studying children's early childhood experience through the lens of seeds of algebraic thinking and (2) studying algebraic thinking processes at later points in their development to illustrate the coordination of seeds into chains of algebraic reasoning.

With respect to the goal of uncovering seeds of algebraic thinking in early childhood experience, one approach is to engage in ethnographic studies of children's activities and family life outside of school (for example, see the Early Learning Across Contexts project, Keifert & Stevens, 2019, or the work of Helenius, et al., 2016 who studied the features of preschool children's playful situations). We propose that viewing children's activities through the lens of "algebraic habits of mind" (Cuoco et al., 1996; Driscoll, 1999) could be a useful heuristic for finding seeds of algebraic thinking in ethnographic data. Examples of algebraic habits of mind could include doing/undoing (using a process to get to a goal and being able to run the process backwards), recognizing patterns and organizing data to represent situations in terms of relationships between inputs and outputs, abstracting from specific instances. These habits of mind, like seeds, are found in multiple algebraic contexts and beyond. Prior work on habits of mind has focused on the places in the school curriculum where algebraic habits of mind can be developed. We conjecture that the perspective could be usefully extended to observing children's pre-instructional experiences in order to identify examples of seeds of thinking that one may expect to be drawn upon in later algebraic thinking processes. For example, the habit of mind "doing and undoing" is active when getting dressed and undressed, loading and unloading a car, and when playing musical scales.

4.1　Strengths and limits of the seed metaphor

We have found the metaphor of calling students' pre-instructional knowledge resources "seeds" to be helpful in communicating key ideas about the nature of student thinking that we want to draw attention to. For example, the idea of "seeds" foregrounds the idea that students' early experiences are potentially productive for later learning, the grain-size that we are interested in is very fine (much smaller than what are typically identified as algebraic concepts), and that

therefore, the student thinking we observe may look quite different in the earlier forms than what it "grows" into later. That said, as with any metaphor, it has limits that should be acknowledged. For example, in our systems view on learning as tuning knowledge systems towards expertise, the same knowledge elements that are productive early on, can also participate in later chains of more refined and algebraic reasoning processes. Physical seeds would not literally work this way.

5　Implications for instruction

The field has turned towards the development of early algebra curricula (e.g., Blanton et al., 2015; Blanton et al., 2019a, 2019b), and we hypothesize that our seeds perspective can help support teachers' practice as they engage with early algebra lessons. We believe that teachers' awareness of the seeds perspective—that students have sub-conceptual ideas gained from experience that can be drawn upon in formal algebraic reasoning—can help them teach in ways that leverage students' prior knowledge.

We do not expect teachers to delve into the theoretical aspect of seeds, nor do we imagine hunting for seeds will be productive for teachers. Instead, what we believe is that teachers, with this perspective, may shift how they engage with student thinking in the classroom. In particular, we imagine our perspective can support teachers in attending and responding to students' thinking, as they engage with early algebra in formal contexts. The ideas a teacher attends to and how she interprets them have implications for classroom instruction (e.g., Luna & Sherin, 2017). For example, if a teacher only attends to fully-articulated, correct (or incorrect) student ideas, with an evaluative lens, different learning opportunities will be available than if a teacher attends to partially-articulated, nascent student ideas, with an exploratory lens. The latter offers opportunities for multiple entry points into the discussion, and for more students to engage.

In pilot work with teacher video-based professional development (PD), we have early evidence that the seeds perspective does impact the ways teachers engage with student thinking. Teachers viewed videos of students in mathematics classrooms and tagged moments of interesting or productive student thinking (Walkoe et al., 2020). After teachers were introduced to the seeds framework, they shifted to identify fragments of ideas, rather than more fully-formed conceptions, as ideas worth discussing. Additionally, teachers worked to understand *where* the idea may have originated, based on the students' life experiences, rather than commenting on the correctness of the idea. In some cases the idea was not productive or not correct in that context, and instead of focusing on what the correct answer would be, teachers worked to understand where the student might have

gotten the idea or how the idea could seem correct to the student in that context. In addition to looking at *how* teachers participated in video PD, we conducted interviews with teachers, in which we introduced them to the seeds perspective and asked them about student algebraic thinking. In one teacher interview, we saw the teacher shift from a focus on complete answers and symbol manipulation to a view that was more open to the potential of ideas to be algebraic, after learning about the seeds perspective.

6 Conclusion

Guided by Knowledge in Pieces, the seeds perspective that we elaborated in this paper, posits that algebraic thinking is a large complex systems phenomenon that involves the coordination of multiple types of knowledge resources, including cognitive resources gained through early experiences. From this perspective, the development toward formal algebra expertise is a process of refining and reorganizing one's algebraic knowledge system so that cognitive resources, including those formed in early experiences, are more consistently appropriately applied in particular contexts. The seeds perspective highlights the potential productivity of pre-instructional knowledge and experiences and has implications for how algebraic thinking is conceptualized by researchers and teachers. For early algebra researchers, the seeds perspective offers a way to broaden how the development of algebraic thinking may be conceptualized. For example, rather than conceptualizing the development of algebraic thinking in terms of student ideas moving through levels of increasing sophistication, we offer an alternate and complementary perspective in which learning is viewed as a process of restructuring existing ideas in new and more context-appropriate ways. In terms of the implications for teaching, early evidence from video-based PD using the seeds perspective with teachers reveals that adopting the seeds perspective helps teachers recognize and value nascent student algebraic thinking.

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