ZDM (2019) 51:1069–1082
https://doi.org/10.1007/s11858-019-01087-z

ORIGINAL ARTICLE

Student perspectives on proof in linear algebra

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Accepted: 6 September 2019 / Published online: 10 September 2019

Abstract

Proof has a prominent place in the linear algebra curriculum, teaching and learning but in first-year courses it continues to be challenging for both instructors and students. While an introduction to new concepts through definitions and theorems adds to the complexity of the course, proof remains the number one hurdle for many students. How do students view proof in linear algebra? Do they distinguish argumentation and proof, and if so how? are among many questions that are still unanswered. Although research on proof in mathematics education is increasing, systematic studies on proof in linear algebra are still scarce. In this study, we examined responses to a set of interview questions on proof by a group of 16 first-year undergraduate students shortly after their final examination. This paper opens the case for a pedagogy of proof in linear algebra and examines students' reactions to, and voices on, proof in a first-year course in linear algebra. In particular, it addresses areas such as student views on understanding of proof, the purpose of a proof, and when and how proofs communicate to them. We employed Tall's Three Worlds as well as Harel's intellectual need to analyse the data. Although, these models are often applied to what students construct, we argue they can also be applied to how students perceive proofs. The results revealed that understanding a proof in order to gain personal conviction was a major concern of students.

Keywords Proof · Linear algebra · Convincing · Comprehension

1 Background

Proof is considered by mathematicians to be central to doing mathematics (Thurston 1995). Hence, there has been much research into student ability with respect to proof, comprising three broad strands: constructing proofs; validating proofs; and proof comprehension. In this paper we examine student perspectives on the purposes of proof and their preferences for the kind of proofs they think meet these purposes. In order to set the scene, we first review what the literature tells us about undergraduate students' reading, comprehension and construction of proofs in mathematics in general, and then consider their role in the teaching of linear algebra.

Since proof is held in such high regard, Weber and Mejía-Ramos (2015, p. 15) note that, often, "a primary goal of mathematics instruction is for students to adopt the standards for proving and conviction that mathematicians hold."

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This may explain why considerable effort has been put into research on how proof construction (Mejía-Ramos and Inglis 2009), such as students' ability to reproduce or construct certain proofs (Lockwood et al. 2016). Three requirements for successful engagement with proof given by Stylianides and Stylianides (2007) are: to recognize the need for a proof; to understand the role of definitions in the development of a proof; and the ability to use deductive reasoning. However, among the conclusions from research is that students do not have the experiences to support building rigorous, deductive arguments (Stylianou et al. 2015). While mathematicians may employ different strategies for proof construction (Lockwood et al. 2016) research has suggested some possible ways to assist students to construct proofs. These include the use of conjectures (Pedemonte 2008), strategic examples (Lockwood et al. 2016), and counterexamples (Zazkis and Chernoff 2008).

In addition to learning how to construct proofs researchers such as Harel (1997) have proposed that students should be encouraged to learn how to read proofs. Recently studies on student reading of proofs have focused on the manner in which they read proofs (Inglis and Alcock 2012; Panse et al. 2018) and how this compares with the ways mathematicians see proofs (Harel and Sowder 2007; Weber and Mejía-Ramos 2011). It has been observed that, in general, students read proofs either to validate the proof (Selden and Selden 2003), that is to decide if it is true or not, or to comprehend, or understand, a proof. Using eye-tracking technology, Panse et al. (2018) found no difference in reading behaviour between reading for validation and reading for comprehension in either undergraduates or mathematicians. However, studies have shown that undergraduate students tend to validate proofs differently from mathematicians (Inglis and Alcock 2012). It seems that mathematicians may expend considerably more effort inferring implicit between-line warrants, understanding the key ideas, the structure and the techniques employed (Weber and Mejía-Ramos 2011), while students may be more concerned with processing algebraic manipulations and focus proportionately less on words and logical relationships (Inglis and Alcock 2012). Weber (2008) also showed that mathematicians tend to use formal, informal deductive and example-based reasoning during proof validation. In distinguishing the role of proof, Hersh (1993, p. 396) asserted that "In research its role is to convince. In the classroom, convincing is no problem. Students are all too easy convinced…What a proof should do for the student is provide insight into why the theorem is true."

Research on reading for comprehension has often focused on task-based interviews, tests and assessment models (Mejía-Ramos et al. 2017). While it has been proposed by Mejía-Ramos and Inglis (2009) that reading for comprehension may differ from that for validation, since in the former the proof is assumed to be valid, both types of proof reading appear to be related to the students' desire to gain a personal conviction, the first level of Mason et al. (1982) three levels of conviction. In addition, this conviction has been described as either relative or absolute, although few students will obtain absolute conviction (Weber and Mejía-Ramos 2015). To gain a personal conviction an individual has both to understand what the proof is saying and believe that it contains a valid presentation of its truth. Such an understanding of a proof involves more than understanding each of the proof's steps but also requires an overview of it (Harel 1997).

1.1 Pedagogy of proof in linear algebra

Linear algebra is a core subject for mathematics students and it is often recommended, or required, for many STEM majors. By the time students arrive at a linear algebra class they have had some exposure to university-level mathematics, but despite this, many struggle to grasp the theoretical aspects of the course, especially proof. One reason for this, observed by mathematicians, is that many calculus courses tend to emphasise formulas and processes (Uhlig 2002) rather than concepts, justifications and proofs (Vinner 1997), which may not be sufficient for the axiomatic nature of linear algebra. In particular, according to Uhlig (2002), calculus does not help because "our current calculus textbooks develop very few proofs—which are often skipped in class—the students of a first linear algebra course generally have had no experience with math proofs for many years" (p. 336). In search of a possible resolution to the question of "what causes the linear algebra fog?", Britton and Henderson (2011, p. 964) claim that conceptual understanding is at the root of the problem.

> …when we expect students in a linear algebra course to use and understand many new definitions and to construct proofs similar to those they have seen, we should acknowledge that success in these tasks is inextricably linked to their level of conceptual understanding, and that in effect we are assessing that conceptual understanding right from the outset.

It may be that linear algebra proofs need to evolve naturally in a course and Harel (1997, pp. 119–122) has offered four recommendations for achieving this goal.

> 1. Students should take an active part in the construction of relations between ideas and in the production of their justifications; 2. Students should be helped to build proofs on their intuitions; 3. Students should be encouraged to read proofs; 4. Students should learn that understanding a proof is more than understanding each of the proof's steps.

While Harel acknowledges that this model of teaching requires many hours of class time, he believes there is no other alternative.

In pursuit of a natural evolution of proofs in a course it is reasonable to consider the role of Tall's (2008) embodied, symbolic, and formal worlds of mathematical thinking, and so a possible question is whether there is a preferential order of these for linear algebra concepts to be taught. In a study by Hannah, Stewart, and Thomas (2014), most students believed that the formal aspect of linear algebra should come last. Others disagree and Harel (1999) cautions that starting with the geometry of the embodied world may hinder students from learning the true concepts of linear algebra. Nevertheless, Harel is not suggesting eliminating geometry from linear algebra. His concern is that the simple geometric examples that are understood by most students will "form an extremely powerful concept image that it is hard for many to relinquish." (*ibid*, p. 613). According to Gueudet-Chartier (2006, p. 190): "…geometric models must be used carefully in linear algebra courses. Geometry cannot be the only starting point for linear algebra, other domains must intervene to justify the need for a general theory". One of the difficulties in moving between Tall's worlds is highlighted by Dias and Artigue (1995, cited in Dorier 1998, p. 158), who found that “students do not spontaneously change frameworks or points of view and that, if they are forced to do so, they have great difficulty”. Hence, they propose that addressing the flexibility to interpret results in different frameworks must be explicit in teaching.

2 Theoretical framework

In this research we employed two major theoretical frameworks to inform our data collection and analysis. The first is Tall’s (2008) Three Worlds of Mathematics framework, which Tall and Mejía-Ramos (2006) have applied to illustrate how proof types change as students become more mathematically sophisticated. The notion of a warrant for mathematical truth (Rodd 2000) defined as that which secures someone’s knowledge in what is claimed to be known (Inglis, & Mejia-Ramos 2008), will be used to distinguish the warrants in each of the worlds. For example, in the embodied world physical experiments lead to visual and embodied reasoning, while in the symbolic world, arguments move from having a numerical to an algebraic warrant, supported by the backing of symbolic manipulation. However, in the formal world proof is warranted by deductive reasoning. This movement through the three worlds, with their changing warrants characterises student development and relates to production of proof schemes, as students move from perception and action, through operation and symbolism, to reason and formality (Tall et al. 2012).

Our second framework is Harel’s (2008a, b, 2013, 2018b) ideas on ways of understanding and ways of thinking. This includes the notion of proof schemes (Harel and Sowder 1998), which builds on Hanna’s (1990) two categories, of proofs that prove and proofs that explain (for the individual). Thus, for Harel and Sowder proving is a process that removes or creates doubts about the truth of an assertion and comprises the two sub-processes of ascertaining (removing one’s own doubts through understanding) and persuading (removing the doubts of others), so that “A person’s proof scheme consists of what constitutes ascertaining and persuading for that person.” (Harel and Sowder 1998, p. 244). The concept of ascertaining corresponds with Mason et al.’s (1982) first level of conviction, mentioned above, which is to convince yourself. This emphasis on proof schemes constitutes a shift towards ways of thinking rather than simply ways of understanding (Harel 2008a, b), where ways of understanding are a generalisation of the idea of proof, and ways of thinking generalises the notion of proof scheme. It also includes problem solving approaches and beliefs about mathematics. A key implication of Harel’s theoretical approach is that the elements of mathematics comprise both ways of understanding and ways of thinking. Furthermore, Harel (2013, 2018a, p. 9) suggests that “for students to learn what we intend to teach them, they must have a need for it, where ‘need’ refers to intellectual need”. Harel (2018b, pp. 36–37) breaks down this intellectual need into the following five categories:

> *(1) Need for certainty*. This is the need to prove, to remove doubts. One’s certainty is achieved when one determines, by whatever means he or she deems appropriate, that an assertion is true. (2) *Need for causality*. This is the need to explain—to determine a cause of a phenomenon, to understand what makes a phenomenon the way it is…. (3) *Need for computation*. This need includes the need to qualify and to calculate values of quantities and relations among them by means of symbolic algebra. (4) *Need for communication*. This consists of two reflexive needs: *the need for formulation*—the need to transform strings of spoken language into algebraic expressions—and the *need for formalization*—the need to externalize the exact meaning of ideas and concepts and the logical justification for arguments. (5) *Need for structure*. This need includes the need to re-organize knowledge learned into a logical structure.

According to Harel (2018a), “the first two needs are complementary to each other: understanding cause brings about certainty, and certainty might trigger the need to determine cause” (p. 15). In this context, a major role of proof is to provide certainty for the individual that an assertion is true. However, it also provides understanding of cause, what makes a phenomenon the way it is, and a way of communicating, satisfying “the need to externalize the exact meaning of ideas and concepts and the logical justification for arguments” (Harel 2018b, p. 37).

While a good deal is known about students’ ability to construct proofs and how they read them, less is known about students’ conceptions of proof and their perspectives on the purpose of proof in mathematics (Stylianides and Stylianides 2007). This research was situated in the context of obtaining the student voice on the purposes of proof, with the idea that this could contribute to Harel and Sowder’s (1998) aim to “map students’ cognitive schemes of mathematical proof” (p. 237). Specifically, we anticipated revealing, which of Harel’s intellectual needs were more noticeable among the students. Furthermore, our primary goal was to investigate students’ perceptions of, and reactions to, proof in a first-course in linear algebra and see whether the embodied or symbolic worlds provided greater understanding for them (Tall 2013).

3 Method

3.1 Participants and settings

As part of a larger project, in this case study research, a group of 16 first-year linear algebra students from four different classes (in different semesters) attending a research university in Southwest of the US, were interviewed and asked a list of open questions regarding proofs in linear algebra. The interviews with students (four females and 12 males) took place shortly after the final examination in their linear algebra course and were independently conducted by a colleague in the mathematics department who had experience in interviewing from previous research collaborations. The students were majoring in Mathematics (2), Physics (1), Meteorology (1) and Engineering (12). This was a typical first year linear algebra course in the US, covering topics such as: systems of linear equations; determinants; finite dimensional vector spaces; linear transformations, eigenvalues and eigenvectors, and more. The course was taught by the first named author and proof was integrated into the lectures and regular homework as well as being examined in the tests and the final examination of the course. The concepts were generally introduced with a combination of definitions, theorems, examples and sometimes pictures.

In general, the level of proof exposure in this course is dependent on the instructor who is teaching it. Some instructors believe that this course should cover many proofs and others prefer less exposure at this level. The assigned textbook by Kolman and Hill (2008), is more inclined toward definitions, theorems and proofs, and throughout the teaching of the course some of the theorems were proved and some were just quoted as results. The instructor hoped that knowing the properties of objects in the proofs and having concept images of the results would work as a base for future building of the proofs not covered. Unlike some calculus proofs, in this course it would rarely be the case that a picture would have been deemed sufficient as a proof.

3.2 Data collection and instruments

The interviews were semi-structured and took about 20–30 min each. They were audio recorded and later transcribed for analysis. The interviewer first explained Tall's (2008, 2013) framework and showed each student three sample linear algebra examples in each world (see Fig. 1). The students were also shown two theorems and their proofs from the course (see Fig. 2). These proofs were selected since they were set as homework assignments and later discussed in class in order to show how to construct proofs. The proof of Corollary 6.2 is short and elegant and brings many definitions and results together in one place. Similarly, the proof of part (b) of Theorem 6.4, relies on knowing a number of previous results and combining and fitting them all together in this proof. Finally, both theorems are significant results that are used frequently, so naturally we want our students to know their proofs.

The interview questions used were: What is the purpose of a proof? Do you find proofs convincing? What kinds of proofs, if any, do you prefer? Did [a specific proof—see Fig. 2] help you to understand why such-and-such was true, or why such-and-such always happens? Are there other ways that would have done a better job of helping you understand, or of convincing you? (For example, a picture or even a series of random examples?) Do you think we should offer a separate course on how to prove results in mathematics?

3.3 Data analysis

A provisional or hypothesis coding, which was based on the theoretical ideas underpinning the research (Miles et al. 2014), was carried out on the transcribed interview data. In this kind of coding some codes, arising from the theory, are assigned before reading the data and others emerge later from the data. Based on Harel's (2018b) theoretical framework, we considered students' intellectual need for proof, particularly to provide certainty about assertions, to aid understanding of cause and to communicate. The initial coding then led to emergent themes or categories of responses. Thus, a categorical analysis

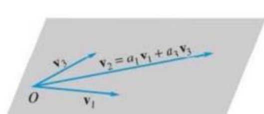
Embodied World	Symbolic World	Formal World
Pictures and geometry  (a) Linearly dependent vectors in R^3.	Examples Let $L: R^3 \to R^3$ be defined by $L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ (a) Is L onto? (b) Find a basis for range L. (c) Find ker L. (d) Is L one-to-one?	Definitions and Theorems Let $L: V \to W$ be a linear transformation of a vector space V into a vector space W. The **kernel** of L, ker L, is the subset of V consisting of all elements $\mathbf{v}$ of V such that $L(\mathbf{v}) = \mathbf{0}_W$.

Fig. 1 Examples of linear algebra in Tall's Worlds

Corollary 6.2 If L: $V \rightarrow W$ is a linear transformation of a vector space V into a vector space W and dim V= dim W, then the following statements are true:

(a) If L is one-to-one, then it is onto.
(b) If L is onto, then it is one-to-one.

Proof. By Theorem 6.6 we have dim Ker L + dim range L= dim V

(a) If L is 1-1, then Ker L = $\{\mathbf{0}\}$, so dim Ker L= 0. Hence, dim range L=dim V= dim W, so L is onto.
(b) If L is onto, then range L = W, so dim range L= dim W= dim V. Hence, dim Ker L= 0 and L is 1-1.

Theorem 6.4 Let L: $V \rightarrow W$ be a linear transformation of a vector space V into a vector space W. Then:

(a) Ker L is a subspace of V.
(b) L is one-to-one if and only if Ker L= $\{\mathbf{0}_V\}$

Proof

(a) We show that if $\mathbf{v}$ and $\mathbf{w}$ are in ker L, then so are $\mathbf{v}+\mathbf{w}$ and $c\mathbf{v}$ for any real number c. If $\mathbf{v}$ and $\mathbf{w}$ are in ker L, then $L(\mathbf{v}) = \mathbf{0}_W$, and $L(\mathbf{w}) = \mathbf{0}_W$.

Then, since L is a linear transformation,

$$L(\mathbf{v}+\mathbf{w}) = L(\mathbf{v}) + L(\mathbf{w}) = \mathbf{0}_W + \mathbf{0}_W = \mathbf{0}_W.$$

Thus $\mathbf{v}+\mathbf{w}$ is in ker L. Also,

$$L(c\mathbf{v}) = cL(\mathbf{v}) = c\,\mathbf{0}_W = \mathbf{0},$$

so $c\mathbf{v}$ is in ker L. Hence ker L is a subspace of V.

(b) Let L be one-to-one. We show that ker $L = \{\mathbf{0}_V\}$. Let $\mathbf{v}$ be in ker L. Then $L(\mathbf{v}) = \mathbf{0}_W$. Also, we already know that $L(\mathbf{0}_V) = \mathbf{0}_W$. Then $L(\mathbf{v}) = L(\mathbf{0}_V)$. Since L is one-to-one, we conclude that $\mathbf{v} = \mathbf{0}_V$. Hence ker $L = \{\mathbf{0}_V\}$.

Conversely, suppose that ker $L = \{\mathbf{0}_V\}$. We wish to show that L is one-to-one. Let $L(\mathbf{v}_1) = L(\mathbf{v}_2)$ for $\mathbf{v}_1$ and $\mathbf{v}_2$ in V. Then

$$L(\mathbf{v}_1) - L(\mathbf{v}_2) = \mathbf{0}_W,$$

so that $L(\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{0}_W$. This means that $\mathbf{v}_1 - \mathbf{v}_2$ is in ker L, so $\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{0}_V$.

Hence $\mathbf{v}_1 = \mathbf{v}_2$, and L is one-to-one.

Fig. 2 Theorems and proofs on transformations (Kolman and Hill 2008)

of the data into potential themes arising from the theoretical framework was conducted by the second named researcher and validated by the first researcher. In order to illustrate this process, Fig. 3 provides examples of some of the codes and themes for the questions 'What is the purpose of proof?" Other codes employed included: no personal value; explanatory; to build more mathematics; and to communicate.

Other examples of codes, arising from responses to the question ‘What kinds of proofs, if any, do you prefer?’ were: Short; medium length; not based on definitions; ones understood; logical; step-by-step; word-based; symbol-based; and combining words and symbols.

4 Results

In this section we examine student reactions to the proofs in this linear algebra course, presenting evidence for their perspective on the truth of proofs (need for certainty) as well as their explanatory nature and ability to provide personal understanding (need for causality) (Harel 2013; 2018b). In addition, whether the underpinning of students’ proof convictions resided more in the embodied or symbolic world is considered.

4.1 Need for certainty

In response to the question, “What is the purpose of a proof?” nine of the 16 students said that proofs were written to establish the validity, correctness or truth of the given result, or theorem. Many of the responses were in line with Harel’s (2013, 2018b) notion of need for certainty. This is seen in S8’s comment that the theorem will

Code	Idea	Illustrative Data Text Examples
Understanding of ideas	The purpose of a proof is to help one understand the mathematical ideas in the theorem	“I feel like the purpose of writing a proof is to really understand how it is not just memorize that this is how it is but just like see the workings of it” “it helps us to understand where they are coming from, whenever they derive those equations. So ah I guess in the context of the concept it will just help us to understand it more.”
Correct idea	The purpose of a proof is to establish the correctness of the mathematical theorem	“I feel the purpose of a proof is to make sure that there’s no contradictions, um, within the theorem and also to make sure that it’s valid, that it makes mathematical sense.” “I guess to show that this proof works in every case or you know yeah it’s, it’s true for any such.”
Memorising	Students memorise the content of a theorem’s statement or its proof	“…kind of show step by step how they got what they got and it helps you better understand it, instead of just memorising the end result.” “I don’t see how learning the proof of something and memorizing it in a way so I can take it on an exam will help me in the long run….” “…for students I feel like the purpose of writing a proof is to really understand how it is not just memorize that this is how it is but just like see the workings of it...”

Fig. 3 Examples of codes and themes arising from the data

be "true in all cases" and "it's a law" (S16) or a "fact" (S4) that doesn't have to be done again.

> S4: …this is not an elegant answer but I mean showing I guess proofs of how we keep track of our progress in mathematics so if something is a theorem or a conjecture or whatever can be proved then that allows us to say okay you know we can now use this fact and to go further in mathematics and lay more of the ground work so I suppose they are a little connections maybe that allow us to keep going onwards so these things don't have to be you write them down and record them I guess and publish them in a textbook and then we don't have to do them again and again hopefully they will be live on forever
> S8: As a student or like in mathematics? I guess you said in mathematics. In mathematics the purpose of the proof as I see it is to um back up or, I can't believe I'm going to use the word prove, or make … so theorems in mathematics are statements that are given to be held true. A proof is backing up the fact that they're held true, making it apparent that this is true in all cases. That's what I would describe it as.
> S16: Prove that what you're doing is not just fallacy or it's not just something that somebody made up. It's real. It's a law.

Yet, when the students were asked 'Do you think we should offer a separate course on how to prove results in mathematics?', only a small number of students were in favour of having a separate course, the majority felt that realistically this might not be beneficial for them or that proofs should be integrated into other courses in linear algebra.

> S8: I think it would be a very effective use of time because you are going to be proving, especially if you are a mathematics major, you are going to be proving for the rest of your life
> S10: I probably wouldn't take a proof course, but I would take abstract linear algebra or linear II, I would take a second course over you know more of a concepts, probably not proofs.

We also noted that the students were mainly concerned about the requirement for their major, rather than their own *need for certainty*. S16 also questioned the need for proof, since instructors in some mathematics classes expect proofs, whereas others do not.

One of the students (S4) did not see the purpose of proof as strongly, making the point that engineers do not need proof, if they can get away with doing some computation instead. This demonstrates use of a computational warrant for argumentation where the veracity of a result is often shown by computation; as S4 says "they don't need proof but and they would probably say that [i.e. computation] is enough".

> S4: …about engineers as long they just want to know how to compute these intervals you know and do whatever maybe use a matrix for some calculation, but they don't need proof but and they would probably say that is enough, but I think mathematics absolutely not what I would say is more pure mathematics, no proof is absolutely essential.

4.2 Need for causality

Seven of the students, including some who commented above, focussed on the explanatory nature of proofs in line with Harel's (2018b) *need for causality*, "or to understand what makes a phenomenon the way it is" (p. 37). They talked about the role of proof in helping them to understand and hence gain a personal conviction, not just of the result at hand, but of mathematics in general. Some of them contrasted this with a common practice of simply memorising proofs for tests and examinations, without really understanding them. For S2 it is clear that "the purpose of a proof is to argue that this is correct", however she qualifies this by adding that "the purpose of writing a proof is to really understand how it is not just memorize". This shows that she wants to understand why it is correct. Similarly, S10 emphasized that definitions and examples are nice, but "knowing why it works, it helps you solve all those steps", and thus it can be applied to more general cases. S9 and S15 express a similar thought of the contrast between understanding a proof and simply memorizing it.

> S2: I mean, obviously the purpose of a proof is to argue that this is correct. That this is real and I like presented to somebody else and say that this is how it is and for them to be able to read it and understand it, for students I feel like the purpose of writing a proof is to really understand how it is not just memorize that this is how it is but just like see the workings of it I guess so
> S9: …just kind of show step by step how they got what they got and it helps you better understand it, instead of just memorising the end result. Um, if you understand the process and the method it will help you not only like understand what's going on but it will help you remember the end result if you know how to get there.
> S10: Kind of what I mentioned, I like knowing the proof because, yeah it's nice to have a definition or just an example but knowing why it works, it helps you solve all those steps in between that you solve in the proof to get to your final theorem, all those steps are not just, I mean in some cases I guess they are but

most of the time they're not just exclusive to that proof so these different maths steps combined that can all be used in practice with other maths steps you need.
S15: I think you just kind of do it to build up your foundation of the math and understand where it's coming from… I don't see how learning the proof of something and memorizing it in a way so I can take it on an exam will help me in the long run.

These two types of responses correlate well with the idea of proofs that prove and proofs that explain (for the individual) (Hanna 1990) or proofs related to ascertaining or persuading (Harel and Sowder 1998).

As noted above, a few of the students (e.g. S2, S9 and S15) contrasted understanding with simply memorising a proof in order to be able to reproduce it at a later date, such as in an examination. This resonates well with Skemp's (1976) distinction between instrumental and relational understanding. In this context the former would be the ability to learn a theorem or proof and reproduce it without understanding why it works, whereas the latter involves deductive reasoning, the ability to deduce the theorem (and proof) from the underlying principles and relationships. As Skemp noted, while instrumental understanding is quicker to attain and has more immediate benefits, relational understanding produces longer term benefits, since it more adaptable to new tasks, or as S10 put it "All those steps are…not just exclusive to that proof", that is, key ideas in proofs often have wider application in mathematics.

4.2.1 Formal proofs versus examples within the embodied and symbolic worlds

One of the main goals of this study was to investigate students' reactions to confronting the formal mathematics, specifically formal proof, and whether they gained more understanding within the embodied or symbolic worlds. Hence, we asked if they were convinced by proofs alone or would have gained more conviction through a series of examples and pictures. The contribution of examples to understanding proofs, was strongly emphasised in the responses.

S3: Examples always help…I was just going back to the embodied world [of Tall] or giving symbolic examples but it gets you in the right direction and thinking the right way…you might not be able to connect these ideas until you get a, example, it helps you combine and understand what it's trying to tell you.
S6: …my assumption is that examples always help me and I'm assuming that examples help me understand this.

Both students believed that since examples are always helpful, so they must be helpful with proofs as well. S3 believed examples will point you on the right directions and help with connecting, combining and understanding ideas in proofs.

In a similar manner some felt that sometimes visualisation might help, either in addition to, instead of, or within examples, but with the need for understanding once again at the centre of their thinking. However, it appears that they were not using the embodied world (Tall 2008) as a warrant for truth, but more as a support for understanding the proof content (Lew et al. 2014), in line with Harel's *need for causality*.

S2: …sometimes I would say a picture would be helpful…With a proof I don't think examples would be really necessary but aside like, I like those little pictures that come in the on the sides of the textbooks.

Others gained their understanding, and hence a personal conviction from the proof as presented to them using deductive reasoning as a warrant for truth and were much more sceptical about the pictures. For example, both S4 and S16 believed that for proofs given them during the interview (see Fig. 2) it may have not even be possible to come up with any pictures. S8 made it clear that "a series of examples does not convince me…I prefer proofs". However, he acknowledged that examples and an embodied 'geometric argument' do have a place in mathematics. S12's comments indicated that although examples are helpful for understanding, since theories work in "any situation", in some ways proofs are "self-sufficient".

S4: Sure. I really I don't think so I mean I suppose you could come up with a picture for this or maybe an example but I guess for myself if I was teaching this course I would not be.. to do that because this is hopefully what we are trying to lean towards even in this case I don't think that would have been necessarily more help for me as a student I can't speak for others but this was fine this I think conveyed all of the understanding.
S8: Well I'm kind of a sceptical person by nature and so if you just gave me a bunch of subsets for L and showed that they were subspaces of V, I'd be like alright that's neat, but I wouldn't believe it was true in all cases. I guess a geometric argument might sway me depending on how effective it was. If it wasn't just a series of examples but you know like if you use the geometric definition of a span or if you look at span, geometric ones, it's pretty easy to see some of the properties of span. So that would maybe convince me but generally speaking a series of examples does not convince me that something is true in all cases or true in many cases or true in X cases. I prefer proofs. I'm more of a questioning type. But there is definitely a

place in mathematics if you just want to get the work done, so.
S12: Examples of pictures helps me to try and understand but I mean but the proof is more like the most definite thing because you can give an example, I mean how I make sure that it didn't give that specific example that works with this theorem but if you give me a proof that doesn't have specific example that make sure that this theorem works with any situation.
S16: I don't think so. I mean if you could visually show it to me it might. But I can't see that visually. I don't know how that would be represented. So I'm not sure how to, if … if you could show it to me visually I think that I could understand it better. But it's not to say that I don't feel like I understand it, cause I do.

4.2.2 Proofs and personal conviction

As Harel (2018a) states, the *need for certainty* and the *need for causality* are complementary, "… understanding cause brings about certainty, and certainty might trigger the need to determine cause" (p. 15). We found the relationship the students perceived between being personally convinced and understanding was seen in their responses to when they were asked "Do you find proofs convincing?" These responses show that for them understanding and making sense are prerequisites for gaining personal conviction.

S7: Yes if I understand them. They're pretty convincing.
S11: I'm not too good of a proof writer but I like them, I think they're useful. There's some proofs that have convinced me. If they haven't convinced me then I probably don't understand them.
S13: I do generally go through each step of a proof trying to fully understand it but I do find them rather convincing once I understand them.

So these students linked being convinced by a proof with first understanding and making sense of the content. However, at least six of them acknowledged that they can occasionally get lost in proofs, since forming understanding is not always easy for them. We see that S2 says "It depends on the proof…I get lost easily", while S16 seems to have problems with proofs that "just throw things in from all different angles" or use other proofs to prove and links his personal conviction with proofs that are "more definite" and "make things make sense" for him.

S1: If it's just words, no. It's just a block of text, my eyes glazed over.
S2: It depends on the proof. These, the linear algebra ones definitely help. In some other courses I feel like I don't know, sometimes I get lost easily but the ones that we've done in our text book seem really well, well to read and easy to read.
S12: Most of the time…Well sometimes some steps I need to think a bit more to convince myself about it. I mean some proof they use specific number of something so I'm not sure sometimes. Most of the time yeah, if it's convinced me.
S16: Some proofs seem to kind of just throw in things from all different angles and kind of go in a big circle to do what you just showed. Um I think there's a lot of proofs that just use other proofs to kind of prove a proof. But there are some proofs that, you know, are more definite and do, can make things make sense for sure. But when you do geometry and you do some of the proofs that we did for like when you use an inverse to prove something, it's kind of like yeah I can use inverse to prove it.

Hence, some reasons why certain proofs were more difficult to understand and so were not convincing included those that were poorly structured (Harel's *need for structure*), those written in a manner that is difficult to read (Harel's *need for communication*), those hard to follow certain steps in the proof and those relying on too many prior proofs/theorems (Harel's *need for structure*). The lack of a clear structure in many proofs and what might be done about it is something that has been noted for many years (Alibert and Thomas 1991).

4.2.3 Student preferences of proof type

To learn about the kind of information in proofs that the students thought would assist them in ascertaining, convincing themselves or gaining relational understanding of the proof they were asked "What kinds of proofs, if any, do you prefer?" some students' responses showed their lack of exposure to different types of proof. For example, S2 said "I'm not sure what do you mean by different.", S9: "What do you mean by kinds?" and S10: "What different types of proof are there?"

As a whole, there was no consensus on preference, with several themes emerging from their comments. However, their preference was often linked to understanding (Harel's *need for causality*) and shed some light on what the students employed as a warrant for mathematical truth (Rodd 2000). Some preferred to gain understanding through symbolic world proofs, with symbolic algebra being their preferred warrant for truth, rather than those proofs with more words, and it is noteworthy that S15 did not seem to associate words with mathematics, as if these are insufficient for use as a backing.

S5: I would say definitely see symbolic cause I'd rather have, I would rather look at the symbols and be able to, yeah, I like, I like the symbolic ones better.
S15: I like proofs that use the math and not the words.

The link to understanding proof was seen in one reason given for preferring symbols to words, expressed by S7. This was the pyramid structure of mathematical proofs and definitions, where one construct is built upon one or more others. As he states it, such proofs require full understanding of all the definitional building blocks. He seems to recognise the importance of a deductive backing that employs definitions even though he prefers a symbolic algebra warrant.

S7: It's easier to see symbols than it is to read words in my mind, for like, especially if I don't understand the definition that they're using in the first place because then you have to refer back but…I won't necessarily understand you know a definition without having to look back and then especially if that definition has another word that I don't understand in it.

Another student (S8) liked symbolic proofs but recognized that he could do some of these proofs without necessarily understanding the ideas behind them, simply by applying other symbolic results. Hence, he added that he likes integrated proofs that rely 'more on properties than just algebra' as long as he understands those properties and so can establish a personal conviction.

S8: Well I always love algebraic proofs. So in linear algebra there are lots of proofs that like if you have algebraic properties you can kind of just like toss things together and really quickly create an interesting proof. But they are kind of unsatisfying in terms of the fact that you can kind of reach a conclusion without the full knowledge of the subject matter underneath it. Just because you have this one property…I don't know. I prefer if I have the knowledge to do more of a written proof or proof that relies more on properties than just on algebra.

Others had a clear preference for words over symbols, with S6 who mentions the need for 'reconnecting' variables to ideas in context, which relates to Harel's *need for communication*, and S13 again linking a preference for words with understanding of proofs. This may be because in their proof schemes they are moving towards acceptance of a deductive warrant and its backing of formal mathematics.

S6: I like the English, English words preferably cause I've taken calculus and I've had difficulty with reconnecting all of the variables to certain things and because some variables mean different things in certain situations. So I prefer the linear algebra format.

S10: …what I'm comfortable with as far as like when I'm learning a new proof, I'd much rather have the words. Just the paragraph of why it works, I like that better than all of the step-by-step algebra or something that when I'm learning a proof. I don't necessarily like the math…I definitely like a paragraph and words I think is what I like better.
S13: I generally prefer primarily text. I like seeing like the symbolic representation of what they're doing but I can usually understand it from primarily text.

The length of a proof is also a factor that contributes to students' ability to understand it, with shorter rather than longer proofs preferred, since longer proofs "make it harder to follow' or cause one to get 'lost'. These two students wanted proofs to be short and preferably fit on a single page.

S4: …but I mean the shorter proofs I suppose. A proof is like two pages long it is likely I am sure that is not nearly that but it does make it harder to follow.
S7: Maybe about three or four lines and then you get like a three page derivation I'm just like I don't know what you said anymore, I'm lost.

Other related factors leading to a preference for certain kinds proofs, again linked to understanding, is their structure, particularly whether they lead students through step-by-step or have a clear, explanatory logical progression for students (the need for communication). They refer to these proof properties being necessary for them to 'understand 'what's happening', to 'see like the connection' or so it 'makes lots of sense'. Their focus is on the need for understanding and sense-making, although there is an important distinction between these two that we will return to below.

S1: The ones that, um, examples where you can see step by step along with the phrases. Ok this group of words means here's what's happening.
S6: The proofs that I don't prefer are the ones that it seems like A equals A through this iteration. I mean it feels like it just repeats itself almost like it, I don't, sometimes, usually I don't see like the connection, so.
S15: I like proofs that use the math and not the words… symbols are…they're not hard to read… I don't know how to say this, to me it doesn't quite compute all the time. So if it's here, this is the proof and then you go eight steps, A, B, C, D, E, you know what I mean, very clear step by step, makes lots of sense instead of a whole bunch of jumbled paragraphs that you kind of have to haul it out.

Even S9, who expressed no preference for a particular type of proof, emphasised the need for causality as the primary consideration, twice stating 'as long as I understand'. This again highlights the student *need for communication*, as well as the *need for structure*, for re-organizing the knowledge learned in a logical manner (Harel 2013, 2018b).

S9: I mean as long as I can see what they're starting with and as long as I understand the method that they are using I don't think that I really have a preference. Just like as long as I understand and I can see where they're starting with and the direction they are going, I don't think it really matters that much to me.

It seemed that the very nature of linear algebra proofs made it difficult for some students to tackle them. Most students were not used to seeing proofs that relied on other theorems as results to be used, including many words and symbols that were taken from definitions or other theorems.

4.3 Two specific proofs

As most of our interview questions were more general and did not involve solving linear algebra tasks, we considered that, in order to examine students' thinking, it would be beneficial to consider their responses to two familiar, specific proofs from the course. Hence, the students were asked about their understanding and conviction with regard to two specific theorems (see Fig. 2) and their proofs. The question was: "Did [this specific proof] help you to understand why such-and-such was true, or why such-and-such always happens?"

Both Corollary 6.2 and Theorem 6.4 and their proofs were set as homework and later were assessed in an examination. The proof of Corollary 6.2 required knowing the definitions for one-to-one and onto and linking them to other theorems to deduce the desired result. This Corollary was also useful in solving other problems. Part (b) of Theorem 6.4, was a key result in the course and was often used in proving other theorems (e.g. Corollary 6.2) and solving various exercises involving matrices. Moreover, the proof for part (b) revealed the definition of one-to-one in action. In a way, both these proofs were meant to help students to work with definitions and to practice with their key elements in order to arrive at the result. During the course most students found these proofs difficult. One main reason is that many students were still struggling to understand the definitions and meanings of the terms such as *Kernel* and specially *Range*. Most students also struggled to compute and find the *Kernel* and *Range* of a matrix.

This was the first time we initiated the word "understanding" within our interview questions. Although, we were interested in students' views, we tried to limit the boundary for the meaning of understanding to "why such-and-such was true, or why such and such always happen".

At least one student (S1) missed the point of this exercise completely and thought that understanding proofs such as these required the use of examples.

S1: These only really help me after I've seen an example of them worked first where I know what the terms are trying to tell me.

However, another student (S2) could clearly see a pattern, referring to the proof of Corollary 6.2 as "a circle", or "like a domino thing".

S2: I don't feel like I've ever needed help understanding it but like, um, I don't know, it shows like it shows that it's a circle I guess and then it's like a domino thing. So yes it helped.

Several students noted that reaching understanding of these proofs in the formal world required some considerable effort on their part, describing it as 'daunting', having a 'steep learning curve' and occasionally requiring a 'giant jump of understanding' for both reading and writing proofs.

S5: The Theorem 6.4, I think it's a little wordy…it's real wordy, it's kind of like daunting and a little off putting to me. Uh, it, it made sense going through it after a couple of times but it's definitely easier to follow the first one [Corollary 6.2 in Fig. 2].
S8: I do think with proofs there's a really steep learning curve. I think that's probably the biggest problem with the formal world of mathematics is the learning curve with a proof is extremely steep. You can stare at a proof for 25 min and get nothing out of it.…there's like a, there's almost like a wall that you run into at a certain point and you can puzzle it out for a long time… Sometimes you just won't get there and it takes this giant jump of understanding to get to the end of it. Especially if you are writing proofs, mostly when you are writing proofs, but reading them too.

Some students claimed that little or no understanding was gained, in one case since it did not 'make sense' to them, and hence there was no personal conviction for the proofs provided.

S10: I don't know if it necessarily helped me understand why …I guess this could be a case where I'd want more algebra instead of just a couple of sentences or maybe I need like four more sentences.
S13: I agree that it is, it does show that. I don't know like my actual understanding of why that is or why that should make sense is really there but I can see that it is.

For Corollary 6.2, working from a dimension equation ($\dim \ker L + \dim \text{range } L = \dim V$), and understanding its importance, even if they did not precisely remember the equation, helped two students to remember or reconstruct the proof. This too was related to understanding, with S9 stating that the equation, even one they seem to have wrongly remembered, 'definitely helps me actually understand why the corollary is true for sure'. The use of this equation as a central plank for understanding appears to imply that, in line with Harel's (Harel 2018a, b) *need for computation*, they were using symbolic algebra as a warrant for truth.

> S7: I know it involves the different dimensions, how has this got to work to get it right…I wanted to play it safe so I just wrote words. So obviously you know this dimension is zero then obviously these two are equal, so I knew that based on.. I mean it had to be that way because I knew the final result, but I mean it was kind of regurgitation in a way. I had to think about it though because I didn't remember the equation. But I mean this one still makes perfect sense to me.
> S9: I know it definitely helps just 'cause you have this symbol equation dimension [kernel] L plus dimension range L plus [equals] dimension of V and you make one assumption, then you can solve for the other one… But then looking at this equation, given Theorem 6.6, it shows clearly and simply like how they got there. So yes, so that definitely helps me actually understand why the corollary is true for sure.

Overall, students' responses indicated that even the proofs that were well discussed and elaborated on in class were not necessarily viewed as straightforward and many concerns still remained in students' minds.

5 Discussion

The goal of this study was to investigate students' perception of proof in a first-course in linear algebra. It is clearly not possible to make generalisations from such a small sample of students in a single university setting. However, our results do show that many students take a different view of proof from mathematicians. Unlike, Hersh's (1993) perception that students are perhaps easily convinced, students often see them as something they need to understand, with understanding, or comprehension of proofs a crucial step in gaining a personal conviction of the mathematics. Although, we are aware that the students' view of understanding may be closer to Skemp's (1979) concept of instrumental understanding and some way from his relational understanding. In this study, while we do not know exactly what the students meant by understanding, we do have some evidence of what they believed non-understanding is, which include rote learning or memorizing a proof. A second vital step in gaining a conviction is that the content has to make sense to the student. Sense-making is gaining more attention in recent years and we agree with Klein et al. (2006) that sense-making is about forming and understanding connections between ideas. Thus, if to understand something is to connect it to a relevant schema (Skemp 1979) then to make sense of it is to form relevant connections between any appropriate schemas. For example, in linear algebra we have previously shown (Thomas and Stewart 2011) that students can make sense of the algebra of eigenvectors by connecting appropriate embodied world visual imagery to algebraic formulations. Thus, embodied and symbolic world thinking can be helpful for some in forming understanding and making sense of proof. So, our contention is that understanding and making sense will need to precede forming a personal conviction. In the absence of these students may tend to memorise proofs, proceeding instrumentally and some also admit to having only instrumental understanding when using symbolic manipulation in proofs.

Tall (2010) points out the different roles that embodiment, symbolism and formalism may play and that these need to be made explicit in teaching. For example, in some situations not only is symbolic manipulation easier than thinking through the embodiment it also "enables a more compressed form of thinking that is supportive in building formal proofs" (p. 25). In his view, "formal mathematics clarifies issues by specifying explicit axioms that are 'rules of the game' and formal proofs deduced using these rules are proven once and for all in any situation where the rules are satisfied" (p. 27).

Our analysis showed some evidence of changing proof schemes illustrating students' ways of thinking about proof, and at least two of them (S10, S14) expressed their interest in knowing "why things work the way they work". S4 also gave the impression that, although this was his first exposure to proofs, once he spent the time studying, he could see the thinking behind them.

Our results also align well with the observation of Alibert and Thomas (1991, p. 215), that it is important to obtain the "students' view of whether proof is a necessary mathematical activity, their understanding of the need for rigour, and their preference for one type of proof over another", which is sometimes neglected in order to preserve the rigour of mathematics. Failure to recognise the disjunct between the mathematician's perspective on the rigour of proof and that of students risks alienating them early on in linear algebra, as Harel (2018b, p. 38) notes "there is an inherent pedagogical inconsistency in instruction that emphasizes rigor without attention to the origin and need of that rigor…As a consequence, students feel aliens in knowledge construction." We need to accept that students may reach understanding and conviction through different warrants for truth.

In addition, to prepare students for a second course in linear algebra, they need to be comfortable with some level of rigour in the first course. The nature of this rigour and the teacher's expectations are best clarified as the proofs are introduced in class. It appears that for some students the structure of a proof required by mathematical rigour may tend to obscure understanding. This is something that could be attended to in a course specifically addressing proof. Around half of university personnel in an ICME survey (Thomas et al. 2015) said their university had a separate course on proving, and agreed (averaging 3.7 out of 5) that learning how to read a proof, working on counterexamples, building conjectures and constructing definitions would be useful components of such a course. However, our research suggests that a proof course could also address examples, which can play an important role in learning concepts. In this regard, the research of Lockwood et al. (2016) suggests four pedagogical ways to assist students' use of strategic examples in proof activity that could be incorporated into linear algebra courses, namely not discouraging example use, encouraging awareness and discussion of example use, explicitly highlighting example use in proving and shifting norms about what it means to be a competent or successful prover. In addition, such courses could include activity addressing the structure of proofs.

Pedagogy of proof in a first-year linear algebra is certainly a complex endeavour. Many students are not mathematically equipped to construct their own proofs. As student S8 put it, "the methodology of proving is very precise and specific. It has to work a certain way or it's not well accepted." Despite their exposure to two semesters of calculus and, for some, a course in discrete mathematics, our analysis showed that many students still struggled with proof in linear algebra. One reason is that each component in a proof is packed with meaning and conceptual ideas. Many students struggle with understanding the basic concepts in linear algebra. It is almost unrealistic to expect students to logically piece together many definitions and other theorems and results together inside a proof. As S8 summed it up "in this class when we were asked to write proofs it was kind of assumed that we knew the direction we would want to go and the methodology we needed to know". Although, we did not study the effect of emotions closely in this paper, some students' negative attitudes toward proof was apparent. For example, S7 said: "I kind of want to stay far away from that as possible, that's why I'm only a math minor".

Regrettably, in many traditional settings linear algebra students may not be given the opportunity to develop Harel's (2008a, b) ways of thinking and ways of understanding that can assist students to be successful. As Tall (2013, p. 414) reminds us, "unlike an apprentice carpenter, who can observe the actions of an expert applying his trade, an apprentice mathematician cannot see what is going on inside an expert mathematician's head". We believe that teaching linear algebra proofs well certainly requires more class time and an awareness and nurturing of individual's intellectual needs. It seems that ignoring students' need for causality and sense-making (Harel 2018b) will only add more obstacles to their learning.

Our research results point lecturers more toward the student perspective of the learning of proof in linear algebra than has previously been the case. Equipped with this, we hope that it will assist lecturers to contemplate strategies used by students and so have a more specific focus, making their noticing more productive by modifying their instructional decisions to match student difficulties (Choy et al. 2017).

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