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SURVEY PAPER

Empirical research on problem solving and problem posing: a look at the state of the art

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Abstract
Problem solving and problem posing have long been of interest to the mathematics education community. In this survey paper we first look at some of the seminal moments in the history of research on the important topics. We then use this history to position the state-of-the-art research being done in both problem solving and problem posing, before introducing the presented state-of-the-art developments in problem solving and problem posing. We then use this work as a backdrop against which to introduce the 16 empirical papers that make up this special issue. Together these 16 papers add nuance to what is already known about problem solving and problem posing; this nuance is the result of attending to very specific contexts and purposes in which these activities are embedded. We end the paper by discussing the future directions these fields can take.

1 Introduction

The field of mathematics education has been focused on problem solving for well over 50 years. In that time, much research has been done and much has been written about problem solving, the sum of which has created a taken-as-shared belief that problem solving is, and should be, an important part of what it means to teach and learn mathematics. And indeed, in that time problem solving has woven itself into curricula around the world both as a skill to be taught and a vehicle through which mathematics is learned. Yet, problem solving is still a source of great difficulty for learners of all ages (Verschaffel et al., 2020). So, the work goes on. In this survey paper we look at the state of the art of research on problem solving as well as its younger sibling, problem posing, and use this research as a backdrop to position the empirical work presented in this special issue.

This survey paper is divided into two main sections—problem solving and problem posing. In each section, we begin by surveying the research that has been conducted in each area. Each section then finishes with an introduction to the state-of-the-art research represented in this special issue. We conclude the paper with a brief commentary about the future needs for research into problem solving and problem posing.

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2 Problem solving

2.1 Looking back: problem solving through the ages

Problem solving has held the attention of the mathematics education research community for well over 50 years (English & Sriraman, 2010; Frensch & Funke, 1995; Kilpatrick, 1969, 1985, 1992; Lester, 1980, 1994; Lesh & Zawojewski, 2007; Lester & Cai, 2016; Schoenfeld, 1992; Silver, 1985, 1990). In the 1960s, this research was focused almost exclusively on textbook problems and problem-solving behaviors (Kilpatrick, 1969). The 1970s saw a shift towards a focus on heuristics (Kilpatrick, 1978) and task variables (Goldin & McClintock, 1979). In the 1980s the emphasis was on a comparison between novice and expert problem solvers (Charles & Lesh, 1985; Schoenfeld, 1985; Silver, 1988), the relationship between beliefs and attitudes and problem solving (Lester et al., 1989; Schoenfeld, 1987), and metacognitive training (Lesh, 1982; Schoenfeld, 1982; Silver, 1982).

Stanic and Kilpatrick (1989) summarized the research on problem solving to this point as falling into three main categories. The first of these categories looked at problem solving as a cognitive activity and included research that provided descriptions of what problem solvers do. The second category positioned problem solving as a learning goal—*an end unto itself*—and looked at how to improve problem-solving competencies. The third category pertained to research that viewed problem solving as an instructional approach—*as a means to an end*—and looked at the use of problem solving to teach mathematics, to develop skills, and to shift beliefs and dispositions.

The 1990s saw problem solving research make a social turn with the realization that it is "an extremely complex form of human endeavor that involves much more than the simple recall of facts or the application of well-learned procedures" (Lester, 1994, p. 668). Problem solving, it turns out, is the coordination of several interdependent and overlapping factors such as knowledge, control, beliefs and affect all situated within a sociocultural context (Lester, 1994). In short, problem solving is "far more socially constructed and contextually situated than traditional theories have supposed" (Lesh & Zawojewski, 2007, p. 779). And the field of mathematics education has been grappling with this "Pandora's box" (Lester, 1994, p. 669) ever since.

The ZDM double issue on *Problem Solving Around the World: Summing Up the State of the Art* (Törner et al., 2007) is an exemplification of the diversity resulting from this social and contextual turn in problem solving, as are various reviews of literature (Cai, 2003a, 2010; Lesh & Doerr, 2003; Lester & Cai, 2016) and compendiums of research (Felmer et al., 2016, 2019; Liljedahl & Santos-Trigo, 2019; Liljedahl et al., 2016) published since then. So, the work goes on.

2.2 Contemporary research into problem solving: the state of the art

Although it is still true that research into problem solving, past and present, can be sorted into the three categories proposed by Stanic and Kilpatrick (1989)—as a cognitive enterprise, as something to be taught, and as something to teach through—such a categorization does not quite capture the complexity that problem solving researchers have been grappling with for the last 25 years. Even the research emerging in the last five years has continued to work at uncovering the nuances behind problem solving as a contextual and socially constructed activity. Out of this research have emerged a number of themes that encapsulate the contemporary research into problem solving. In what follows, we exemplify these themes through the state-of-the-art research into problem solving as well as use these themes to provide background for the problem-solving contributions in this special issue.

2.2.1 The role of collaboration in problem solving

Much of the early research on problem solving was done with individuals solving problems on their own, relying only on the resources—their knowledge and experience—that they brought into the situation with them (Sekiguchi, 2021). Lester (1994) made an urgent call to broaden the problem-solving research to also consider how problem solving functioned within collaborative groups. And that call was heeded. In some cases, the research on group problem solving was done explicitly for the purpose of better understanding what collective problem solving looked like (Clark et al., 2014; Ryve, 2006; Sekiguchi, 2021). In these cases, the focus of the research was about the human resources that the group relied on in solving the problem (Koichu, 2015, 2018). In other cases, collaboration was simply part of the environment in which the problem solving took place (Ng, 2021). In these cases, the focus was often on the way the group made use of resources external to themselves, such as technology (Ng, 2021) or an external source of knowledge (Koichu, 2015, 2018).

In this special issue, there are two papers that focus on what students do in collaborative problem-solving situations when they have access to resources that go beyond the bounds of the group. In their paper, Rott et al. (2021) challenge the idea that real problem solving (as opposed to ideal) does not follow the prescriptive sequences of normative problem-solving models put forth in many of the phased and heuristic models of problem solving (Dewey, 1910; Mason etal., 1982; Pólya, 1945; Schoenfeld, 1985; Wilson et al., 1993; Yimmer & Ellerton, 2010). Instead, they propose a more descriptive model of problem solving that honors the errors, detours, and cycles that occur in real collaborative problem solving. Upwards of 200 preservice elementary teachers were video recorded working in small groups on five non-routine problems, some with the aid of dynamic geometry software (DGS) and some without. Analysis of these recordings, along with the written work of these students, produced a descriptive cyclic model that accounts for both the pencil-and-paper and the DGS problem solving processes as well as capturing the idiosyncratic nature of collaborative problem solving with all of its errors, wrong turns, detours, and cycles.

Likewise, Pruner and Liljedahl (2021) look specifically at an example of what happens when collaborative groups have access to resources from outside the group. Working in a choice-rich problem-solving environment (Koichu, 2018) called a thinking classroom (Liljedahl, 2020) wherein students worked in collaborative groups of three while standing at vertical whiteboards, Pruner and Liljedahl (2021) look at what high school students do when the problem-solving resources in their group run out. Results show that, in these choice-rich environments (Koichu, 2018), groups will seek out more resources from the visible work of other groups around them and will engage with that work either passively (looking) or actively (discussing). These results give insights into what the problem-solving processes look like when problem solving is liberated from the artificial collaborative (but isolated from other groups) problem-solving environments most often seen in the classroom.

Moving away from the construct of resources, collaborative problem solving can also be located within the larger milieu of collaborative learning, which research has shown to have a positive effect on achievement, attitudes, and perceptions (Kyndt et al., 2013). This body of research also shows that individuals in collaborative settings benefit not just from the resources they gain from groupmates but also from sharing their own ideas, explaining their thinking, and justifying their solutions (Pijls et al., 2007), and that their progress as individuals is furthered by the positive feedback they receive from groupmates (Dahl et al., 2018). Of course, these aspects of collaboration are all predicated on the quality of the interaction and communication within these collaborative learning settings, something that has been shown to be equally important to the collaborative problem-solving environment (Barron, 2003), whether the medium of communication is verbal (Dahl et al., 2018), non-verbal (Liljedahl & Andrà, 2014), or textual (Koichu & Keller, 2017).

Through a socio-cultural lens, collaborative problem solving involves the negotiation of meanings, rules, expectations (Voigt, 1994), construction of authority (Langer-Osuna, 2016) and the interpretations of each other's actions and intentions against a backdrop of socio-mathematical norms (Cobb, 2000; Rassmusen et al., 2003). For example, Koichu (2019) conceptualized collaborative problem solving as a socio-cultural process shaped by the coaction between the individual and the group while working towards a goal, and resulting in a solution that is negotiated and endorsed by the members of the group. Such a conceptualization helps us to see how a group is shaped by, and shapes, the contributions of an individual.

In this special issue, Salminen-Saari et al., (2021) use eye-tracking technology to look closely at the ways in which some of this coaction takes place within collaborative problem solving. Through the affordances of recent developments in mobile eye-tracking technology, Salminen-Saari et al. (2021) are able to focus on when students in a collaborative setting learn from each other—something that is central and vital to understanding the effectiveness of collaborative problem solving. By using the eye-tracking technology to identify moments of *joint attention*—the simultaneous focus on the same thing by all members of a collaborative group—the researchers were able to identify the phases of the problem-solving process when students were most in tune with each other. These were the *understanding the problem, watching and listening*, and *verifying* phases. This research showcases how students in a collaborative problem-solving situation are able to build understanding through the recursive processes of displaying, verifying, and repair of ideas (Roschelle, 1992).

2.2.2 The role of professional development in problem solving

> Simply put, I do not believe that any problem-centered mathematics curriculum has a chance of success unless the teacher's role in the curriculum is clearly and unambiguously spelled out. However, very little of the literature on mathematical problem-solving instruction discusses the specifics of the teacher's role, and just as little of the research literature on teaching deals with problem solving. In my view, attention to the teacher's role should be the single most important item on any problem-solving research agenda. (Lester, 1994, p. 672)

Despite an abundance of over 50 years of research, the enactment of problem solving continues to be a challenge for teachers (Chapman, 2016). For some, this challenge is the result of hesitancy brought on by their fear of unpredictable outcomes of problem solving (Russo & Hopkins, 2019). For others, the challenge stems either from their beliefs about what it means to know mathematics (Rott, 2020) or their beliefs about what problem solving is (Son & Lee, 2021). And for others, it stems from their personal experiences as students solving problems or their professional experiences as teachers enacting problem solving (Berk & Cai, 2019). Regardless of the source, teachers need help developing and sustaining their problem-solving practices, and one source of this help comes from professional development.

This professional development can occur through mentorship relationships between teachers. For example, Masingila et al. (2018) did a self-study wherein they reflected on their own mentorship/mentee experiences of learning to teach a mathematics content course for preservice teachers through problem solving. Results showed that the mentorship relations, coupled with an inquiry stance, allowed the two novice teachers to set goals effectively for both the lesson and the course as a whole, to choose and use problem-solving tasks effectively, and to develop their ability to scaffold problem-solving processes in their classes. Alternatively, professional development can take place in more structured programs. For example, Wake, Swan, and Foster (2016) used the lens of cultural-historical activity theory to look at the development of problem-solving competencies through a lesson study model of professional learning. Their findings showed that the making of artefacts played an important part in shaping the teachers' problem-solving processes in both the classroom and lesson-study activity systems.

In this special issue, Mellone et al. (2021) look at the changes in prospective teachers' conceptions of teaching through problem solving inside of a structured professional development program. Using a theoretical framework of cultural transposition (Mellone et al., 2018), Mellone et al. (2021) examined the unconscious beliefs that Italian prospective teachers took for granted until they were immersed in a thinking classroom (Liljedahl, 2020), which, for them, was culturally a very different learning environment (Bartolini Bussi & Funghi, 2019). From a combination of survey, interviews, and case studies, the results showed that the juxtaposing of culturally different educational contexts promoted changes in the prospective teachers' beliefs about the teaching and learning of mathematics, methods of teaching mathematics, and the central role of problem-solving tasks and mathematical content.

Also embedded within a formal professional development setting, Saadati and Felmer (2021) looked at the effect of teachers' participation in a nine month long professional development program focused on the promotion of collaborative problem solving in the classroom. These sessions promoted the use of non-routine problems and collaborative problem solving and did not teach or encourage the teaching of any problem-solving heuristics. Through the lens of student improvement, Saadati and Felmer (2021) compared students whose teacher participated in the series of workshops with students whose teacher did not. The results showed that there was a statistically significant improvement of problem-solving performance among the students whose teachers participated in the workshops. Interestingly, this improvement was despite the fact that there was no difference in the two groups with respect to the variety of strategies used. This result means that the improvement in problem-solving performance of the experimental group was due entirely to their experiences in solving non-routine problems.

2.2.3 The role of task variables in problem solving

The impact of task variables on problem solving has been studied since the 1970s (Goldin & McClintock, 1979; Lester, 1994). In the early days, this line of inquiry was focused entirely on understanding the factors that contributed to problem difficulty. Initially, this research was focused on the relationship between variance in tasks and the degree to which individuals were able to correctly solve the tasks (Goldin & McClintock, 1979). This focus eventually shifted away from the characteristics of the solution and towards the characteristics of the solver (Lester, 1994).

Since then, much of the research on task variability has been done in the area of problem posing and is not discussed here. But there is still some research into task variability in problem solving happening. For example, Vörös et al. (2021) found that task complexity had an impact on time-on-task, which, in turn, had an impact on problem-solving performance. If this complexity resided within the technical proficiencies required (for example, the use of specific software), then it affected different individuals than if the complexity resided within the cognitive and metacognitive demands of the task. Alternatively, Di Mascio et al. (2018), in the context of creative problem solving, found that teachers interpreted problem-solving instructions such as 'be creative' very differently from what the researchers intended, leading to a variety of results. In particular, they found that variables as nuanced as whether the 'be creative' instruction came before or after the task had an effect on the novelty of the solutions.

In this special issue there are two papers that focus on task variability. The first, by Carotenuto et al. (2021), looks specifically at how the structure of a problem affects how students solve a problem. The original multiple-choice problem, taken from a grade 5 national assessment in Italy, asked students to figure out how many car transporters (each capable of carrying 10 cars) are needed to transport 62 cars. Using an experimental cycle wherein variations to this problem are made depending on the results of previous experiments, the researchers varied the words used in the problem (changed the number of transporters needed to the number of trips), the format (open problem versus a multiple choice problem as well as varying the possible choices to include decimal answers), and picture (changing the picture from a transporter that can carry 10 cars to one that can carry 8). The results of their research showed that variations in the original problem changed the students' sense of realism of the situation, which, in turn, changed their approaches to solving the problem. In essence, the problem created a context from which the students approached the problem, and the context made a difference.

In addition, Koichu et al. (2021), in their research, examined high-school students' collaborative problem solving on who-is-right (WIR) tasks. These tasks consisted of a problematic situation accompanied by multiple and contradictory solution narratives (representing partial or full solutions). Working in groups, students had to decide which of these narratives to endorse, and provide an argument for why endorsement was warranted. Results showed that, in solving these WIR tasks, the students exhibited processes that align perfectly with Pólya's (1945) looking-back stage of his four-part heuristic. Further, they found that the looking-back process was enhanced by the explicit need for justification (Why is this one right?) that exists only implicitly when solving a problem on one's own (Am I right?). This result, coupled with the expanded problem-solving resources available in a group and mobilized by the discursive demand of the WIR tasks, gives greater insights into how students solve mathematical problems collaboratively.

2.2.4 The role of technology in problem solving

Recent developments in technology have resulted in the emergence of a series of computational, modelling, and programming tools that can support students in solving problems (Carreira & Jacinto, 2019) and in expressing their solutions to problems (Santos-Trigo, 2019), and can guide their discovery processed (Jacinto & Carreira, 2017), enrich modelling processes (Greefrath et al., 2018; Greefrath & Siller, 2017), and help transfer knowledge from problem solving to different mathematical contexts (Rodríguez-Martínez et al., 2019). For example, Amado et al. (2019) found that spreadsheets, through their ability to give constant and instant feedback, are an effective way to help students explore the relationship between variables and to build a bridge between their informal algebraic thinking and the formal representation of this thinking necessary to communicate with and through the technology. Likewise, Santos-Trigo et al. (2016) found that GeoGebra allowed students to check examples, explore special cases, make conjectures, and look for counterexamples while problem solving. In short, the use of digital technologies is fundamentally changing the way we solve problems (Gros, 2016).

This realization has given rise to a new set of competencies that need to be developed. That is, digital tools do not just help students with problem-solving competencies. The use of digital tools to solve problems is a competency in its own right (Carreira & Jacinto, 2019 Forgasz et al., 2010). Digital problem solving requires the user to have mathematical competencies and technological competence, both of which are interlaced with and inseparable from the user (Borba & Villarreal, 2006; Jacinto et al., 2016).

A specific form of digital problem solving has recently emerged out of the world-wide maker movement (Hughes et al., 2017) which is built on the idea that students learn through the crafting of technologically enhanced artefacts (Chu et al., 2015; Ng & Chan, 2019) in a social environment. Within this movement, problem solving is closely aligned with Papert's (1980) idea of "learning as making" (Ng & Chan, 2019; Ng & Ferrara, 2019) and involves an iterative process of conjecturing, testing, and revising in order to find ways to produce a deliverable that meets the required constraints of a task.

In this special issue, Ng and Cui (2021) situate their research within this milieu of the digital maker space and look at the collaborative problem-solving practices of a group of grade 5 and 6 students (ages 11–13) participating in a digital making summer camp. These students were asked to solve a number of mathematical problems (such as whether or not 7,081 is a prime number) using a block-based programming language. Studying their problem-solving processes across three days (and three problems), Ng and Cui found that the digital making environment supported the students' modeling and algorithmic thinking while at the same time necessitating and facilitating testing and debugging processes. In particular, Ng and Cui (2021) found that the problems, coupled with the tangible nature of making, promoted the students' flexible and non-procedural approaches to making the programs work (as opposed to trying only to find the right answer).

2.2.5 Problem solving as a cognitive activity

Although the aforementioned state-of-the-art research into problem solving was sorted into the themes of collaboration, professional development, task variables, and technology, the research also could have been sorted into the three categories proposed by Stanic and Kilpatrick (1989)—as a cognitive enterprise, as something to be taught, and as something to teach through. In such a sorting, seven of the papers in this special issue would have fallen into the first of these categories in that they all looked, in one way or another, at what happened inside of a problem-solving environment, whether that environment was collaborative, was influenced by the nature of the task, or used technology. The fact that these papers also fell into the themes described above does not negate the fact that our curiosity around what problem solving looks like in different contexts has not been satiated by the last 25 years of research into problem solving.

To some degree, the reason for this complexity is that ever since problem solving was liberated from the constraints of an individual working on a problem alone, the endless variances in problem solving environments has kept this area of problem-solving research relevant, important, and interesting. In this regard, this special issue has one paper that fits best into the category of problem solving as a cognitive activity. Cirillo and Hummer (2021) looked at the proof-related competencies used by high-achieving high school (ages 13–17) students while solving geometric proofs, as well as what these students did in the absence of these competencies. Using smartpen technology to audio-record their think-aloud explanations and to capture pen strokes, 23 students completed a series of triangle congruence proofs. The results showed that there was a well-defined set of competencies that influenced the degree to which the students were successful on the proofs, with the largest impact being the ways in which students attended to proof assumptions (making valid assumptions about diagrams or extracted relevant mathematical information from the givens), attended to warrants in their proofs (postulates, axioms, definitions, and theorems), and demonstrated logical reasoning (through logical connectives such as *next*, *then*, and *so*). These results give insights into what students do when successfully solving proof problems.

3 Mathematical problem posing

3.1 A brief history of problem-posing research

Like problem solving, problem posing has been of interest to the mathematics education community for the past several decades. However, in the history of mathematics, problem posing has been viewed as sitting at the heart of mathematical advances for much longer than this (Cai & Mamlok-Naaman, 2020). For example, the set of 23 influential mathematical problems posed by David Hilbert inspired a great deal of progress in the discipline of mathematics (Hilbert, 1901–1902). Einstein even claimed that "to raise new questions, new possibilities, to regard old problems from a new angle, requires creative imagination and marks real advance in science" (Einstein & Infeld, 1938, p. 95). Both Cantor and Klamkin held a firm view that posing a problem has as high or higher value than solving it (Cai & Mamlok-Naaman, 2020). In addition, because of the importance of generality and flexibility in posing problems, researchers studying creativity have for quite some time used problem posing as a measure of creativity (Getzels, 1979; Guilford, 1950).

Yet in mathematics education, problem posing has really drawn the community's attention only since the early 1980s. Certainly, one could argue that aspects of problem posing could be found in work in mathematics education before the 1980s. For example, Pólya's (1945) last step of problem solving, the 'looking back' step, essentially involved posing new/related problems (Silver, 2003). Similarly, the idea that posing a problem properly matters gained some attention (e.g., Butts, 1980). In particular, Butts (1980) pointed out that the way in which a problem is posed has a significant impact on the problem solver's motivation to solve it, as well as his or her understanding of key underlying concepts of the problem.

Problem posing did not draw the mathematics education community's attention widely, in a coordinated way, until Brown and Walter's[1] seminal book, *The Art of Problem Posing*, was published in 1983. In that book, the authors synthesized a case for engaging learners with problem posing and described ways of thinking and pedagogical techniques such as the 'what-if-not' strategy that could encourage and support students to pose their own problems. However, this book did not address problem-posing research per se.

From a research perspective, Jeremy Kilpatrick's (1987) chapter on problem formulating might have provided the first arguments for research that makes problems and their origins the object of study. That is, instead of studying the solving of mathematical problems, researchers could focus on how problems might be posed (by students, for example) in a variety of situations. Kilpatrick explicitly argued that problem posing should be viewed not only as a *goal* of instruction but also as an instructional approach. He advocated that the experience of discovering and creating one's own mathematics problems ought to be part of every student's education.

Around the same time, important developments in the curriculum sphere were also beginning to reflect some attention to problem posing. The 1989 NCTM *Curriculum and Evaluation Standards for School Mathematics* and the subsequent 1991 *Professional Standards for Teaching Mathematics* very explicitly called for providing opportunities for students to pose mathematical problems in classrooms. These NCTM standards documents brought attention to using problem posing to facilitate students' learning and advocated the inclusion of more mathematical problem-posing activities in mathematics curricula. Because of the inclusion of problem posing as a key idea in the influential standards movement, the idea of students' posing problems quickly spread in the mathematics education community.

The publication of Kilpatrick's chapter was soon followed by a USA National Science Foundation-funded research project led by Edward Silver in 1989. That project could be considered the very first funded empirical work on mathematical problem posing. Silver's work was summarized in his widely cited paper, "On Mathematical Problem Posing," published in 1994, as well as in a few other papers (e.g., Silver & Cai, 1996; Silver et al., 1996). Silver's work was intended to try to understand the kinds of problems students and teachers were able to pose and the kinds of cognitive processes that were involved.

Although the empirical research and standards documents discussed above were quite seminal and influential for the line of research on problem posing in the United States, researchers in other countries were also forging new paths in this domain (e.g., Brink, 1987; Cai, 1998; English, 1998). For example, Ellerton (1986) examined the mathematical problems posed by eight high-ability and eight low-ability young children, asking each to pose a mathematical problem that would be quite difficult for his or her friends to solve. She found that that the high-ability children posed problems that were more challenging than those posed by the low-ability children.

Since that early work, there has been much research activity in the domain of problem posing. This blooming of research has been reflected in journal special issues (Cai & Hwang, 2020; Cai & Leikin, 2020; Singer et al., 2013) and books (e.g., Felmer et al., 2016; Singer et al., 2015). In fact, scholars from more than 40 countries and regions

[1] It is with great sadness that, as we write this historical survey on the emergence of problem solving and problem posing in mathematics education, we learn that Marion Walter passed away at the age of 92. Her contributions to mathematics education in general and problem posing in particular are significant, valuable, and much appreciated.

have contributed to the work represented in these journal special issues and books on problem posing. In turn, these journal special issues and books have themselves further stimulated international discussion of problem posing. The present special issue serves to add even more empirical work on problem posing to the developing corpus.

Not only have scholars from many countries engaged in problem-posing research, but also the scope of the research has expanded greatly. In a review chapter, Cai et al. (2015) examined 10 areas of research concerning mathematical problem posing, as reflected in the following questions: (1) Why is problem posing important in school mathematics? (2) Are teachers and students capable of posing important mathematical problems? (3) Can students and teachers be effectively trained to pose high quality problems? (4) What do we know about the cognitive processes of problem posing? (5) How are problem-posing skills related to problem-solving skills? (6) Is it feasible to use problem posing as a measure of creativity and mathematical learning outcomes? (7) How are problem-posing activities included in mathematics curricula? (8) What does a classroom look like when students engage in problem-posing activities? (9) How can technology be used in problem-posing activities? (10) What do we know about the impact of engaging in problem-posing activities on student learning outcomes? This list of questions clearly shows the breadth of the current scope of research on problem posing, although recent efforts have started to zoom in on special areas such as affect in mathematical problem posing (Cai & Leikin, 2020) and teachers learning to teach mathematics through problem posing (Cai & Hwang, 2020).

Despite these varied research directions, the state-of-the-art research on problem posing, like the state-of-the-art research onto problem solving, can still be organized into the three perspectives that parallel those Stanic and Kilpatrick (1989) proposed for problem solving, namely, problem posing as a cognitive activity, problem posing as a learning goal unto itself, and problem posing as an instructional approach.

3.2 Advances in problem-posing research

There are several recent reviews that have presented the state-of-the-art in problem-posing research (Cai et al., 2015; Cai & Hwang, 2020; Cai & Leikin, 2020; Singer et al., 2013; Weber & Leikin, 2016; Silver, 2013). These reviews showcase that, unlike the contemporary research into problem solving, these aforementioned perspectives on problem posing (as a cognitive activity, as a learning goal, and as an instructional approach) continue to nuance the state-of-the-art research, including the seven empirical papers published in this special issue. Table 1 below shows the primary (and secondary, if applicable) perspectives on problem posing that each of these studies has taken.

Table 1 Focus of each problem-posing article

Studies	As a cognitive activity	As a learning goal	As an instructional approach
Elgrably & Leikin	Primary	Secondary	
Guo et al.	Primary		
Hartmann et al.	Primary		Secondary
Jia & Yao		Secondary	Primary
Silber & Cai	Primary		Secondary
Yao et al.	Primary		Secondary
Zhang & Cai			Primary

Five of the papers (Elgrably & Leikin, 2021; Guo et al., 2021; Hartmann et al., 2021; Silber & Cai, 2021; Yao et al., 2021) viewed problem posing as a cognitive activity. In particular, the authors of these papers used problem posing to assess students' or teachers' mathematical thinking.

Silber and Cai (2021) examined the problem posing of 45 college students enrolled in a developmental mathematics course. These students were underprepared to take the usual college-level mathematics courses. In particular, this study was designed to understand better the kinds of problems these students can pose and the kinds of mathematical ideas they exhibited when posing problems. It is the very first study to explore underprepared undergraduate students' mathematical problem posing. The findings clearly support that underprepared undergraduate students are capable of posing problems that are mathematical, meaning that they call for mathematical or quantitative reasoning, and that are solvable with the information provided in the posing task or in the student's posed problem itself. This result provides a foundation for the possibility of using mathematical problem posing to help underprepared undergraduate students learn mathematics through problem posing.

In contrast, Elgrably and Leikin (2021) used mathematically quite mature students in their study: participants of the Israeli International Mathematical Olympiad team (MOs) and college mathematics majors who excelled in university mathematics (MMs). These people engaged in Problem-Posing-through-Investigations (PPI) using a dynamic geometry environment. One of the interesting findings of this study was that university mathematics courses do not develop creative mathematical abilities and skills. On PPI tasks, the lowest scores exhibited by MOs on almost all the examined criteria were higher than the highest scores achieved by MMs on those criteria. This result resonates with the finding of Silber and Cai (2021) that students' problem-posing performance was not related to their course grades. The study by Elgrably and Leikin shows that problem-posing tasks appear to facilitate the further development of mathematically mature students.

Moreover, the study by Silber and Cai shows that problem-posing tasks appear to have the potential to be a context within which even underprepared undergraduate students can engage successfully with key mathematical ideas. Together, these findings support the potential of problem posing in providing learning opportunities for all students, a quality of problem posing that Cai et al. (2015) have referred to as “a low floor and high ceiling.”

Guo et al. (2021) investigated middle school mathematics students’ problem posing from a developmental perspective. Although this was a cross-sectional study and not a longitudinal one, it is interesting to look across students in different grades with an eye to their problem posing. Overall, Guo et al. found that the developmental trajectory for middle school students’ problem posing was both irregular and context-dependent. This result parallels findings from an early study of Cai (2003b) that explored Singaporean fourth, fifth, and sixth grade students’ mathematical thinking in problem posing. The results of his study showed that the overall statistically significant differences across the three grade levels were mainly due to statistically significant differences between the fourth and fifth grade students. Between the fifth and sixth grade students, there were no statistically significant differences in most of his analyses.

Hartmann et al. (2021) looked at problem posing in a modeling context. Problem posing is an integral part of mathematical modeling. In their study, Hartman et al. asked students not only to pose problems but also to solve them. The findings verified the connection between problem posing and problem solving, although it should be noted that the study did not examine problem posing throughout the entire modeling cycle. Nevertheless, this study lays a foundation for how to study problem posing in the complete modeling cycle in the future. They suggested the potential of fostering modelling through problem posing. In fact, it is quite possible that problem posing based on authentic situations from the real world might be a promising approach for fostering modelling.

Yao et al. (2021) examined problem posing and problem solving within the specific context of division of fractions. Focusing on preservice teachers’ understanding of fraction division, Yao et al. found that the preservice teachers more frequently exhibited conceptual understanding on a problem-posing task than on a problem-solving task involving division of fractions. This result provides further evidence of potential for problem posing to open up learning opportunities, as suggested by the studies of Elgrably and Leikin (2021) and Silber and Cai (2021) in this special issue. Yao et al. also found that the provision of a cue designed to draw the preservice teachers’ attention to a conceptual interpretation of fraction division was critical for preservice teachers to exhibit their conceptual understanding on both problem-posing and problem-solving tasks.

Two of the papers (Jia & Yao, 2021; Zhang & Cai, 2021) viewed problem posing as an instructional approach. The study by Jia and Yao (2021) is unique in the sense that it takes a historical approach, examining the inclusion of problem posing in the number and algebra strand of mathematics curriculum over 70 years of Chinese textbooks published by the same publisher. In total, Jia and Yao analyzed six different versions of the textbooks, clearly identifying how problem posing appeared in these materials over the years. As expected, only in recent years have problem-posing activities been systematically and purposefully included in textbooks. Yet even now, textbooks include very few problem-posing activities. As indicated by Cai and Jiang (2017), analyses such as this lay a foundation for conceptualizing how to include problem-posing tasks more prominently in school mathematics as well as in classroom instruction.

Zhang and Cai (2021) analyzed 22 teaching cases of teachers using problem posing to help their students learn mathematics. This is the very first paper to present and analyze such problem-posing teaching cases. The analysis focused on instructional tasks and how the teachers handled students’ posed mathematical problems. This paper directly paints an initial picture of lessons that use problem posing as an instructional approach. The study not only contributes to our understanding of the design of problem-posing tasks, but also to our understanding of how teachers can deal with student-posed problems in the classroom. Most importantly, with more successfully implemented teaching cases using problem posing as a resource, teachers can learn from the cases how to teach using problem posing, despite the paucity of problem-posing tasks in current textbooks and other curriculum materials.

4 Looking ahead

Looking across the nine problem solving papers and seven problem posing papers, some collective themes about these fields begin to emerge. In this section we explore some of these themes and their implications for future research.

4.1 Problem solving: the role of context

As noted by Lesh and Zawojewski (2007), “the development of problem-solving abilities are highly interdependent and far more socially constructed and contextually situated than traditional theories have supposed” (p. 779). This statement is supported by the nuanced and contextualized nature of all nine problem-solving papers in this issue.

Two of the papers clearly demonstrated that the nature of the task highly impacts problem solving behavior and, with it, the mathematics that can be accessed. For example, Carotenuto et al. (2021) demonstrated just how sensitive this connection is and how even slight changes to a relatively straightforward task elicit very different responses. Likewise, Koichu et al. (2021) showed that simply shifting a task from is-this-right to which-is-right enhanced students' mathematical justifications. These results tell us that if we want to teach through problem solving, then further research into the role of the task is warranted and needed.

Likewise, five of the papers clearly showed that what problem solving looks like is also highly dependent on the context in which it takes place. Whether problem solving is done in collaborative settings (Pruner & Liljedahl, 2021; Salminen-Saari et al., 2021), completed within a digital maker space (Ng, 2021), or in an immersive environment (Mellone et al., 2021; Saadati & Felmer, 2021), context matters. And it matters a lot. This, coupled with the nature of the task, shows us that problem solving is highly situated and socially constructed. Consequently, when problem solving is embedded in more authentic problem-solving contexts, what we have long assumed problem solving to look like becomes more nuanced, challenged, and expanded (Koichu et al., 2021; Pruner & Liljedahl, 2021; Rott et al., 2021; Salminen-Saari et al., 2021). Further research into these contextually nuanced environments is needed in order to understand better what needs to be accounted for when teaching through problem solving (Liljedahl, 2020).

4.2 Problem posing: an opportunity for teaching mathematics

As noted above, the Silber and Cai (2021) and Elgrably and Leikin (2021) papers together support the argument that problem posing may be able to create learning opportunities for all students. Guo et al.'s (2021) findings of development varying across grade levels suggest that more research is needed to inform how problem-posing instructional tasks should be designed for students at different grade levels. There has not yet been a longitudinal study that has followed a group of students to see how they grow mathematically in terms of their learning though problem posing. Jia and Yao (2021) study emphasized the significant role of textbooks on students' learning and teachers' teaching. In order to truly implement problem posing in classrooms, it is clear that more resources in textbooks are needed for teachers, both in terms of problem-posing tasks and in terms of guidance on how to teach using problem posing. More research is needed to understand how to integrate problem posing into textbooks in ways that facilitate teachers' teaching through problem posing. On a positive note, Hartmann et al. (2021) have already framed a way to integrate problem posing into teaching. In particular, they examined how to facilitate the modeling process and how students may pose problems in that process. This is a promising direction for future work in this area.

The Yao et al. (2021) paper also points to a promising direction for future work. Given the capacity for problem-posing tasks to elicit conceptual understanding from preservice teachers, a natural question is whether engaging them in more problem-posing activities may have an effect on their problem solving in the area of fraction division. The authors have suggested that this is a direction they will pursue, and future findings should further illuminate how to teach mathematics through problem posing. Their study also suggests the need to examine the role of problem-posing task variables in students' or teachers' problem posing.

Finally, the Zhang and Cai (2021) paper directly addresses models of teaching through problem posing and specifically the need to develop instructional models to implement this approach in the classroom. More effort is needed to accumulate teaching cases in problem posing. In fact, teaching cases could potentially serve as physical artifacts for storing and improving professional knowledge for teaching (Cai et al., in press). Thus, there is not only a need for research on developing problem-posing teaching cases, but also a need for facilitating teachers' learning to teach through problem posing.

4.3 Looking ahead: Teaching mathematics through problem solving and problem posing

Despite the fact that both problem solving and problem posing have long been regarded for their potential to teach mathematics effectively, only three of the papers in this special issue attend directly to this theme. The rest of the papers position the teaching of mathematics as a broader and more global goal, which can be attained only if we have better understand of some of the very situated and contextual variables associated with problem solving and problem posing. Together the results of these 16 papers tell us that there is still much work to be done in order to understand better what problem solving and problem posing look like in various contextualized situations, how we can improve our problem solving and problem posing competencies in these contexts, and how problem solving and problem posing can be used to teach mathematics in these contexts.

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