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ORIGINAL PAPER

Strategy creativity and outcome creativity when solving open tasks: focusing on problem posing through investigations

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Abstract

One of the well-known approaches to creativity differentiates between creative person, process, product, and press. In the study presented in this paper we focus on creative process and product associated with Problem Posing through Investigation (PPI) by experts in mathematical problem solving. We link the creative process to creativity of PPI strategies and the creative product to PPI outcomes (i.e., strategy creativity and outcome creativity). Furthermore, we draw a connection between the openness of tasks and their power for the evaluation of strategy creativity and outcome creativity, demonstrate the aptness of PPI tasks for the evaluation of both types of creativity, and examine the connections between them. The model for the evaluation of creativity that we used in this study, was initially designed and validated using analysis of problem-solving strategies when solving multiple solution tasks. We previously extended the model to evaluation of PPI outcomes, and we here demonstrate its implementation to evaluation of creativity of PPI strategies. To examine connections between creativity of PPI strategies and creativity of PPI outcomes, we focused on PPI by eight experts in mathematical problem solving who were members or candidates of the Israeli IMO team. We present empirical evidence for the distinctions between strategy creativity and outcome creativity, and for the connections between them. We analyzed strategy creativity as a unique characteristic of problem-solving experts. We found that higher strategy creativity does not necessarily lead to higher outcome creativity, and that a high level of strategy originality correlates with outcome flexibility. We conclude that creative product and creative process are two distinct characteristics of cognitive processing linked to creativity-directed problem solving.

Keywords Problem posing through investigations · Mathematical Creativity · Strategic creativity · Outcome creativity · Geometry proof problems · Mathematics expertise

1 Introduction

Nowadays the importance of creativity is unquestionable both as a means to an end of any educational process, and as the end in itself. The importance of creativity as a means to an end is rooted in vygotsky's (1930/1982, 1930/1984) axiom that creativity and imagination are among the main mechanisms of any learning process. This axiom implies that creative individuals are better learners since they have the power to use their knowledge in new situations flexibly and in an original manner, as well as to connect pieces of existing knowledge and skills with newly acquired information. Thus, development of creativity is essential for the development of better learners. As for creativity as an end in itself, it is almost trivial to claim that creativity is an educational goal since it is one of the central twenty-first century skills. Its essentiality is associated with the development that happens continuously in all areas of life as a result of exponential progress in technology and science (Griffin et al., 2012; Pellegrino & Hilton, 2012). These developments require individuals to be able to adapt to new environments, and to comprehend and implement novel ideas. Consequently, creativity has been identified as being of eminent importance for economic development, healthy psychological functioning, and academic success (Plucker et al., 2004).

Consequently, mathematics educators and researchers broadly recognize the centrality of the development of mathematical creativity (Amado, Carreira, & Jones, 2018; Leikin & Sriraman, 2016; OECD, 2019). However, due to the wide range of perspectives on creativity, there is still no consensus on the types of mathematical tasks that promote

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the development of creativity, and allow evaluation of creativity, or on which elements of creativity can and should be developed and evaluated. The focus of the works in this area varies among the following contrasts: divergent and convergent reasoning, flexibility and originality, associative reasoning and insight, and creative processes and products.

The study described in this paper is based on the observation that studies that analyze creative processes in general, and in mathematical activity in particular, are less common than studies that analyze creative products. Moreover, we find that the relationship between creative process and creative products in mathematics has barely been explored systematically. Thus, the goal of this paper is twofold. First, we draw theoretical distinctions between strategic creativity, reflecting a creative problem-solving process, and outcome creativity, which indicates creativity of product. To achieve this goal we used *problem-posing-through-investigation (PPI)*.

The PPI employed in this study is a mathematical activity in which participants are given a geometric figure and are required to pose geometry problems related to the figure and solve them (Leikin & Elgrably, ; Elgrably & Leikin, 2021). PPI is conducted in a dynamic geometry environment (DGE) through performing auxiliary constructions, measuring, conjecturing and proving or refuting the conjectures. The proven conjectures constitute new posed problems. As we explain later, PPI allows the evaluation of the creative components of process (PPI strategies) and products (the posed problems).

We implemented a model for the evaluation of creativity using Multiple Solution Tasks (Elgrably & Leikin, 2021; Leikin, 2009; Leikin & Elgrably, 2020). The model first suggested analyzing problem-solving strategies (i.e., process) and later was elaborated for the evaluation of creativity of PPI outcomes (Elgrably & Leikin, 2021; Leikin & Elgrably, 2020). In the current paper we demonstrate further elaboration of the model for the analysis of PPI strategies.

In the earlier study that employed PPI tasks (Leikin & Elgrably, 2015) we hypothesized that while discovery skills can be developed in people with different levels of problem solving expertise, the range of this development depends on the problem solving expertise. Using the thought experiment methodology, we suggested that the discovery process is rooted in previous problem-solving expertise. The research described in this paper is a part of the bigger study focused on PPI by participants with different levels and types of mathematical expertise. Previously we demonstrated that PPI is an effective tool for the development of creativity and proof skills in MM participants—prospective mathematics teachers holding a BSc degree in mathematics (Leikin & Elgrably, 2020). Elgrably and Leikin (2021) demonstrated significant differences between the two kinds of expertise in mathematics, i.e., MM experts, MMs with high achievements (above 90) in university mathematics courses, and MO experts, participants in or candidates for the Israel IMO (International Mathematical Olympiad) team. We showed that MO expertise significantly influences the quality of PPI as reflected in proof skills and creativity components linked to the PPI outcomes. Unfortunately these studies demonstrated that university undergraduate mathematics courses are not necessarily directed towards the development of mathematical creativity.

In contrast to the previous publications, in this paper, we provide systematic analysis of PPI strategies along with a description of analysis of PPI outcomes. This analysis allows us examine differences and connections between strategy creativity (the process) and outcome creativity (the outcome). In publication by Leikin and Elgrably (2015, 2020) we noticed that PPI strategies used by the MM participants involved trial and error, independently of their experience in tackling PPI task. Thus, consistently with Star and Newton's (2009) argument that strategy flexibility in problem solving is a characteristic of mathematical experts, in this paper we report on our analysis of PPI strategies and outcomes of 8 MOs who participated in our study.

2 Background

2.1 Different approaches to creativity

The educational and research literature covers a wide range of models of creativity that differ in terms of the meaning and role assigned to the concept of creativity and the corresponding focal components of creativity. According to Wechsler et al. (2018), divergent thinking is the measure most often used to assess creative thinking. Guilford (1956) suggested that a combination of divergent and convergent thinking is a necessary condition for creative processing. The Torrance (1974) figural and verbal tests are the most popular measure of creativity based on divergent thinking as assessed by fluency, flexibility, originality, and elaboration, as suggested by Guilford. According to Kleiman (2005), a "creative process receives the most attention by far of writers and researchers" (p. 7), while many of the suggested models for a creative process integrate the seminal four-stage model—preparation, incubation, illumination and verification—of the creative process developed by Wallas (1926). Among these four steps, the illumination stage is connected to the conversion of the incubation process and insight.

Besemer and O'Quin (1987) defined creativity in terms of three dimensions, namely, novelty (the product is original, surprising and germinal), resolution (the product is valuable, logical, useful, and understandable), and elaboration (the product is organic, elegant, complex, and well-crafted). This view is consistent with Sternberg and Lubart's (2000) view on creative product as new and useful and with the work of Beghetto and Kaufman (2015), who argued that creativity requires both originality and effectiveness (in the economy context). Moreover, the definitions of originality include different criteria, such as rarity, low probability of occurrence, and surprising effect.

Hadamard (1945) demonstrated that the process of mathematical creation that leads to mathematical discoveries at the absolute level fits Wallas's (1926) four-stage model of the creative process. Usiskin's (2000) 8-tiered hierarchy of creativity clarified the degrees of giftedness and creativity in mathematics. However, Sriraman (2005) addressed the vagueness of this hierarchy and argued that "in the professional realm, mathematical creativity implies mathematical giftedness, but the reverse is not necessarily true" (p. 21). Ervynck (1991) considered mathematical creativity to be one of the characteristics of advanced mathematical thinking, and defined it as the ability to formulate mathematical objectives and find inherent relationships among them. Leikin et al (2017) found a strong relationship between creativity and giftedness in school students and demonstrated that flexibility is a more expertise-related trait, while originality is linked to mathematical giftedness.

In the educational context, understanding the relative nature of creativity becomes essential for studying and developing creativity (Liljedahl & Sriraman, 2006). Leikin (2009) suggested that viewing creativity in school children requires a distinction between *relative* and *absolute* creativity, with a*bsolute creativity* associated with discoveries at a global level ("historical works" as termed by Vygotsky (1930/1982, 1930/1984) while *relative creativity* refers to mathematical creativity exhibited by school students. This distinction is similar to the distinction between objective and subjective creativity (Lytton, 1971) and that of Big C and Little C creativity (Csikszentmihalyi, 1988). Kaufman and Beghetto (2009) proposed an elaborated model that adds more fine grading of levels of creativity: mini-c (the creativity involved in learning), little-c (the creativity of everyday activities), Pro-C (the creativity involved in professional activities), and Big-C (the revolutionary creativity that transforms culture and society), and adds more fine grading of levels of creativity. Mathematical creativity in school mathematics is usually associated with problem solving or problem posing.

There are different approaches to the study of creativity in school (Amado, Carreira, & Johns, 2018; Leikin & Sriraman, 2016). Mathematical creativity in school mathematics is often associated with problem solving and problems posing. Our work follows that of Silver (1997), who utilized Torrance's (1974) components of creativity in the development creativity through problem solving and problem posing, associated with generating multiple mathematical ideas (fluency), generating different mathematical ides (flexibility), and generating new ideas (originality). Based on these ideas, Leikin (2009) suggested a model for the evaluation of creativity using Multiple Solution Tasks (MSTs), which we lately adapted for the evaluation of problem posing (Elgrably & Leikin, 2021; Leikin & Elgrably, 2020).

Plucker et al. (2004) distinguished between focuses on creative *person, process, and product*. The person dimension consists of intellectual potential comprising cognitive, affective, and personality traits, such as knowledge, skills, and motivation, linked to attaining a goal or solving a problem. The process dimension refers to the creative actions performed to attain goals and creative ways of solving problems. The product dimension comprises the creative output of the process performed by the person. According to Kleiman (2005) a creative product is evidence that creativity has occurred and the tangibility of the product "makes it the easiest element to assess" (p. 8). Kleiman also argues that a creative process is difficult to assess and a 'good' creative process leads to a 'good' creative product. In the study reported in this paper, we used Leikin's (2009, 2013) model for the evaluation of creativity using Multiple Solution Tasks to evaluate mathematical creative process linked to PPI strategies, creative products linked to PPI outcomes, and we examined the relationship between them.

2.2 Aptness of problem posing through investigations (PPI) for the evaluation of strategy creativity and outcome creativity

Mathematical investigations and problem-posing are included in the class of open mathematical tasks (cf. Pehkonen, 1995). Leikin (2018) analyzed a group of creativity-directed activities described by Amado, Carreira, and Johns (2018) and classified them as open-start, open-end, and combined problems. In this classification open-start tasks are defined as multiple solution tasks (MSTs) that require solvers to use multiple strategies to solve a problem, with all the strategies leading to the same solution outcome (2009). Open-end tasks are linked to multiple outcomes tasks (MOTs), which require solvers to find multiple answers to a problem (Klein & Leikin, 2020). Combined open-start and open-end tasks require solvers to both solve problems in multiple ways and find multiple solution outcomes.

Surprisingly, Klein and Leikin's analysis of the tasks posed by the teachers led to the observation that whereas MSTs are inherently open-start tasks, MOTs are not always open-end tasks. We illustrate this observation using Problems P1 and P2 (Leikin, 2018).

P1: C*onstruct multiple polygons with an area of 15 sq. units* (Tabach & Levinson, 2018)

P2: *Construct all possible rectangles for which the area is 120 sq. units and the sides' lengths are whole numbers of units* (Gontijo, 2018)

Problems P1 and P2 are potentially open-start tasks since they allow (but do not require) the constructing of geometrical figures of different types using available resources (e.g., paper and pencil, and dynamic geometry environments—DGE) P1 is also an open-end problem. The discussion of the solution space for P1 can be focused on the variety of the outcomes and their rareness. For example, Tabach and Levinson (2018) pointed out the figure transformation strategy in DGE that led to the sets of triangles and quadrilaterals having a common side and equal altitudes. Alternatively, they demonstrated how participants in their study drew convex and concave polygons constructed of 15 unit squares. P1 allowed the use of different problem-solving strategies as well as the finding of different problem-solving outcomes, the rareness of which determined their originality and the differences between which determined their flexibility. The participants were not supposed to provide either a whole set of solutions (that seems to be unlimited) or a generalized solution. And thus P1 is both MOT and open-ended.

P2 is also both a MOT and potentially a MST (no requirements for multiple ways of solutions were presented to the participants). The solution strategies directed at constructing a rectangle of a given area and length of sides (whole number of units) can vary meaningfully as the factorization of 120 into 2 factors can vary from writing all possible pairs of numbers the product of which is 120 (e.g., 1×120, 2×60, $3\times 40, \ldots, 10\times 12$) or using a fundamental theorem of arithmetic as that can be considered the basis for all the produced solutions. Finding the number of rectangles can be a combinatorial problem when considering the product $2^3 \times 3\times 5$. The open start nature of this task is obvious: P1 is a MST and an open start task and allows for evaluation of strategy creativity. At the same time, if the whole set of the rectangles is not found (drawn) the solution is incomplete. Thus, P2 exemplifies that MOT cannot necessarily be considered open-end tasks, since some require (or allow) a complete set of outcomes.

This observation about the differences between MSTs and MOTs, is especially important to the study presented in this paper. Evaluation of strategy creativity demands using open-start tasks, whereas the evaluation of outcome creativity requires employing open-end tasks. Evaluation of both strategy and outcome creativity involves using tasks which are both open-start and open-end tasks. This observation highlights the importance of using PPI tasks in this study.

There are different types of problem posing. Stoyanova and Ellerton (1996) introduced three categories of problem posing in mathematics education: free, semi-structured and structured. Another possible distinction is between problem posing through or for another mathematical activity (e.g., proving, investigations); that is, problem posing can be a goal or a means of the activity (Leikin, 2014). We argue here that independently of these distinctions, problems can be posed using different strategies and can lead to the posing of different types of problems. As such, problem posing activity has high potential for dual openness: It is open-start since problem posing can be performed in different ways, and open-end while there are explicit requirements for posing a number of problems.

Leikin and Grossman (2013) and Klein and Leikin (2020) analyzed strategies that teachers used when asked to transform regular proof or computational problems from mathematics textbooks into open (including investigation) tasks. Both studies demonstrated that teachers used different task transformation (problem-posing) strategies to pose different types of open tasks. These findings were similar to those of Silver et al. (1996), who demonstrated that problem posing strategies are expressed in variations in goals and givens of a given problem. Klein and Leikin (2020) also demonstrated that affective characteristics associated with opening regular problems to pose open tasks differed with respect to different types of open tasks.

As presented in the introduction section, the PPI geometry tasks required participants to pose as many problems as they could, through discovering properties related to a given geometric figure. The open-end character of the PPI task is related to the explicit requirement of posing multiple problems (i.e., discovering multiple non-trivial properties). Due to the multiplicity of the strategies that can be used during PPI, the PPI task is an open-start one. Thus, since PPI tasks allow the evaluation of both products and processes, they are suitable for the evaluation of both and the relationships between them.

3 The study

3.1 The study goal and the PPI task

The goal of the study presented here was to examine the relationship between the strategy creativity and the outcome creativity of expert problem solvers (MO) as reflected in their performance on the PPI Task. We asked: What are the strategies used by MOs in the PPI process? Are there mutual relationships between strategy creativity and outcome creativity of PPI, and if so, what are they? We hypothesized that higher strategy creativity would imply higher outcome creativity.

Figure 1 depicts the PPI task used in this study.

In what follows we use the terms 'posed problem' and 'discovered properties' interchangeably.

PPI Task 1

Given: ABC is a right triangle ($AC \perp AB$), $\angle ABC = 60°$
ABD is an equilateral triangle, CD intersects AB at point E.
(a) Find at least two non-trivial properties by investigation in DGE
(b) Formulate new problems and solve them.

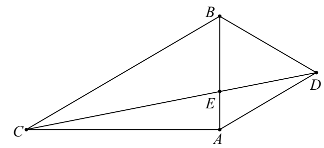


Fig. 1 PPI task used in this research (presented also in Elgrably & Leikin, 2021; Leikin & Elgrably, 2020)

3.2 Participants and data collection

Eight participants (16–18 years old) who were candidates for, or members of, the Israel National Olympiad team are the problem solving experts in this study. All these participants passed the problem-solving training for the IMO (International Mathematical Olympiads), which is the most prestigious mathematics competition nowadays (Koichu & Andžāns, 2009). The training that includes solving problems from classical content areas and those that usually are not studied in school or university is directed at the development of the highest level of problem-solving skills and strategies. The 8 participants volunteered to participate in our study upon our request.

Following is a detailed list of the mathematical competitions won by each participant: David participated in four international Olympiads, and won 3 bronze medals and 1 silver medal. Yuval won 2 silver medals when participating in 2 IMO competitions and a gold medal in a national student Olympiad. Avi won a silver medal in IMO, and a silver medal in the national student Olympiad. Moran won a bronze medal in IMO, and 2 silver medals and one gold medal in an international girls' Olympiad. Amir won a number of national competitions, and won a gold medal in the student Olympiad. All the participants, including Dekel, Rami and Erez, participated in a number of national competitions, and won or finished among the top competitors up to third and second place. All the names used in the paper are pseudonyms.

None of the participants had training in solving PPI tasks prior to their participation in the study. Thus, they first received a preliminary, very short introduction to PPI tasks and the ways of working with DGE, and then they were asked to solve Task 1 during individual interviews. The interviews were recorded using Camtasia software to allow analysis of each action during the investigation process and formulation of the posed problems. Participants were engaged in the PPI for about 90 min and stopped the interviews by themselves when they felt that they did not have additional ideas. All the interviews were transcribed and content analysis was performed to identify the PPI strategies used and PPI outcomes found by the participants. We analyzed proof skills and creativity-related components of the PPI process and products. In this paper we present a detailed analysis of PPI creativity-related components only.

3.3 Evaluation of creativity of PPI outcomes and strategies

Evaluation of creativity implies use of operational definitions and a scoring scheme for evaluation of creativity. As mentioned above, in our previous publications (Elgrably & Leikin, 2021; Leikin & Elgrably, 2020), we utilized the decimal scoring scheme introduced by Leikin (2009, 2013) for the evaluation of Fluency, Flexibility, and Originality of PPI outcomes. In this study we further elaborated the use of the model for the evaluation of Fluency, Flexibility, and Originality of PPI strategies and outcomes as defined in Table 1.

The use of the decimal scoring scheme and the edges of 10% and 40% were introduced and justified by Leikin (2009, 2013). Further validation of the scoring scheme was performed in multiple studies since 2009 (e.g., Leikin et al., 2017; Levav-Waynberg & Leikin, 2012). Within the space limitations of this paper, we do not repeat here justification of the edges. However, we provide a short explanation of the effectiveness of the decimal scoring scheme, since that is important for an understanding of the results of the study.

The major advantage of the decimal scoring scheme is the reversibility of the flexibility, originality and creativity final scores with respect to the mental processes involved in the solution. For example, the number of tens in the overall flexibility score, reflects the number of flexible switches between the strategies used or outcomes attained. The number of tens in the overall originality score demonstrates the number of original ideas implied or attained. No less important, the number of ones in the final scores of flexibility and originality reflects the number of strategies/outcomes with a lower level of flexibility or originality. The number of tenths in the flexibility score demonstrates repeated strategies/outcomes. When (as in this study) the requirement for producing *different* strategies or attaining *different* outcomes is stated explicitly, the number of tenths in the flexibility score reflects lack of critical thinking directed at the analysis of the differences between the solutions. The number of tenths in the originality score reflects the number of algorithmic, common or trivial PPI strategies/outcomes. Evaluation of creativity of each one of the solutions (either for strategy or outcome) using the product of flexibility and originality is equivalent to the statement that the strategy/discovery is creative when it is original and attained flexibly. In this way,

Table 1 The Model and accompanying scoring scheme for the evaluation of an individual space of problems posed through investigations

Score for a particular solution Score for individual space of posed problems		10	1	0.1
Fluency n	Number of posed problems			
Flexibility $Flx = \sum_{i=1}^{n} Flx_i$	PPI Strategy	• Each strategy the participant used as the first one or • A strategy that differs from those used in the course of previous discoveries	• Use of similar strategy but related to different concept or theorem	• Repeated use of a strategy
	PPI Outcome	• Each property the participant discovered as the first one • Property of another type	• Property is similar to a previously discovered property but with a change in type (for example ratio of length and ratio of areas) • Repeated property after additional auxiliary construction	• Repeated type of discovered property
Originality $Or = \sum_{i=1}^{n} Or_i$	PPI strategy PPI outcome	Properties and strategies that are • New for the participant (in the PPI context) • Insight-based • Rare type used by less than $P < 10\%$ participants (in a sample of above 30 participants)	• Less rare property or strategy that is used or identified by • $10\% \le P \le 40\%$ participants	• Trivial (previously learned) property • Property of frequent type ($P \ge 40\%$) in the collective space, following directly from the data given
Creativity: $Cr = \sum_{i=1}^{n} Flx_i Or_i$				

hundreds in the strategy creativity score indicate the creative process reflected in the number of creative strategies. The hundreds in the outcome creativity score reflect the product creativity. Evaluations of outcome creativity or strategy creativity require careful analysis and sorting of either outcomes or strategies. Evaluation of outcome creativity and its components using the PPI task 1 was described in detail by Leikin and Elgrably (2020) and Elgrably and Leikin (2021).

As mentioned earlier, we extended Leikin's (2009, 2013) model for the evaluation of creativity using MSTs since PPI tasks allow the use of multiple PPI strategies. Our definition of PPI strategy is based on Wallas's 4-stage model (preparation, incubation, illumination, and verification) of the creative process, as follows. While performing PPI task the participants are engaged in investigation with a DGE by performing auxiliary constructions, measuring, comparing and observing properties that are immune to dragging (Leikin & Elgrably, ; Elgrably & Leikin, 2021). This process is involved in the preparation and incubation stages of PPI. At the moment of illumination during the PPI, a conjecture is raised. We define the PPI strategy as the way in which the conjecture is raised, as reflected in participants' explanations of how the conjecture is raised, and we consider the PPI strategy to be a heuristic leading to hypothesis generation (Amabile, 1983; Batey, 2012). We argue that PPI strategies are inherently creative due to the open nature of PPI activities (Lubart, 2001).

We identified 8 PPI strategies which are consistent with observations presented by Leikin and Elgrably (2015). The strategies are as follows: using a theorem (to arrive at the conjecture), raising a conjecture through logical inference, raising a conjecture using symmetry considerations, association with (solution of) a familiar problem or of a theorem leading to the conjecture, raising a conjecture while searching for a proof or a new hard problem, intuitive discovery of a conjecture, and a trial and error strategy. In Sect. 5.1 we describe in detail and exemplify the discovered strategies.

Note here, that the verification stage is requires by the PPI task, since it requires proving the discovered properties. The proof skills, complexity of the posed problems, and the auxiliary constrictions performed, are described and analyzed in the paper by Elgrably and Leikin (2021). Within the space limitations of this paper, we omit the analysis of the preparation, incubation and verification PPI stages. Note here that the 8 MO participants posed overall 141 problems, and proved all of them, except 2 that seemed to be obvious to one of the participants (Elgrably & Leikin, 2021).

4 Findings and discussion

4.1 PPI strategies used by the experts in problem-solving

In this section we describe the PPI strategies identified in our study and exemplify them using excerpts from the interview with Amir who used 7 of 8 of the devised strategies. The experts from the interviews are linked to Fig. 2, which

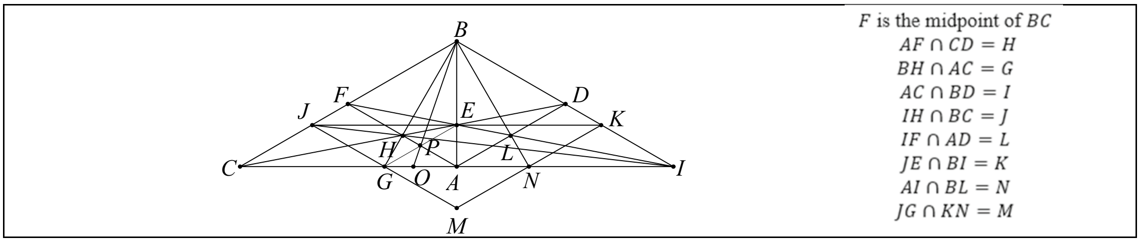


Fig. 2 Auxiliary constructions by Amir

presents an integrated diagram of Amir's auxiliary constructions in the course of the PPI. Each strategy leads to a posed problem (denoted PP). The descriptions of the strategies also make reference to the excerpt from Yuval's interview, which is analyzed in Sect. 5.2.

Using a Theorem strategy embraces discovering a property that can be inferred directly from a theorem or using a construction to fit the theorem. The theorems could be curricula-based (e.g., the ratio of segments based on the theorem about intersection of median lines in a triangle) or be extracurricular (e.g., Ceva's or Menelaus' Theorems). [coding: Th] (Yuval: 1–3, 7, 9, 14, Fig. 2).

Making an association to a theorem or a familiar problem involves discovering a property through analogy to a previously solved problem, or through association with a theorem proof [coding: ATP] (Yuval: Y5, 6, Fig. 3).

Searching for hard problem embraces discovering a property while aiming to formulate a challenging problem (for example for fellow math experts) [coding: SHP].

Amir [I do the construction] because now I want to use Ceva's theorem. Meaning, here I have two Cevians [AF, CE], and I want to create the third [BG],

PP: $\frac{GC}{GA} = 2$

Building Logical Inference strategy encompasses discovering a property through inference from a previously discovered property(s) or by applying several geometry theorems [coding: LI] (Yuval: Y4, 10–14).

Amir Here [CJ:JB] there's a ratio of 1 to 2, and here [CG:AG] there's a ratio of 2 to 1. Here [JG∦AB] it's not going to be parallel. Okay, we can also see that JG isn't really parallel to AB, but there is, there is a triangle here [ΔJEG]. JE is parallel to AC and EG is parallel to BC, so it's a parallelogram - JEGC is a parallelogram.

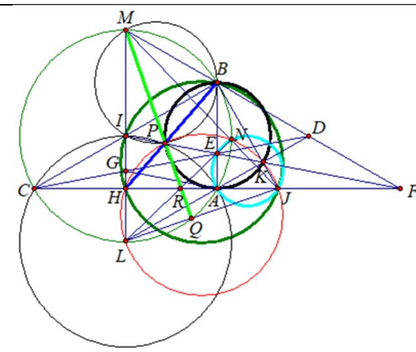
PP: JEGC is a parallelogram (Figure 3)

Using Symmetry Considerations refers to discovering a property based on symmetry of the image, or a symmetric auxiliary construction [coding: Sym].

Amir I marked [AD and IF in L] and now I got such an image that is quite symmetrical, ... if I do symmetry around AB I will get that J is symmetrical to K and ... now H is Symmetrical to L. The point G is symmetrical to the point N, etc., and we have a BGN triangle in the middle. And $\sphericalangle BGA = 60°$, yes, yes, okay.

Okay, ... I tried to find if $\sphericalangle BGA = 60°$. If $\sphericalangle BGA = 60°$ then $\sphericalangle GBA = 30°$ And BG should be an angle bisector.

PP: $\sphericalangle GBA = 30°$ (Figure 3)



$\frac{CE}{DE} = 2$ **[Y1- Th]** That's using the <u>angle bisector theorem</u>. There's 60° here and 60° here (points with the mouse to $\measuredangle ABC$ and $\measuredangle ABD$), so $\frac{CE}{DE} = \frac{BC}{BD}$ [proof]. And $BC = 2AB$ because that's a property of a 30-60-90 triangle, so $BC = 2BD$, and so there's a ratio of 2 here. [Builds the intersection point between BD and CA and marks it as F] Measure $\measuredangle AEC = 79.11°$ Ah, *not a special angle*.

$\frac{AE}{EB} = \frac{1}{2}$ **[Y2- Th]** There's an <u>intersection of median lines</u> here, or just because of the parallel lines [$AD \parallel BC$]. [marks the middle of the segment CE as G]

Is the *$\measuredangle BGA$ interesting?* [measures and exits] 54.79°. OK, *I'm not having luck with angles.* [G midpoint of CE] OK.

But $AG = CG = GE = DE$ **[Y3 –Th]** *is also trivial* because here there's the <u>midpoint of a hypotenuse in a right triangle</u>. Let's try to construct a circle [meaning, to circumscribe the triangle ABC, The midpoint of BC is the center of the circle, marks it as I]. Ah, *I think that this midpoint is actually interesting*.

I see that there's a **parallelogram $AEIG$ [Y4-LI]** here, be<u>cause</u> there's IG, that's a <u>midpoint</u>, this is a midline. So $IG = \frac{1}{2}BE$, <u>and</u> $AE = \frac{1}{2}BE$. *That was really not intentional*.

So now I want to construct here using a specific angle. OK. [Constructs the point J on AF such that $\measuredangle BJA = 60°$, by creating a line that bisects $\measuredangle ABF$]. So this is $\measuredangle JDA = 30°$. $\measuredangle JFA = 30°$ OK, there's a tangent line here [**AD**], there's a **tangent to a circle** [the circle circumscribing the triangle **JDF**] **[Y5-Ass]**. **BJ is tangent to a circle [ABC] [Y6-Ass].** <u>That's true for any right triangle with an altitude (to the hypotenuse)</u>. OK, **I, E, F are on one straight line [Y7- Th]** <u>because there are median lines here</u>. Ah, and it also passes through here [point K] **I, E, K, F are on one straight line [Y8-LI]**. *That's interesting*. *I found something non-trivial*. E, I and K are supposed to be on one line. That's what we got. Apparently there's a Ceva's theorem that needs to be done. Clearly it's enough to prove that K, E and F are on one straight line. That's just an equivalent statement. And here there's an immediate Menelaus $\frac{BE}{AE} = 2$.

OK, so let's draw this line [AD] that intersect with the circle L. OK, **L is the middle of the arc [AC] [Y9- Th]**. Really, *it's pretty obvious* why. <u>Because there's an angle of $30° = \measuredangle LCA$</u>, which <u>is really half of 60°</u>, so the inscribed angle on CL [arc] is half of the inscribed angle on [arc] AC, and so L is the middle of the arc. <u>And</u> it's also on the line IG **[I, G, L are on one straight [Y10-LI]]**. I want to know, for example, what is $\frac{IG}{GL}$ does that give anything good? yes, and I think that it can even be calculated without a calculator. <u>$IL = BA$, $EA = IG$</u>, there's a ratio of 2 [$\frac{BE}{AE}$] to one that we already calculated. And also **$BILA$ a parallelogram [Y11-LI]**. So I know that $\frac{LG}{GI} = 2$ **[Y12-LI]**. OK, there's also the midpoint of the big arc, that is M [BF intersect the circle in M]. [MJ intersect the curcle in N] I want to know if [N] it's really the middle of the arc BA, that's my hypothesis. OK, its equivalent to $\measuredangle BMJ = 15°$, that will be the way to confirm it [measures] . No, that's false, 16.10° not 15°. Maybe $\measuredangle BMK = 15°$? No. OK, it was an attempt. OK, other than that, **M, I, G and L are on the same line [Y13-LI].** *The truth is it's pretty obvious,* <u>it's the same line that is perpendicular to AC</u>. Although it's possible to explain it.

OK, let's draw the line EN, it goes through L **N, E, L are on the same line [Y14-Int].** <u>I thought that would happen. Because it just looks good</u>. *The question is if it's really true, or if it's just very close*. How do I verify that in the image? In order to verify it we need to verify that $\measuredangle ENJ = 90°$. That's how to verify it. *That's nice*. Because then, if it's 90°, then if we continue, it means that it's equivalent to $\measuredangle MNE = 90°$, and a 90°, if we took a segment and drew a perpendicular then it would go through the opposite point [The end of the diameter ML]. *Good, that's also a property*. <u>If it's true</u>, it means that in particular, **J, K, N and L are on the same circle**, <u>because</u> $\measuredangle JKL = 90°$. If it's true. And not just, not just that, on this circle there's also another point, H [intersection between ML, AC]. That means that H is also on the same circle. **There are 5 points [J, K, N, H, L] on a circle [Y15-LI]**. $\measuredangle ANJ = \measuredangle MCA = 60°$ There's a reverse similarity [$\Delta JNA \sim \Delta JCM$]. Then that's equivalent to showing that $\measuredangle ENA = 30°$… I see that E is on the radical axis of the black circle [ML diameter] and the red one [JL diameter]. It's on the common chord [LN], so it's really on the radical axis, that means that E has an equal power. So it's equivalent to proving that E is on the radical axis. I need to show that $\measuredangle ENA = 30°$. What else is equal to 30°? 60°…I said that $\measuredangle JNA = 60°$, OK, <u>that's equivalent to showing that</u> **J, N, E and A are on a single circle [Y16-SP]**, because of the 90°. $\measuredangle JNA = 60°$, so it's equivalent to show that $\measuredangle JEA = 60°$. OK, that's equivalent to showing that $JE \parallel BF$ and that's correct. Why is that correct? Here we have a $\frac{BE}{AE} = 2$, and $\frac{JF}{AJ} = 2$ because BJ angle bisector.

Now we can discover something else, if we mark this point as P [intersection between IF with the circle that JL diameter], so L, P ,B should be on one line. [checked] No. It's false.

The text in squared parentheses presnts the authors' additions and explanations. The text in Italic presents affective components of the PPI process. Underlained text present moments utterances from which the IPP strategies were deturmined. The bold text denoted the discovered properties.

Legend:
Th – Using theorem, LI – Logical Inferene, Ass – making association with a familiar problem, SP - Searching for a proof, Int – Intuitive discovery

Fig. 3 Excerpts from the interview with Yuval

Amir Okay, what's hard to prove?... Are there good constructions I could build?

Haim And what are 'good constructions' to you?

Amir ..That is, maybe I'll get to something that's not very trivial, but… Okay. I'll try it. … Maybe I'll add this too [adds the lines KN and JG in figure 3]. Okay. Obviously they're going to intersect at one point. Now I can ask if it's a rhombus, because it looks a little like a rhombus. Actually, now that I look at the lines I added, I see that JG is parallel to AF, so yeah, it's got to be a rhombus that's like… That is, it [BJMK] came out a rhombus that's similar to the rhombus BFAD I found there earlier.

PP: BJMK is a rhombus (Figure 3)

Searching for a proof: A property found while searching for a proof for a different discovery, or during one of the stages of proving a different discovery [coding: SP] (Yuval: Y16, 20–25, Fig. 2).

Intuitive conjecturing: A conjecture about a property is raised intuitively and then tested using measurement or dragging [coding: Int] (Yuval: Y8, 17, 18, Fig. 2).

Amir Okay, that looks like it [BGMN] is inscribed in a circle. Meaning it looks like this angle [$\measuredangle\mathsf{GMN}$] is 120, let's check that before I try to prove something that isn't true. ....

I'm just checking [with DGE] so I don't try to prove something wrong, and then… that is, there's nothing wrong with trying to prove something that isn't correct, it's just a bit of a waste of time, *it [DGE] is educational because it shows you that things aren't correct*, but… okay.

PP: Quadrilateral BGMN is inscribed in a circle

Trial and Error: Discovery of a property that results from auxiliary constructions (if performed), observation and measurement. In some cases dragging is used to verify the measurement [coding: TE].

Amir … We could ask what this ratio is [$\mathsf{AE{:}BE}$], if it's constant? So let's check.

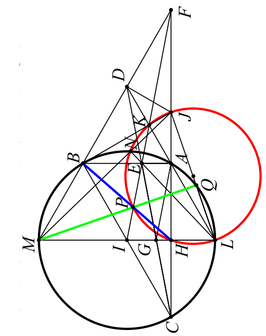
How do I do it? [I measure the] distance [of segments AE, BE]. Okay, and then we do Calculate. The ratio between them. ...I need to do this divided by that [BE by AE] this is 2. It really doesn't vary. Okay [drags].

PP: $\mathsf{AE{:}BE} = 2$ (Figure 3)

Table 2 Strategies and the frequency at which they were used by different participants

	Participants (pseudonyms)	David	Yuval	Amir	Moran	Dekel	Avi	Rami	Erez
Number of posed problems		12	25	26	16	14	18	14	19
Number of different strategies observed per participant during the investigation		5	5	7	4	2	4	5	5
Number of times a strategy was used	Using a theorem	2	4	5	7	9	12		
	Logical inference	3	6	6	3		5	3	3
	Symmetry			1	2			1	5
	Association with a familiar problem		4					3	
	Searching for a proof or a hard problem	2	2	7				3	3
	Intuition	2	6	6	4	5	1	4	7
	Trial and error	1		1					1

Table 3 Yuval's PPI outcomes and strategies and evaluation of the corresponding creativity components

	The discovery	Components of outcome creativity			Strategy	Components of strategy creativity			Auxiliary construction—illustration with all constructions
		Flexibility	Originality	Creativity		Flexibility	Originality	Creativity	
	Total	85.3	129.4	507.22	8	58.3	223	483.1	
Y1	$\frac{CE}{DE} = 2$	10	0.1	1	Th	10	10	100	
Y2	$\frac{AE}{BE} = \frac{1}{2}$	0.1	0.1	0.01		1	10	10	
Y3	$AG = GE = GC = DE$	0.1	0.1	0.01		1	10	10	
Y4	$AEIG$ Parallelogram	10	1	10	LI	10	10	100	
Y5	AD tangent circle JDF	10	10	100	Ass	10	10	100	
Y6	BJ tangent circle ABC	1	10	10		0.1	10	1	
Y7	I, E, F colinear	10	1	10	Th	0.1	10	1	
Y8	I, E, K, F colinear	0.1	1	0.1	LI	1	10	10	
Y9	*Arc* AL=*Arc* LC	10	10	100	Th	1	10	10	
Y10	I, G, L colinear	1	1	1	LI	0.1	10	1	
Y11	$GL = 2IL$	1	0.1	0.1		0.1	10	1	
Y12	$BILA$ Parallelogram	1	1	1		0.1	10	1	
Y13	M, I, G, L collinear	1	1	1		0.1	10	1	
Y14	L, E, N collinear	1	1	1	Th	0.1	1	0.1	
Y15	J, K, N, H, L on one circle	10	10	100	LI	1	10	10	
Y16	J, N, E, A *on one circle*	1	10	10	SP	10	10	100	
Y17	M, P, Q colinear	1	1	1	Int	10	1	10	
Y18	The lines MP, EN, AC *intersect in* R	10	10	100		1	1	1	
Y19	$\sphericalangle BPF = 30°$	1	10	10	SP	1	10	10	
Y20	B, P, H *colinear*	1	1	1		0.1	10	1	
Y21	M, B, P, I *on one circle*	1	10	10		0.1	10	1	
Y22	C, I, P, A on one circle	1	10	10		0.1	10	1	
Y23	$\sphericalangle APF = 30°$	1	10	10		0.1	10	1	
Y24	B, P, A, K on one circle	1	10	10		0.1	10	1	
Y25	H, I, B, J on one circle	1	10	10		0.1	10	1	

Th Using theorem, *LI* logical inferene, *Ass* making association with a familiar problem, *SP* searching for a proof, *Int* intuitive discovery

Table 2 summarizes the number of problems posed by the participants, the number of strategies used by them and the number of times the participants used the strategies. It depicts that larger strategic flexibility, as determined by the number of different strategies used by the participants, does not necessarily lead to outcome fluency, defined as the larger number of posed problems. Four participants who used 5 different strategies discovered different numbers of properties: 12 (David), 14 (Rami), 19 (Rami) and 25 (Yuval). Six of 8 strategies (except the discoveries in the course of searching for a proof, and the trial and error strategy), are related to the participants' previous knowledge and proof skills. All the participants raised intuitive conjectures, which were examined for correctness by dragging. This is particularly interesting in light of our discovery—as part of a bigger research project—that participants without Olympiad experience performed PPI using solely the trial and error strategy. Consistently with the results of Elgrably and Leikin (2021), these findings emphasize the importance of a high level of problem-solving expertise for PPI creativity.

4.2 Evaluation of the PPI creativity of Yuval

In this section we describe the PPI process and product of Yuval, who attained the highest creativity score for the creativity of PPI process. We start with the almost complete excerpt from his interview (Fig. 3) and demonstrate the way in which the PPI outcomes and strategies were evaluated. The analysis and evaluation of creativity components of Yuval's PPI outcomes and strategies yielded the following results.

Overall Yuval discovered and explicitly formulated 25 properties, thus his fluency score was 25. The properties that Yuval discovered (see Fig. 3 and Table 3) were of 7 different types: Y1-3, 11 two segments ratio, Y4, 12—special quadrilaterals, Y5, 6 a line is tangent to a circle, Y7, 8, 10, 13, 14, 17 and 20—three points on a straight line, Y9, 19, 23 equality of arcs or angles, Y15, 21, 22, 24, 25 four or five points on a circle and Y18 three lines intersect in one point. Properties Y1, Y4, 5, 7, 9, 15 and 18 were scored with 10 points for flexibility as these properties were of different types appearing for the first time in the course of the PPI. Y3 was scored with 1 for flexibility since like Y2 it was a two segments ratio but with an additional auxiliary construction of point G. Y2 and Y8 were scored with 0.1 points for flexibility since they were repeating types relative to Y1 and Y7 respectively. Y2, 6, 8, 10–14, 16, 17, 20–25 were also repeating properties discovered in different locations of the figure based on a series of auxiliary constructions, and thus received a score of 1 for flexibility. Originality of PPI outcomes was evaluated based on the frequency of the property, as determined by the number of participants who discovered the property. The frequency was calculated based on the problems posed by the participants in the bigger study (see details in Leikin & Elgrably, 2020). For example Y5, 6, 9, 15, 16, 18, 19, 21, 22, 24 and 25 were scored with ten points for originality because under 10% of the participants found these properties. The creativity of each posed problem within an individual space of posed problems was evaluated as the product of the flexibility and originality of the associated discovered property (Leikin, 2009).

Yuval used 5 strategies overall: Using Theorems, Logical Inference, Association with a familiar problem, Search for a proof, and Intuition. Discoveries Y1-3, 7, 9 were based on Yuval's curricular knowledge since he connected each of these discoveries to theorems or definitions from the curriculum. For example, when he discovered that $\frac{CE}{DE}=2$ [Y1 – Th] he connected this discovery immediately to the angle bisector theorem. Logical Inference was used when discovering Y4, 10–13, 15. Discoveries Y8, 14, 17 and 18 were based on Yuval's Intuition. For example an Intuition strategy for Y14 was reflected in Yuval's utterance "I thought that would happen". Discoveries Y5, 6 were based on Yuval's Association with a familiar problem since he connected each of these discoveries to a problem that he recognized. For example, when discovered that *BJ* is tangent to a circle [ABC] [Y6-Ass]. He connected this discovery to "that's true for any right triangle with an altitude (to the hypotenuse)".

Table 4 Connection between discovery strategies and types of discoveries by expert participants

Expert participants (pseudonyms)	David	Yuval	Amir	Moran	Dekel	Avi	Rami	Erez
Number of posed problems	12	25	26	16	14	18	14	19
Number of different strategies used	5	5	8	4	2	3	5	5
Creativity-related components of PPI outcomes								
Flexibility of discoveries	74.1	85.3	113.3	78.1	76.1	105.3	58.1	98.2
Originality of discoveries	74.1	129.4	41.3	53.8	33.8	57.6	53.6	40.6
Creativity of discoveries	524.01	507.22	356.2	351.6	331.6	463.2	333.4	275.3
Creativity-related components of PPI strategies								
Flexibility of strategy	53.4	58.3	81.8	51.1	21.2	51.3	52.7	51.4
Originality of strategy	102.1	223	196.1	124	95	171	104	117.1
Creativity of strategy	344.1	483.1	624.5	329.2	118.4	423	418.1	319.6

Table 5 Correlations between components of outcome creativity and strategy creativity

Spearman correlations		Components of outcome (product) creativity	Components of outcome (product) creativity	Components of outcome (product) creativity	Components of outcome (product) creativity	Components of strategy (process) creativity	Components of strategy (process) creativity	Components of strategy (process) creativity	Components of strategy (process) creativity
		Fluency	Flexibility	Originality	Creativity	Fluency	Flexibility	Originality	Creativity
Components of outcome creativity (product)	Fluency	1							
	Flexibility	**0.862	1						
	Originality	−0.012	−0.071	1					
	Creativity	−0.012	0.048	0.881**	1				
Components of Strategy creativity (process)	Fluency	0.580	0.307	0.282**	0.344	1			
	Flexibility	0.778*	0.476	0.143	0.095	0.847**	1		
	Originality	0.874**	0.857**	0.143	0.167	0.393	0.690	1	
	Creativity	0.766*	0.595	0.357	0.357	0.663	0.881**	0.881**	1

*$p<0.05$, **$p<0.01$, ***$p<0.001$

Discoveries Y16, 19, 20, 21–25 were based on search for proof. For example, when searching for a proof for the conjecture that J, N, E and A are on a single circle [16-SP] he discovered first that there are 5 points $[J, K, N, H, L]$ on a circle [15-LI] and then stated: “that’s equivalent”.

Flexibility associated with each PPI strategy was analyzed with relation to previously activated PPI strategies. For example, Y1 was based on use of a theorem which was evaluated with 10 for flexibility as the first PPI strategy used. Y2 was posed using a different theorem and thus strategy flexibility was evaluated with 1, whereas Y14 was posed using the same theorem as Y2 (intersection of the midlines) and thus received a score of 0.1 for flexibility. After using theorems for posing problems Y1-3, Yuval designed a logical chain (inference) when discovering Y4. Thus Y4 was evaluated with 10 for strategy flexibility. Yuval first used a PPI strategy in Y5-Ass, Y8-Int, and Y16-SP which were scored with 10 for strategy flexibility. Strategy originality was evaluated according to the frequency of a strategy with respect to the number of participants who discovered the property. Frequencies were examined in the bigger (N = 76) group of participants (described in Elgrably & Leikin, 2021). Trial and error strategy was used by 73 of 76 participants and thus received a score of 0.1 for originality. Intuition was used by more than 10% of participants and thus was evaluated with 1. All other strategies used by MOs were rare (used by less than 10%) and thus were evaluated with the score of 10. This evaluation is depicted in Table 3 that summarizes all the problems posed by Yuval and the evaluation of the associated creativity-related components of PPI strategies and outcomes.

4.3 Connection between discoveries and discovery strategies by problem-solving expert participants

Our hypothesis that higher strategy creativity implies higher outcome creativity was not confirmed, as follows from the findings presented in Tables 4 and 5.

Table 4 demonstrates that David, who demonstrated the highest outcome creativity, received an outcome flexibility score lower than that of 6 other participants. Yuval and Amir posed 25 and 26 problems respectively, that is, they exhibited the highest outcome fluency, while the strategy fluency of Amir was higher than the strategy fluency of Yuval. Amir’s strategy creativity was higher that Yuval’s strategy creativity whereas Amir’s strategy creativity was the highest among the participants, however, his outcome creativity was lower than that of David, Yuval and Avi.

Spearman non-parametric correlations (Table 5) between the elements of strategy creativity and outcome creativity demonstrated statistically significant correlations between fluency and flexibility of both kinds ($r_s = 0.862$, $p < 0.01$

$r_s = 0.874$, $p < 0.01$) and originality and creativity of both kinds ($r_s = 0.881$, $p < 0.01$). A statistically significant correlation between flexibility and creativity was found for strategy creativity only.

We found statistically significant correlations between some components of strategy creativity and outcome creativity. While there were no significant correlations between strategy fluency and outcome fluency, strategy flexibility and originality significantly correlated ($r_{s\text{-}flex} = 0.778$, $p < 0.05$; $r_{s\text{-}or} = 0.874$, $p < 0.01$) with outcome fluency. The statistically non-significant correlation between strategy and outcome fluency may be explained by the fact that finding more discoveries makes it difficult to increase strategic fluency, because there are only nine different types of strategies. In addition, it was interesting to see that there was a significant correlation between strategy originality and outcome flexibility ($r_s = 0.857$, $p < 0.01$, $n = 8$). That is, the more the MOs made use of original strategies, the higher their flexibility of discovery.

It is important to note that the original strategies MOs used led to an increase in the flexibility of their discoveries, rather than various strategies leading to flexibility of various outcomes. It can be said that for both PPI outcomes and strategy, there is a strong, significant connection between originality and creativity.

5 Conclusions

This study makes several contributions. First we explain the unique aptness of problem posing tasks in general, and PPI tasks in particular, for the evaluation of strategic and outcome creativity. Second, we identify PPI strategies and describe them in detail. Third, we introduce a model for the analysis of creativity of PPI strategies and outcomes and the relationships between them. Fourth, we perform a study with an exclusive group of participants—MOs, who are rarely interviewed—in order to analyze their problem-posing performance.

Participants who are experts in solving proof problems warrant a deliberate investigation, based primarily on pre-existing knowledge, and involving both creative thinking and critical thinking. This finding was expressed in those investigation strategies that only problem-solving experts used—they showed a tendency to 'look for familiar things', suggested hypotheses without using the dynamic environment, and checked their hypotheses using the dynamic environment. In addition, the experts used the dragging tool carefully, mostly to confirm that there was a need to prove the conjecture.

As reported previously (Elgrably & Leikin, 2021), we found that the high level of mathematical expertise of MO participants was reflected in the significant correlation between proof skills and outcome creativity skills, and that problem posing performed by MOs and proving by MOs were inseparable. We found these findings to be consistent with Duncker's (1945) argument that raising a hypothesis (problem posing) is an intrinsic part of the problem-solving process in mathematical experts. In this part of the study we additionally demonstrated that PPI strategy creativity involves proving and search for proofs. Thus, while problem posing has previously been considered a tool for proving, here, proving appeared to be a tool for problem posing.

The study suggests that a distinction be made between strategy creativity and outcome creativity. This distinction leads to the discovery of interesting connections between PPI strategies and PPI outcomes, and between the elements of creativity and proving skills. The study indicates seven systematic (different from trial and error) PPI strategies used by MOs that can be divided into two larger groups. The first group—strategies that employed existing knowledge and skills—included the following: using theorems, constructing logical inference, using symmetry, association with a theorem or familiar problem and intuitive discovery. The second group—looking for the 'unknown'—included looking for a hard problem and looking for a proof for a discovered property.

Analysis of the ways experts solved investigation tasks indicates that there were statistically significant correlations between outcome fluency and strategy flexibility and originality; that is, flexibility and originality of the creative process lead to fluency of creative products. Moreover, there was a statistically significant correlation between strategy originality and outcome flexibility; that is, product flexibility was connected to the originality of the creative processes the MOs used in solving problems.

Interestingly, the power of DGE was explicitly recognized by MOs (who had not had experience with DGE prior to our interviews). After performing a number of auxiliary constructions Yuval understood the power of DGE for PPI and asked whether he is allowed 'to build whatever he wants to'. After this moment of the interview, he performed a variety of constructions in the course of tackling the PPI task. Similarly Amir (in the excerpt that illustrated an intuitive PPI strategy) argued that use of DGE helps in refuting conjectures in order to avoid unnecessary attempts to prove incorrect conjectures.

Finally the study highlighted the importance of the affective component involved in the PPI process performed by MOs (see the text in italics in Fig. 2): The participants were searching for *interesting* discoveries and found some of them to be *nice*. Moreover some of them were searching for *interesting* and *difficult* properties. Some of the discoveries appeared to be *obvious* and while conjectures appeared false they argued that they *had no luck*. Critical thinking was apparent when MOs evaluated whether the properties were *trivial or not* and to which extent they were complex.

This study presents many new findings that are summarized here, but even more it opens directions for future research on problem posing, mathematical investigations, creativity, affect and many other interconnected aspects associated with PPI.

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