ZDM – Mathematics Education (2021) 53:507–519
https://doi.org/10.1007/s11858-021-01270-1

SURVEY PAPER

Ideas foundational to calculus learning and their links to students' difficulties

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Accepted: 22 April 2021 / Published online: 13 May 2021

Abstract
Existing literature reviews of calculus learning made an important contribution to our understanding of the development of mathematics education research in this area, particularly their documentation of how research transitioned from studying students' misconceptions to investigating students' understanding and ways of thinking per se. This paper has three main goals relative to this contribution. The first goal is to offer a conceptual analysis of how students' difficulties surveyed in three major literature review publications originate in the mathematical meanings and ways of thinking students develop in elementary, middle and early high school. The second goal is to highlight a contribution to an important aspect that the articles in this issue make but was overshadowed by other aspects addressed in existing literature reviews: the nature of the mathematics students experience under the name "calculus" in various nations or regions around the world, and the relation of this mathematics to ways ideas foundational to it are developed over the grades. The third goal is to outline research questions entailed from these articles for future research regarding each of them.

Our article makes two contributions: first, it builds from surveys of calculus research to address an issue not addressed by any: the impact of mathematical understandings students develop in early grades on their learning of calculus. We examine research on students' early learning of algebra both in terms of what it says about what students do and do not learn about ideas foundational to calculus. We then provide conceptual analyses of these same ideas with an idea of how they might be fostered in early grades so students' understandings of them are relevant to learning calculus.

The second contribution is to introduce the articles in this issue. Some of them address issues related to our first part; some do not. That is understandable—our call for papers simply requested that articles attend to the case of calculus in their country or region. Our hope was, as a collection, the issue would provide a picture of calculus learning and teaching around the world.

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1 Surveys of research on calculus learning and teaching

Inquiries into calculus learning, teaching, and curricula have happened for decades. Smith (1970) gave an anecdotal report of a calculus curriculum designed around computer usage and students' responses to it. Tall and Vinner (1981) employed Vinner's ideas of concept image and concept definition to delve into students' difficulties with limits and continuity. Orton (1983a, 1983b, 1984) produced a series of landmark studies of students' understandings of differentiation, integration, and rate of change. These studies were just the first of many investigations of students' understandings of ideas of calculus and reports of different approaches to the reform of calculus curricula and calculus instruction.

Rasmussen et al. (2014) edited a special issue of *ZDM Mathematics* aimed at "taking stock" of calculus research. They identified four trends in the calculus research literature: studies identifying misconceptions; studies investigating the processes by which students learn particular concepts; studies dealing with instructional treatments, and studies in teacher knowledge, beliefs, and practices.

Bressoud et al. (2016) produced an ICME topical survey on calculus research. Their explanation of the survey's purpose was that it.

> … aims to give a view of some of the main evolutions of the research in the field of learning and teaching Calculus, with a particular focus on established research topics associated to limit, derivative and integral. These evolutions are approached with regard to the main trends in the field of mathematics education such as cognitive development or task design. (p. 1)

Bressoud et al. also (2016) highlighted various theoretical frameworks within which studies were conducted and their influence on the natures of reported research.

Larson et al. (2017) surveyed research on learning and teaching calculus and also surveyed recent reform attempts in calculus curricula that take research into account. It would be at least redundant for us to survey the same literature as these three collectively did. Our strategy instead will be to summarize them and add a dimension unaddressed by all three—research on ideas foundational to students' calculus learning and possible links to students' difficulties reported in research already surveyed.

Our first section will summarize the surveys by Bressoud et al. (2016), Larsen et al. (2017), and Rasmussen et al. (2014). The second section will provide a conceptual analysis of variation, covariation, accumulation, and constant and average rate of change. We will use this conceptual analysis as context for discussing research on learning and teaching the same at grade levels prior to when students first meet calculus. The third section will introduce the articles in this issue.

1.1 Research surveyed by Bressoud et al., Larsen et al., and Rasmussen et al.

All three surveys addressed the question of what fundamental concepts and ways of thinking are crucial to students' understanding of calculus. Each of them identified the concepts of limit, derivative, and integral as fundamental calculus concepts.

Bressoud et al. (2016), as well as Larsen et al. (2017), cited Cornu (1981, 1983, 1991) as finding that everyday meanings of terms involved the mathematical definition of limits impacts students' conceptualization of limit. As an example, they discuss students' interpretation of the term "approaching" as getting progressively closer to an unreachable value, while the term "limit" as boundary not to be surpassed. Bressoud et al. and Larsen et al. indicate both interpretations are akin to the notion of limit as a process rather than as process encapsulated into a number (Cottrill et al., 1996; Tall & Vinner, 1981), which is the conceptualization consistent with the formal definition of limit.

As to ways of thinking impacting students' understanding of the concept of limit, Larsen et al. (2017) cite Szydlik's (2000) finding that students who view calculus as a set of procedures and facts to be memorized possessed increased misconceptions of limit as an unreachable value or a boundary, whereas students who view calculus as a body of meaningful knowledge they are capable of figuring out were more likely to demonstrate understanding of limit consistent with the formal definition.

Both Bressoud et al. (2016) and Larsen et al. (2017) discuss Oehrtman's (2009) typology of metaphors, or concept images, governing students' conceptualization of limit, the most useful in supporting students' reasoning involves the image, which can be expressed as follows: the measures of two quantities, A and B, are in a state of a rule-governed covariation proceeding reflexively in the following manner: a change in the measure of A brings about a change in the accumulation of the measure of B; and as the accrual's increments to a particular measure of A progressively diminish to zero, B reaches the measure corresponding to f(B) in the covariation.

Larsen et al. (2017) found Oehrtman's classification of limit as approximation as highly useful, most notably in the interpretation of Taylor series and Riemann Sum. Rasmussen et al. (2014) cite Kouropatov and Dreyfus (2014) observation regarding students' ability to leverage such images of approximation, in combination with the notion of accumulation, to develop a perceptual understanding of the integral as a foundational background for learning the Fundamental Theorem of Calculus.

Other common concept images students construct for the concept of limit identified by Oehrtman (2009) include the proximity image (as x gets closer to y, $f(x)$ gets closer $f(y)$), infinity-is-a-very-big-number image, and physical limitation image (a smallest positive number exists). Collectively, researchers have concluded that these images—whether they are the effect of narrow instruction or of cultural and linguistic references—are a source of students' difficulties with the formal concept definition of limit. Larsen et al.'s survey found that such images are hard for instruction to alter, let alone eradicate.

1.1.1 Derivative

Both Bressoud et al. (2016) and Larsen et al. (2017) cite Orton's (1983a) article which showed that while students are capable of carrying out procedures to differentiate functions, they do not understand the quantitative meaning of derivative in terms of average and instantaneous rate of change; nor do they understand differentiation symbolically as limit at a point, or graphically as the slope of a tangent line at a point. Students' difficulties with the concept of rate of change as well with the concept of slope was found by Orton to be rooted in their impoverished understanding of the concept of ratio as a multiplicative relation—a concept students should have mastered in early grades.

A less elementary source of students' difficulties with the concept of derivative is their inability to coordinate between covariations—specifically, between the covariation expressed in the function being differentiated and the covariation between the slopes of secant lines and the input points of the function, as the secant lines progressively approach the tangent line. Furthermore, according to Nemirovsky et al. (1991), students' weak understanding of the underlying quantitative meaning of derivative was found to account for their rigid focus on superficial visual features common to a graph of function and the graph of its derivative, rather than attention to quantitative relations between the two graphs as covariations.

Both Bressoud et al. (2016) and Larsen et al. (2017) found a study by Bingolbali et al. (2007) revealing. The study explores the influence of the departmental affiliation of students on their conceptualization of derivative—while mechanical engineering students tend to focus on rate of change, mathematics students incline towards tangent-oriented aspects.

Attention to rate of change is more consistent with understanding of a function as a covariation (as a process, using APOS theory terms) whereas tangent oriented view is more consistent with understanding a function as a static object. Larsen et al. (2017) surveyed the work by Confrey and Smith (1994) and Thompson (1994b) indicating that the latter understanding of function is a source of problems students have with the interpretation of derivatives. Thompson and Carlson (2017) documented the importance of covariational reasoning to the development of conceptual understanding of the derivative.

In sum, these surveys suggest that quantitative reasoning with functions expressed dynamically, analytically, and graphically; ratio as a multiplicative relation; and coordination among covariations, are underlying cognitive roots for meaningful understanding of derivative.

1.1.2 Integral

The research on students' understandings of integrals surveyed by Bressoud et al. and Larsen et al. makes a strong case that students' understanding of integrals is largely procedural, lacking quantitative meanings of accumulated change. Furthermore, textbooks typically focus on definite integrals, and do so symbolically as Riemann sums and graphically as area under a curve. It is not surprising, then, that Sealey (2008, 2014) found that students can compute the area under a curve but are not able to relate their computations to a Riemann sum as an accumulation. Definite integrals are a number. The upper limit must vary for an integral to be a function of one variable. But the upper limit does not vary in a definite integral, nor does the interval over which a Riemann sum approximates a definite integral.

According to Larsen et al.'s (2017)' discussion of Jones' (2013) article, the most successful conceptualizations for dealing with the integral involved identifying the integrand as the derivative of some function, and "adding up pieces"—the latter being closely related to the concept of accumulation. We note that Jones (2013) did not find students who understood "derivative of some function" to mean "rate of change of some function". Rather, students understood "derivative" as the result of operating symbolically on an antiderivative.

Larsen et al. (2017) pointed to Thompson's and Silverman's (2008) observation that even though the concept of accumulation is rooted bodily in students' day-to-day experience, in calculus students experience difficulty conceptualizing integration as an accumulation of multiplicative measures created in the process of covarying two quantities. It is critical to emphasize that key to this observation is the ability to coordinate schemes for, multiplicative measures, accumulation, and covariation of the two quantities. Larsen et al. noted also that the multiplicative measures in Thompson's model are not static representative rectangles; rather, they are "bits" of accumulations, each involves an average rate of change over an interval, the accrual of which is the net accumulation from some reference point. Larsen et al. (2017) say in sum that it is this network of constructs and their coordination that are crucial to understanding the fundamental theorem of calculus operationally.

2 Conceptual analysis of ideas foundational to calculus

As noted by Larsen et al. (2017), early research on understanding ideas of calculus was on students' "misconceptions"—ways students' behavior departed from what authors took as normative understandings of topics like derivative, integral, and limit. Later research focused on students' understandings per se, understandings that expressed themselves in what prior observers classified as misconceptions. Larsen et al. then described several efforts to design entire courses taking into consideration ways students tend to understand ideas of calculus. We believe it will be profitable for the research community to take a new perspective on students' difficulties in calculus—that many student difficulties in calculus are due to the meanings and ways of thinking at the root of variable, function, and rate of change that students develop in elementary, middle and early high school.

2.1 Meanings and ways of thinking foundational to understanding variables and functions in calculus

Our conceptual analyses focus on foundational meanings and ways of thinking for variable, function, rate of change, and accumulation in separate subsections. At the same time, it is important we acknowledge that they form a web, each drawing on aspects of others. It is also important we say our conceptual analyses are founded in scheme theory, where by "scheme" we mean "an organization of actions, operations, images, or schemes—which can have many entry points that trigger action—and anticipations of outcomes of the organization's activity" (Thompson et al., 2014, p. 11). While this definition of scheme might seem circular, it is not. It is recursive. A person's scheme for constant rate of change, for example, could call upon her schemes for variation, accumulation, and proportionality.

We also alert readers that while discussing schemes for variable, function, rate of change and accumulation that are productive for understanding ideas of calculus we will speak at greater length about variable than about the others. Our motive is twofold: (1) we claim the field's understanding of ways students do and should understand variables is underdeveloped; and (2) we will argue that it will be highly productive for calculus students' learning of function, derivative, and integral were they to have particular ways of understanding variables.

2.1.1 Variable

Schoenfeld and Arcavi (1988) shared ten definitions of "variable", each reflecting a different usage of symbols in mathematical statements—ranging in formality from a quantity whose value varies to any symbol whose meaning is not determined. They go on to say, in line with Freudenthal (1983), "the dynamic aspects of the variable concept should be stressed whenever it is appropriate and feasible" (Schoenfeld & Arcavi, 1988, p. 426). In other words, they promote the idea that values of variables vary. However, to say values of variables vay begs the question of what it means for a person to understand "vary". One meaning is you can substitute one value for another: "Suppose the value of x is 2. Now let the value of x be 3." This meaning does not entail an image of continuous variation that is critical for understanding ideas of calculus.

Castillo-Garsow (2012) reported a teaching experiment in which high school students displayed different ways of envisioning continuous variation. The first he called "chunky variation". The second he called "smooth variation". A person thinking with chunky variation envisions a variable's value varying in "chunks"—the variable's value goes from one value to another like measuring sticks being laid. Intermediate values are there, but the person does not think of the variable attaining them. One student in Castillo-Garsow's study repeatedly referred to a bank accounts balance growing each year, each month, and each day—but never spoke of the balance growing within a year, month, or day. This same student, after saying 65 miles per hour means you go 65 miles in one hour, initially said "no" when asked, "Can you travel 65 miles per hour for one second?", then hesitated and said you could if you changed one hour to seconds.

A person thinking with smooth variation envisions change in progress within intervals of change. Castillo-Garsow described a second student who described a bank account's value growing constantly.

> Smooth thinking, in contrast, is inherently continuous … When Derek imagined change in progress from zero to one year, he could not imagine jumping directly to one year, because in order for time to get to one year, it has to pass through every moment of time before that year. (Castillo-Garsow, 2012, p. 62)

Castillo-Garsow et al. (2013) went on to describe smooth thinking more explicitly in terms of envisioning change in the flow of time.

> Ongoing change is generated by conceptualizing a variable as always taking on values in the continuous, experiential flow of time. A smooth variable is always in flux. The change has a beginning point, but no end point. As soon as an endpoint is reached, the change is no longer in progress. (Castillo-Garsow et al., 2013, p. 34)

Consequences of students not thinking that variable's values vary can be severe.[1] White and Mitchelmore (1996) reported their effort to emphasize conceptual understanding of derivative as rate of change.

> The number of students who could symbolize rates of change in noncomplex situations increased dramatically. However, there was almost no increase in the number who could symbolize rates of change in complex items or in items that required modeling a situation using algebraic variables. Detailed analysis revealed three main categories of error, in all of which variables are treated as symbols to be manipulated rather than as quantities to be related. (White & Mitchelmore, 1996, p. 79)

White and Mitchelmore's statement that students saw symbols to be manipulated as distinct from representing quantities' values points to students' not reasoning quantitatively about situations to be modeled. More to the point, however, students did not see symbols as representing values of quantities whose values varied, which was essential for them to think of quantities having a rate of change with

[1] We readily admit the power of using symbols to represent "unknowns". For example, Schoenfeld (1988) posed this problem: *Reverse the digits in 39 and 62. Then 39×62=93×26. Are there other pairs of numbers having this property?* By representing digits' place value you get $(10a+b)(10c+d)=(10b+a)\ (10d+c)$, which reduces to $ac=bd$.

respect to each other. In our judgment, White and Mitchelmore's observations point less to the ineffectiveness of their treatment and more to the effect of meanings for variable students developed in prior schooling.

While there is an abundance of research on the development of elementary and middle school students' concept of variable, there is little research on ways students envision variables' values varying. One reason is that many researchers themselves presumed "varying" meant replacing one value by another (e.g., Ayalon et al., 2015; Bardini et al., 2005; Duijzer et al., 2019; Graham & Thomas, 2000; Küchemann, 1978, 1981, 1984; Moss et al., 2019; Radford, 1996; Schliemann et al., 2003; e.g., Trigueros & Ursini, 1999; Weinberg et al., 2016).

Other researchers investigated the development of algebraic reasoning, but defined it as generalizing from operations on numbers—which most often were whole numbers (Blanton, 2008; Blanton et al., 2015, 2019; Brizuela & Earnest, 2007; Carpenter et al., 2003; Carraher et al., 2000; Franke et al., 2007; e.g., Schifter et al., 2007), which leads naturally to students' taking for granted that variables have natural number values (Van Hoof et al., 2014). Letters still stood for unknown or to-be- known numbers in students' experience when generalizing from arithmetical operations on numbers. Other researchers investigated students' algebraic reasoning, defining algebraic reasoning as using letters to represent numerical patterns. To researchers investigating children's meanings of variables letters stood for replaceable numbers—one number could be replaced by another in the general statement. We see early algebra researchers' common use of "vary" to mean drawing from a replacement set as explaining the dearth of research on school students' images of ways variables' values vary.

What meaning for variable's values would we have students develop in lower grades that would support later learning in calculus? Certainly, meanings that entail smooth variation. But even that would not be enough. If calculus students are to understand something akin to instantaneous rate of change, they must envision that smooth variation happens in bits. Understanding the expression

$$\frac{f(x+h)-f(x)}{(x+h)-x}$$

as a quotient of variations requires students to first understand the value of x as a variation from 0, the value of $(x+h)-x$ as a variation from x to $x+h$, the value of $f(x)$ as a variation from 0,[2] the value of $f(x+h)-f(x)$ as a variation from $f(x)$ to $f(x+h)$, and the quotient to be a measure of

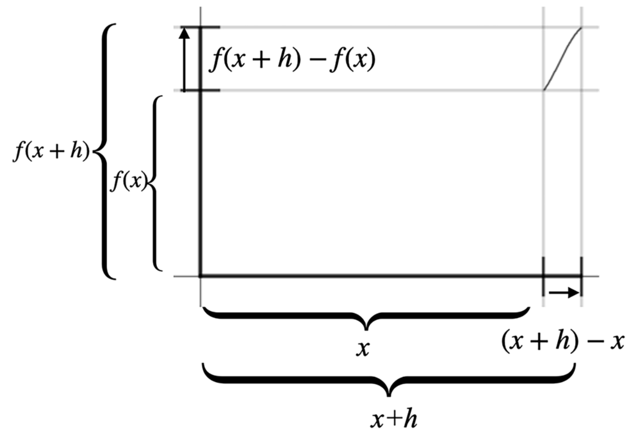

Fig. 1 Variation in the value of x and the corresponding variation in the value of $f(x)$

the relative size of the two differences. But *what* should they understand actually varies? We cannot say that the value of x varies from x to $x+h$. We are forced to create a mental object which can have a value of x and have a value of $x+h$. Then the value of this mental object can vary from x to $x+h$ and the value of another mental object can have a value that varies from $f(x)$ to $f(x+h)$. We argue that students would be more successful in creating such mental objects if they had experiences in earlier grades thinking about and speaking of actual quantities with actual values that vary.

For example, if x represents the height of water in a tank in centimeters and $f(x)$ represents the volume of water in the tank at height x centimeters, then $(x+h)-x$ is the variation in the water's height from x to $x+h$ and $f(x+h)-f(x)$ is a corresponding variation in the water's volume as the height varies from x to $x+h$ (Fig. 1). It is important to see in Fig. 1 that values and variations are represented on *axes*. This emphasizes that it is variation in quantities' values that students must relate.

The value of h is the "bit" by which the value of the water's height varies after varying from 0 by x centimeters. The value of $f(x+h)-f(x)$ is the "bit" by which the water's volume varies from $f(x)$ to $f(x+h)$ centimeters after varying from 0 by $f(x)$ centimeters, and the water's volume covaries (varies simultaneously) with its height as the height varies from x by h centimeters. Not only does the height's value vary smoothly from x to $x+h$ after varying from 0 by x centimeters, it varies smoothly by bits within any interval contained in x to $x+h$. This is the motive for Thompson and Carlson (2017) defining smooth continuous reasoning, the highest level of variational thinking, recursively:

> [A person reasons with smooth continuous variation when] the person thinks of variation of a variable's … value as increasing or decreasing … by intervals *while*

[2] Here we finesse the matter of function notation, which is notoriously confusing for students and teachers (Carlson, 1998; Even, 1990; Sajka, 2003).

anticipating that within each interval the variable's value varies smoothly and continuously. The person might think of same-size intervals of variation, but not necessarily. (Thompson & Carlson, 2017, p. 440; emphasis added)[3]

We see little evidence that students in lower grades are supported in building images of large variations in a variable's value happening in small amounts. The default change is + 1, which students can assimilate easily into a "replacement" meaning of varying—x is 0, then 1, then 2, and so on. We suspect that textbook authors and teachers may feel that students need to actually calculate changes in order to reason about them. A practice of calculating all changes would interfere with students building an image of a variable's value varying from, say, 2 to 3 in 10,000 bits. A more natural approach might be for teachers to speak consistently about changes happening smoothly and to manage reflective discussions of animations used in problematic contexts showing small bits of variation as they make large variations (like grains of sand being poured into a cup). The teacher could pause the animation at several moments to discuss the accumulated variation and its measure.

2.1.2 Function

Students conceiving variation productively affords them the possibility to think of covariation productively. Carlson et al. (2002), Kruger (2019), Nemirovsky (1996), Oehrtman et al. (2008), Thompson (1994a), and Thompson and Carlson (2017) make a compelling case that understanding functions covariationally—as an invariant relationship between two quantities' values as they vary simultaneously—is the most important meaning of function for students learning calculus. But it is too much to ask students to develop this meaning for function in their study of calculus. It is entirely appropriate to emphasize covariation of quantities' values in middle school and lower high school (Confrey & Smith, 1995; Ellis et al., 2016; Smith & Confrey, 1994).

The science education research communities are seeing the importance of smooth continuous variational and covariational reasoning in physics (Boudreaux et al., 2020; Brahmia et al., 2021; Carli et al., 2020; Christensen & Thompson, 2012; Doughty et al., 2014; Fuad et al., 2019; Lucas & Lewis, 2019; McDermott et al., 1987; Sokolowski, 2020; Thompson, 2006), chemistry (Rabin et al., 2021; Rodriguez et al., 2018), biology (Bressoud, 2020; Lehrer et al., 2020; Roth & Temple, 2014), geosciences (González, 2021), and economics (Feudel & Biehler, 2020). This is not surprising to us. The ideas of variation in quantities' values and covariational relationships among quantities' values are central to understanding ideas in science and economics. They are also central to students' developing powerful understandings of accumulation and rate of change—foundational ideas for students to recognize the utility of calculus in scientific fields. We discuss accumulation and rate of change in the next section.

2.1.3 Accumulation and rate of change

The importance of students conceiving integrals as accumulation functions is gaining wide acceptance (Carlson et al., 2001; Doughty et al., 2014; Jones, 2013; Kouropatov & Dreyfus, 2013, 2014; Palha & Spandaw, 2019; Sealey, 2014; Swidan, 2020; Swidan & Naftaliev, 2019; Thompson & Silverman, 2008; Thompson, 1994a). It is also clear from this same research that the idea of an accumulation function is nontrivial for students. Our reading of the literature is that students' understandings of accumulation can break down at many points. First, they must be able to reason about variation of quantities' values happening cumulatively, in bits, where the bits are at least chunky (Castillo-Garsow, 2012) as distinct from envisioning variation happening as if a rubber band is being stretched. With the latter image, nothing accumulates even though the quantity's value varies. Further, for students to conceptualize an accumulation of one quantity to have a rate of change with respect to accumulation of another, students need to envision variations happening within bits—at least smoothly and at best smoothly and continuously (Byerley 2019).[4] Jones (2013), Sealy (2014), and Thompson (1994a, 1994b) documented difficulties students have conceptualizing integrals as Riemann sums when they do not envision variations in variables' values happening in bits and do not conceive the variations covariationally.

Our characterization of conceptualizing functions as covariational relationships between quantities' values as they vary, and variations in quantities values as accumulations, highlights the strong connection between calculus mathematically and calculus in scientific applications. Thompson and Ashbrook (2019) leveraged this connection by building approximate accumulation functions with reference point a as Riemann sums with variable upper limits. They then define exact accumulation functions as$A_f(a,x) = \int_a^x r_f(t)dt$, where r_f is a function whose values $r_f(x)$ are the rate of

[3] Thompson and Carlson (2017) were careful to define "variable" as a symbol which represents the value of a quantity (concrete or abstract) whose value varies. They distinguished among using a symbol as a variable, as a parameter, and as a constant, where the distinction resides in what the person intends to represent.

[4] Two quantities conceived as having fixed values cannot be thought to have a rate of change. It would be like asking, "What is the rate of change of 5 with respect to 3?".

change of a function f at each moment of its domain. The interpretation they offer students is, as explained by Ely (this volume), that $\int_a^x r_f(t)dt$ means as the value of x varies, t varies smoothly from a to x, $r_f(t)dt$ calculates an infinitesimal bit of accumulation for each value of t, and the integral is the hyper-sum of the bits of accumulation.

For Thompson and Ashbrook (2019), values of the function f and x can be measures of any quantities that change in relation to each other. They treat area bounded by a curve as just a special case among many quantities, showing that $f(x)$ is the rate of change of accumulated area as the value of x varies in rectangular coordinates, and that $f(\theta)^2/2$ is the rate of change of accumulated area as the value of θ varies in polar coordinates. Physical quantities are treated likewise. Density is a rate of change of mass with respect to volume, concentration is a rate of change of solute in relation to solvent, force is the rate of change of momentum with respect to time, pressure is a rate of change of force with respect to area, and so on. This approach addresses the problem students in science courses have with understanding how integrals-as-areas apply in physical situations (Nguyen & Rebello, 2011).

In summary, research suggests many students have little opportunity to build meanings for variable, function, accumulation and rate of change in early grades upon which they can build productively for understanding ideas in calculus. Our conceptual analyses suggest students will have a greater chance to conceive ideas in calculus productively when ideas of variation, covariation, function, accumulation, and rate of change are addressed thoughtfully in lower grades so their symbolic representations of them retain their imagistic foundations. As Thompson (2008) insisted, "It takes 12 years to learn calculus". We urge the calculus research field to consider this adage seriously and investigate the content of students' learning in earlier grades in relation to their successes and difficulties in calculus.

3 Articles in this special issue

Articles in this special issue address a broad range of concerns related to calculus learning and teaching: students' preparation for calculus; learning and teaching calculus per se; societal, political, and educational aspects of changes in high school and university calculus; and learning and teaching multi-variable calculus.

3.1 Preparation for calculus

Frank and Thompson (this issue) delve into one aspect of issues we raised in our introduction. They illustrate ways curricula and instruction in the US pre-calculus curricula fail to support students' understandings of important ideas foundational to the calculus. Toh (this issue) probes a different connection between students' early mathematics and their study of calculus. He demonstrates that in Singapore, even with its vaunted international standing on TIMSS and PISA, the calculus curriculum is remarkably unconnected to its highly conceptual elementary and middle school mathematics curricula. Bressoud (this issue) wonders whether US students taking calculus in high school would be "better served with more time spent on algebra and precalculus rather than studying calculus".

Bressoud's comment goes to the heart of the question, what does "knowing calculus" mean? He quotes the recommendation by Steen and Dossey (1986) that calculus should not be taught in US high schools unless algebra had been taken in 8th grade. This recommendation can be and has been interpreted misguidedly, that what students require is the same algebra as normally taught, earlier. We suspect, as demonstrated by articles in this issue, that students need a different kind of algebra—one founded in generalizations of reasoning quantitatively—and a different calculus, one that builds on a new algebra. Harel (this issue) develops a strong argument that, even in multivariable calculus, for students to develop ideas meaningfully they must develop a strong foundation in reasoning quantitatively about covariation, rate of change, and accumulation. The articles by Frank et al., in our eyes, reveal the underlying issue in calculus curricula and instruction that seems common across nations. It is that derivatives and integrals, the major topics in calculus, even when developed "informally", fail to engender an intellectual need for them, where "intellectual need" is as described in (Harel, 2013).

We distinguish preparation for calculus from what others have called preparation for university mathematics (e.g., Thomas et al., 2015). Ghedamsi and Lecorre (this issue) address the latter case. Their article includes analyses of (a) commonalities, not just discrepancies, between high school calculus and university calculus; (b) university instructors' "calculus knowledge for teaching", which includes both the content of high-school calculus as well as pedagogical considerations employed by high-school teachers in teaching that content; and (c) possibilities for high school teachers to readjust their calculus teaching actions for advancing students' preparation for university calculus. The study shows that attention to theoretical foundations and deductive coherence is a common feature to high-school calculus and university calculus. However, despite this, high-school students enter university calculus with major shortcomings in dealing with formal definitions and theorems.

The distinction between preparation for calculus and preparation for university mathematics is needed when looking across international borders; Ghedamsi and Lecorre's study (this issue) is a case in point. In contrast, by "calculus" we mean integral and differential calculus presuming the real continuum and presuming nearly-smooth functions (smooth except for a finite number of discontinuities). Students in many countries meet ideas of calculus in high school—some studying just differential calculus and some studying both differential and integral calculus. Analysis, the study of real numbers and real-valued functions, is commonly this group's first mathematics course in university. As explained by Bressoud (this issue), a minority of US high school students take either differential or differential + integral calculus in high school. Many of these students take the same content in university. For other students in the United States, university-level differential and integral calculus is their first contact with calculus.

3.2 Issues pertaining to calculus learning and teaching per se

Several articles in this special issue expand upon research surveyed in Rasmussen et al. (2014) and Bressoud et al. (2016) to focus on what students learn or what they might learn more productively. In this regard there is a difference of perspectives among authors from which they examined issues of learning and teaching calculus.

To say what students learn in calculus depends highly on what you assess. Tallman et al. (this issue) make a compelling argument that US calculus instructors focus far more on what students are able to do and far less on the quality of students' understandings. Tallman et al.'s argument resonates with that of Frank and Thompson (this issue) in that focusing on students' retention of procedures and skills does not provide students with a repertoire of meanings and ways of thinking upon which they can build for future mathematics learning. On the other hand, the case of Singapore (Toh, this issue) also reminds us that solid preparation in pre-calculus mathematics does little to aid students in learning calculus if the calculus they are expected to learn is disconnected from that rich preparation.

Ely (this issue) describes the history and role of the idea of infinitesimal in calculus, including its replacement by ideas of limit and resurrection in Robinson's (1966) development of non-standard analysis. He argues that differentials as infinitesimal quantities can be more meaningful to students in calculus than are ideas of limit. Ely also argues compellingly that the central issue is not just the relative efficacy between a limit-based calculus and a differential-based calculus, but the relative efficacy between student-centered-meaning-laden calculus and teacher-centered-procedure-laden calculus.

Moreno (this issue) shares a personal story of teaching a course on calculus pedagogy that has general implications. His course is for teachers who teach calculus in high school and college. His approach, in our view, helps teachers gain a clearer distinction between calculus (as we defined it) and analysis. He does this by giving them grounded experience in the historical dilemmas in calculus that gave rise to analysis, such as the need for rethinking ideas of rate of change and accumulation in calculus to accommodate differentiation and integration of monster functions (borrowing this term from Lakatos) like those created by Dirichlet et al. Moreno's approach is highly humanistic in that it focuses on the experience of teachers in his course so they can have a better understanding of what is properly the domain of calculus and what is properly the domain of analysis. In Harel's terms, Moreno aims to establish intellectual need for ideas of analysis by having his teachers experience the dilemmas in calculus mathematicians faced that gave rise to it. Bressoud's (2007) *A Radical Approach to Analysis* is a comprehensive implementation of a similar approach to undergraduate real analysis. Scucuglia et al. (this issue) take a humanistic approach to instruction of future calculus teachers similar to Moreno's. They report their approach to engaging future teachers within a "humans with media" perspective. They focus not on "using computers in teaching calculus", but on the change in teachers' thinking that happens in the learning process when computing technology becomes "ready at hand" (Winograd & Flores, 1986), when computing technology becomes an instrument of thinking for them (Artigue, 2002; Drijvers, 2002; Hickman, 1990; Vérillon & Rabardel, 1995).

Hitt (this issue) examines students' performance in the reformed calculus curriculum in Quebec, Canada. A significant element of this reform is an emphasis on modeling activities involving kinematics problems as a source of and motivation for the generation of calculus concepts and ideas, on the one hand, and a deemphasis on symbolic manipulations on the other hand. The examinations centered around students' representations as they attempted to model dynamic situations and the evolution of their representation as they work collaboratively. An important observation Hitt makes in his study is that while students experienced difficulty constructing and articulating coherent representations for these problems when working individually, their collaborative work effort led to convergent thinking that facilitated successful solutions of the problems.

Other articles in this issue worked from the perspective of interpreting students' performance on common calculus tasks through the lens of competent performance, examining the fit between students' thinking and what authors take as institutionalized thinking. Greefrath et al. (this issue) identify four "basic mental models" of the integral which they consider to be institutionalized in the mathematics community. They then assessed the prevalence of these models among students. Their perspective begins with ideas they presume are held by mathematicians and look to see the extent to which students also hold them. They conclude that the mental models they identified cover the spectrum of students' understandings of integrals.

3.3 Societal, political, and educational aspects of changes in high school and university calculus

Curriculum reform at a national level is heavily context dependent. Stakeholders often collectively exert political and intellectual pressure on the group producing the reforms. An extreme example is what were called "the math wars" in the United States (Bishop, 1999; Jacob, 2001; O'Brien, 1999; Schoenfeld, 2004).

Two articles in this issue take very different perspectives on calculus reform in their nations. Yoon et al. (this issue) examine the process of calculus reform in South Korea from the outside, giving an account of the societal pressures on calculus reform, taking into account stakeholders' motives for their support or opposition to the work of reform as it developed. Dreyfus et al. produced an insiders' account of Israel's first curriculum document for high school mathematics, focusing specifically on its calculus component.

The effort in South Korea happened in the context of larger educational reforms that aimed to lessen students' and parents' anxiety produced by a highly rigorous and demanding high school mathematics curriculum. Competing pressure came from groups concerned that relaxed rigor might affect the quality of what South Korean students learned. The effort in Israel was motivated by the disorder that arose naturally over time from the historical lack of mathematics curriculum documents. Dreyfus et al.'s account of this work focuses on the ways mathematics education research informed their decisions (largely indirectly) and the overall design principles of the new mathematics document, which aim to support students having positive mathematical experiences while learning a more coherent mathematics. They also speak of a major constraint on the calculus curriculum document itself—teachers' capacities to teach a more conceptually coherent calculus than the ones to which they had become accustomed.

3.4 Learning and teaching multi-variable calculus

Martinez-Planell and Trigueros (this issue) and Harel (this issue) address learning and teaching multivariable calculus. Martinez-Planell and Trigueros provide a literature review focusing largely on the learning and teaching of calculus of two-variable functions, particularly the conceptual demand involved in its acquisition as students transition from singe variable calculus to multivariable calculus. The study concludes that while understanding the calculus of one-variable functions is essential to understanding the calculus of two-variable functions, it is by no means sufficient to successfully deal with the demand to coordinate among the fundamental planes comprising the 3-dimensional representation of a real-valued function on R^2. At the heart of this conceptual demand, Martinez-Planell and Trigueros argue, is spatial reasoning abilities, communication, and mathematical representation skills in 3D space. Instructional implications drawn from the research review include the need to underline the role of local approximation (i.e., linearization) and the need to introduce 3D geometry early.

Harel, too, addresses the issues of linearization and geometrical representation in the learning and teaching of multivariable calculus, albeit his analyses are not restricted to the calculus of two-variable functions. He demonstrates how current instruction pays scant attention to linear approximation despite the central role it plays in the study of calculus, and how the prevalent treatment of linearization in current instruction is through graphical representation with insufficient attention to phenomenological experiences involving quantitative covariation. Harel theorizes the potential consequences of current multivariable instruction against those entailed from instruction guided by the DNR framework. His analysis reveals advantage of the latter over the former in the quality of understanding the concept of function as a covariational processes; in understanding the concept of derivative as a linear approximation; in understanding the relation between composition of functions and their Jacobian matrices; in understanding the idea underlying the concept of parametrization and the rationale underpinning the process of implicit differentiation; as well as in thinking in terms of structure, in constructing coherent mental representations; and in making logical inferences.

4 In closing

The articles in this special issue speak collectively to calculus learning and teaching around the world. Some articles report national efforts to take research on student learning into account as one factor in redesigning national curricula. Other articles examine efforts to improve students' learning of specific ideas. Several articles portray calculus in disarray in the United States—it being largely a university course except for a small number of high school students who take calculus in high school and move directly to a higher calculus at university, and university courses being assessed in contrast to instructors' stated goals. Calculus in Europe is commonly a high school subject, but the state of students' preparation for it is largely unexamined. Calculus in Singapore, at least as suggested by Toh, is largely disconnected from students' conceptual preparation for it in prior grades. One thing we haven't learned is the extent to which calculus is being taught in line with the issues we articulated at the beginning of our article—the extent to which calculus emphasizes variation, rate of change, and accumulation.

These considerations lead us to three questions for future research.

1. What prior-to-calculus mathematical meanings and reasoning abilities might students develop in elementary, middle, and secondary grades to enhance their transition to calculus? To what extent must prior-to-calculus curricula and instruction target those meanings with an eye to the ways they are important for learning calculus? To what extent must calculus textbook writers and instructors have an image of ways pre-calculus meanings and ways of thinking are foundational for ideas in calculus to leverage them productively?
2. Successful curricula require constructive and robust interactions among a triad of elements: (a) production of national mathematics curriculum documents, (b) mathematics education research, and (c) teachers' understandings of content, cognition, and pedagogy. Investigating these interactions in a way that respects differences among nations and cultures requires sociological and cognitive methodologies outside of traditional mathematics education research. How can these interactions be investigated systematically at an international level?
3. Differentials-based calculus is not accepted widely. Why? Is it an interaction between what textbook publishers believe textbook adopters want and attitudes and beliefs among textbook adopters about what students need to learn? Might it be that the tension is not the relative efficacy between a limit-based calculus and a differential-based calculus, but instead between beliefs about the relative efficacy of student-centered-meaning-laden calculus and teacher-centered-procedure-laden calculus?

These questions by themselves are overly broad, but they provide an umbrella for studies with more focused questions that contribute to answering them.

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