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ORIGINAL PAPER

The Use of Digital Technology to Estimate a Value of Pi: Teachers' Solutions on Squaring the Circle in a Graduate Course in Brazil

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Abstract
In this research we investigate how mathematics teachers, as graduate students, estimate the value of π by exploring the problem of *squaring the circle* using digital technology. Initially, we mention some aspects of teaching and learning of calculus in the literature, emphasizing studies that use the notion of humans-with-media to highlight the role of technology in mathematical thinking and knowledge production. Insights on the history of mathematics in calculus are also discussed. We developed a qualitative study based on three different solutions created by groups of teachers using the software GeoGebra and Microsoft Excel. All the teachers' solutions improved the approximation $(8/9)^2 \approx \pi/4$, by determining p and q for $(p/q)^2 \approx \pi/4$. The first two solutions with GeoGebra emphasized experimentation and visualization, improving the approximation from one to three decimal places. The third solution with Excel pointed out the elaboration of a formula and improved it up to six decimal places. We emphasize how media shaped the strategies and solutions of the groups. Based on these solutions, we explore an approach for *cubing the sphere*, discussing approximations for π, highlighting the role of media in enhancing conceptual complexity in the solution of mathematical problems. The nature of the strategies for solving a problem is discussed, especially regarding different ways of thinking-with-technology developed by collectives of teachers-with-media. Although we acknowledge an alternative design for the proposed task, the exploration of problems using aspects of the history of mathematics contributes to the state of the art concerning the studies in calculus developed by the Research Group on Technology, other Media, and Mathematics Education in Brazil.

Keywords Calculus · History of Mathematics · Humans-with-media · GPIMEM

1 Introduction

Several issues in the teaching and learning of calculus have been investigated in mathematics education (Thompson et al. 2013; Rasmussen, Marrongelle, and Borba 2014), such as students' low achievement at the undergraduate level (Edge & Friedberg, 2015) and students' difficulties in understanding the concepts of limit, continuity, derivative, and integral (Bezuidenhout, 2001; Tall, 1993). Other topics that have been explored include the complexity of mathematical language introduced to mathematics majors, in contrast to the level of mathematical symbolism and rigor in elementary and high school (Tall, 1991), and teachers' beliefs and knowledge about teaching calculus (Eichler & Erens, 2014). In order potentially to overcome these issues, which are also matters of affect and attitudes (Sierpinska et al. 2008), alternate methodologies for research regarding the use of digital technology and the history of mathematics have been designed (Sousa & Gomes, 2020; Thompson & Ashbrook, 2019).

Since the 1980s and 1990s, researchers have investigated the use of technologies such as graphing calculators and computer software in the teaching and learning of calculus (Kaput & Thompson, 1994; Rochowicz, 1996). More recently, since the 2000s, the literature has emphasized the role of visualization and experimentation/simulation (as is possible using GeoGebra) in the comprehension, understanding, and production of meanings and knowledge about concepts of calculus (Hohenwarter et. al. 2008; Thompson, 2016; Byerley & Thompson, 2017). Technology also has

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the potential to reorganize connectivity and power relations in classrooms, providing educational alternatives to traditional pedagogic dynamics, since the heuristic nature of digital technology offers different types of problems to solve, including open-ended tasks (Honorato & Scucuglia, 2019).

The history of mathematics, and in particular the history of calculus (Baron, 1969), has also provided conceptual elements that can be used to reorganize aspects of the teaching and learning of calculus. Katz (1993) stated that "[a]n historical approach to the teaching of calculus provides the students with a better understanding of the material than the standard approach and helps as well to introduce them to the relationship between mathematics and other aspects of our culture" (p. 243). The way problems about quadrature and problems about tangents were historically explored, or the contrast between approaches and notations used by Newton and Leibnitz, may offer pedagogic elements to conceptualize alternate perspectives in the planning for calculus classes. Such teaching may provide students with the joy of discovery in learning calculus. It is also important to highlight that, according to Tall and Katz (2014), "[t]he calculus today is viewed in two fundamentally different ways: the theoretical calculus used in applications, often based on the symbolism of Leibniz, enhanced by the work of Cauchy using infinitesimal techniques, and the formal mathematical analysis of Weierstrass" (p. 97). Thus, the history of mathematics has conceptual and pedagogical significance for the teaching and learning of calculus.

In Brazil, the teaching and learning of calculus usually emphasizes symbolic-formal approaches. The strict focus on proofs and demonstrations has been highlighted in undergraduate courses in the exact sciences in general. Historically, since the decade of the 1950s, the prevalence of courses in mathematical analysis in mathematics curricula at the university level has mostly transformed calculus courses in 'pre-analysis' endeavours. The level of formalism and rigor required from first year majors has contributing to increasing the failure of students in calculus. In contrast, the variety of calculus textbooks in Portuguese has offered pedagogic alternatives to strict formalist pedagogic scenarios, by including the use of digital technology. Thus, researchers on teaching and learning of calculus in Brazil have explored computer-based experimental approaches.

In the research discussed in this paper, we explored the use of digital technology, lightly regarding aspects on the history of calculus as a starting point for teachers' mathematical activity and learning. We present different teachers' solutions to the 'squaring the circle' problem (Baron, 1985), emphasising estimations for π. The participants of the study were masters and doctoral students in graduate programs at São Paulo State University in Brazil. The overall aim of the study was to investigate how mathematics teachers, as graduate students, use digital technologies to explore, and potentially to extend, problems from the history of calculus. Specifically, we discuss how the use of different technologies may offer ways to explore different strategies and solutions concerning the 'squaring the circle' problem (Baron, 1985); that is, in a particular scenario, we analyze pedagogic aspects of thinking collectives of teachers-with-media in the interface between calculus and the history of mathematics in terms of trends in mathematics education (D'Ambrosio & Borba, 2010).

2 Research on calculus at GPIMEM related to humans-with-media

The notion of humans-with-media highlights that "cognition is not an individual enterprise, but rather collective in nature… Media have usually been considered useful to support learning and improve teaching, but [it is important] to emphasize that media transform and reorganize those activities" (Villarreal & Borba, 2010, p. 51). More recently, Engelbrecht et al. (2020) have shown how a perspective that recognizes agency in non-human-entities is essential for the discussion of different technologies and mathematics education. In the introduction to this special issue on digital technology, it is argued that such a perspective is also important in understanding the current pandemic, in which a virus would have agency (Engelbrecht et al. 2020).

Since Borba's (1993) work on multiple representations of functions, the Research Group on Informatics, other Media, and Mathematics Education (GPIMEM) at São Paulo State University in Brazil has investigated several aspects of the role of technology in the teaching and learning of calculus (e.g., Villarreal, 1999; Scucuglia, 2006; Barbosa 2009). The research produced in GPIMEM highlights ways to think-with-technology, in which visualization and experimentation/simulation with digital media reorganize thinking (Rabardel & Bourmaud, 2003; Tikhomirov, 1981) and condition the production of knowledge of thinking collectives (Levy, 1993).

According to Borba and Villarreal (2005), knowledge is not produced only by humans, but by collectives of humans-with-media. This metaphor emphasizes the technological dimension of thinking and intelligence. From this perspective, problems designed for the use of paper and pencil have a different conceptual nature than those designed for the use of computers, for instance. In fact, orality, writing, and informatics may be combined to solve problems: a mathematical problem may be solved through oral dialogue, paper and pencil, and software. Pragmatically, the way one explores functions with paper and pencil is qualitatively different from the way one explores functions with GeoGebra (Borba et al., 2014).

The genesis of the idea of humans-with-media is also related to the research developed by Villarreal (1999) on the teaching and learning of calculus, in which he investigated how majors in Biology explored the concept of derivatives using the software Derive. Through the development of teaching experiments, Villarreal (1999) proposed activities or tasks related to the study of functions and derivatives, such as the following: graphing the derivative from the graph of a function, discussing the relationship between the monotonicity of the function and the sign of its derivative, exploring relations between the zeros of the derivative and the roots of the functions, and plotting the graph of the function from the graph of the derivative, and the graph of tangent lines to the function. In this case, functions and derivatives were content already covered by the class in the applied mathematics course in which the participants were students.

Villarreal (1999) observed that there was a conflict between the concept of the derivative of the function and the tangent line to the graph of the function, and that the form used by students to resolve a conflict generated by the computer, in general, was algebraic. The visual approach provided by the computer was not natural for the students, who frequently resorted to pencil-and-paper to resolve some conflicts. However, students started to think more about the concepts and not just to repeat algorithms. The computer favoured multiple representations, as it offered the opportunity to think about and solve problems in new ways. These provided a greater understanding of the concepts of calculus, illustrating and reinforcing basic concepts. With Derive, it was possible to establish interrelations between graphical, numerical and analytical representations in the exploration of functions and derivatives. The computer offered images and resources that would otherwise be inaccessible to students in communicating new ideas visually and experimentally. The combination of computational-graphing and algebraic representations engaged the thinking collectives formed by students-researcher-with-other-media in relevant processes of mathematical thinking such as elaboration of conjectures and refutations.

Barbosa (2009) also developed teaching experiments to investigate how undergraduate students explore the chain rule using the software Winplot. The research highlighted students' difficulties in comprehending the notion of function composition, which revealed conceptual weaknesses in students' previous mathematical knowledge (in high school). One of the initial tasks laid out by Barbosa (2009) offered an experimental approach in introducing a graphing exploration of composition of functions with computers. Based on the of Winplot, the task offers ways for students to animate the coefficient of functions, explore graphically combinations such as f ← g, and explore similarities and differences between plotted graphs.

Based on analysis conducted by Barbosa (2009), it is important to highlight the nature of the design of proposed task, which refers to an important aspect of humans-with-media. First, the activity offers students specify ways to think-with-Winplot since the design of the task indicates the use of particular resources offered by the software used. Software without the *combination tool*, for instance, would not offer the same pedagogic possibilities in terms of mathematical exploration and thinking. Second, the task has an open-ended design, since the questions and pedagogic guidance offer ways of experimentation/simulation with technology and development of different strategies of solutions to the problem. Therefore, from a humans-with-media perspective, the design of the tasks with computers conditions the spectrum of mathematical knowledge produced by thinking collectives. Problems are problems of collectives of humans-with-media (Borba, 2012).

The Fundamental Theorem of Calculus (FTC) has been deeply explored in mathematics education (Thompson, 1994). In GPIMEM, Scucuglia (2006) conducted teaching experiments to investigate how undergraduate students-with-graphing calculators (TI-83) would explore the FTC. In this study, based on the notion of humans-with-media, Scucuglia (2006) not only emphasized the establishment of functional similarities between a program named *SOMA* that explored Riemann sums and a command named $\int f(x)dx$ for students' mathematical understanding, but also highlighted the processes of mathematical proof regarding the use of technology. Based on the use of specific resources of the TI-83, Scucuglia (2006) proposed a task in which students could generalize patterns obtained from the calculation of the area limited between a polynomial curve and the x-axis in a certain interval [a, b]. The identification of patterns and the generalization of results such as $(b^3 - a^3)/3$ for $f(x) = x^2$, $(b^4 - a^4)/4$ for $f(x) = x^3$, and so on, offered elements for students-with-TI-83 to conjecture that (i) derivative and integral have an 'invertible relationship': $F'(x) = f(x)$, and (ii) $\int_a^b f(x)dx = F(b) - F(a)$. Based on the knowledge that students produced with technology, shaped by affordances of the TI-83, Scucuglia (2006) introduced them to formal proofs of the FTC.

Other studies developed by researchers in GPIMEM, such as Araújo (2002), Javaroni (2007), and Soares (2012), have also been mentioned in the literature related to calculus and the use of technology. Particularly, Domingues and Scucuglia (2019) explored the interface between the history of calculus and the use of GeoGebra.

They revealed the qualitative methodological scope of GPIMEM's studies, involving not only teaching experiments, but scenarios such as knowledge mobilization courses, work in classrooms at different levels, and interfaces to mathematical modeling. They also reinforced the main issues discussed here related to the humans-with-media perspective (Borba & Villarreal, 2005): the role of visualization and experimentation in exploring concepts of calculus (and pre-calculus), the role of technology in thinking-with-media, the technological and open-ended nature of the design of tasks in calculus, etc. Studies such as those of Borba (2012) highlight specificities of online environments designed for teachers to explore varieties of strategies and solutions of problems about functions using computers (e.g., different solutions for $2^x = 5$ based on visual approaches, mostly combined with other types of representations). The study presented in this paper is part of the scope of research on pre-calculus and calculus developed by GPIMEM. This scope is representative of the mosaic of research on calculus and technology developed in Brazil.

3 Methodology

This study is qualitative in nature (Bicudo, 1993; Borba et al. 2018), as the aim is to investigate a profound understanding of the way teachers think-with-technology. The overall research question is as follows: How do mathematics teachers explore problems of the history of calculus using digital technology? Particularly, in this paper, we report on a study in which we investigated how mathematics-teachers-with-media explored the problem of squaring the circle.

The site of investigation and data production for this research was a 90-h graduate course named "Genesis of Differential Thinking", offered as part of two graduate programs at São Paulo State University (UNESP) in 2019. The participants were master's students in the Graduate Program in Teaching and Formative Processes and master's and doctorate students in the Graduate Program in Mathematics Education. Each of the twenty students had a background in mathematics and worked or was working as a mathematics teacher in different school levels and/or as an instructor in a college or university. In this course, they organized themselves into 5 groups. Thus, we consider the participants in this research to be mathematics teachers.

Audio-visual registers of the classes (recordings of the videoconferences), field notes, and registers of asynchronous interactions (e.g., use of WhatsApp in groups) make up the research data. The digital constructions created and shared by the teachers during the course (e.g., dynamic constructions with GeoGebra and other software) are also considered part of the data. For the analysis of the data, we make use of the notion of the qualitative case study (Ponte, 2006), the model of video analysis proposed by Powell et al. (2004), and the notion of triangulation for trustworthiness (Araújo & Borba, 2004).

We made use of the notion of a *convenience sample* in qualitative research (Marshall, 1996), and in this paper we discuss one episode that refers to the exploration of the 'squaring the circle' problem, regarding the Egyptian approach according to Baron (1985). In fact, we analyzed all the videos, online interactions, and teachers' solutions (computer files) that were registered/produced during the first two classes of the course. The analysis was particularly conducted considering the question *How do mathematics-teachers-with-media explore the problem of squaring the circle?*, that is, based on the notion of humans-with-media, we analyzed the role of digital technology in conditioning teachers' mathematical thinking and knowledge production regarding their strategies for solution of the problem.

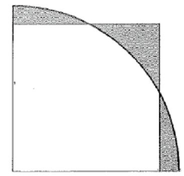
We present three different solutions created by 3 groups of teachers (Groups 1, 2, and 3), namely, two solutions using GeoGebra and one using Microsoft Excel. Although Groups 4 and 5 also explored some possibilities of strategies for solution, they did not present a complete solution for the problem. We also present an extension of this problem, the cubing the sphere, based on the solutions developed by the groups. Teachers' computational solutions are available at https://drive.google.com/open?id=13JkIR4eClN3Xzwmd148AhMh3Ki_brfSh, accessible upon request and with permission of the authors.

4 Squaring the circle

According to Baron (1969), the calculation of curvilinear areas first appeared in attempts to approximate to the area of the circle. "The use of 3 as an approximation for π seems to have been common to most ancient civilisations and arose through taking the mean of the inscribed and circumscribed squares" (p. 11). "The Egyptians used a value equivalent to $\pi/4 = (8/9)^2$" (p. 12). This approach is related to the genesis of differential thinking, since this principle of calculation of areas (and volumes), such as in the method of exhaustion, is present at several moments in the history of calculus.

In Baron's writing (1985), the following statement was presented as a task. In fact, we literally presented this quote to the teachers in the course as a starting point to exploring the squaring of the circle:

> Draw a circular quadrant (eye estimate) and build a square with an area equal to the quadrant you drew. Finally, by measuring the side of the square and expressing it in terms of the radius of the circle, you can see the results compared with Egyptian approximations, that is, side of the square = 8/9 of the radius. (Baron, 1985, p. 13)



Based on the problem/task proposed by Baron (1985), an investigation was introduced in the graduate course based on the following questions: Why is $(8/9)^2$ a 'good' approximation for π /4? Using current notation and considering the radius of the circle equal to 1, that is, r = 1, we have $A_{quadrant} = \frac{\pi \times 1^2}{4^2} = \frac{\pi}{4} \cong 0.785$ and $A_{square} = \left(\frac{8}{9}\right)^2 = \frac{64}{81} \cong 0.790$. Thus, $(8/9)^2$ is a 'good' or, at least, an 'acceptable' approximation for $\pi/4$, to one decimal place. In particular, the specific questions posed in the course, for the teachers to get engaged in discussion and experimentation, were as follows: How can Baron's statement be explored using digital technology? How can we improve the approximation using GeoGebra? That is, rather than 8/9, what would be p/q, with p and q as integers, for a better approximation for $(p/q)^2 \approx \pi/4$?

4.1 Solution 1 with GeoGebra

Group 1 decided to use GeoGebra to explore the squaring of the circle. The decision making involved in the decision to use this digital technology, in particular, is already a matter to investigate in the humans-with-media perspective. At this point, we assumed that GeoGebra's affordances shaped the mathematical thinking of Group 1 in their attempts to solve the problem. The tools available in GeoGebra conditioned the actions of Group 1 in constructing strategies to square the circle and, consequently, to estimate the value of π.

Teachers-with-GeoGebra suggested that a better approximation for π/4 would be 78/88 squared, instead of 8/9 squared. They plotted the linear function $g(x) = (x\sqrt{\pi})/2$, regarding the fact that $p = (q\sqrt{\pi})/2$ was a key equation in

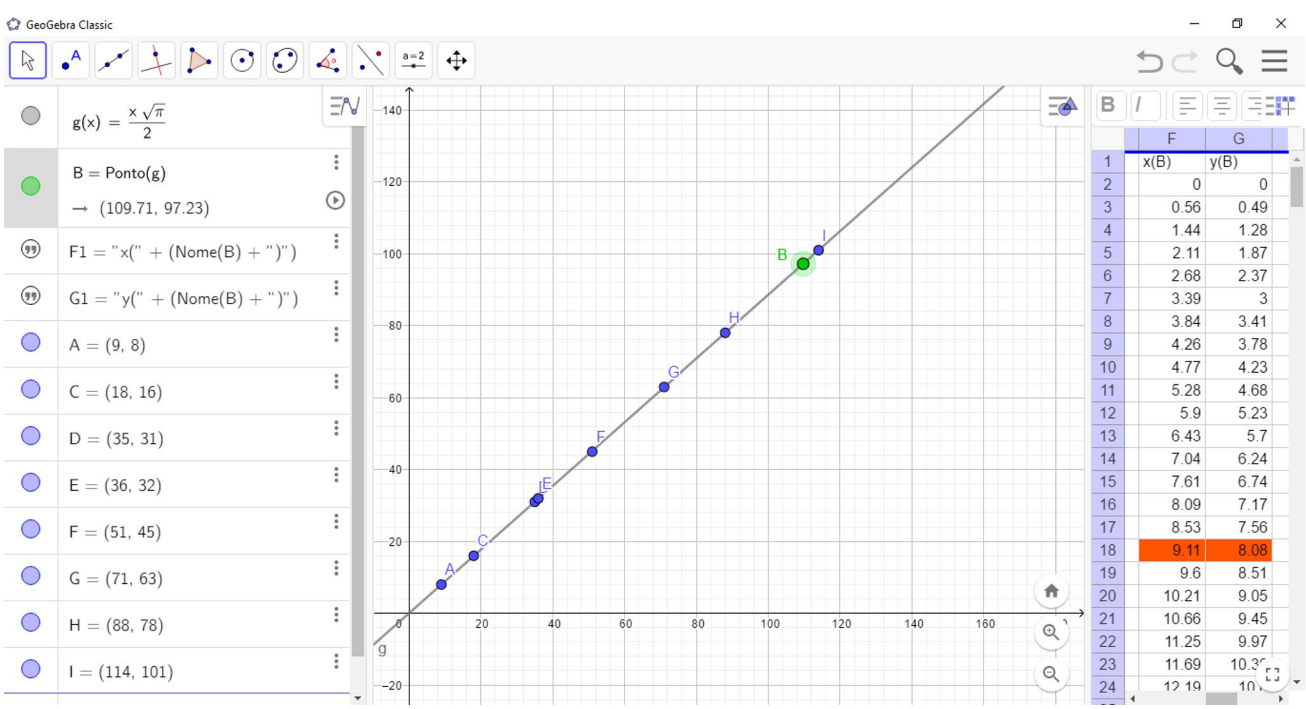


Fig. 1 Figuring out (p, q) from $g(x) = \frac{x\sqrt{\pi}}{2}$

their strategy for solution. Based on GeoGebra's potentialities, their idea was to build the graph of g(x), where they would obtain a line and, using another tool of GeoGebra, they could insert points on that line. Moreover, the collective of teachers-with-GeoGebra would move or animate this point to obtain different ordered pairs (q, p) that would satisfy $p=(q\sqrt{\pi})/2$. The thinking collective initially found several decimal numbers for *p* and *q*, but they then considered those that had a better approximation to the whole numbers. Group 1, thinking-with-GeoGebra, actually performed the construction of two objects to come up with their solution. In the first approach, shown in Fig. 1, they plotted the graph of the linear function $g(x)=(x\sqrt{\pi})/2$, constructed points on the line and then created a spreadsheet. Thus, the coordinates of those points were displayed while animating it. The use of all those resources of GeoGebra shaped the way the teachers developed their strategies to solve the problem, conditioning their mathematical thinking. Their decision making in using graphing and tabular representations of a linear function and other affordances of GeoGebra shaped the way they conducted a solution for the problem.

Using the spreadsheet available in GeoGebra, the teachers-with-GeoGebra selected coordinates that had a good approximation, taking into account the condition of *p* and *q* being integers. Thus, they noticed that A = (9, 8); C = (18, 16); D = (35, 31); E = (36, 32); F = (51, 45); G = (71, 63); H = (88, 78) and I = (114, 101) are points that have integral coordinates very close to the line of $g(x)=(x\sqrt{\pi})/2$.

Analysing the strategy conducted by Group 1, it is important to highlight from a humans-with-media perspective that, initially, the teachers plotted the function $g(x)=(x\sqrt{\pi})/2$. Since π is an irrational number and they were trying to find a pair of natural numbers *p* and *q*, in such a way that $(p/q)^2$ would be an approximation to the value of the area of the first quadrant of the circle with radius equal to 1, they were conceptually dealing with a frontier or tension between rational and irrational numbers. They would not find *p* and *q* in such a way that $(p/q)^2=\pi/4$, although that is the notation used by Baron (1985). They were exploring approximations such as $(p/q)^2 \approx \pi/4$. The use of the spreadsheet or x–y table available in GeoGebra conditioned the way the group selected candidates from pairs (p, q). They identified that 8/9 was a candidate because (9.11, 8.08) was one of the pairs displayed by the software—the most convenient involving integers. Analogously, considering the integral part of the numbers displayed as 'good approximations', the group selected the pair (88.15, 78.12) as a candidate. Thus, 78/88 was the p/q conjectured by group 1. Although Group 1 used GeoGebra's value of π, plotting the graph of $g(x)=(x\sqrt{\pi})/2$ was significant for teachers' (visual and experimental) thinking in the exploration of the task's solution. This is a significant aspect regarding a perspective anchored in the construct humans-with-media: GeoGebra had agency in the choice of the path taken by the group-with-media. The group would not have come up with this particular solution without GeoGebra's affordances.

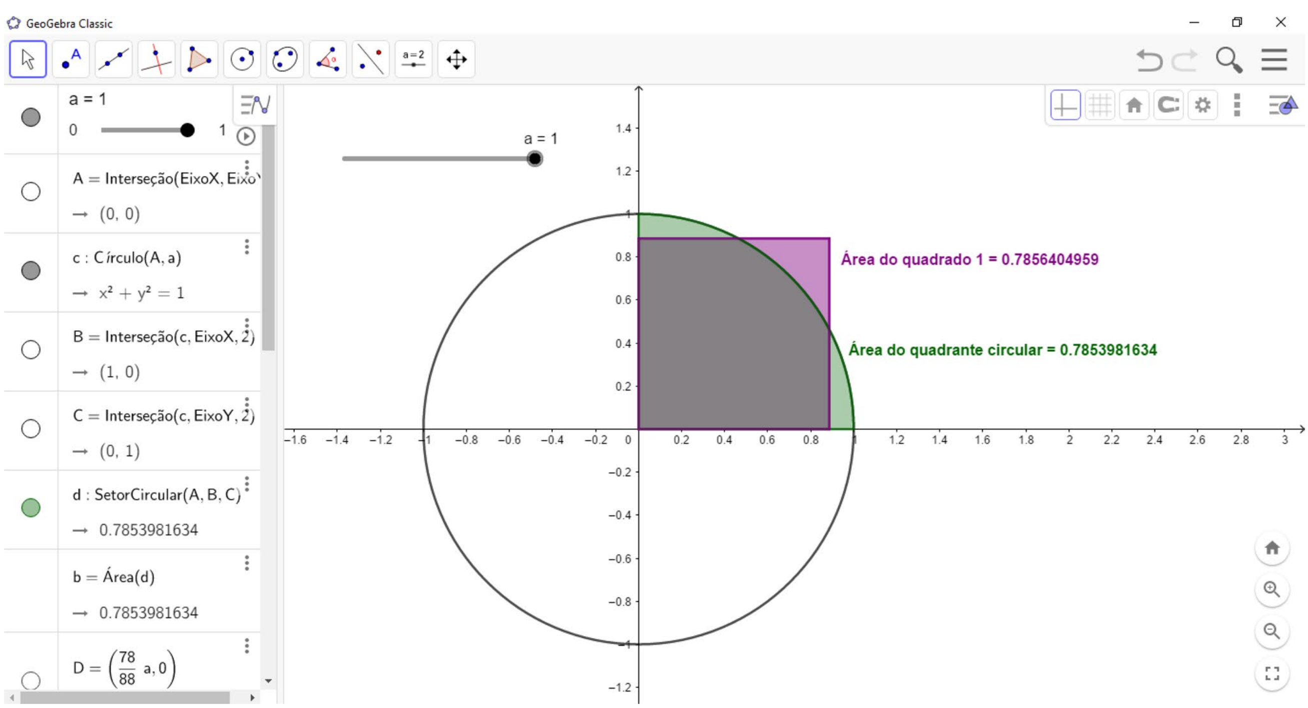


Fig. 2 Verifying that $(78/88)^2$ is a good approximation for $\frac{\pi}{4}$

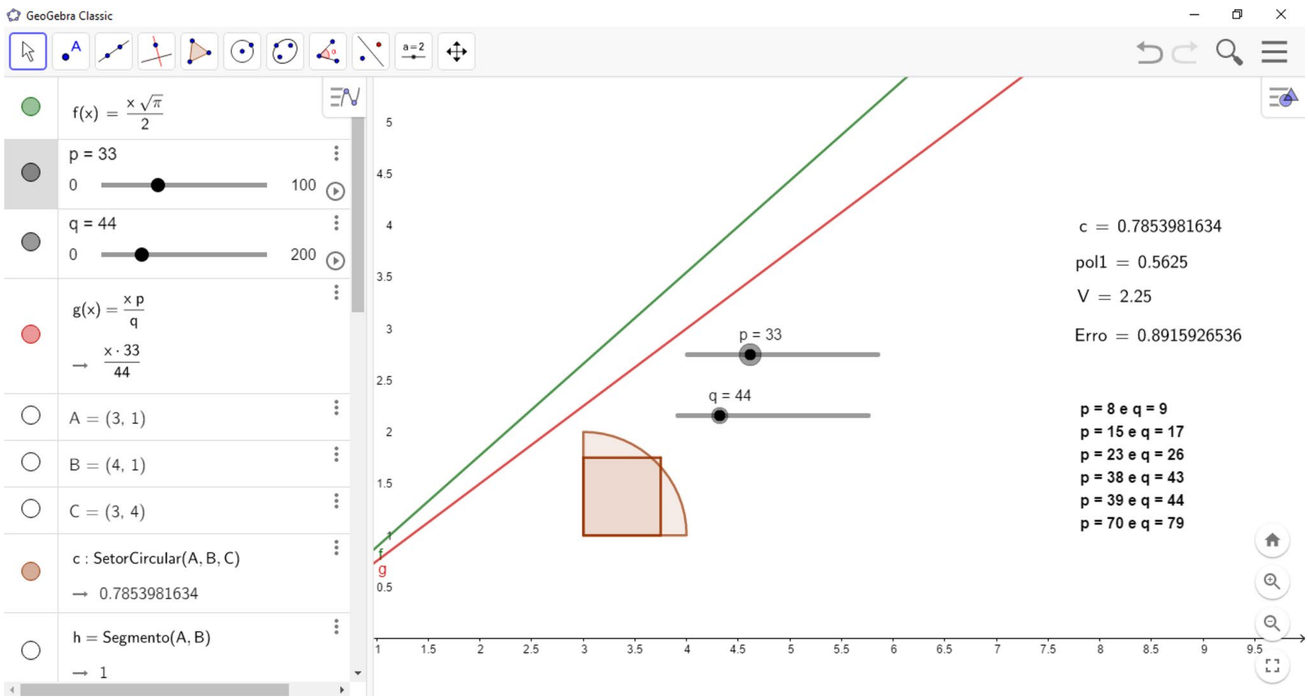


Fig. 3 Construction proposed by Group 2—part 1

To further explore the problem, Group 1 created a new object with GeoGebra. In this second construction (see Fig. 2), teachers-with-media created a slider tool involving a circle (radius equals to *a*) and a square whose side measure was (78/88)*a*. They also calculated the area of the square (*area do quadrado*) and the area of the first quadrant of the circle (*area do quadrante circular*).

Group 1-with-GeoGebra decided that the point H (see Fig. 2) was a good candidate for (p, q) and, thus, $(78/88)^2$ is a good approximation of π/4, to three decimal places. That is, the area of the square of side 78/88 is an approximation of the area of the first quadrant of the circle of radius 1, better than the square of side 8/9. In this case, the value of π was used to determine and display the value of the area of the first quadrant, that is, π/4. In fact, conceptually, it is important to notice that the teachers had a value of π. The GeoGebra actually offered ways for the teachers to devise a method or strategy in which they could get whole numbers whose quotient squared was close to π/4.

4.2 Solution 2 with GeoGebra

Group 2 proposed another solution for squaring the circle based on the use of GeoGebra as well (see Fig. 3). Initially, Group 2 used an idea similar to that of Group 1, that is, they considered $p=(q\sqrt{\pi})/2$. However, they explored the slope of $(\sqrt{\pi})/2$. In fact, contrasting the irrationality of $(\sqrt{\pi})/2$ with the actual rationality of p/q, group 2-with-GeoGebra constructed another straight line with a rational slope and then this pair, p/q, would be exactly the fraction they were looking for. Thus, p/q would approximate to $\sqrt{\pi}/2$ regarding the approximation of the slopes. When they changed the values of p or q in the construction, they obtained another line. In GeoGebra, the red line was the rational slope and the green line was the irrational slope. Therefore, animating values for *p* and *q*, they identified candidates for (p, q) when the lines would closely overlap.

As can be seen in the figure, the value of *p* varies from 0 to 100 and the value of q from 0 to 200. In this dynamic construction, teachers-with-GeoGebra built sliders and moved around to get the lines closer. When they achieved lines that were very close to one another, they considered that they had a fraction that could be a good approximation. They did several tests, started with 8/9. Using the resource of zoom, they could identify when the lines were closely overlapping. Through a process of experimentation-with-GeoGebra, they did some tests and considered 15/17 as a good approximation, which also gave them an approximation considering one decimal place. In addition, the experimentation offered ways to consider 23/26 a good result with one decimal place as well. With two places, they managed to find an approximation of 39/44, by visualizing the overlap of the lines. It is important to highlight that Group 2's solution involved the use of the notion of slope, which is central in the geometric exploration of the concept of derivative. Regarding the exploration of slopes, we note that the dynamic properties

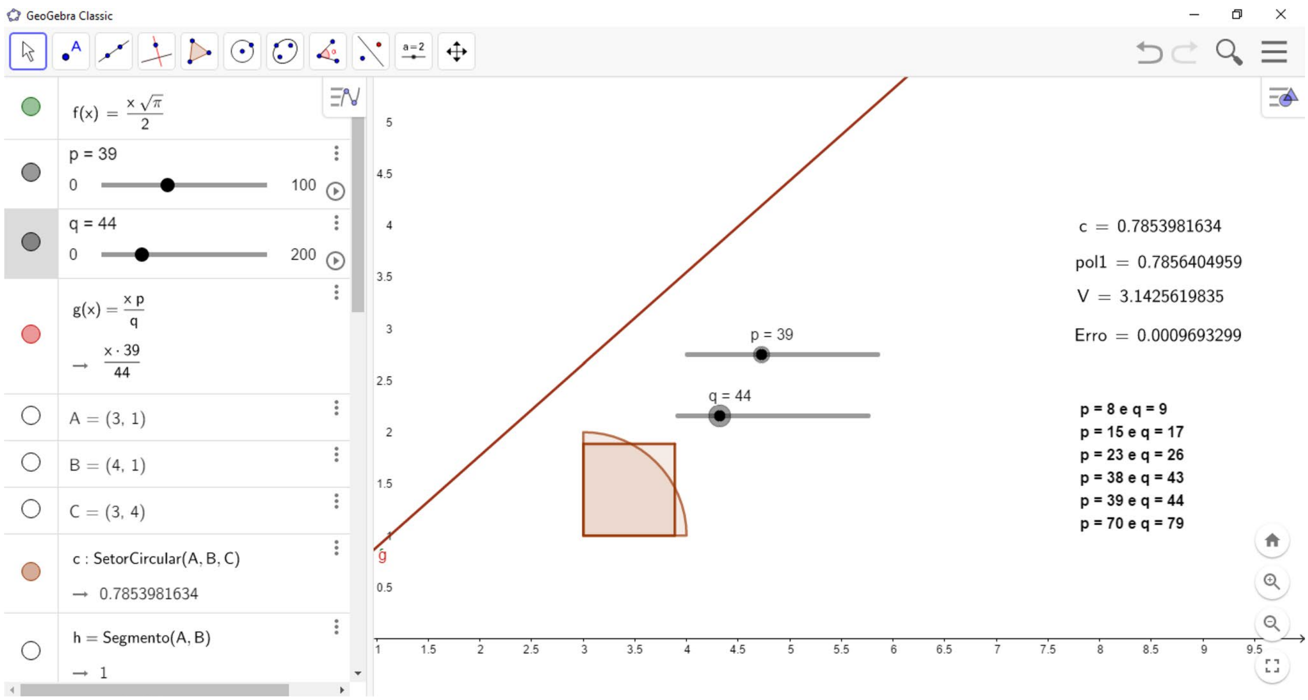


Fig. 4 Construction proposed by Group 2—part 2: lines visually overlapping

of GeoGebra offered ways for the teachers to obtain insights in solving the problem. The notion of slopes and the possibilities of exploring them based on GeoGebra's affordances, shaped the teachers' strategies. In addition, as pointed out by Hegedus and Kaput (2004), slopes are very significant for the comprehension of ideas in algebra and calculus in multiple representation environments. The final solution proposed by Group 2 is shown in Fig. 4.

In this case, teachers-with-GeoGebra considered several values for p and q, such as the following: p = 15 and q = 17; p = 23 and q = 26; p = 39 and q = 44; p = 70 and q = 79. To compare the values and analyze which pair best matched a potential solution, the group created the function "Error = | π-V |" which shows the difference between the value of π and four times the area of the square, since the problem is explored in the first quadrant of the circle. Thus, they realized that $(39/44)^2$ is a good approximation, to three decimal places, of π/4.

The final solution presented by Group 2 is the same as that presented by Group 1, since 39/44 = 78/88. However, the nature and quality of the strategies for solving the problem are different. Group 1 explored approximate coordinates (p, q) considering $g(x) = x\sqrt{\pi}/2$ from $p = (q\sqrt{\pi})/2$, and Group 2 compared the slopes of $f(x) = (x\sqrt{\pi})/2$ and $g(x) = xp/q$. These qualitative differences are significant from a humans-with-media perspective, since the natures of the strategies of solutions were numerically, visually, and dynamically distinct. Although in both cases the value of π was considered as given to determine the error of the estimated value, and 39/44 = 78/88 squared was found as a better approximation for π/4 than 8/9 squared, different collectives of teachers-with-GeoGebra were formed since the natures of the solutions were disparate.

4.3 A solution with Microsoft Excel

Group 3 started exploring the problem using the GeoGebra software. However, one of the teachers realized that a solution using GeoGebra would require visualization and supposed that they might not be able to find a good approximation to the values of p and q using only visual information. Thus, Group 3 proposed a solution using Microsoft Excel. To justify their choice, they claimed that the exploration with GeoGebra would be very dependent on visualization and they supposed that they would not be able to find a very good approximation using only visual perception. As they were familiar with using Excel spreadsheets in their classes, and since they had to experiment with several pairs and compare to find which pairs give the best approximation, they first compared all pairs p over q to 100 and identified whether or not the approximation was improved in relation to 8/9.

For this purpose, teachers-with-Excel created a spreadsheet (Fig. 5) consisting of a matrix with rows representing values for *p* from 1 to 100 and columns representing values for *q* also from 1 to 100. Each cell contained the respective

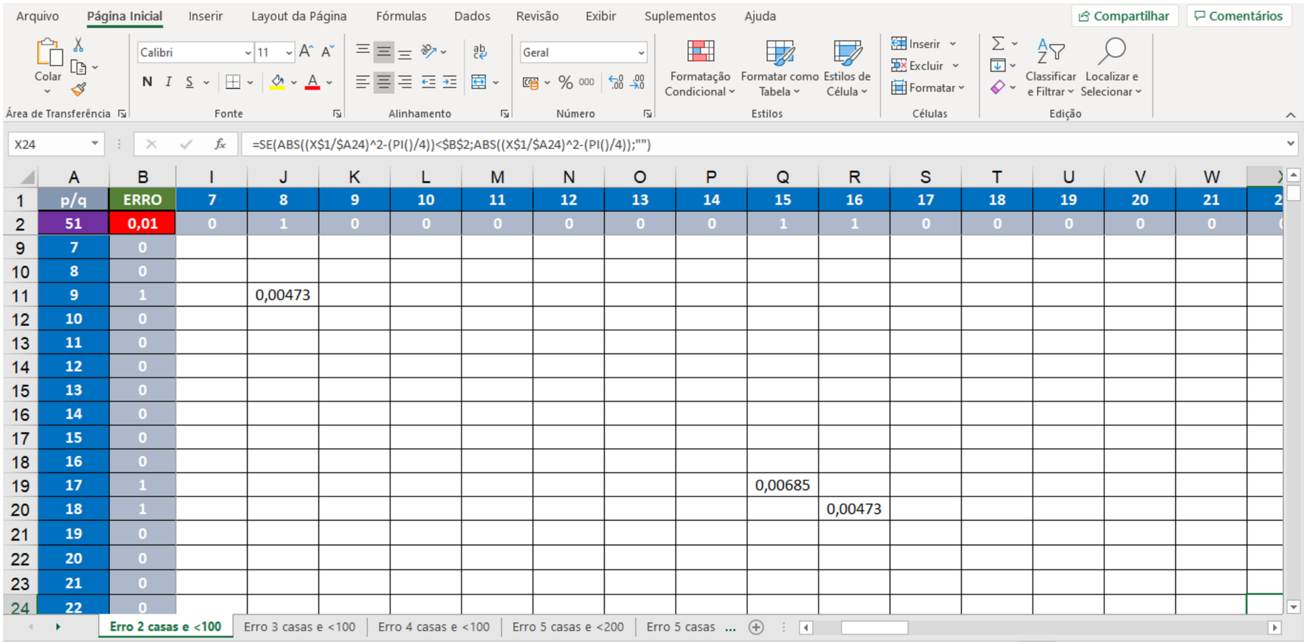


Fig. 5 Spreadsheet created by Group 3 (two decimal places)

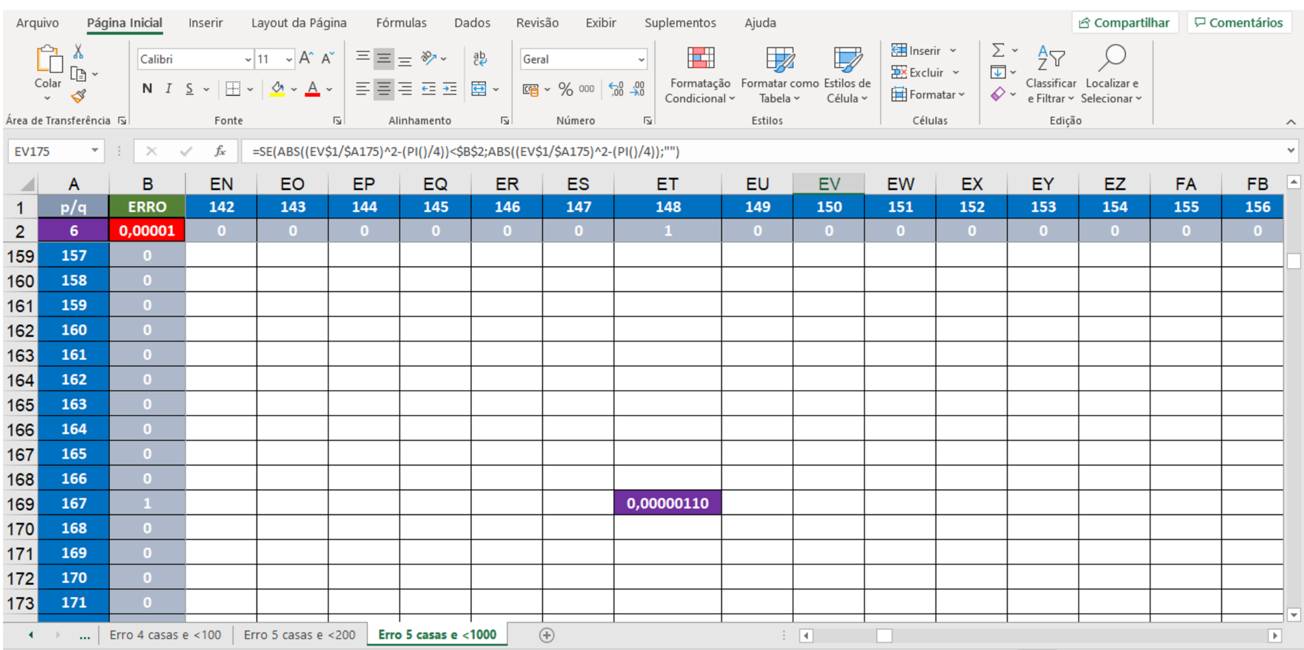


Fig. 6 Spreadsheet created by Group 3 (five decimal places)

formula, such as the following in C1:=*IF(ABS((C$1/$A3)^2-(PI()/4))<B2;ABS((C$1/$A3)^2-(PI()/4));"")*. This formula calculates p/q, squares it and compares it to the value $\pi/4$. If the difference is less than or equal to $(8/9)^2$, the spreadsheet displays the number; if greater, the cell is blank.

Note that at the bottom of Fig. 5 other worksheets have been created. Such spreadsheets follow the same programming logic, but taking into account that only values that have an approximation error less than 8/9 will be displayed. Moreover, the values of p and q can go up to 200 on one of the other sheets, and up to 1,000 in another. Using these additional sheets, Teacher 3A found the pair 148/167, which is an approximation with five decimal places compared to $\pi/4$. The teachers argued that for values of p and q

up to 1000, that is the best approximation. Figure 6 displays the solution.

Since the spreadsheet was configured to show errors of a minimum of five decimal places, one may notice that it displayed an approximation of six decimal places, since $(148/167)^2 = 0.785399261357$ and $\left|\left(\frac{\pi}{4}\right) - \left(\frac{148}{167}\right)^2\right| = 0.000001097960$. One of teachers also argued that the exploration can continue for higher values of p and q, pointing out, '*Of course, it is possible to continue the investigation, just drag and manipulate up to 10,000, up to 100,000, as long as the person has time to experiment; […] there is no need to write [a new] formula, just drag [the spreadsheet cells]'.* However, executing the calculations depends on the performance of the computer. Compared to solutions 1 and 2 with GeoGebra, the solution with Excel did not explore visual-graphic aspects. The affordances of Excel offered ways to explore the problem only numerically, but the proposed solution was more precise. This is a significant aspect of Excel's agency in the collective of teachers-with media.

5 Discussion

A perspective on digital technology based on the construct humans-with-media offers lenses to identify and highlight the role of technology in mathematical thinking. The solutions presented by the teachers reveal ways to think-with-technology. The affordances offered by *GeoGebra* and Excel offered windows to create different strategies and solutions to determine better approximations for squaring the circle than $(8/9)^2$. GeoGebra and Excel are artifacts that are products of human influence and of human design. These digital technologies also shaped the actions of each group of teachers. These pieces of software, together with each group of teachers, made it possible to generate solutions that would not have been generated if paper-and-pencil or some other technology had been used. In this sense, we say that the knowledge produced by the three groups is the result of collectives of humans-with-media.

We acknowledge the possibility of alternate designs for the proposed task. The following redesigned task may be used in future studies with the aim of improving the methodological role of the history of mathematics and the potentiality of the computer-based approaches: *Baron declared there is no record of Egyptians' reasoning. (1) How might Egyptians have reasoned that 8/9 is a good approximation? (2) Without using digital technology that has a built-in value for π, devise a method to produce a fraction that is a better approximation than 8/9 and give an argument for the validity of your method. (3) How could we continually adjust sliders' values to give ever better rational approximations to π?*

One possibility for exploring this task would be to define an area-preserving transformation that transforms one closed shape into another closed shape. Then, one would apply this transformation to change a square into a circle (without using π to define the circle) and superimpose a transformed square (the square having side length a/b) on to a circle with radius 1, adjusting a and b to get an accurate superimposition. This method would probably rely on affordances of the medium and it would not presume that one already knows a value of π, enhancing the experimental and historical dimension of the task. There are yet other methods that would rely on visual comparisons and would not presume a value of π.

5.1 Discussion of Teachers' solutions

The solution proposed by Group 1 improved the approximation $(p/q)^2 \approx \pi/4$ from one to three decimal places. In the original task posed by Baron (1985), which referred to the ancient Egyptian approach, it was suggested that $(8/9)^2 \approx 0.7$ as $\pi/4 \approx 0.7$. The approach conducted by Group 1 was based on $(78/88)^2 \approx 0.785$ as $\pi/4 \approx 0.785$. The strategy to figure out this solution explored aspects of rational and irrational numbers, and specific ways to think-with-GeoGebra. In the first part of the strategy, Group 1 plotted the graph of $g(x) = (x\sqrt{\pi})/2$ and, using the tabular representation x–y in the spreadsheet, identified candidates of pairs (p, q) or (q, p), but considering only integers. The group realized that, since π is an irrational number, it would not yield pairs of (p, q) with p and q as integers, but, thinking-with-GeoGebra, coordinating visual/graphic and numerical/tabular representations of $g(x) = (x\sqrt{\pi})/2$, they could select candidates, seeking for 'good' approximations of $(p/q)^2 \approx \pi/4$.

In the second part, based on the selection of 78/88 as a 'good' candidate for (p, q) or (q, p), Group 1 constructed an arc of a circle, and a square, in the first quadrant, in which the measurement of a side of the square was 78/88 of the radius (see Fig. 3). Then, using resources available in GeoGebra, the group calculated the area of the part of the circle in the first quadrant ('area of the circle/4') and the area of the square. Configuring the settings of GeoGebra to display 10 decimal places, and comparing the results displayed by the software concerning the values of the areas, Group 1 figured out that $(78/88)^2$ was a good approximation for π/4 to three decimal places.

The humans-with-media perspective is also relevant for identifying qualitative differences between solutions of mathematical problems. Group 1 and Group 2 used the same software and figured out the same answer for the problem, that is, in both cases $(78/88)^2$ and $(39/44)^2$ were considered a good approximation for π/4. However, when we analyze the role of technology in developing the solution, we can see that the nature of each solution is different. Instead of plotting $f(x) = (x\sqrt{\pi})/2$, Group 2 plotted $f(x) = (x\sqrt{\pi})/2$ and g(x) = xp/q, obtaining the graph of two lines. Visually and dynamically, the group compared the slopes of the curves. By animating the parameters *p* and *q* using the slider tool of GeoGebra, they overlapped the lines.

Avoiding potential misconceptions of visualization in dynamic mathematics (Hoffkamp, 2011), Group 2 created a parameter to evaluate the '"fairness' of approximations for the problem. The group created the tool named 'Error = | π − V |', in which π is the value of the area of the circle with radius equal to 1 and V is the value of the area of the square constructed over the circle, $(p/q)^2$. Although the squaring the circle problem is usually explored in the first quadrant, the 'Error' tool created by Group 2 in conjunction with the possibilities of GeoGebra is interesting not only because it refers to aspects of $(p/q)^2$ instead of (p/q), but also because it refers to an approximation for π (circle) instead of π/4 (part of the circle in the first quadrant). The 'Error' tool, which in the construction involves different ways of thinking-with-GeoGebra, is potentially relevant to evaluate the quality of the candidates (p, q) for good approximations of $(p/q)^2 \approx \pi/4$, that is, minor errors indicate better approximations. Although Group 2 indicated (39/44) as their candidate for the moment, they claimed that their construction with GeoGebra would allow users to improve that answer. One could change the range of the slider tools of p and q from [0; 100] to [0, 1.000], for instance, and, through experimentation-with-GeoGebra, figure out a new candidate (p, q) with a smaller error than (39/44), which is E = 0.0009693299.

The solution of Group 3, based on the use of Excel, did not explore visual and dynamic elements because the design of the software does not offer these types of potentialities in the same way as GeoGebra does. In contrast, Group 3 came up with a 'better' approximation for $(p/q)^2 \approx \pi/4$ with the candidate 148/167, which has an accuracy of five decimal places instead of three decimal places as was the case in solutions 1 and 2 with GeoGebra. The design of the spreadsheet created by Group 3 may be highlighted in several ways that reveal possibilities of thinking-with-Excel. The formula created by the group not only executes the calculation of $(p/q)^2 - \pi/4$ for each cell considering 100 × 100 and 1000 × 1000; it also only displays the results with values less than 0.01 (two decimal places) or 0.00001 (five decimal places) as configured in the cell B2. The cell A2 = (COUNTIF(C2:ALN2;" < > 0")) is also interesting—it displays the number of results, because C2(= COUNTIF(C3:C1002;" < 0.00001")). Therefore, the process of generalization with the software was significant in the development of the mathematical thinking of teachers-with-Excel formed in Group 3.

5.2 From 2 to 3D

Teachers' solutions were important prompts for us, the authors, to engage in experimentation-with-technology and produce further data that would also be significant to inform our research question. Although Baron (1985) did not present specific extensions to the squaring the circle problem in the ancient Egyptian or Babylonian context, we decided to explore the 'cubature of the sphere' using the solutions presented by the teachers, considering π as given.

We explored the problem regarding the irrational context, that is, we did not figure out *p* and *q* for the approximation of $(p/q)^3 \approx \pi/6$, because considering r = 1, the volume

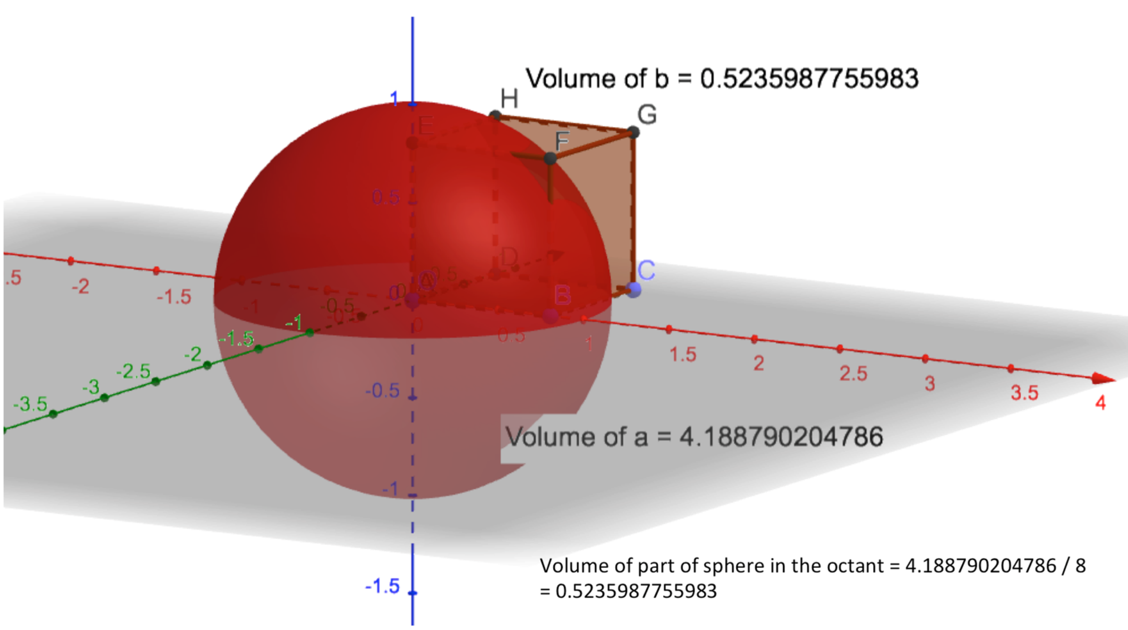


Fig. 7 Cubature of the sphere

of the part of the sphere in the octant is determined by $\frac{4}{3}\pi r^3 \times \frac{1}{8} = \frac{\pi}{6}$. Firstly, we used GeoGebra to compare the following visually and numerically: (a) the volume of the cube such that the measurement of its edge is π/6 and (b) 1/8 of the volume of the sphere with radius equals to 1. Secondly, we combined the use of GeoGebra 3D and Excel to determine p = 457 and q = 567, and to suggest an approximation between the volume of the cube with edge equal to 457/567 and the area of the part of the sphere in the octant.

Analogously to the two-dimensional context, we verified experimentally with GeoGebra 3D that the volume of the cube with edge π/6 is equal to 1/8 of the volume of the sphere of radius 1. That is, $\left(\left(\frac{\pi}{6}\right)^{\frac{1}{3}}\right)^3 = \frac{1}{8} \times \frac{4}{3}\pi.(1)^3 = \frac{\pi}{6}$. In Fig. 7, we present an image of the construction considering fifteen decimal places (the maximum number of decimal places available in GeoGebra Classic 6).

In the three-dimensional context, we can also explore different values for p and q in order to determine a 'good' approximation for $\left(\frac{p}{q}\right)^3 \approx \left(\frac{\pi}{6}\right)$. Similarly to Group 1, we could plot the graph of the function $f(x) = x\left(\frac{\pi}{6}\right)^{\frac{1}{3}}$ and, using the spreadsheet, figure out pairs of (p, q) from the domain and the image of f(x). We could also, like Group 2, plot the graph of the functions $f(x) = x\left(\frac{\pi}{6}\right)^{\frac{1}{3}}$ and g(x) = xp/q and, using the sliders in GeoGebra, we could also identify and select candidates for p/q visually, comparing the slopes of f(x) and g(x). However, these approaches do not specify the nature of the error, as in the design of the spreadsheet created by Group 3 using MS Excel. Adapting the formula from squaring the circle, and letting it be = IF(ABS((QQ$1/$A569)^3(PI()/6)) < B2;ABS((QQ$1/$A569)^3(PI()/6));""), with 0 < p, q ≤ 1,000, we figured out 457/567 to 6 decimal places.

Using this candidate, we suggest the following approach for the cubature of the sphere using GeoGebra 3D:

1. Open the 3D viewing window of GeoGebra and construct a sphere given center A (0, 0, 0) and a radius of 1.
2. From the origin A (0, 0, 0), create a segment with a given length of 457/567. It will determine B (457/567, 0, 0).
3. Insert a cube determined by A (0, 0, 0) and B (457/567, 0, 0).
4. Calculate the volume of the cube.
5. Calculate the volume of 1/8 of the sphere (in the octant).

Another aspect must be highlighted regarding the connections between solutions, which is the approximation for π. In Table 1 we present for each solution the value of p/q, the 'error' in relation to the area in the first quadrant, that is $\left|\left(\frac{\pi}{4}\right) - \left(\frac{p}{q}\right)^2\right|$, the approximation for π, which is $(4(p/q)^2)$, and the error for π, that is $\left|\pi - 4\left(\frac{p}{q}\right)^2\right|$. In Table 2, we present the analogous parameters for the cubature of the sphere.

It is interesting to notice that the error between π/4 and the value of the areas in the squaring the circle is bigger than the error between π/6 and the value of the volume of the cube in the cubature of the sphere. However, the best approximation for π is the solution developed by Group 3 with Excel. This solution is more accurate because to calculate π when squaring the circle, one multiplies by 4, and in the cubature of the sphere, one multiplies by 6. We conjectured that, in order to have a better approximation for π through the cubature of the sphere, such as$\pi \approx 3.14159$, a candidate for p/q would be 3.2223.983/4.000.000, because $\left(\frac{p}{q}\right)^3 = \frac{3.14159}{6} \to \left(\frac{p}{q}\right) = \sqrt[3]{\frac{3.14159}{6}} \approx 0.80599575 = \frac{80,599,575}{100,000,000} = \frac{3,223,983}{4,000,000} = 0.805999575 \to (0.805999575)^3 \times 6 = 3.141589999$. In future research we intend to explore this problem regarding Leibniz's approach based on the concept of transmutation (Baron, 1985), in which the following series is proposed: $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$. We expect to explore this series with GeoGebra, extending the problem for the series of π/6 in the three-dimensional context.

Table 1 Errors: two-dimensional approach

Solution	Technology	p/q	Area of the square: $(p/q)^2$	Error: (π/4)—Area	Approximation for π	Error for π
1 and 2	*GeoGebra*	78/88 = 39/44	0.785640495867	0.000242332470	3.142561983471	0.000969329881283
3	*Excel*	148/167	0.785399261357	0.000001097960	3.141597045430	0.0000043918403046

Table 2 Errors: three-dimensional approach

Solution	Technology	Value of p/q	Volume of the cube $(p/q)^3$	Error: (π/6)—Volume	Approximation for π	Error for π
4	*GeoGebra + Excel*	457/567	0.523599741575	0.000000965975	3.141598449450	0.0000057958605578

6 Conclusions

In this research we investigated teachers' use of technology in the exploration of the squaring the circle problem. Specifically, we discussed teachers' ways of thinking-with-technology by presenting different solutions for the problem based on the use of different digital technologies. The use of GeoGebra and Microsoft Excel was significant in visually, numerically and experimentally exploring the mathematical problem.

In fact, the problem proposed in this study involves introductory aspects concerning what Easterday and Smith (1991) and Williamson (2013) call "the calculating of Pi using Monte Carlo Method" (p. 1). In general, we argue that the idea explored in this paper deals with an alternative approach to the teaching and learning of calculus for the following reasons: (a) it explores an open-ended problem involving the quadrature of the circle, which refers to aspects of the genesis of differential thinking (Baron, 1985); (b) teachers' solutions in this research revealed ways of thinking and computational strategies that are relevant to learning calculus with technology (approximation, estimation, errors, graph of functions, slopes, sliders, etc.); (c) in the scope of the course in which the study was conducted, teachers' approaches to estimating π were significant for their consequent further explorations of both the method of exhaustion using GeoGebra, and the series for π/4 in the context of Leibniz's transmutation (Baron, 1985).

Digital technology reorganizes the teaching and learning of calculus conceptually and pedagogically. According to Swidan and Yerushalmy (2014), "[w]hen considering the artifact as a constitutive part of thinking rather than a mere aid or simplifier in accomplishing a mathematical mission, the role of teachers and students must be modified" (p. 530). "The emergent knowledge from this digital medium is different from the knowledge emerging from a paper-and-pencil medium because the mediator is not epistemologically neutral. The epistemology of mediation is another primitive. […] Educational Technology is no longer 'education using technology' but a metaphor for new cognition" (Hegedus & Moreno-Armella, 2009, p. 398).

The design of the tasks is a significant aspect of the humans-with-media perspective. It opens windows into mathematics, conditioning ways of thinking-with-technology. Most tasks with sequential tutorial information or high levels of guidance (step-by-step description of procedures) and questions related to strict answers are closed and do not explore the heuristic nature of digital technology. In contrast, research produced by GPIMEM tends to create and explore activities with an open-ended or semi-open-ended design, that is, tasks that offer ways for multiple strategies and solutions (Honorato and Scucuglia 2019). In particular, the design of the task explored in this paper is interestingly flexible because it can be explored with more than one digital technology with different platforms, in this case, GeoGebra and Excel. In fact, the combination of these technologies and the solutions constructed with them offered ways to extend the problem three-dimensionally.

Moreover, as highlighted by Moreno-Armella (2014), "[t]here is a problem that goes through the history of calculus: the tension between the intuitive and the formal" (p. 621). Teachers' solutions in this research revealed different intuitive aspects. The empirical nature of the historical mathematical problem in consonance with the experimentalism of the digital technology amplified the heuristic nature of the conceptual and pedagogical approach developed by the teachers. Particularly, considering the outline of the course, the formalization of the concept of definite integrals was not conducted at the time the teachers were exploring the squaring the circle, but it was considered later when they were studying Leibniz and Cauchy's symbolism, notations and concepts. Therefore, the interface between the history of calculus and the use of digital technology may intensify aspects of the tension mentioned by Moreno-Armella (2014).

Finally, this study contributes to the mosaic of research produced by GPIMEM. The genesis of the concept of integral was explored by the teachers in terms of quadrature and cubature based on problems conceived in the history of mathematics. In particular, the current concept of functions was important in different ways for the elaboration of all teachers' solutions. In fact, the interface between calculus and history is a novel aspect of the research produced by GPIMEM on calculus and technology. In future research, we will explore teachers' solutions of other problems, such as Proposition 4 of the Method of Archimedes and its relation to the volumes of cylinders, cones and paraboloids of revolution, using the concept of integer to calculate the volumes of solids of revolution.

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