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SURVEY PAPER

The relationship between mathematical practice and mathematics pedagogy in mathematics education research

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Abstract

This paper introduces two central questions (1) what are available methodologies for making claims about mathematical practice and (2) when and how do claims about mathematical practice influence mathematics instruction. To motivate these questions, we critically analyze the relationship between research into mathematicians' practice and the design of mathematics instruction in three ways. First, we describe three influential research programs in mathematics education and illustrate how each research program uses claims about mathematical practice to inform their instructional goals. Second, by examining these important works, we highlight intrinsic difficulties in investigating mathematical practice. Our conclusion is that every research methodology for investigating mathematical practice is fundamentally limited and we require triangulation from multiple methods and theoretical lenses to fully understand mathematical practice. Third, we highlight reasons for why mathematical practice sometimes should not inform mathematics instruction. We conclude this paper by discussing how the articles in this special issue address our two central questions.

Keywords Instructional design · Mathematics instruction · Mathematical practice · Methods

1 Introduction

Many mathematics educators believe that mathematicians' practice should inform the way that mathematics is taught to students. In particular, many mathematics educators maintain that classrooms should be organized so students participate in similar activities and engage in similar interactions to those that mathematicians do (e.g., Ball & Bass, 2000; Lampert, 1990; Harel & Sowder, 2007; Schoenfeld, 1992; Sfard, 1998; Weber, Inglis, & Mejía-Ramos, 2014; see Skovsmose, 2020, for a dissenting viewpoint). As such, some mathematics educators believe that understanding how mathematicians engage in mathematical practice has implications for how mathematics classrooms should be organized, and thus they believe that research into mathematicians' practice is relevant for mathematics education. For a more comprehensive discussion, see Schoenfeld (1992).

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Because some mathematics educators perceive a link between mathematicians' practice and mathematics instruction, research into how mathematicians practice their discipline exists as a legitimate branch of mathematics education research, at least to the extent that articles investigating mathematical practice appear in leading mathematics education journals. In the last two decades, the journal *Educational Studies in Mathematics* has published inquiries into how mathematicians solve problems (Carlson & Bloom, 2005), learn new mathematics (Wilkerson-Jerde & Wilensky, 2011), generate proofs (Kidron & Dreyfus, 2014), read proofs (Mejia-Ramos & Weber, 2014; Weber & Mejia-Ramos, 2011), choose research questions (Misfeldt & Johansen, 2015), use examples (Alcock & Inglis, 2008), and evaluate conjectures (Inglis, Mejia-Ramos, & Simpson, 2007). Similarly, the *Journal for Research in Mathematics Education* has published articles on mathematicians' writing (Burton & Morgan, 2000), how mathematicians check proofs for correctness (Inglis & Alcock, 2012; Weber, 2008), whether mathematicians skim proofs before reading them closely (Inglis & Alcock, 2013; Weber & Mejia-Ramos, 2013), and how mathematicians conceptualize proof (Czocher & Weber, 2020). Other researchers have written influential articles with pedagogical implications based on philosophers' analyses and the reflections of professional mathematicians (e.g., deVilliers, 1990; Hanna, 1990; Hanna & Mason, 2014).

If mathematical practice should indeed influence how mathematics is taught, this raises two fundamental questions: (a) how should mathematics educators generate and verify claims about mathematical practice and (b) how and when should claims about mathematical practice inform instruction? Regarding the first question, we have observed that mathematics educators often disagree on the nature of mathematical practice. For instance, consider the following divergent views with respect to mathematicians' practices regarding the notion of mathematical proof. Harel and Sowder (1998) claimed that mathematicians gain psychological conviction in mathematical claims predominantly through proof while deVilliers (1990) believed that mathematicians are usually convinced a claim is true before seeking a proof. Selden and Selden (2003) asserted that mathematicians usually agreed on whether a particular argument is a proof while Dreyfus (2004) emphasized a lack of uniform standards. While some disagreement is inevitable in any field of inquiry, we are not aware of critical analyses of how these disagreements should be resolved in mathematics education research. In this paper, and in this special issue, we begin a conversation toward this end.

The second question reflects our skepticism about which professional mathematicians' practices should be present in mathematics classrooms. As Staples, Bartlo, and Thanheiser (2012) have argued, the needs of the professional mathematical community and the needs of a classroom community often differ, implying some practices of the professional mathematical community might not be useful in mathematics classrooms. As Ball (1993) noted, some professional mathematical practices may not be worth emulating. In this special issue we closely examine how some professional mathematical practices can productively inform pedagogy and why other professional mathematical practices should not.

In this paper, we explore three different patterns of research using mathematical practice to inform mathematical pedagogy. Each uses accounts of how mathematicians accomplish particular goals to identify or justify goals for instruction. The first type forms models of how mathematicians engage in mathematical tasks (e.g., how do mathematicians solve problems?) to inform instruction regarding how students engage in analogous mathematical tasks. The second type forms models of mathematicians' epistemological beliefs (e.g., what is the relationship between proof and certainty to mathematicians?) to inform instruction on how students justify and use mathematical claims. The third type forms models about how mathematical communities develop concepts (e.g., how do mathematical communities form definitions?) to inform instruction regarding concept-formation. We choose three representative and highly influential research programs to illustrate each type of research—Schoenfeld's (1985, 1992) work on problem solving, Harel and Sowder's (1998, 2007) work on proof schemes, and Lampert's (1990, 1992) work on classroom design.

Although the work is dated, other scholars continue to pursue research with similar goals using similar methods to the authors we cite, so the issues raise remain pertinent to the field today. For instance, just as Schoenfeld (1985) used task-based interviews with mathematicians to understand how mathematicians solve problems, more recently Lockwood, Ellis, and Lynch (2016), Karunakaran (2018), Knuth, Zaslavsky, & Ellis (2019), and Samkoff et al. (2012) used task-based interviews with mathematicians to understand how mathematicians construct proofs. The issues we raise with regard to Schoenfeld's work are just as pertinent to these studies as well. Similarly, like Harel and Sowder (1998, 2007), researchers continue to make assertions about the nature of proof and its relationship to conviction and use these assertions to draw pedagogical consequences (e.g., Cirillo, Kosko, Newton, Staples, & Weber, 2015; Czocher & Weber, 2020). We illustrate how Lampert (1990, 1992) drew upon the philosopher Lakatos (1976) account of mathematical practice to design classroom instruction. Scholars continue to use Lakatos (e.g., Komatsu et al., 2017; Larsen & Zandieh, 2008) and other philosophers (Hanna & Larvor, 2020) for this purpose.

In this paper, we will carefully scrutinize the argumentation schemes that Schoenfeld, Harel and Sowder, and Lampert used to warrant claims about mathematical practice and use these claims to make pedagogical recommendations. To avoid misinterpretation, we will neither endorse nor refute the findings from the research we survey or the theoretical assumptions that undergird this research, which in some cases are still being debated today by mathematics educators. Instead, our purpose is to use this research to articulate and contextualize difficulties common to many mathematics educators who want to make valid claims about mathematical practice. Finally, we discuss how the papers in this special issue, *The role of mathematicians' practice in mathematics education research*, address these issues.

2 From mathematical practice to pedagogy: the case of mathematical competencies

2.1 Schoenfeld's problem-solving research program

One goal of mathematics instruction is to prepare students to successfully complete mathematical tasks, such as solving mathematical problems, posing conjectures, or proving theorems. To prepare students to complete these tasks, many mathematics educators desire to have a firm understanding of precisely how these tasks should be completed. One way that mathematics educators have developed an understanding of how to successfully complete a task is to observe mathematicians completing the task and identify the competencies that mathematicians have (e.g., strategies, metacognition) that allow them to be successful on these tasks. Once these competencies have been identified, helping students develop these competencies can become an explicit goal of instruction.

Perhaps the most successful instance of this type of research is Schoenfeld's work in mathematical problem solving. In *Mathematical Problem Solving*, Schoenfeld (1985) set the stage for the book by wondering how mathematicians' experience allows them to solve mathematical problems better than students, comparing his work to biomedical research on senility:

> "What causes senility? Of course senility occurs with aging, but saying so really begs the question. The implied and more precise question is, Can we explain the underlying biological and chemical processes that comprise the phenomenon known as senility. Clarifying this question, elaborating on it, and exploring it lead one into deep questions of medical science. In a similar vein, we can formulate questions about [mathematicians'] experience with greater precision. When the faculty members sit down to work on the problems, there are fewer resources immediately accessible to them than to students. Yet the faculty manage, somehow, to see what makes the problem tick, to come up with a variety of plausible directions for exploration where the students do not, and so on—all with some sense of purposefulness and efficiency. To specify the nature of these skills and to elaborate the means by which they are achieved is to begin to provide a real explanation of what the faculty's experience means" (p. 3–4).

Schoenfeld added that although research into the nature of mathematicians' problem solving was pure research, the result of the research had pedagogical implications:

> "The book is devoted to the exploration of mathematical processes, mathematical strategies, and tacit mathematical understandings that constitute mathematical thinking. The focus here is primarily on research, although there is a strong 'applied' component to the work as well. Much of the work is inspired, and informed by, classroom applications" (p. 4–5).

This rationale is why Schoenfeld's pioneering work studying mathematicians has proven enormously influential in mathematics education. To Schoenfeld, how mathematicians think can, should, and does inform mathematics instruction (Schoenfeld, 1992). There is a final premise that undergirds Schoenfeld's work on this subject—namely that there is some uniformity in the way that mathematicians solve mathematical problems. In the passage above, it makes sense to speak of "mathematical thinking" of a unitary construct that can be decomposed into processes, strategies, and understandings. Schoenfeld (1985) explicitly makes this presumption clear when describing problem solving strategies and heuristics:

> "While the development of strategies… is idiosyncratic, it is also somewhat uniform. That is, there is a substantial degree of homogeneity in ways that expert problem-solvers approach new problems. (This is not terribly surprising, in that a relatively small number of techniques will allow one to be successful in mathematics. In a sense, one can say that successful problem-solvers were trained by the discipline). That is not to say that two experts will approach the same new problems in precisely the same ways. Rather the argument is that, if two experts grapple with an extended series of problems, there will be substantial overlap in the problem solving strategies that they try" (p. 71).

The same assumption about homogeneity is made by Schoenfeld in his investigations of metacognition, beliefs, and practices; he elaborated on the point further in Schoenfeld (1992). The point about homogeneity is important. If different mathematicians approach the same mathematical problems in radically different ways, it will be unclear which approaches students should be taught to use and whether there is an imperative to introduce any particular one into the classroom.

In *Mathematical Problem Solving,* through a detailed analysis of mathematicians' verbal protocols while solving problems, Schoenfeld (1985) identified a number of competencies that mathematicians used to successfully solve problems. For instance, when solving problems, Schoenfeld discussed metacognitive actions such as planning, monitoring, and evaluating progress that mathematicians exhibited but students did not. He then set as an instructional goal for students to exhibit the same metacognitive actions in their problem solving attempts (for more detail, see Schoenfeld, 1987). In our interpretation, Schoenfeld is advancing the following argument:

(i) mathematicians take specific metacognitive actions (e.g., planning) in problem solving;
(ii) these metacognitive actions contribute to mathematicians' success in problem solving;
(iii) with training, students are capable of engaging in similar productive metacognitive behaviors;
(iv) therefore, students should be encouraged to take similar metacognitive actions when they solve problems.

2.2 Are the descriptions of mathematical practice accurate?

Schoenfeld's research program was fruitful. His attempts to lead students to improve their metacognition was successful and led to an improvement in their problem-solving performance (Schoenfeld, 1985, 1987). In the remainder of this section, we use Schoenfeld's research program to highlight inherent difficulties in identifying mathematicians' behaviors and the complicated links between mathematical behavior and instruction. Do mathematicians engage in the metacognitive behaviors that Schoenfeld identified? A study by DeFranco (1996) suggests that the answer might be no—or at least that the answer is not so straightforward. DeFranco (1996) compared the problem solving behavior of two groups of eight mathematicians, which we will refer to as *internationally recognized mathematicians* and *ordinary mathematicians*. As our titles suggest, the first group of mathematicians was extremely successful. The second group of mathematicians was less successful, but nonetheless each ordinary mathematician earned a Ph.D in mathematics and published papers in mathematics journals. A key finding from DeFranco's (1996) study was that the "ordinary mathematicians" were not observed engaging in the metacognitive behaviors that Schoenfeld (1985) had identified as characteristic of mathematical thinking.

One way to resolve the discrepancy between Schoenfeld's (1985) finding and DeFranco's (1996) finding is to say Schoenfeld was not actually studying mathematical practice but "mathematical thinking", where mathematical thinking does not mean the thinking of professional mathematicians but "the thinking of expert problem solvers." Schoenfeld's focus on professional mathematicians could be viewed as merely a methodological heuristic to identify subjects who would likely be expert mathematical problem solvers. The fact that Schoenfeld's students' problem-solving improved when they learned metacognitive strategies vindicates Schoenfeld's decision. Therefore, what DeFranco's work reveals is the counterintuitive finding that many professional mathematicians do not engage in "mathematical thinking," when so-defined. Such an account is internally coherent, but it still raises two questions for mathematics educators who choose to study mathematical practice.

First, when mathematics educators say that "mathematicians" engage in a certain practice, to whom exactly are they referring? If DeFranco's (1996) "ordinary mathematicians"—with their Ph.D's and publications—are not full members of the mathematical community, then the number of bona fide mathematicians is very small indeed. We refer to this issue as the *mathematical community identification problem*. We note that when mathematics educators speak of the professional mathematical community, what they often refer to research-active, pure mathematicians. Further, mathematicians in these studies are often sampled from elite universities, usually from North American and European universities. We question whether the mathematical community should be limited to this narrow population (especially when studying this community's practices to identify *generalizable* tools for the classroom).

If we do say that DeFranco's "ordinary mathematicians" are members of the mathematical community, then we must resolve the discrepancy between his findings and Schoenfeld in a different way. We must call Schoenfeld's homogeneity assumption into question, which we shall call the *heterogeneity problem*. Other scholars have challenged the claim that mathematicians engage in mathematical practice in similar ways. Notably, based on her interviews with 70 mathematicians, Burton (2004) identified stark differences in the ways that mathematicians solved problems. For instance, many mathematicians would regularly invoke visual reasoning to solve problems while other mathematicians would rarely do so. If mathematical practice is heterogenous as Burton and other scholars (e.g., Weber, Inglis, & Mejía-Ramos, 2014) have claimed, then this implies that researchers should examine a large number of mathematicians to make generalizable claims about mathematical practice[1] (although small-scale studies can offer existence proofs that at least some mathematicians engage in a certain type of behavior).

Moving from Schoenfeld's claims to more general considerations, we observe two general issues that arise when scholars want to research how mathematicians complete tasks in their own mathematical practice (e.g., Carlson & Bloom, 2005; Inglis & Alcock, 2012; Kidron & Dreyfus, 2014; Misfeldt & Johansen, 2015; Weber, 2008; Weber & Mejia-Ramos, 2011). There is the *advanced content problem*—when mathematicians conduct research, they typically investigate advanced concepts that would take a mathematics educator several years of advanced study to comprehend. This makes it inherently difficult for a mathematics educator to analyze or report on what it is that mathematicians actually do. There is the *time scale problem*—it may take a mathematician days, weeks, or even years to perform an authentic mathematical task such as proving a theorem or refereeing a paper, which poses great impediments to studying these practices (not to mention extending them to the classroom). These problems threaten the validity of task-based interviews (such as those used by Carlson & Bloom, 2005; DeFranco, 1996; and Weber, 2008). One paper in this

[1] In our experience, researchers who have studied a small number of mathematicians are careful to qualify their claims as hypotheses or existence proofs, and perhaps say that more research with a larger number of mathematicians needs to be conducted before making a general claim. However, this poses challenges for those who want to use the tentative or modest findings from smaller-scale studies to make instructional recommendations.

special issue (Mejia-Ramos & Weber, this issue) explores task-based interviews in depth. In task-based interviews, mathematicians are rarely working on truly authentic tasks, but rather laboratory tasks on relatively simple mathematical material that could be completed in a short time period. Any observed behaviors on these tasks could be due to the artificiality of the tasks and the laboratory environment and not indicative of mathematical practice. For instance, perhaps the mathematicians in Schoenfeld's studies constantly monitored their progress because they knew that a simple solution to the task existed and they were only given 20 min to find it. Lester and Kehle (2003) summarized the problem of using artificial tasks as follows:

> "One can certainly give expert mathematicians textbook problems to solve, and compare their strategies and mental representations to those of novices. But expert mathematicians do not solve textbook problems for a living. Our point is that even if cognitive scientists have studied the right people, they may have studied them doing the wrong tasks" (p. 504).

Finally, there is the *accuracy problem*. Put simply, the cognitive psychology literature documents that the ways in which mathematicians describe their activities may not accurately reflect what they actually do (see Inglis & Alcock, 2012, for further discussion). However, some mathematics educators who seek to understand how mathematicians accomplish tasks do so by asking mathematicians to describe how they complete these tasks in open-ended interviews (e.g., Misfeldt & Johansen, 2015; Weber & Mejia-Ramos, 2011) or by referring to the written reflections of famous mathematicians (see Schoenfeld, 1992, on the influence of George Polya's writings in mathematics education). The accuracy problem challenges the validity of drawing inferences in these manners.

2.3 Are these practices appropriate goals for mathematical classrooms?

For the sake of argument, suppose we managed to identify competencies that enable mathematicians to succeed at certain tasks. For instance, suppose we know that mathematicians have the metacognitive competence to monitor their work while solving challenging high school problems. Schoenfeld (1985) highlighted a critical problem on forming pedagogical decisions based on this insight, which we will refer to as the *resources problem*. The successful implementation of a desirable mathematical behavior often requires a vast array of background knowledge and competencies—knowledge and competencies that mathematicians may have and students may lack. For instance, when mathematicians anticipate whether a problem solving plan is likely to be successful or evaluate whether they are making progress on a problem, these metacognitive self-prompts are helpful because mathematicians have the capacity to provide accurate answers to these questions; this capacity is often based on experience and intuition that students may not share. (For a detailed argument about how much background knowledge and metacognitive competence are required to successfully employ problem-solving heuristics, see Schoenfeld, 1985, chapter 3). The resources problem does not mean that insights into mathematical expertise are irrelevant for pedagogy, but efforts to leverage the relationship between mathematical practice and pedagogy should account for this discrepancy.

3 From mathematical practice to pedagogy: the case of epistemology

3.1 Harel and Sowder's proof schemes research program

A second goal that some mathematics educators have is to teach students epistemology: students should learn about epistemology in mathematical practice and behave in a manner consistent with that epistemology. Perhaps the most influential research program designed to teach epistemology in mathematics is Harel and Sowder's *proof schemes* research program. Harel and Sowder's work on proof schemes is rich and nuanced and we cannot do justice to it in this limited space; we encourage the reader to read Harel and Sowder's classic works on this topic (e.g., Harel & Sowder, 1998, 2007). However, a simplified view of their theory is this: an individual's proof scheme is the mental scheme that she uses to seek and obtain psychological certainty in mathematical statements. Professional mathematicians possess analytic proof schemes, meaning that they seek and obtain psychological certainty in mathematical statements via logical deduction. Harel and Sowder (2007) formulated their goal for instruction as follows:

> "[T]he goal of instruction must be unambiguous—namely, to gradually refine current students' proof schemes toward the proof schemes shared and practiced by mathematicians today. This claim is based on the premise that such a shared proof scheme exists and is part of the grounds for scientific advances in mathematics" (p. 809).

As the quotation above clearly illustrates, Harel and Sowder (2007) want students to seek and obtain psychological certainty in a manner similar to how mathematicians seek psychological certainty. By framing the goal of proof instruction in this way, Harel and Sowder privileged the epistemological role of proof in mathematics, desiring that students to develop the same epistemological standards for conviction and certainty that mathematicians hold. We believe Harel and Sowder (1998, 2007) are advancing the following argument:

(i) mathematicians hold analytic proof schemes in which they obtain psychological certainty in mathematical statements via proof;
(ii) this aspect of mathematical practice contributes to the scientific success of mathematics;
(iii) with training, students can be led to develop analytic proof schemes;
(iv) therefore, it is a suitable goal of instruction that students develop the analytic proof schemes that mathematicians hold.

This focus on epistemology has been beneficial for mathematics education, leading to robust understandings of how students and teachers view proof and the design of effective instruction for teaching proof.

3.2 Are the descriptions about mathematical practice accurate?

Let us critically analyze this chain of argument. First, are Harel and Sowder's claims about mathematical practice accurate? Specifically, do mathematicians seek and obtain psychological certainty in mathematical assertions perhaps exclusively via proof? In reading Harel and Sowder's (1998, 2007) works, we find a detailed historical analysis of mathematicians' ontology, epistemology, and proving practices, establishing the importance of proof in verifying mathematical claims. However, we do not find direct support for their claim that mathematicians gained *psychological certainty* from the proofs that they produced. Other scholars argued that mathematicians should not and do not obtain psychological certainty from proofs (see Weber, Inglis, & Mejia-Ramos, 2014, for a review of some of these scholars). This raises an interesting question about which claims about mathematical practice require justification by mathematics educators. It would seem that carefully justifying obvious claims, such as "many journal articles in mathematics contain proofs," would be a waste of time. The idea that proof provides certainty in mathematical statements might seem equally obvious to some scholars.

For the remainder of this section, let us grant the premise that mathematicians prove to obtain psychological certainty and that they usually obtain certainty from the proofs that they produce. We can see two issues raised in the previous section are again relevant here. First, *the mathematical community identification problem* is important here. It is probably true that most *pure* mathematicians engage in proving in their research. However, many applied mathematicians and statisticians justify their claims with empirical evidence (e.g., Monte Carlo tests). Why are we privileging the schemes of pure mathematicians over the schemes of applied mathematicians and statisticians? Second, *the heterogeneity problem* is arguably relevant here. Harel and Sowder (1998, 2007) state it as a premise that a "shared proof scheme exists" amongst the community of mathematicians. Yet many mathematics educators (e.g., Dreyfus, 2004) vigorously contest this premise.

3.3 Are these practices appropriate goals for mathematical classrooms?

For the sake of argument, let us suppose that Harel and Sowder's (1998, 2007) claims about mathematical practice are accurate. That is, let us suppose that most mathematicians seek and obtain psychological certainty in mathematical claims exclusively via logical deduction. Should we expect students to seek psychological certainty in the same way? The *resources problem* suggests the answer might be no. Mathematicians, who have spent years honing their ability to detect errors in proof, may be warranted in gaining certainty from a proof after affirming that the proof is correct. Students who lack the mathematicians' experience may be right to doubt their ability to validate a purported proof (Weber, Lew, & Mejía-Ramos, 2020). Indeed, numerous studies document that when students are asked to determine if a given proof is correct, their performance is quite poor (e.g., Selden & Selden, 2003). Consequently, students are perhaps justified in not gaining certainty from the proofs that they read.

4 From mathematics to pedagogy: the case of authentic mathematical activity

4.1 Lampert-inspired classrooms promoting a Lakatosian view of mathematics

In the 1990s, many mathematics educators shifted away from viewing mathematical learning as the acquisition of competencies and desirable epistemological beliefs and toward participation in authentic mathematical activity (e.g., Lampert, 1990; Schoenfeld, 1992; Sfard, 1998). The critical shift involved not having students engage in mathematical activity to achieve other cognitive goals (e.g., engaging students in defining so they will understand the resulting definition better), but rather viewing the participation in the activity as an end in itself (Sfard, 1998). This research paradigm led to an interest in the nature of professional mathematical communities and their collective activities. If the goal of instruction is to engage students in authentic mathematical activity, then one must have some knowledge of what authentic mathematical activities are. This is often defined as what professional mathematical communities do ("authentic" in this case meaning true to the discipline more than true to the student, though the term is in many cases used to bridge the two senses).

Consider the title of Lampert's (1990) famous article, *When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching*. Our interpretation of the suggestive title is that Lampert is reconceptualizing school mathematical activity less in terms of obtaining correct solutions to problems and more in terms of engaging students in authentic mathematical activity. Such activities should lead students to know mathematics in a more discipline appropriate way. Lampert's aim was to design fifth-grade classrooms where students would participate in two interrelated activities. First, students should engage in "conscious guessing" (i.e., forming reasoned conjectures) that evinced the virtues of courage and modesty. The second was to have students engage in a "zig zag" process in which they formed conjectures and then used refutations of these conjectures to challenge faulty assumptions. Lampert justified her first instructional aim as follows:

> "Why does this teacher think it is appropriate to encourage conscious guessing and the human virtues of courage and modesty? The answer is to be found in Lakatos' analysis of what it means to know mathematics and his ideas about how new knowledge develops in the discipline" (p. 30).

Lampert went on to summarize some themes of Lakatos' (1976) well-known text *Proofs and Refutations*, in which Lakatos presented a fictitious dialogue between teacher and students as they develop Euler's conjecture, "following a 'zig zag' path starting from conjectures and moving to the examination of premises through the use of counterexamples and 'refutations'" (Lampert, 1990, p. 29). Lakatos appended the virtues of courage and modesty to this process of conscious guessing because one's proofs and even one's axioms are up for re-examination and may always be revised or overturned. Lampert stated "it is this vulnerability to re-examination that allows mathematics to grow and develop" (p. 30). Consequently, the virtues of courage and modesty should be exhibited not primarily to build the moral character of the students, but because expressions of these moral virtues are conducive to mathematical growth.[2]

Lampert observed that school mathematics typically did not feature the mathematics that Lakatos described; classrooms typically did not permit space for conscious guessing or 'zig zag' paths in which students were permitted to question prior assumptions. After lamenting the disjunction between school mathematics and Lakatos mathematics, Lampert wondered if the disjunction was necessary. The purpose of her paper was "to examine whether it might be possible to bring the practice of knowing mathematics in the school closer to what it means to know mathematics within the discipline" (p. 30), by producing "lessons in which public school students would exhibit—in the classroom—the qualities of mind that Lakatos … associated with doing mathematics" (p. 33).

Our interpretation of Lampert's argumentation can be summarized as follows:

(i) knowing mathematics in the discipline involves the process of conscious guessing and the 'zig zag' pattern of reasoning as described above;
(ii) this account of mathematical practice is justified by Lakatos' (1976) philosophical-historical analysis;
(iii) the 'zig zag' contributes to the success of the discipline of mathematics;
(iv) in the right mathematical classroom environments, students can meaningfully engage in and appreciate the 'zig zag' in a matter that is both reflective of professional mathematicians' activity and appropriate for school age children;
(v) therefore, it would be desirable to create mathematical classrooms in which students engage in some form of conscious guessing and following a path similar to the 'zig zag' path that Lakatos described.

4.2 Are the descriptions about mathematical practice accurate?

Bruner (1960) called for "a continuity between what a scholar does on the forefront of his discipline and what a child does in approaching it for the first time" (p. 27–28). Lampert's article was an important response to that call, which in turn led to an increased focus amongst mathematics education researchers and practitioners on mathematical practices. Below, we use Lampert's (1990) article to highlight challenges inherent in any attempt to meet Bruner's challenge of having students engage in authentic mathematical activity, some of which were discussed by Lampert (1992) herself in a subsequent article.

Returning to the two recurrent questions of our article: Did Lampert (and Lakatos) accurately describe mathematical practice[3]? Mathematicians certainly make reasoned

[2] Lampert also buttressed her claims about mathematical practice with quotations from George Pólya. However, these claims only support the virtues of conscious guessing. Our concern in this section will be on whether the 'zig zag' pattern described by Lakatos (1976) constitutes an accurate portrayal of mathematical practice.

[3] Although Lakatos' (1976) volume only discussed two case studies, he claimed that he was describing a "very general heuristic pattern" in the discovery of mathematical ideas (p. 127). However, as Larvor (1998) emphasized, Lakatos claimed his account does not apply to all of mathematics. This raises an interesting version of the

conjectures, but do they regularly use these conjectures to overturn definitions, axioms, and other assumptions? Are proofs frequently refuted and amended? Selden and Selden (2003), two former mathematicians turned mathematics educators, claimed that this is not often the case in contemporary mathematics. The difference, according to Selden and Selden, was that Lakatos' analysis of eighteenth century mathematics occurred when definitions were synthetic (i.e., descriptive definitions, like dictionary definitions). Contemporary mathematics uses analytic definitions (i.e., precise, unambiguous, and stipulative definitions, reducible to foundational terms), so Selden and Selden claimed that the 'zig zag' phenomenon that Lakatos highlighted seldom occurs today:

> "Today's use of analytic definitions renders validations (and proofs and theorems) very reliable, whereas earlier uses of synthetic definitions left them problematic … *Proofs and Refutations* provides a fairly accurate description of the way that some mathematics developed historically. It illustrates how definitions and results can co-evolve, but the compression of time involved in its historical narration, together with its synthetic treatment of definitions, may suggest that validations, proofs, and theorems are far less reliable than they really are today" (p. 8).

We believe that Selden and Selden's (2003) critique should be taken seriously. Indeed, many philosophers and historians have questioned whether Lakatos' technique of "rational reconstruction" (i.e., taking literary license to transform historical mathematical episodes into fictional dialogue) provides an accurate account of history. The philosophers Musgrave and Pigden (2016) wrote a survey on the work of Lakatos and summarized the criticisms of his historical work as follows:

> "This device [of rational reconstruction], first necessitated by the dialogue form, became a pervasive theme of Lakatos' writings. It was to attract much criticism, most of it centered around the question of whether rationally reconstructed history was real history at all. At one point in *Proofs and Refutations* a character in the dialogue makes a historical claim, which according to the relevant footnote, is false […] Horrified critics protested that rationally constructed history is a caricature of real history, not actual real history at all but rather 'philosophy fabricating examples.'"

Our point in bringing up these criticisms is not to say that Lakatos' portrayal of mathematical practice is inaccurate. Rather, we highlight that many mathematics educators, philosophers, and historians have questioned the validity and relevance of Lakatos' work. These criticisms were neither acknowledged by Lampert (1990) nor by the other mathematics education studies using Lakatos cited earlier in this paper. Mathematics educators often accept Lakatos' conclusions as a reasonable account of mathematical practice.

To us, these criticisms highlight a significant challenge with the ways that mathematics educators justify claims about mathematical practice. We refer to this problem as the *interdisciplinary problem*. To form and justify claims about mathematical practice, mathematics educators frequently appeal to scholarship from other disciplines, such as philosophy, history, and sociology. Schoenfeld (1985) approved of the interdisciplinary approach to understanding of mathematical practice, but highlighted an inherent challenge within it:

> "While this situation [reading work from diverse disciplines] is natural and healthy, there are ways in which it makes for difficulties. Papers within any disciplinary tradition are usually written under the assumption that readers will share the author's paradigms, assumptions, and language. Readers from outside the discipline can find it hard to penetrate the barrier of assumptions and language, to uncover relevant results, and to see connections that might otherwise be seen" (p. 5–6).

We agree with Schoenfeld about mathematics educators' difficulties *interpreting the meaning* of the work from other disciplines from which they draw upon. We highlight a further point: Mathematics educators will also have difficulty *analyzing the validity or correctness* of this work and *contextualizing it* as one viewpoint within an ongoing discourse. This danger of misinterpretation is not simply hypothetical. In a review of a mathematics education volume that drew heavily upon classical philosophers, Larvor (2019), a philosopher conversant in mathematics educational research, remarked that standard philosophical practices for citing the views of classical philosophers were not practiced and he questioned whether the attribution of positions to classical philosophers was accurate. [e.g., "I suspect that Kant has been tagged with a view that did not even exist in his day" (p. 321)]. However, Larvor conceded that, "perhaps this does not matter for the purposes of research into mathematics education, because the gesture in the direction of canonical philosophers is simply for orientation and throat-clearing" (p. 321). If we hope to use philosophical work not just for

Footnote 3 (continued)

"heterogeneity problem" for historical analyses—there may be sharp deviations in the ways that mathematical ides develop historically. Similarly, Lampert (1990, 1992) does not insist that *all* classrooms employ Lakatos' 'zig zag', only that it might be desirable for some classrooms to do so.

throat-clearing but to inform instruction, it is important to do this philosophical work justice.

Philosophers, through their training and their familiarity with the broader philosophical landscape, will be in a good position to know which of Lakatos' insights are accepted, which insights are controversial, and which insights are regarded as refuted. Many mathematics educators will lack the background to make such judgments. Of course, it is possible for mathematics educators to make such judgments accurately—say, by conducting an extensive review of the philosophical literature or collaborating with a philosopher—but at a minimum, this adds a large burden for the mathematics education researcher who wishes to use insights from other disciplines to inform their understanding of mathematical practice.

4.3 Are these practices appropriate goals for mathematical classrooms?

For the sake of argument, let us suppose that Lakatos' *Proofs and Refutations* accurately captures the nature of mathematical practice. This alone would not imply that such practices should be emulated in mathematics classrooms. Lampert desired that her classrooms practice Lakatos' conscious guessing and zig zag patterns not just because this is what mathematicians do, but because these mathematical activities directly contributed to the success of mathematics as a discipline (which is similar to Harel and Sowder's justification for emphasizing deductive proofs schemes). However, it is still not clear that any practice that has contributed to the success of a science would be useful and appropriate to emulate in a classroom. Rather, Lampert used her professional judgment to select mathematical practices that were germane to mathematics education. While we may agree with her judgment, it raises the question of how we render such judgments. Ball (1993) nicely summarized the concern as follows:

> "Constructing a classroom pedagogy on the discipline of mathematics would be in some ways inappropriate, even irresponsible … Certain aspects of the discipline would be unattractive to replicate in mathematics classrooms. For instance, the competitiveness among research mathematicians— competitiveness for individual recognition, for resources, and for prestige— is hardly a desirable model for an elementary classroom. Neither is the aggressive, often disrespectful, style of argument on which much intradisciplinary controversy rests" (p. 377).

Ball's commentary raises two concerns. The first is an *ethical problem*, where it may be unethical to encourage students to fight for credit or status, or to be aggressively disrespectful to their classmates when criticizing their work, even though the drive for credit and status may drive professional mathematicians to great accomplishments.

Ball's second concern relates to the differences in epistemic goals between research communities and classrooms. According to Auslander (2008), many mathematicians will freely apply the published theorems of others without understanding why these theorems are true. If the result appears in print and the trusted experts in the field declare the result to be reliable, this is sufficient for many mathematicians to accept and use the results. Auslander (2008) claimed that this practice is an essential contributor to growth in mathematics. If a mathematician had to verify every published result for herself before she used it, this would slow her productivity to a halt. Weber (2018) argued that we would not want this practice to be normative in mathematical classrooms. Individual students should not accept a result because the authorities in the classroom (the teachers and the best students) understood and sanctioned the results. Ball (1993) made a similar point: In contrast to mathematicians, teachers "are charged with helping all students learn mathematics, in the same room at the same time… The best and seemingly most talented of the students must not be alone in developing mathematical understanding and insight" (p. 377). In short, mathematicians' practices of distributing expertise throughout the community may be essential for the growth of the discipline, but undesirable in a classroom. We refer to this difficulty as the *different community goals problem*.

5 Discussion

We have written this paper to make three contributions:

(i) we have illustrated that mathematics educators frequently use claims about mathematical practice to inform mathematics pedagogy;
(ii) we have identified several reasons why it is intrinsically difficult to make accurate claims about mathematical practice;
(iii) we have shown why the relationship between mathematical practice and pedagogy is not straightforward. Classroom communities should not engage in a given activity solely because professional mathematical communities do.

With regard to (i), we have described three different, highly-influential research programs in which leading scholars invoked claims about mathematical practice to warrant their pedagogical decisions.

With regard to (ii), we have identified a number of problems with making claims about mathematical practice. We summarize these claims below:

- *The mathematical community identification problem* Accounts of mathematical practice will depend upon choices about who is and is not a mathematician (e.g., pure, applied, or experimental mathematicians). The truth-value of some claims about mathematical practice will therefore be ambiguous, as they will depend on who counts as a mathematician.
- *The heterogeneity problem* Any reasonably large mathematical community will likely be heterogeneous in at least some of their practices, making general claims about how mathematicians approach a task problematic.
- *The advanced mathematical content problem* Mathematicians typically study mathematical content that mathematics educators lack the background to comprehend, making it difficult for mathematics educators to investigate and interpret genuine mathematical practice;
- *The time-scale problem* Authentic mathematical activities can take months or years to complete, making it difficult for mathematics educators to investigate such activities;
- *The accuracy problem* Mathematicians may have distorted views about how they engage in mathematics, making their reflections on their participation in mathematical activities untrustworthy;
- *The interdisciplinary problem* Drawing on historical or philosophical literatures that provide accounts of mathematical practices requires being aware enough of those bodies of scholarship to know when particular (even famous) claims have been deeply challenged or even refuted.

As a consequence of these difficulties, we maintain that every known methodology for investigating mathematical practice is problematic in some way. Task-based interviews are vulnerable to the advanced mathematical content problem and the time-scale problem. Mathematics educators who conduct task-based interviews are not observing mathematicians complete authentic tasks, but rather laboratory tasks designed to approximate mathematical practice. Any observed behavior could plausibly be an artifact of the design of the task or the laboratory setting. Quoting the reflections of famous mathematicians is especially sensitive to the heterogeneity and accuracy problems: what the quoted mathematician describes might not be reflective of the practices of the broader mathematical community, or even her own practices. Open-ended interviews are subject to the same threats to validity.

We believe triangulation and interdisciplinary collaboration are necessary to paint an accurate picture of mathematicians' practice. Mathematics educators can use multiple methodologies from different research traditions to gain a better understanding of mathematical practice. However, this raises the *interdisciplinary problem* according to which some mathematics educators may understandably lack the background to interpret or evaluate the results or apply the methods from other disciplines.

The first set of papers in this special issue was written to address these issues. We invited these papers from disciplinary or methodological experts in philosophy (Hamami & Morris, 2020), history (Barany, 2020), task-based interviews (Mejía-Ramos & Weber, 2020), experimental methods (Inglis & Aberdein, 2020), and crowdsourcing (Pease, Martin, Tanswell, & Aberdein, 2020). They were asked to:

1. provide a useful primer on their disciplines and methods,
2. describe the questions about mathematical practice that their approach can answer while highlighting limitations in their approach, and
3. provide specific examples of how research in their domain has informed our understanding about mathematical practice.

In a final paper, Hanna and Larvor explore the affordances and limitations of relying on the writings of famous mathematicians for understanding mathematical practice. While "quoting mathematicians" is not an explicit methodology per se, this is a common means that mathematics educators use to warrant claims about mathematical practice and we believe that mathematics educators will benefit from this critical analysis thereof.

Regarding (iii), we identified a number of reasons why it might be challenging or even inappropriate to import professional mathematicians' practices to the mathematics classroom. Below we summarize the issues that we identified:

- *The resources problem* Students may not have the knowledge, experience, or ability to engage in the activities that are productive for mathematicians;
- *The ethical problem* Some practices of the mathematical community that contribute to the growth of the discipline may be ethically problematic to replicate in classrooms.
- *The different community goals problem* Activities or practices that may be useful for the growth of mathematical knowledge within the professional discipline may be inappropriate or detrimental for promoting student growth in the classroom.

We remain confident that, at least to some extent, mathematical practice can and should inform mathematics pedagogy. However, the issues above highlight that the relationship between what mathematicians do and the classroom require care and attention. In the second part of this special issue, authors have analyzed these themes further. The first three articles in this part of the issue explore how mathematical practices *can* inform pedagogy. They look at mathematical practice's roles in instructional situations (Herbst &

Chazan, 2020), consider when mathematical practices can be viable and scalable in mathematics classrooms (Schoenfeld 2020), and explore how mathematical epistemology can be taught (Dawkins, 2020). The remaining three articles raise concerns about why some mathematical practices might be the wrong practices to base instruction upon. They focus on which mathematical practices are ethical (Tanswell & Rittberg, 2020) and equitable (Skovsmose, 2020) and whether pure mathematicians' practices reflect the mathematics that is done by most individuals (Stillman, Brown, & Czocher, 2020). The authors of these three papers differ in their stances on how mathematics educators should regard mathematicians' practice, but they all raise concern that any mathematics educator should consider when justifying instructional choices around mathematical practice.

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