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ORIGINAL PAPER

Yes, mathematicians do X so students should do X, but it's not the X you think!

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Abstract

Educators often argue that students should engage in activities such as conjecturing, proving, and deductive reasoning. The underlying principle is that learning mathematics means doing as mathematicians do. However, "mathematician" implicitly refers to a pure mathematician at university. *The aim of our paper is to critically question the logic model underpinning these premises in order to suggest which aspects of mathematicians' practice could be salutary for students in schools and university and which are not*. We argue that aligning learning with these practices might not meet the broader educational goals of pragmatism and vocationalism. We show that activities attributed to pure mathematicians are largely ignored by biologists, engineers, and physicists and in workplace settings. In contrast the practices of professional modellers are highly valued. We argue that such practices are desirable for learning to use and apply mathematics. Next, we illustrate the suitability of practices from studies of professional modellers and applied mathematicians for classroom learning using empirical data. We conclude that the interpretation of mathematicians' practice to be emulated must be broadened to include professional modellers' practices to better serve meeting educational goals for more students.

Keywords Citizenship · Goals · Mathematicians' practices · Mathematical modelling · Occupational preparation · Professional modellers

1 Introduction

In many countries, influential guiding documents such as *Adding It Up* (Kilpatrick et al. 2001), *Principles and Standards* (National Council of Teachers of Mathematics 2000), the *Common Core* (National Governors Association Center for Best Practices and Council of Chief State School Officers 2010) and *The Stockholm Declaration* (Centre for Curriculum Redesign 2013) seek to shift how teachers, educational researchers, and the public view mathematical knowledge and achievement. Watson (2008, p. 3) articulates these expectations well:

> school students should be introduced to authentic mathematical activity such as is practiced by professional mathematicians, and those forms of exploration that contribute to the development of the subject. Through this kind of activity, students get a sense of mathematics as a human invention, as certain habits of mind, that is more engaging and meaningful than learning a procession of given facts, methods, and question-types.

The logic model that we see as supporting the shift we have unpacked as the following premises:

1. Imitation of experts' practice in scholastic settings leads to development of expertise of that discipline (Zhou and Gou 2016).
2. Learning the discipline of mathematics means learning it as professional mathematicians (are perceived) to know and practice it (Fernández-León et al. 2020; Harel 2008).
3. Professional mathematicians should be taken to mean research-active pure mathematicians.
4. The core practices of research-active pure mathematicians are conjecturing, proving (Alibert and Thomas 1991), and deductive reasoning (Harel 2008).

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5. Therefore, to learn mathematics, students should engage in conjecturing, proving and deductive reasoning in the classroom.

The first premise of the model is underpinned by a pedagogical assumption so commonly held as to have garnered support from students themselves (Zhou and Gou 2016). The second premise follows immediately, and can be expanded as "students should participate in the classical practices of mathematicians approaching to the way mathematicians do it" (Fernández-León et al. 2020, p. 1). These ideas are not limited to mathematics education. In many countries, students are expected to learn science by practising scientific method (Bauer 1994; Williams 2018), for example. In essence the premise is, in order to learn X, students should do X by emulating how professionals do X. From here there remain two further details to work out: *what is X?* and *who are the professionals?* Determining the appropriate X's for mathematics requires some philosophical reflection on the nature of mathematics. Judging from statements in policy and curriculum documents from around the world (e.g., Queensland Curriculum and Assessment Authority 2019), the field has achieved the following consensus: mathematics is a science of patterns (Steen 2017) and it is essentially a human activity (Hersh 1997) that is as much a way of sanctioning and organizing knowledge as it is a body of knowledge or collection of skills. In keeping with these views, educators often argue that students should learn to conjecture, problem solve, and prove. Indeed, Schoenfeld (2016, p. xii) recently prefaced a volume on teaching mathematics to elementary school students by stating that, "If problem solving is the 'heart of mathematics', then proof is its soul".

The fourth premise about the core practices constituting *doing mathematics* that students should emulate rests upon assumptions about who are the correct disciplinarians, that is premise #3. Often, the "professional mathematicians" are taken to be pure mathematicians teaching at research universities. We infer this from the kinds of skills that are suggested for students to learn (deduction and proof) (see Davis 2014), the mathematicians who are commonly cited such as George Polya and William Thurston both of whom endorse pure mathematical goals, and from the samples of experts whose opinions are typically sought by mathematics educators conducting research on what mathematics should be taught (e.g., Harel 2008) or professional mathematicians' practice (e.g., Fernández-León et al. 2020; Misfeldt and Johansen 2015; Smith and Hungwe 1998). For example, educators argue that in order to be successful problem solvers and provers, students should learn to reason deductively and be convinced by deductive arguments, as research mathematicians do and are (Harel 2008); that students should learn to work from givens to goals, even when the goal is not immediately apparent (Goldin and McClintock 1984); and develop the syntactic–analytic skills (i.e., render propositions into symbols and operate logically on them) that will allow them to communicate their ideas to a critical audience (Zazkis et al. 2016). In addition, the tasks investigated with mathematicians usually involve problem solving (e.g., a difficult geometry problem, Schoenfeld 1992; fraction order and equivalence, Smith and Hungwe 1998) or proof (e.g., Weber 2008) in pure mathematics. Implicitly or explicitly, these core practices are derived from research studies with samples constituted by university-based, pure mathematicians.

The aim of our paper is to critically question this logic model, as others have done for parts of the model previously (e.g., Lesh and Zawojewski 2007), *in order to suggest which aspects of mathematicians' practice could be salutary for students in schools and university and which are not*. While we believe there are grounds to question Premises #1 and #2, we specifically seek to cast doubt on, and argue against, Premise #3. Even if we accept that adoption of experts' practice is the appropriate pedagogical approach for students to develop skills within the discipline, it does not follow that research pure mathematicians' practices are the only ones or even the correct ones to emulate. We build our argument by considering two common *goals* of education in general that can be applied to mathematics education specifically: pragmatism, and vocationalism.

Pragmatism in the context of educational goals encompasses Dewey's notion of "civic efficiency or good citizenship" (1916, p. 140) fostering a capacity for good judgment as an effective member of society. *Vocationalism* is a closely related but utilitarian goal incorporating preparation for the world of work as well as managing economic resources efficiently within a democratic society (Gonon 2009). In short, education is undertaken in pursuit of an occupation.

We will cast doubt on Premise #3 by first arguing that using research-active pure mathematicians' practices (Premise #4) as the basis for statements about what should be done in mathematics education does not necessarily fully meet the goals of mathematics education for most students. We will then argue that a modelling or applications-based approach bedded in practices of applied mathematics can easily meet pragmatism and vocationalism goals. We conclude that our classrooms perhaps *should* be emulating the ways that applied mathematicians, statisticians, or inter-disciplinarians from scientific, social, and allied health fields, to name a few, use and know mathematics in addition to some of the ways that pure mathematicians use and think about mathematics.

2 Goals of mathematics education

Our aim in this section is to cast doubt on the ability of an educational paradigm, where teaching creates opportunities for students to emulate the practices of research-active pure mathematicians, to meet the various perceived goals of mathematics education. We consider the two goals identified above.

2.1 Pragmatism

One goal of mathematics education is educating the citizenry to be competent decision-makers. In this view, students learn mathematics to help them reason in order to make good decisions or to promote social justice (Dewey 1916). Pragmatism derives from Dewey's views that teaching children to live in their current environment is society's great responsibility. Pragmatism means "thinking of or dealing with problems in a practical way, rather than by using theory or abstract principles" (Collinsdictionary.com). The interests and needs of those being educated are to be foremost in developing their civic consciousness in preparing them for being critical members of society. Most problems that students encounter in their environments, whether the real-life problems generated within their communities (e.g., flooding of their school and local community, Quiroz et al. 2015) or the problems found in STEM classrooms (e.g., recreating custom colour paint, Carreira and Baioa 2018) do not benefit solely from purely deductive arguments that produce logical consequences from explicit premises. Warrants of deductive reasoning alone cannot resolve authentic, real-world problems completely; analysis of problems with real-world origins benefit from reasoning that may be observation-, evidence-, policy-, or *especially* values-based (e.g., English and Watson 2018). Even if real life and curricular problems did benefit from deductive reasoning, and even if deductive reasoning was successfully applied to mathematical problem solving in a mathematics classroom, there is ample evidence that reasoning does not automatically transfer to make those skills available for use in other domains (Carraher and Schliemann 2002; Inglis and Attridge 2017).

2.2 Vocationalism

The second purpose we consider is perhaps the most familiar contemporary claim in support of teaching mathematics at all: that mathematical knowledge is necessary and desirable for preparation for an occupation (Gravemeijer 2013). The argument derives from the view that a goal of education is to meet societal need and, especially, economic demand. Many research papers, policy papers, and grant proposals appeal to this purpose when they invoke the "STEM pipeline" (see Freeman et al. (2014), for an example of this positioning). Within this purpose lies the greatest distance between the desired outcome (career preparation) and the purported means of achieving it (teaching students to reason like pure mathematicians). We begin with a crude overview of societal need and economic demand in terms of bachelor degrees awarded (2016, USA from datausa.io) and workforce size in the USA for a variety of fields.

Table 1 Societal need and economic demand in terms of bachelor degrees awarded and workforce size

Discipline	2016 Degrees	Average wage	Workforce size (M)
Biology	150,020	$103,208	2.63
Mathematics	28,073	$95,155	0.72
Physical Sciences	51,276	$106,668	1.08
Engineering	180,651	$109,324	4.3
Allied Health	121,752	$74,141	4.1
Social Sciences	194,755	$93,738	3.68

Table 1 shows that economic demand for educating mathematicians is least of the mathematics-dependent disciplines. It is therefore appropriate to question why the occupational training for 4.23% of the population with the highest mathematics requirements in post-secondary education should dominate the K-12 educational paradigm (not even 4.23% of the college-bound population, let alone the general population!). Thus, we should question what mathematical knowledge should be taught and how it should be known in order to benefit students who pursue careers in these other disciplines. In many fields, such as political science, psychology, economics, allied health, sociology, and anthropology, amongst others, there has been a sizable shift in the undergraduate curriculum away from the traditional calculus sequence and towards requirements for statistics, probability, and simulations. We take examples from recent educational scholarship conceptualizing mathematical needs of three disciplines and the general workplace.

Biology is one of the fastest growing STEM fields. Steen (2005) enthused that "after a century's struggle, mathematics has become the language of biology" (p. 22) and explained that biology is no longer a safe haven for the mathematics-averse. Wilson Sayres et al. (2018) surveyed 1260 biology faculty who stressed the importance of modelling biological systems which is facilitated by technical and algorithmic advances. However, they also noted that mathematics education lags behind meeting the needs of contemporary biologists. The field increasingly uses large data sets and requires skills in statistics and computer science with a focus on quantitative reasoning, modelling, and simulating complex systems (Feser et al. 2013). Biologists, then do not have a great need for the content knowledge typically emphasised in mathematics major coursework, like deductive proof. Biology is not unique in this respect.

Even in classical applied fields like physics or engineering, scholars argue that the disciplinary practices of research-active pure mathematicians may not be the ones to emulate (see, e.g., Haines 2011). For example, van der Wal et al. (2017) reported that practising engineers agreed that mathematics should be "taught in context to enhance student motivation" (p. S98). They also characterized the mathematics used by practising engineers as *technomathematical literacies* (Noss and Hoyles 2010) such as developing a sense of number and error, data literacy, and technical software skills. Gainsburg (2007) noted that the justification practices of engineers diverge substantially from those of school mathematics,

> In school mathematics, starting assumptions (the givens of the problem and mathematical axioms and postulates) are accepted as true—the student need not establish their accuracy—and the real challenge is to construct a chain of logic linking them to the desired end statement. For engineers, many of the starting assumptions—simplifications of the design and environmental conditions—were not established, and identifying appropriate ones and justifying their accuracy were typically the main challenges. Once the assumptions were set, proving that the design was structurally sound was usually mathematically trivial. (p. 485)

Elsewhere, scholars have argued that applying or consuming mathematics is of greater importance than capacity to create it (e.g., Douglas and Attewell 2017). Collins (2007) reported that, even for physicists for whom consumption of mathematics is extremely important, 85% of the practising physicists he surveyed reported reading mathematical results only up to a level of stepping through a proof without trying to reproduce it, that is, "they could get by with not much more mathematical ability than the most mathematically weak of their colleagues" (p. 681). Further, he stated provocatively that, although some physicists are highly accomplished mathematicians,

> a high level of mathematics is not used much of the time by most physicists and none of the time by a few physicists opens the way to a new understanding of what it means to be a physicist. Existing undergraduate educational programs imply that to be a physicist is to be a mathematician. (p. 684)

Greca and Moreira (2001) also argued that comprehension of a physics topic is tantamount to predicting phenomena without reference to mathematical formalism. Niss (2012), citing Redish (2005)'s argument that solving physics problems requires something other than what students learn in mathematics, claims that mathematicians and physicists interpret and use equations differently as physicists "combine conceptual physics and mathematical symbolism" (p. 12).

Finally, in the workplace, Douglas and Attewell (2017) found that only a small proportion of workers use school or higher mathematics to carry out their work. Only 2.6 million employees of 123 million in the USA workforce (2%) reported their jobs needed mathematical skills at or above the level of calculating the square footage of a house. Wake (2014) identified specific ways of connecting and using mathematical ideas without appealing to formalism such as producing and using measures or examining the "mathematical structure of measures that are productions of others" (p. 287), echoing Thompson's (2011) discussion about the centrality of quantification and quantitative reasoning to knowing mathematics in a usable way. These skills identified as crucial to technical disciplines, such as number sense or conducting dimensional analysis, are often seen as separate skills from conjecturing, proving and deductive reasoning but often feature in the practices of professional modellers (Drakes 2012). This is not to say that developing competencies in proving or deductive reasoning are incompatible with disciplinary practices and needs, but we have, as yet, very little evidence to suggest that the former results in the latter.

2.3 Summary

Research-active, pure mathematicians regularly engage in practices like proving, logical and deductive reasoning, and problem solving using syntactic–analytic skills to solve problems. As we have argued, adopting these specific ways of understanding and using mathematics does not automatically lead to achieving the goals of mathematics education. In contrast, and for some purposes, emulating them may lead away from the corresponding educational goals. When we consider problems with real world origins, we find that the ways that mathematics is understood and used can be quite different. The goal in these problems is to gain insight or make predictions about phenomena in the world. There is good reason for our educational goals to value empirical insight over deductive rigour.

3 Mathematical modelling and applications

As many others have done previously (e.g., Villa-Ochia and Berrío 2015), we propose mathematical modelling as the key that will allow us to meet many espoused goals of mathematics education. Several mathematics curricular statements (e.g., Argentina, Villareal et al. 2018, British Columbia Ministry of Education, 2020) continue to support the capability to solve problems arising in workplaces, everyday life and society as a major goal of mathematics education. In Ireland, for example, the purpose of a mathematics education in the reform oriented national curriculum is to enable students to "actively participate in their communities and society" (NCCA 2017, p. 3). The *Singapore Mathematics Framework,* for primary to pre-university, has emphasised a core focus on problem solving and more recently applications and modelling as one of the three processes supporting this (e.g., Ministry of Education, Singapore 2012). So, what are mathematical modelling and mathematical applications?

3.1 What are mathematical modelling and applications?

"An application of mathematics occurs every time mathematics is applied, for some purpose, to deal with some domain of the extra-mathematical world" (Niss et al. 2007, p. 3). Modelling is the process of constructing mathematical model(s) through structuring some extra-mathematical domain, using suitable mathematics to work mathematically to describe a situation within the extra-mathematical domain, or to provide answers to questions about that situation. This is done through interpreting and evaluating conclusions about the situation based on results from mathematical working. First, or even, later models might not provide desirable outcomes so the whole process is iterative allowing for both refinement and further extended exploration of the beginning situation.

Often, the process renders an ill-defined real-world problem as a well-defined mathematical problem that can be solved using available mathematical techniques. The solution must then be verified mathematically but also validated against real-world constraints. Since the real-world constraints are often competing values and priorities held by the modeller or the client, or society at large (Pollak 1997), validation often requires judgments that extend beyond the realm of pure mathematics (Gainsburg 2006, p. 31).

Today, many professionals engage in mathematical modelling in academia and in industry (Haines 2011). They engage in practices that allow them to assess risk to banks or assets, understand human-resource dynamics, to predict weather or climate, or plan rush hour traffic patterns, to name a few (Frejd and Bergsten 2018). We agree with Lesh and Zawojewski (2007) that beyond the academic requirements for entry into careers, students have a need to understand, describe, and explain the social and economic systems in which they operate. Teaching designed to meet these goals would emphasize practices like using mathematics to create, construct, refine, adapt, describe, explain, manipulate, predict, simulate, and design in service of a client's (or one's own) needs (Frejd and Bergsten 2016; Lesh and Zawojewski 2007). Clients need models adequate for specific purposes, which often means using mathematical principles and representations to solve ill-defined problems constrained by values or priorities, rather than seeking "correct" answers that could be logically justified.

3.2 What could students learn through mathematical modelling?

Four of the practices that are inherent in the practices of professional modellers as identified in several studies of their practices and are suitable in regular classrooms are: (1) making meaning for decision making in the situational context and mathematical representations of this context (Drakes 2012; Frejd and Bergsten 2018), (2) anticipating/mathematical foresight (Maciejewski and Barton 2016), (3) argumentation based on judgments beyond the realms of mathematics (Ekol 2011; Pollak 1997; Spandaw 2011), and (4) validating or justifying (Drakes 2012; Frejd and Bergsten 2018). In this sub-section, we will (1) argue that mathematics education based in such practices associated with mathematical modelling is within the capabilities of students, and teachers, in regular classrooms, and (2) illustrate these practices with empirical examples.

3.2.1 Student and teacher capabilities in regular classrooms

English (2003) noted that young learners in elementary and middle school years, given appropriate modelling tasks, are capable of engaging in mathematical processes such as describing, analysing, coordinating, explaining, constructing, critical reasoning, argumentation, and generating a generalizable model, during mathematization of objects, relations, patterns, and rules. These are the same practices mentioned above by Frejd and Bergsten (2016) and Lesh and Zawojewski (2007). In a study by Brown and Stillman (2017), 25 Year 6 students (12 years old) being introduced to modelling to develop a modelling conception of mathematics (Houston et al. 2010) began to develop a conception of mathematical modelling not only as a way of handling problems, particularly real-world ones, but also as a broader conception of mathematics being a way to think about life. Recently, Fulton et al. (2019) have reported similar mathematical engagement with students as young as 6 years old in different modelling tasks. Fulton et al. found that teachers were able to design and implement modelling tasks, in the regular classroom, that promoted opportunities for all students to solve problems that mattered to them.

3.2.2 Addressing four key practices promoted by mathematical modelling

In the following we illustrate with paradigmatic examples,[1] the four practices from the modelling of professional modellers that we identified as being of potential benefit to a mathematical classroom. In describing these practices we are moving beyond the notion of trying to directly teach what experts do or prescribing lists of their modelling

[1] Practices are illustrated by episodes from classroom video clips, teacher observation records and group posters from two groups of Year 5/6 students. Names are pseudonyms. The episodes were collected in the TALR project.

behaviours as processes that modellers in schooling or university must learn. We want to capture the essence of what the doing, seeing and making judgments in professional modellers' work entails which can be mined to pursue educational goals in mathematics classrooms. The four practices will be illustrated by points from studies of professional modellers' practices followed by a classroom episode that will be analysed and related to professional practice. The last two practices will be exemplified within the same classroom episode as in reality most of these practices interrelate.

Making meaning. In Drakes' study (2012) of 14 professional modellers' practices, the experts reported they usually started from the perspective they were ignorant of the problem situation they were modelling so the majority identified trying to understand the problem before starting as their initial action. For them, making meaning of the situational context from which the problem is posed could be through exploration, researching literature, gathering data, discussion or simplification. These modellers claimed to use most often the heuristic, *draw a picture*, to clarify understanding of the situation and to represent what the words being used by the problem poser and those the modeller was collaborating with, meant.

Our first example involves a group of three Year 5/6 students trying to make meaning of the *Hay Bale Stacking Task* (see top of Fig. 1) through discussion. At one point they begin discussing whether a triangular haystack, say 5-high, is the same height as five times one hay bale or if when the cylindrical bales intertwine into those below the result is a different height. Simone had simplified the problem to a 2-high stack to explore this further. Figure 1a–f shows the transcript of a video clip of their exchange and diagrams involving rough sketches of bales as approximate circles and freehand drawn lines representing the alternative situations. However, the inaccuracies in these diagrams allowed both the claimant of equal height, Jen, and the challenger, Simone, to support their cases, albeit without a resolution. It was only when the more conventional diagrammatic representation of straight lines and circles of equal radii throughout (Fig. 1 g, h) was used that a resolution ensued. Simone was able to convey the meaning he wanted visually in order to justify his challenge, persuade his two collaborators and confirm what he was seeing mathematically.

This is an example of young students making *mathematical* meaning from, and imbuing meaning into, a particular situation using firstly, inadequate representations to gain a resolution of differing claims based on nothing other than the loudest voice. The final warrants for decision making are not based on purely logical/deductive reasoning but on more conventional mathematical representations. The argument for this final resolution involves both mathematical representations that bring accuracy to the modelling of the situation and real-world experience that allows Simone to bring in the more realistic model of the situation involving representation of the haystack as including intertwining of bales rather than the more simplistic model of separate layers of annuli. In essence, the use of the *draw a picture* heuristic has provided the students with a means to go beyond their current ways of thinking and communicating about the problem so all can see it more mathematically (Lesh and Zawojewski 2007), albeit, geometrically. The basis of their conviction is visual, oft invoked as "a hindrance to progress in proofs" (Dove 2002).

Anticipating Maciejewski and Barton (2016) have confirmed a conceptualisation of what they term *mathematical foresight* by interviews with five mathematicians both pure and applied. Mathematicians engage in a process when faced with a novel mathematical situation that allows them to foresee a path to resolution. "Mathematical foresight is a preliminary, global view of the resolution destination and trajectory and may aid in making strategic decisions along the way" (p. 27). The mathematicians agreed they regularly engaged in this practice and it contributed to their anticipation of how to proceed. There is an analogue in mathematical modelling which Niss (2010) has labelled *implemented anticipation*. Such anticipating can be evident throughout the modelling process. This practice involves repeated foreseeing and feedback onto subsequent actions (Stillman et al. 2015). We illustrate it with respect to reaching an interim goal in modelling.

In this next episode from the *Hay Bale Stacking Task* (Fig. 1), another group foresees an approach to estimate the height of the stack of hay bales which they wrote on their poster in different coloured pens (so each writer was known) and then implemented for a 5-high stack and adapted for a 7-high stack. In the following they are foreseeing the path.

> Katie: We have to figure out the height of 1 bale. [*anticipating height of bale will be helpful*]
> Alia: If we figure out the size of 1, what can we do? What can we do to figure out the height of 5? [*collective questioning*] If we find out how to measure the height of 1 bail (sic) [*interim goal*], we can multiply that by 5. [*plan to reach goal.*]
> Katie: Maybe the height of 1 bale is the same height as the person. [*anticipating how height of bale can be estimated*]
> Alia: How tall is the person? [*collective questioning*]
> Maria: The person is maybe taller than the bale standing up. [*anticipating how to estimate person's height*]
> Katie: The person's body is about the same size as the bale. [*identifying relationship between situational features*]
> Maria: The person in the picture is not as tall as [teacher]. [*identifying relationship between situational features*] We could ask [teacher] to stand next to the door to measure.

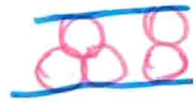
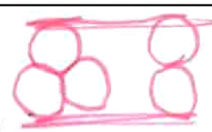
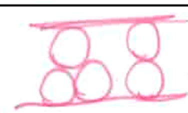
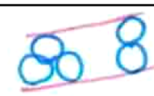
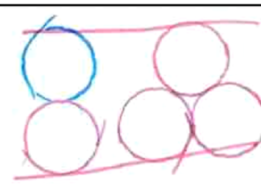
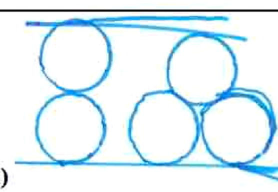
	The photograph shows a stack of hay bales. The bales in the photo are piled up with five bales on the ground, four in the next row, then three, two and on the top one bale. **Task A: How high is this stack of hay bales?** Clearly record all your mathematical thinking and calculations. Convince Jill that all parts of your solution are **reasonable** and make sense – use diagrams, words, **and** calculations to do this. Tell Jill how you came to your conclusion and how your group knows this is a correct solution.
Simone: It's not the same. Jen: It is the same. Oh My God, they are equal. [pointing to a diagram drawn earlier.] [*Using heuristic make a diagram to represent alternative situations.*]	 **a)**
Simone: [drawing a stack of three bales] It is not the same. Jen: It is the same. **It is the same size.** Simone: No [draws a stack of two bales] …. it doesn't, *it doesn't* [shows frustration] because it/ Jen: No way. [aligns top and bottom with her hand, then traces a line with her finger showing first top and then bottom of the stacks]. You have equal, equal! It is the same.	 **b)**
Simone: No, this [two bales] is more than the other [draws line across the top]. Jen: No, not compressed! Simone: I can't understand you. Not equal.	 **c)**
Simone shakes his head. Jen grabs the pink Texta pen and draws over his line at the top. She then adds a line at the bottom-back and forth. Then again draws over the 'top line'.	 **d)**
Jen adds a new drawing. Simone watches her. She starts with the bottom line, then adds the stack of three, followed by the stack of two and finally the top line. Jen: It's the **same height**. [*Inaccuracies in drawing support her claim of equal height.*]	 **e)**
Simone [starts another drawing, in blue]: Look, **now** is better. Jen [again adds pink lines to his drawing, clearly indicating to her equal height]: It's the same height.	 **f)**
Simone finds a glue stick. He draws in blue pen around the base of the cylindrical glue stick. Rebel holds the glue stick to help Simone draw a 'perfect' circle. Jen [looks over at them]: Oh yeah, yeah. For the second circle, Simone holds the glue stick and Jen draws around it to create a circle directly below his. She then draws the stack of three. These stacks are aligned at the top, but clearly not at the bottom. [*More accurate representation allows Rebel and Jen to 'see' Simone's argument.*]	 **g)**
Finally, the group produce another diagram, using the glue stick. This time the separate stacks are aligned at the base. Two lines are drawn at the top, one parallel to the bottom line not touching the top of the second stack, and a second line sloping down to the second stack of intertwined bales. Simone has finally made his point. [*Resolution of impasse with visual justification for new interpretation of situation.*]	 **h)**

Fig. 1 The Hay Bale Stacking Task episode. (Task adapted from *Bale of Straw* Borromeo Ferri 2007)

Voicing one step in their approach triggered the foreseeing of the need for a subsequent one, setting in train a set of thoughts, and their articulation and recording, with one step pre-empting the need for another until an anticipated path to the interim goal was reached. These voicings and recordings occurred in a seemingly haphazard fashion as thoughts were articulated and written down immediately in whatever space was available on the first poster sheet. The actual beginning and end points and the projected probable actions that connected them had to be held in the modellers' minds so they could feedback their foreseen actions onto current actions as they happened. The actual implementation of these anticipated steps requires the carrying through of corresponding decisions and actions to bring those anticipated forthcoming moves into fruition (Stillman et al. 2015). This occurs dynamically, in the moment, and might not lead to successful implemented anticipation of the foreseen path. There was no carefully written step by step recorded plan in the Polya (1945) sense a priori. The actual recorded plan on their second poster sheet was an a posteriori record of what they did recorded after the modelling was completed, with their mis-steps along the way omitted.

Validating and argumentation. Validation in the sense of verifying that the model constructed or adapted was mathematically correct and acceptable for answering the questions posed within the real world situation being modelled was an integral practice in all the studies of professional modellers' practices that we reviewed. The actual techniques used to do so differed depending on the type of modelling being undertaken and the interests and purposes of those commissioning the modelling. Rather than checking they were pursuing "the correct solution", modellers in Drakes' study (2012) highlighted that there was no such thing as "the correct model" but instead the model(s) they were constructing or adapting needed to be consistent and make sense especially if no data were available for verification purposes and the model could not be used to generate predictions or experimental output. For argumentation, as Gainsburg found in her ethnographic study of structural engineers, "mathematical justification alone is insufficient for accepting a model or method or result" (2006, p. 31) in many instances.

In a subsequent episode in the *Hay Bale Stacking Task*, Maria noted that the teacher's height was ¾ of the height of the door in the open classroom space where her group was working, making his height almost 2 m by their estimation. He confirmed he was 196 cm tall (verification). To estimate the height of a bale, they used their estimate of the height of the person on the haystack. They noted the person was shorter than their teacher. Alia suggested this person was 160 cm tall because he was a teen. The implication was he would be taller than the girls who were not teenagers; but Katie said her own height was around 140 or 150 cm and estimated the person's height to be about 155 cm, which they all agreed was their final estimate for the height of one bale. The final result was the product of a series of adjustments in argumentation based on judgments that extended beyond the realm of pure mathematics into their prior experiences for justification.

4 Discussion

In this section, we discuss how (1) mathematical modelling and applications contribute to the learning of mathematical content and positive affect in mathematics classrooms, and (2) mathematics education based in the practices associated with mathematical modelling practices of professional modellers is able to address the mathematics education goals outlined previously.

4.1 Consequences for content and affect

In undergraduate coursework, learning of mathematics content (concepts, definitions, techniques) is enhanced by attending to modelling practices and connections to real-world principles (Czocher 2017). The same holds in secondary and post-secondary levels more broadly (Han et al. 2015) as well as in primary grades (English 2006; English and Watters 2004). Moreover, through modelling, students learn to create significant mathematical ideas that attend to structure, are shareable and reusable, and are effective prototypes for similar problems encountered in the future (Lesh et al. 2000). Through carefully designed modelling tasks, students learn to critique, justify, and generalize. Self-efficacy positively predicts performance when solving modelling problems (Sharma 2013), as it does in mathematics generally (Pietsch et al. 2003), but the association does not end there. Working on reality-based mathematics problems in the classroom and modelling tasks contribute to positive student affect, especially when paired with student-centred modes of teaching (Schukajlow et al. 2012). Likewise gains in self-efficacy have been recorded for students working on extended modelling problems in an extracurricular setting (Czocher et al. 2019).

4.2 Meeting the goals

In addition to individual gains in knowledge and affect, creating opportunities for learners to engage in modelling practices has the potential to meet educational goals for a greater proportion of students.

First, modelling and applications empower learners to use mathematical knowledge to support their decision making as future citizens. An example comes from Gutstein's (2003) classroom where over 2 years, students learned to read and write the world, thus developing "mathematical power" (p. 45). Gutstein (2003) concluded that the learners exhibited capacity to use mathematics to hold and defend critical viewpoints as well as to dissect society and understand inequality, a point made by several mathematics educational researchers investigating practices in their own classroom (e.g., Almeida and Silva 2015; Villa-Ochia and Berrio 2015). This is exactly the pragmatist's argument for the value of mathematics education. Clearly tasks must be designed to include issues and decisions that have ethical, moral, social or cultural aspects and these be taken into account during modelling to ensure or advance "the sustainability of health, education and environmental well-being, and the reduction of poverty and disadvantage" (Niss et al. 2007, pp. 17–18). There have been moves in this direction recently. Maass et al. (2019) have conducted an interdisciplinary international design research study in Europe which developed teaching and professional development materials connecting mathematics and science education to citizenship education which include such modelling tasks. That professional modellers consider ethical aspects in their decision making was demonstrated in Frejd and Bergsten's study (2016).

Second, from the vocationalism standpoint, if capacity for prediction, empirical insight and contextual decision-making are what occupations desire then students should have modelling experiences that cultivate these. Learning to mathematically model problems found in the real world provides a means of gaining insight within a communal environment about how the world works as modelling is done in teams both professionally (Drakes 2012) and in education (Stillman et al. 2015). It also develops a broader conception of mathematics as being a way to think about life (Houston et al. 2010). Deriving, adapting and interpreting mathematical models of phenomena subject to stakeholders' constraints allow prediction of future outcomes in single situations and also simulation of outcomes across many, potentially variable, situations. For example, graphs can provide insight into (and eventually upper bounds for) how quickly rumours can spread in a social network (Doerr et al. 2012). In fact, different kinds of graphs operating under different assumptions about how effectively the nodes exchange information will yield different results. A good model can capture many possibilities at once leading to further insights about how seemingly disparate situations might be closely related in terms of the natural laws governing the underlying phenomena. Thus, mathematical models also have the power to provide insight into phenomena (Bliss et al. 2016), offering explanations for *why* a phenomenon unfolds the way it does.

Providing educational opportunities for learners to engage them in modelling practices valued by STEM disciplines or professional modellers satisfies pragmatism and vocationalism goals. Together, these opportunities contribute to another advantage: the epistemological stance that results—even those arising from mathematical analyses—are not certain or infallible. Since "our model is not the full truth, it is only justified, if certain conditions are fulfilled, if our parameters are reliable; if we forget that, we become responsible for errors in the simulation of finance models, of ash clouds, of evacuation plans" (Neunzert 2013, p. 59) so modellers must be ever vigilant. In contrast mathematical results obtained deductively are regarded as secure. If young students learn through proving and mathematical problem solving that mathematical results are certain and infallible,[2] and that deductive reasoning is the only reasoning that is valued, then either there is no possibility for revision, or these other forms of reasoning necessary to apply mathematics are less valued. Or, perhaps because the disciplines and 'real life' value evidence-based or values-based reasoning, deductive reasoning (at least for students who do not major in mathematics) must be reframed so that students recognise its application in the logic of everyday and scientific argumentation.

5 Concluding remarks

Our aim was to critically question a logic model underpinning the notion that teaching and learning, regardless of level, should be based on the practices of pure mathematicians, in order to suggest which aspects of mathematicians' practice could be salutary for students in schools and university and which are not. Our analysis suggests that the broader educational goals of good citizenship (pragmatism) and occupational preparation (vocationalism) may be more effectively formulated in terms of practices of professional modellers such as making meaning for decision making in the situational context and its mathematical representation, anticipating, argumentation based on judgments beyond mathematics, and validating or justifying. In suggesting this we do not mean to set up a false dichotomy between the kinds of practices employed by research-active pure mathematicians and professional modellers. Mathematics is of course necessary for mathematical modelling. Thus we have suggested that the notion of mathematicians' practice needs to be broadened to include the essence of what the doing, seeing and making judgments in professional modellers' work entails for the professional practices of mathematicians that students could emulate in learning mathematics and meeting educational goals.

Though there is overlap between the practices of professional modellers and mathematical problem solving practices, there are differences in what are acceptable standards of conviction and the uses to which aspects of those practices are put. There is overlap in *justifying,* for example. Modellers must justify their model by validating it against real-world constraints, using evidence in the form of empirical

[2] Even some mathematicians and mathematics philosophers doubt the claim that *proving* a theorem makes the result certain or infallible–Czocher and Weber (2020) elaborate on this position in the context of defining *proof.*

data, through verifying their mathematical analysis, by checking the theoretical consistency, or matching outcomes to stakeholders' (perhaps incompatible) constraints. These tools allow them to establish the limitations of their model, setting and communicating its scope or domain of validity. Though the warrants for justification in modelling may at times differ from those in proving, they are equally important to disciplinarians and applied mathematicians in academia as they are to their counterparts in industry (Drakes 2012; Frejd and Bergsten 2016).

It is clear to us, and to most who advocate for modelling in the classroom, that models are necessarily dependent upon the mathematical knowledge and representational fluency of the modeller. As mentioned above, English (2003) noted that all young learners, given appropriate modelling tasks, can engage in the important mathematical processes that underlie the mathematical practices we advocate. Furthermore, in doing so, they "develop important mathematical ideas and processes that would be largely untapped in more traditional classroom activities" (p. 13). In the ideal classroom, we see a symbiotic relationship between mathematics, the situations it can be mapped to, and the kinds of reasoning that bind the two, advancing mathematical thinking and educational goals.

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