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ORIGINAL ARTICLE

Making connections among representations of eigenvector: what sort of a beast is it?

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Abstract
Many studies provide insights into students' conceptions of various linear algebra topics and difficulties they face with multiple modes of thinking needed for conceptualization. While it is important to understand students' initial conceptions, students' transfer of learning of these conceptions to subsequent courses can provide additional information to structure meaningful curricular materials. This study explores physics students' transfer of learning of eigenvalues and eigenvectors from prerequisite experiences to quantum mechanics. Data analysis focused on three task-based interviews with undergraduate students, observations of physics courses, and students' course artifacts. Existing studies on students' conceptions of linear algebra topics indicate the necessity of developing flexible shifts between different modes of thinking in order to grasp linear algebra. This study's participants, who had initial learning experiences of linear algebra, were also observed to struggle with such shifts prior to quantum courses. It seems that various contexts in quantum courses, and explicit instructional methods, provided opportunities for students to enhance this initial learning of eigenvalues and eigenvectors. In particular, the explicit reasoning of one of the quantum courses' instructors concerning the choice of certain representations during problem solving in class, seemed to facilitate students' construction of similarities, thus providing evidence for actor-oriented transfer. Results of this study align with goals for recently developed instructional materials and interventions that emphasize opportunities for students to inquire and connect multiple modes of thinking.

Keywords Actor-oriented transfer · Eigentheory · Linear algebra · Transfer of learning

1 Introduction

There is a rich body of research from various fields including education and psychology on transfer of learning with a history of over 100 years. One key reason for this research is its direct relation to an important goal of education: providing learning experiences that can be generalized and used by the learner outside the initial learning situation (Bransford et al. 1999).

The generalization of knowledge and understanding of how to enhance transfer are essential in all educational settings as many course designs rely on prerequisite knowledge. For example, students are required to take a linear algebra course prior to studying quantum mechanics. Eigentheory is important prerequisite knowledge for quantum theory. The study discussed here focuses on this area and addresses the following question: what do physics students transfer from various learning experiences of eigenvalues and eigenvectors to interviews?

There are various theoretical frameworks for exploring transfer of learning of knowledge. In this qualitative study, transfer of learning was explored by utilizing one of the contemporary frameworks, namely, actor-oriented transfer (AOT) (Lobato 2003). By focusing on participants' perspectives, rather than those of experts, the AOT helped explore participants' construction of similarities between various learning experiences during interviews.

2 Background and theoretical framing

2.1 Conception of linear algebra topics

Students' conceptualization and sources of difficulties in linear algebra topics, including eigenvalues and

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eigenvectors, are captured in many existing studies. Dorier and Sierpinska (2001) describe two stages in construction of concepts in linear algebra. The first stage, "recognition of similarities between objects, tools and methods brings the unifying and generalizing concept into being" and the other stage is "making the unifying and generalizing concept explicit" (p. 257), in order to form an object through restructuring of prior knowledge. Sierpinska (2000) further describes this construction through three modes of thinking and highlights the need of flexible shifts between them for construction.

Sierpinska (2000) introduces three modes of thinking: synthetic-geometric, analytic-arithmetic, and analytic-structural and states that they are "equally useful, each in its own context, and for specific purposes, and especially when they are in interaction" (p. 233). The synthetic-geometric mode of thinking uses geometrical language and descriptions to visualize mathematical objects. For example, a synthetic-geometric mode of thinking of an eigenvalue could involve treating it as a value describing how much a vector stretches or shrinks under a certain transformation. And, in this mode of thinking, an eigenvector could be viewed as a vector which is not moving when a transformation is applied. In this mode of thinking both eigenvalue and eigenvector descriptions use geometrical language. The analytic-arithmetic mode of thinking refers to using the language of arithmetic, descriptions to carry out certain computations, and linking mathematical objects to a formula. Eigenvalues in this mode of thinking can be described computationally as the values that satisfy the characteristic equation, $\det(A - \lambda I) = 0$ and eigenvectors then satisfy the eigenvalue equation, $A\vec{v} = \lambda\vec{v}$. In other words, analytic-arithmetic modes of thinking for both eigenvalues and eigenvectors invoke the use of arithmetic procedures with given formulas.

The analytic-structural mode of thinking uses abstract language to indicate the underlying structures of a mathematical object through its properties. This mode of thinking could describe an eigenvector as a special non-zero vector that gets mapped into a scalar multiple of itself under a linear operator (which extends to vector spaces beyond R^n). In other words, it is affected only up to scaling by this operator. In this mode of thinking, an eigenvalue could be viewed as a particular scalar that describes the 'impact' of the linear operator on the associated eigenvector. Sierpinska (2000) states that the main difference between the synthetic and analytic modes of thinking concerning a mathematical object centers on how it is treated by "the mind". In the synthetic case, the object is "given directly to the mind" (p. 233) whereas in analytic modes the object is constructed using its properties, "given indirectly" (p. 233). Sierpinska also states that the synthetic mode is an indication of more practical thinking and the analytic mode is an indication of a theoretical one.

Sierpinska's (2000) study results show students' difficulties in shifts between these three modes of thinking. Students had challenges "going beyond the appearance of the graphical and dynamic representations" (Dorier and Sierpinska 2001, p. 263) that they observed and manipulated in activities. Students based their thinking on prototypical examples (i.e., practical thinking) rather than using definitions from the analytic mode. Similar results on students' difficulties with different modes of thinking and representations are reported on the topic of eigenvalues and eigenvectors (e.g., Larson and Zandieh 2013; Stewart and Thomas 2010). Based on the studies they conducted, Stewart and Thomas (2009) report that students could not reason about relationships between a diagram and an eigenvector, lacking a geometric interpretation, but they seemed confident with algebraic and matrix procedures. Thomas and Stewart (2011) state that students employed symbolic manipulation of algebraic representation, which in turn did not help them to develop formal, conceptual thinking concerning these concepts.

Students' practical or intuitive thinking, nevertheless, can help in developing instructional tools to move students' thinking forward to an abstraction. Wawro et al. (2011), for example, showed that students' intuitive ideas about span and linear independence could be leveraged to build formal thinking. Similarly, in studies regarding students' interpretation of the equation $A\vec{x} = \lambda\vec{x}$ prior to studying eigentheory, students were observed to use symbolic, numeric and geometric interpretations to make sense of this equation. Students employed these interpretations within their existing three views: a linear combinations view, a system of equations view, and a linear transformation view (Larson and Zandieh 2013). Researchers (e.g., Zandieh et al. 2017) then developed an instructional sequence that focused on multiple interpretations of eigenvalues and eigenvectors, to assist students' shifting among them. (For more examples of instructional materials, see http://iola.math.vt.edu/ and Stewart et al. 2018.)

Students from various disciplines are required to take linear algebra as a prerequisite course. Understanding of linear algebra plays an important role in developing new concepts in subsequent courses. Thus, it is important to know students' conceptions and difficulties with linear algebra. However, focusing only on whether a student has (or does not have) particular content knowledge is not a sufficient model for knowing what students can *do* and *act on* in a subsequent course. Investigation of students' conceptions and processes in subsequent courses, with a framework that privileges the students' perspective, could enhance the existing body of research in this area. There is a need to explore *what students do know* when they enter a subsequent course, students' processes such as *how they utilize and progress* with such knowledge, and what pedagogies afford opportunities for development of conceptions and processes. As discussed in the next section, transfer of learning frameworks, particularly the contemporary ones, provide opportunities for such explorations.

2.2 Transfer and actor-oriented transfer framework

Transfer of learning has traditionally been defined as the ability to apply knowledge learned in one context to new contexts (Mestre 2003). Earlier research studies utilizing this broad definition explored whether or not a learner transfers knowledge from one setting to another. Studies with traditional transfer research designs report conflicting results (summarized in Karakok 2009) and do not necessarily provide insights on the mechanisms of transfer. Given that in everyday experiences learners *can* perform successfully in new situations by finding similarities from previous situations, understanding how such 'transfer' happen needs further exploration. The actor-oriented transfer (AOT) framework (Lobato 2003) aims to address this need by leveraging the learners' perspective.

The AOT views transfer as "the personal construction of relations of similarity between activities, or how 'actors' see situations as similar" (Lobato and Siebert 2002, p. 89). The main focus of this framework is the learner (actor) and how the learner sees the target situation in relation to the initial learning situation. Obtaining evidence for actor-oriented transfer differs from traditional transfer approaches. In traditional perspectives, *successful* application of knowledge on a transfer task after the initial one is considered as evidence. Meanwhile, in the AOT framework, regardless of successful performance, any influence of prior learning experiences is considered as evidence for transfer. More specifically, the AOT analysis focuses on participants' processes, what they *do* and what they 'use' in target situations. In AOT studies, the evidence is gathered "by scrutinizing a given activity for any indication of influence from previous activities and by examining how people construe situations as similar" (p. 89).

One example provided by Lobato and colleagues is finding the steepness of a wheelchair ramp (target task) after participants were introduced to finding the slope of a line (initial task). These tasks, according to experts (or researchers) share the same structural features and can be solved using a similar approach of rise over run. These tasks' surface features (contexts) however are different. When Lobato and Siebert (2002) examined one student's reasoning through such tasks, they observed that this student did not transfer (in the traditional sense) the slope formula (rise over run) from the initial learning. However, examination of the student's reasoning on the wheelchair task revealed his progress that included identifying the related two quantities (height and length) contributing to steepness and developing a multiplicative relationship between them, all of which were directly linked to finding the slope. Researchers then explored the student's experience in a ten-day teaching intervention prior to an interview. They noticed the student was most probably using reasoning in the interview that he created for an in-class task. They postulated that the student demonstrated transfer between these two situations by creating his own similarities between these two situations—rather than what researchers expected the student to transfer. Examination of in-class activities together with the student's reasoning process during the transfer task provided insight into the mechanisms (e.g., building similarities) of this particular student's actor-oriented transfer.

The current study utilizes the AOT framework to understand undergraduate physics students' transfer of learning of eigenvalues and eigenvectors from various learning experiences to tasks in interviews. The AOT framework, in particular, provides a lens to gain insight into students' construction of similarities between these learning experiences. Students' conceptions of and difficulties in linear algebra topics in general and eigenvalues and eigenvectors in particular are reported by other researchers, as noted in the previous section. In their description of two stages of construction of concepts in linear algebra (see Sect. 2.1), Dorier and Sierpinska (2001) refer to actions such as "recognition of similarities," "making the unifying and generalizing concept explicit," and "reorganization" (p. 257) of prior knowledge. The AOT framework has the potential to unpack actions of recognition of similarities, making connections, and reorganizing prerequisite knowledge from students' perspectives, by examining their processes in transfer tasks. This examination is particularly important as linear algebra serves as prerequisite to many other courses both in mathematics and other fields. Understanding *students' processes* of building similarities (i.e., what they 'see' as similar) in learning experiences and what they choose to connect from learning experiences, could help us not only to build more student-oriented and student-related learning experiences but it would also help us to re-assess what is important in prerequisite learning experiences.

3 Methods

This section provides a brief outline of the setting, the data sources, and analysis. The study in this paper is from a larger phenomenological study that was designed to explore transfer from students' perspectives (Karakok 2009). A phenomenological study design was suitable for this investigation as it is "particularly effective at bringing to the fore the experiences and perceptions of individuals from their own perspectives, and, therefore, at challenging structural or normative assumptions" (Lester 1999, p. 1).

Participants in the study were third year undergraduate students enrolled in third-year physics courses at a large, public university in the United States. The third-year physics courses considered in the study were Symmetries (S) during fall term, Quantum Measurements and Spin (QMS), Waves (W), and Central Forces (CF) during winter term. The study started at the beginning of the fall term with students who were in the Symmetries course and planned to take QMS, W, and CF courses in the winter term. Starting the study in fall provided the opportunity to interview students prior to their taking quantum courses (QMS, W, CF) in the winter term, and also to observe both fall and winter term physics courses. Prior to QMS, W, and CF, students were required to take a one-week linear algebra review course [referred to as Linear Algebra Week (LAW)] at the beginning of the winter term. The schedules of the courses and three interviews are summarized in Table 1.

The winter term started with reviewing prerequisite linear algebra topics in LAW that met for 7 h in total over a week. Some of the topics were matrix operations, determinants, symmetric matrices, linear transformations, eigenvalues and eigenvectors, properties of Hermitian matrices, and vector spaces. Students had two assignments and two quizzes this week. A pre-quiz was given on the first day and a post-quiz was given on the last day of LAW. These quizzes tested students' computational skills concerning matrix operations, determinants and eigenvalues and eigenvectors. Students were also asked to describe eigenvalues and eigenvectors for the purpose of this study.

After LAW, students took the QMS, W, and CF courses all of which focused on quantum theory and met for 7-h a week for 3 weeks. Topics discussed in these courses included postulates of quantum mechanics with demonstrations for the simple spin ½ Stern-Gerlach experiment (QMS), Schrödinger's equation (QMS), the terminology to describe waves in different contexts including electrical circuits, waves on ropes and the matter waves of quantum mechanics (W), central forces in classical mechanics, the separation of variables in Schrödinger's equation and related equations (CF).

Professor Clay (pseudonym) taught the Symmetries in the fall term, LAW and CF courses, and two other professors taught the QMS and W courses in the winter term. Prof. Clay was one of the faculty who was involved in the redesign of these three junior level quantum courses and others that students were required to take before and after QMS, W, and CF.

12 students out of 20 volunteered in the fall term and seven of them completed all three interviews. In this paper, I discuss results of the larger study by providing examples from Gus's and Milo's (pseudonyms) data analysis. Gus, a junior engineering physics major, did not take any required linear algebra course as his engineering courses satisfied this requirement. Milo, a junior physics major, took two courses focusing on linear algebra topics. He took Math X prior to his junior year, and was also enrolled in Math Y, a linear algebra course, in the fall term. Math X covered many topics including complex numbers, matrices and linear systems, linear transformations, eigenvalue problems, infinite and power series, and convergence tests. Math Y was designed to be the first course in linear algebra and was a course required by many client disciplines, and was compulsory for mathematics majors. It covered matrix-oriented topics including matrix operations, systems of linear equations, determinants, eigenvalues and eigenvectors, and diagonalization.

Gus and Milo were chosen for this paper for the following reasons: (1) they had different prerequisite linear algebra experiences which exemplified the range of initial learning experiences of students in the QM courses; and (2) even though Gus did not have a traditional prerequisite linear algebra course, his case demonstrates compelling and somewhat similar AOT experiences to those of Milo.

Data sources were three audio- and video-recorded interviews, video recordings of the physics courses (S, LAW, QMS, W and CF), the researcher's observation notes from these courses, and participants' course artifacts. All interviews were semi-structured and task-based and were at most 90 min. The interviews were transcribed in their entirety, serving as the main data source. In the first interview, students were asked questions regarding their background knowledge of linear algebra topics including eigenvalue and eigenvector. In the second and third interviews, questions were mostly on eigenvalues and eigenvectors.

Table 1 Course and interview schedules

	Fall term (September–December)	Winter term (January–March)	Spring term (April–June)
Observed courses	Symmetries	LAW (week 1) QMS (weeks 2–4) W (weeks 5–7) CF (weeks 8–10)	No course observation
Data	Interview 1 (in November)	Pre- and post-quizzes in LAW (week 1) Interview 2 (weeks 2 and 3)	Interview 3 (weeks 2 and 3)

Methods of data analysis, as summarized in Fig. 1, were chosen to fit the phenomenological study design, in order to capture themes that emerged from data aiming to understand the phenomenon of students' transfer of learning. Specifically, analyses in this study design focused on capturing "descriptions of what [students] experience and how it is that they experience what they experience" (Patton 2002, p. 107), which align with the theoretical perspective of the AOT framework that emphasizes "the personal construction of relations of similarity between activities" and how students "see situations as similar" (Lobato and Siebert 2002, p. 89).

Data analysis was an ongoing process in which a thematic analysis method (Auerbach and Silverstein 2003) was utilized first in phase 1 (Fig. 1). To create themes, the initial coding focused on identifying relevant texts which were excerpts from transcripts when students talked about eigenvalues and eigenvectors. These relevant texts were then coded to capture students' processes and thinking. Some codes were formula use, visual representation, geometrical language, computation, physics example, etc. These codes were further organized into repeating ideas (i.e., repeated actions such as using a formula), which was a process similar to axial coding. Codes and repeated ideas from all participants were used to create themes to describe students' conceptions. Some themes were algebraic reasoning, geometric reasoning, and context-depended reasoning. In the second iteration of analysis in phase 1, these themes were refined by utilizing the existing research studies on students' conceptions. In particular, Sierpinska's (2000) three modes of thinking were instrumental as an analytical tool.

This phase was followed by identifying episodes in which more than one mode of thinking was used to describe students' thinking, and shifts were observed in students' utterances or written work. These episodes were crucial for the AOT analysis in phase 2 (Fig. 1). Students' processes in these episodes were explored by checking what they were doing against physics-course data "for any indication of influence from previous activities" (Lobato and Siebert 2002, p. 89). For example, at first glance it may seem normative that a student uses two different modes of thinking to describe an eigenvector. However, the *process* in which such modes of thinking are used can reveal the student's construction of similarities between activities (e.g., see Gus's case in Sect. 4.1.3). Such episodes were compiled and categorized according to different types of potential sources, included instructor interactions, classroom interactions (e.g., what students were doing), course activities and notes. Episodes within these categorizations for each participant constituted evidence of their actor-oriented transfer.

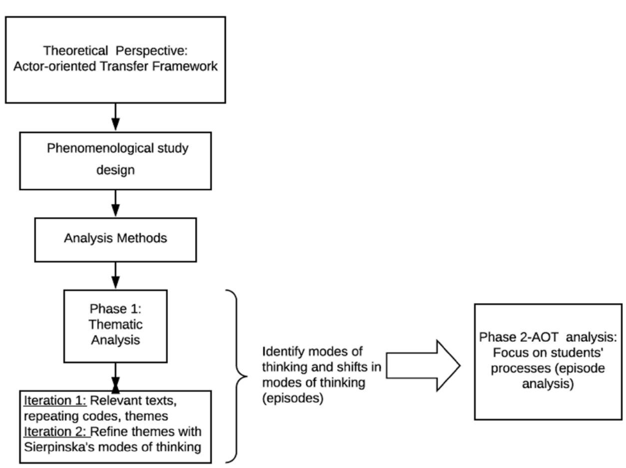

Fig. 1 Framework of the study

4 Results

This section first outlines the results on students' conceptions of eigenvalues and eigenvectors before and after LAW experiences. Descriptions of students' conceptions, through the lens of Sierpinska's (2000) three modes of thinking, align the results within the realm of existing linear algebra studies on students' conceptions (Sect. 4.1). The results from the actor-oriented transfer analysis provide more insights on potential influences of prior activities on students' processes, and their possible constructions of similarities between interview tasks and prior activities. Prior activities include LAW, QMS, W and CF course experiences. Examination of students' processes (e.g., how they communicate, how they approach questions) as shifts in modes of thinking observed (i.e., episodes) reveals students' constructions of similarities between observed learning experiences in the study. In other words, the action of integrating of and shifting in modes seems to be the process of construction with reference to the observed learning experiences. Some examples from the analysis of episodes are provided in Sect. 4.2.

4.1 Modes of thinking

The refined themes helped to describe students' conceptions through and expansion of Sierpinska's (2000) three modes of thinking. This analysis suggested two new subcategories, synthetic-arithmetic and analytic-geometric modes, to describe students' thinking further. Briefly, the subcategory of synthetic-arithmetic mode of thinking encapsulates thinking in which arithmetic language is used but in a synthetic manner. Specifically, the mathematical objects are described as they are "given directly to the mind" (Sierpinska 2000, p. 233) in a formula, rather than computed or constructed through their properties. The subcategory of analytic-geometric describes thinking in which more visual-geometric language is used not only to describe mathematical objects but also to construct them from their properties. Table 2 summarizes Gus's and Milo's initial modes of thinking and the shifts to the others, where the category QM-Courses refers to LAW, QMS, W and CF experiences.

4.1.1 Before LAW: synthetic- and analytic-arithmetic modes

Gus was the only participant who did not take a required linear algebra course but had other learning experiences in which eigenvalues and eigenvectors were discussed. When asked, "what is an eigenvalue and an eigenvector?" in the first interview, he mentioned that eigenvalues and eigenvectors were briefly covered in an engineering course he took prior to fall; however, he did not recall how to find them nor what they meant, other than that the variable λ was "used somewhere." Identifying the variable λ was coded as part of the synthetic-arithmetic mode of thinking.

On the pre-quiz of LAW, Gus was able to find eigenvalues of the matrix $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ correctly and wrote $\begin{pmatrix} 0-\lambda & 2 \\ 2 & 0-\lambda \end{pmatrix}$, and the equation $\lambda^2 - 4 = 0$, followed it by $\lambda = \pm 2$. It seemed that prior to the winter term (after the first interview), Gus learned how to find eigenvalues. However, his answers to the other questions on the pre-quiz indicated that he still did not know what they meant. For "What is an eigenvalue?" he again wrote λ, and for "What is an eigenvector?" he wrote $\vec{\lambda}$. His responses indicated the synthetic mode with recognition of arithmetic procedures demonstrating a 'practical' approach with computation. For these reasons, Gus's initial conception of eigenvalues and eigenvectors were coded as part of the synthetic-arithmetic mode of thinking.

Milo was enrolled in Math Y at the time of the first interview. Eigenvalues and eigenvectors had not been discussed yet, but they were covered in Math X that he took prior to fall. At the first interview when asked, "What is an eigenvalue and an eigenvector?" Milo said, "if we have a matrix A, eigenvalues are such that $Ax = \lambda x$, where λ is the eigenvalue." He could not remember what eigenvectors were. When asked to elaborate, Milo said that he thought x was "maybe" a vector and decided to look at an example where A was a specific two-by-two matrix and x was a generic two-by-one vector. While working on his example, he noticed that scalar multiples of the vectors he used were the same as the ones he obtained from the matrix multiplication. He said, "Huh, I never put that together during class, I just now taught myself that."

To understand his thinking, Milo was asked, "if you have a linear transformation that reflects the vectors over x-axis, what are the eigenvalues and eigenvectors of this transformation?" He first drew some vectors and applied the linear transformation to draw the transformed vectors. After checking his work, he represented the linear transformation with the correct matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then, to find eigenvalues and eigenvectors, he tried to implement his previous, *self-taught* idea, stating that the eigenvalue "would have to be some type of a constant that I could multiply this vector to get this outcome [pointing to the expression he wrote $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$]." He claimed that there would be no such values, as he was focusing only on specific vectors he picked prior to forming the matrix. Hence, I, the researcher, asked if there were vectors other than the ones he used that would satisfy his self-taught idea. He explored his drawings of transformed vectors and stated, "Well, if there was no x component, so if you have the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, this transformation would be the same thing as multiplying this [pointing to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$] by negative one."

He explained further by stating that vectors along the x axis would not be affected by this linear transformation. However, throughout his explanation he did not utter the word eigenvector nor claim that the vectors on the x axis were eigenvectors. He summarized his thinking, "Well, I guess you can say that an eigenvalue is somehow a condensed version of a matrix, or a transformation… So you are finding a way to transform or alter a vector using a constant instead of a matrix."

Milo's reasoning process about eigenvalues in this first interview seemed to include aspects of the synthetic-arithmetic mode as he reasoned with the operational aspect of his equation, and the mathematical object, eigenvalue, appeared

Table 2 Participants' modes of thinking

Participants	Synthetic		Analytic		
	Geometric	Arithmetic	Geometric	Arithmetic	Structural
Gus					
Before QM-courses		X			
During & after		X	X	X	
Milo					
Before QM-courses		X		X	
During & after		X	X	X	X

as it was "given directly to the mind" (Sierpinska 2000, p. 233). Even though his visual representation showed synthetic-geometric thinking for eigenvectors, he did not explicitly state that vectors he observed were eigenvectors.

On the pre-quiz of LAW, Milo found the eigenvalues and eigenvectors of $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ correctly by utilizing the equation $(A - \lambda I)\vec{x} = 0$. On the second question, Milo described eigenvalues as "values of λ that satisfy det $(A - \lambda I) = 0$; they take the place of a matrix operation by turning it into a scalar multiplication for a given eigenvector." For an eigenvector description, he wrote: "Vectors for which a matrix operation may be replaced by a scalar multiplication." His description of eigenvalues on the quiz included a formula, with arithmetic language, and his description of eigenvectors was based on his arithmetic thinking concerning eigenvalues. Overall, his thinking (together with correct computations) seemed to indicate an analytic-arithmetic mode of thinking.

4.1.2 End of LAW: inclusion of geometric mode

On the post-quiz of LAW, Gus could not find the eigenvalues and eigenvectors of $\begin{pmatrix} 2i & 3 \\ 0 & -7 \end{pmatrix}$. He wrote det$(\lambda I - A)$ and used it to find eigenvalues. His error in the characteristics polynomial (he wrote, "$(\lambda - 2i)(\lambda + 7) + 3 = 0$") yielded incorrect roots. However, he wrote a short note for the *process* of finding eigenvectors using the eigenvalue equation, $A\vec{v} = \lambda\vec{v}$:"$\begin{pmatrix} 2i & 3 \\ 0 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$; choose an x, then find y."

On the second question of the post-quiz that asked what eigenvalues and eigenvectors are, Gus wrote, "$Av_1 = \lambda v_1$, [eigenvalues] scalar multiples used to find eigenvectors," and "[Eigenvector is] a vector that has an unchanged direction (except in the opposite direction) when operated on by a transformation matrix." In his eigenvalue description, Gus seemed to employ synthetic-arithmetic thinking due to his language of "scalar multiple" without referring to a geometric object, focusing on the multiplication operation as it appears in his equation. On the other hand, Gus seemed to use an analytic-geometric mode to describe an eigenvector. This description was considered to be part of the analytic mode in which the object of thought (in this case eigenvector) "is seen as conceived or constructed in a language or a conceptual system" (Dorier and Sierpinska 2001, p. 264). His visual-geometric language usage of "a vector that has unchanged direction" indicated geometric thinking and his inclusion of "(except in the opposite direction)" hinted that his thinking was beyond synthetic mode.

Milo, on the post-quiz, was able to find eigenvalues and eigenvector of $\begin{pmatrix} 2i & 3 \\ 0 & -7 \end{pmatrix}$ correctly, using the eigenvalue equation $A\vec{x} = \lambda\vec{x}$ instead of $(A - \lambda I)\vec{x} = 0$. On the second question, he again used arithmetic language to describe an eigenvalue, stating, "A scalar that accomplishes the same thing when multiplied to a specific eigenvector as is accomplished by the given matrix operation." However, it appears that his arithmetic language use was coming from synthetic, 'practical', thinking. He described an eigenvector as "A vector whose 'direction' is unchanged by a given matrix operation-it is merely scaled along its direction by its associated eigenvalue." In his description of eigenvectors, he used more visual-geometric language for eigenvectors. It seems that his thinking concerning eigenvectors included aspects of analytic-geometric thinking.

At the end of LAW, Gus's and Milo's modes of thinking already showed some changes to include additional ways to describe eigenvalues and eigenvectors. Specifically, they both incorporated aspects of an analytic-geometric mode of thinking for eigenvectors.

4.1.3 After LAW: integration of other modes

At the beginning of the second interview, a week after LAW, Gus mentioned that the geometric interpretation of eigenvectors was one of the most interesting things he learned in LAW. To explain what eigenvalues and eigenvectors meant, he wrote the matrix $\begin{pmatrix} 1 & 5 \\ 2 & 10 \end{pmatrix}$ and operated on a variety of vectors. Then, he determined that input vectors were all mapped on to the line $y = 2x$. He claimed that the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ was an eigenvector because it was already on the line and provided further reasoning that "the slope of the vector is the same as this line." He stated that this meant the vector did not change its direction when it was operated on by the matrix. His reasoning process seemed to have both analytic-geometric and analytic-arithmetic modes of thinking.

Similarly to Gus, in the second interview Milo stated that the geometric interpretation of eigenvectors was one of the interesting ideas he learned in LAW. His descriptions of eigenvalues and eigenvectors were very similar to what he wrote on the post-quiz. He stated that he could revise his description of eigenvectors on the post-quiz to describe eigenvalues in a similar manner. He further commented about his experience in LAW,

> Well, definitely the discussion in class [LAW] that showed us the geometric interpretation of the eigenvector made it a lot easier to understand this little formula here [pointing to $A\vec{v} = \lambda\vec{v}$] and there is certainly something concrete now that I have attached in my head. I remember [at] that interview [referring to the first interview] thinking that this is something I should know, and I know I have been taught this. But

> I didn't remember, and I am pretty sure I am going to remember at least the very basic concept of what an eigenvector is. Maybe there is a lot more to it; I am sure there is, but, yes, I have something to hold on to now.

In the third interview, approximately 12 weeks after LAW and 3 weeks after the last QM course, participants were asked to describe an unknown operator M which had two eigenvalues (1 and -1) and corresponding eigenvectors $\left(\begin{pmatrix}1\\1\end{pmatrix} \text{and} \begin{pmatrix}1\\-1\end{pmatrix}\right)$ that were associated with these eigenvalues, respectively. Gus first wrote the eigenvalue equation, $A\vec{v} = \lambda\vec{v}$, and then plugged in the given eigenvalues and eigenvectors. After trying a couple of random matrices with entries 0 and 1, he stated that M was one of the Pauli spin matrices that they had in QMS course. He tried to construct M via a couple of Pauli spin matrices. When he got stuck, Gus changed his approach and wrote M as a generic two by two matrix $\begin{pmatrix}a & b\\c & d\end{pmatrix}$ and used the eigenvalue equation to solve for the entries. After correctly finding M, he further identified that it was a reflection transformation over the $y = x$ line. He checked his claim, once again, using a geometric interpretation of eigenvectors. He said the eigenvectors of the matrix made sense, because the first eigenvector $\begin{pmatrix}1\\1\end{pmatrix}$ was on the reflection axis, and if the second eigenvector $\begin{pmatrix}1\\-1\end{pmatrix}$ was reflected over the line $y=x$, it would become the multiple of itself by -1, hence the vector $\begin{pmatrix}1\\-1\end{pmatrix}$ was not changing its direction. Gus's reasoning process seemed to indicate shifts between analytic-arithmetic to analytic-geometric modes of thinking, as he employed both arithmetic and geometric languages.

Milo, for this third interview question, said, "I'm trying to think if I can get from here to actually building the matrix… I mean, these are [pointing to the eigenvectors] linearly independent eigenvectors, they have their own eigenvalues." He then continued to explain what linearly independent meant and concluded his explanation with, "I think M is not a rotation matrix. I think it would be a flippy guy." When asked how he knew, he said, "Because the only way for both of these vectors to be changed only by a scalar, I think everything is flipped around one of them, so that one of them is totally unchanged, the other [is] scaled by a negative." He also stated that if M was a rotation then both given vectors would change directions, that they would not be eigenvectors anymore. His reasoning in this question showed aspects of the analytic mode with utterances of visual-geometric and arithmetic languages with inclusion of structural properties such as linear independence to construct the mathematical object of eigenvector.

Similarly to Gus, Milo also wrote a generic two by two matrix and then used the eigenvalue equation to find entries of the matrix. While working, he said the matrix M looked "an awful lot like one of those" Pauli spin matrices, which was very similar to Gus's reaction but unlike Gus, he did not proceed with this idea. After finding the matrix $M = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$, he checked his conjecture of M being a reflection. After correctly identifying the matrix M, he explicitly mentioned that eigenvalues were "stretching or shrinking" factors and they were special in physics contexts, representing measurables. These spontaneous (i.e., not requested by the researcher) thought processes capture Milo's integration of the analytic-structural mode of thinking.

The last question on the third interview was: "Give the general solution to the differential equation $i\frac{d}{d\varphi}f(\varphi) - af(\varphi) = 0$, subject to the condition $f(\varphi) = f(\varphi + 2\pi)$." Gus was stuck at the beginning of this problem; as he looked at the equation, he pointed to $f(\varphi)$ and whispered "what kind of a beast is this?" He said, "I am trying to [inaudible] how to solve a differential equation. I just had a whole course on it." After writing $cos(x) = \frac{e^{ix}+e^{-ix}}{2}$ and manipulating the given differential equation, he stated, $f(\varphi) = c_1e^{-ia\varphi}$ is a general solution. However, he stated that he was not convinced with his answer because he was not certain if his answer would satisfy the given condition. Gus tried to reason with several trigonometric functions. Before he was about to give up, I asked him if there was a way to change the problem into an eigenvalue problem. Gus said,

> I believe that—I know that this is an eigenvector, or an eigenfunction [pointing to f(φ) on the question sheet], rather, because when I want to take its derivative times i, it is just giving you constant out front. So, f(φ) is an eigenfunction of that operator. So it is a solution to the differential equation, which I think that it would have to be, in order for it to be a solution, you could call it an eigen-solution. I'm sure you can. It is hopefully part of the definition even.

When Gus said "the definition", he wrote the equation, $\left[\frac{d^2}{dx^2} + \frac{d}{dx} + 1\right] f(x) = cf(x)$, and connected the symbols $f(x)$ on both sides of the equation with an arrow and draw a box around the letter "c" in his equation. His engagement with this question provided some insight into the possibility of Gus having an early analytic-structural mode of thinking, shifting his previous modes of thinking of the eigenvalue equation, $A\vec{v} = \lambda\vec{v}$, to a theoretical one that focuses on a given operator's properties.

Milo, for this question of the third interview, stated that the function $f(\varphi)$ has to be periodic so it can either be sine or cosine or an exponential function. He 'guessed' the answer would be an exponential function because in the given equation there was an i and differentiating the exponential would give him another i to cancel out. His reasoning seemed to focus on making sense of the equation, rather than focusing on solving a differential equation. He used the exponential function $e^{-ia\varphi}$ for $f(\varphi)$ in the equation, $i\frac{d}{d\varphi}f(\varphi) - af(\varphi) = 0$ to test his conjecture. While he was doing his computation, I asked if this problem could be turned into an eigenvalue problem. He seemed to be surprised with this question and replied, "That just blew my mind. An eigen—well, we can have eigen-functions, I believe. So this [pointing to $f(\varphi)$] could somehow be an eigen-function." I asked him to elaborate more.

> Yeah, because this [pointing to $\frac{d}{d\varphi}$] would be the operator. Ohh, this [equation in the problem] rewritten [pointing to what he wrote $i\frac{d}{d\varphi}f(\varphi) = af(\varphi)$], so instead of an eigenvector, you have the eigen-function, and this would be an eigenvalue [pointing to the constant a] and this is the operator [pointing again to $\frac{d}{d\varphi}$]… and it can be represented as a matrix, I think. I think that sort of thing happens.

Even though he did not make the connection to eigenvectors on his own, his reaction and further elaboration seemed to indicate that his reasoning in the context of this problem showed some aspects of analytic-structural thinking.

Overall, synthetic- and analytic-arithmetic modes of thinking observed before the LAW experience shifted to others to include analytic-geometric and structural ones. Both Gus and Milo seemed to integrate analytic-geometric and analytic-arithmetic modes of thinking in their processes in the second and third interviews. Milo additionally incorporated analytic-structural thinking, using abstract properties in his process of communicating and working with eigenvectors.

4.2 AOT analysis

In the previous section, I provided some examples from Gus's and Milo's data analysis to discuss their incorporation of various modes of thinking in the interview tasks. Recall Gus's process of describing an eigenvector with the matrix $\begin{pmatrix} 1 & 5 \\ 2 & 10 \end{pmatrix}$: he operated it on some vectors, geometrically represented the output vectors, and found the projection line. Then, he described what he thought eigenvectors were using this particular example. Without access to Gus's prior learning experience, this projection matrix might seem random. Especially, his process of inquiry, just to describe eigenvectors, might seem an idiosyncratic one. Similarly, Milo's switch between the equations $(A - \lambda I)\vec{x} = 0$ and $A\vec{x} = \lambda\vec{x}$ on the pre- and post-quizzes of LAW, respectively, might seem to be a minor nuance. However, when such processes and shifts were explored together with students' prior learning experiences within the scope of this study, we can gain additional insights into these processes. Analysis using the AOT framework helps identify potential influences of prior learning experiences on students' processes and shifts. In other words, what constitutes evidence for the actor-oriented transfer is "indication of influence from previous activities" (Lobato and Siebert 2002, p. 89) and examples of such evidence from Gus's and Milo's data analysis are provided in this section.

4.2.1 Instructor's interactions

As part of the instructor interaction category, I specifically focus on Prof. Clay's instructional moves. The AOT analysis of episodes (1) the switch from the equation, $(A - \lambda I)\vec{x} = 0$, to the eigenvalue equation, $A\vec{v} = \lambda\vec{v}$; (2) the use of the question, "What kind of a beast is this?" (3) Gus's use of the matrix $\begin{pmatrix} 1 & 5 \\ 2 & 10 \end{pmatrix}$ in the second interview; and (4) Gus's and Milo's processes to find the matrix M in the third interview, all indicated that both Gus and Milos could have been attempting to construct similarities between the interview tasks and their prior learning experience with respect to Prof. Clay's instructional moves.

On the second day of LAW, eigenvalues and eigenvectors were introduced with a small-group activity. Each group was assigned a different 2×2 matrix and asked to "operate on" five given vectors on the worksheet, and to graph both initial and transformed vectors. They were also asked to find the determinant of the matrix and "make note of any differences between the initial and transformed vectors. Specifically, look for rotations, inversions, length changes, anything that is different." Milo's group worked on $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and Gus's group worked on $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$. Each group presented their answers as Prof. Clay facilitated the discussion and posed questions.

During this whole-class discussion time, Prof. Clay focused students' attention on different ideas. She explained that her reason for using the word "operate on" was to get them familiar with the phrase for abstract linear operators in QM courses. Use of this phrase was observed in Gus's and Milo's interviews (e.g., Gus's post-quiz). She asked each group to write the equation $A\vec{v} = \lambda\vec{v}$ with their group's matrix and unchanged vectors. She used this moment to introduce the terminology of eigenvector for these vectors.

Prof. Clay showed students how to find eigenvalues and eigenvectors in a brief lecture with examples on the fourth day of LAW. She started with the equation $A\vec{v} = \lambda\vec{v}$ on the board and asked, "What kind of a beast is this?" while pointing at each symbol in the equation. She posed this question frequently in the Symmetries course. She explicitly provided her reason for asking this question by stating that before solving an equation, identifying elements in the equation would help to get an insight into "What's happening on both sides of the equation?" As she demonstrated each step of finding eigenvalues and eigenvectors, she made explicit suggestions such as using the eigenvalue equation, $A\vec{v} = \lambda\vec{v}$, instead of the equation, $(A - \lambda I)\vec{x} = 0$ to find eigenvectors after obtaining the eigenvalues. Her reasoning was that this latter equation would not be as meaningful in other contexts where operators were not matrices. As noted in both Gus's and Milo's work on pre- and post-quizzes of LAW, they seemed to be influenced by this explicit recommendation and made shifts. Furthermore, Gus's utterance of "What kind of a beast is this?" and Milo's reasoning with his "guessed" function during the last question of the third interview seemed to indicate that they were both trying to make sense of the equation, in their own way, and were influenced by Prof. Clay's explicit suggestions.

LAW's fourth day continued with a small-group activity in which groups were assigned a matrix and were asked to find its eigenvalues and eigenvectors. During group presentation time, Prof. Clay asked the whole class what each matrix represented geometrically and to recall their observations from the second day of LAW on unchanged vectors. This seemed to be Prof. Clay's explicit way of connecting geometric and algebraic representations of eigenvectors. She further suggested that students could first examine the given matrix geometrically and explore unchanged vectors. A similar recommendation was also provided during her CF course. She wanted students to get a "feel for" the operator first, either geometrically or algebraically. In the second interview, when Gus provided the example $\begin{pmatrix} 1 & 5 \\ 2 & 10 \end{pmatrix}$, his process of reasoning of what eigenvectors were followed the processes suggested by Prof. Clay, using the geometric interpretation of the matrix first.

When both Gus and Milo were trying to find the matrix M in the third interview, they both wrote M as a generic two by two matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and solved for the entries using the eigenvalue equation. This approach was again one of Prof. Clay's explicit suggestions. This particular suggestion was given on the third and fifth day of LAW; first when students were discussing the properties of a rotation matrix A and then later when they were asked to find a condition for a matrix A to be Hermitian.

4.2.2 Class experiences

For the classroom interactions category of the AOT analysis, I share results from Gus's and Milo's processes in which indication of influences of certain in-class activities and discussions were observed. As noted before, Gus provided the matrix $\begin{pmatrix} 1 & 5 \\ 2 & 10 \end{pmatrix}$ in the second interview. His process of describing eigenvectors not only followed Prof. Clay's explicit suggestion but it also indicated his way of constructing similarities between in-class activity from the second day of LAW. On that day, more time (compared to other group's presentations) was spent on understanding the matrix, $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$, after the group's presentation, through a whole-class discussion. Prof. Clay posed questions to guide the discussion, which was centered around finding the slope of the projection line and the concept of linear dependency. It seems that this particular classroom experience influenced Gus's reasoning in that he not only implemented the same process of inquiry but he also used a matrix that was similar to $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$, his group's matrix.

Milo, as shared in previous sections, discussed why M cannot be a rotation matrix and suggested that it was a "flippy guy," and gave the meaning of eigenvalues in other contexts in the third interview. These utterances of Milo and integration of multiple modes of thinking of eigenvalues suggested influences from LAW-class discussions. For example, on the second day of LAW, during the group presentation time, students decided to use the phrase "flippy guy" for the reflection transformation, and this phrase was repeated on the fourth day of LAW. Prior to each group's presentation of computations of eigenvalues and eigenvectors of their group's assigned matrix on this fourth day, Prof. Clay kept asking the whole class to recall what the matrix represented from day two of LAW. At this point, students again used the phrase "flippy guy." In other words, this phrase seemed to be part of this community's negotiated word use. Also, concerning Milo's process of integrating of multiple modes of thinking, "stretching or shrinking" factors were discussed on the second and fourth days of LAW. Students had several QM experiences in which eigenvalues represented certain measurements corresponding to eigenstates. For example, one of the postulates in QMS course-summary notes from class, "The only possible result of a measurement of an observable is one of the eigenvalues a_n of the corresponding operator A," was discussed during the ninth day of the course.

Both Gus and Milo referred to Pauli-spin matrices during the third interview while they were working to find the matrix M, but neither one of them continued with this line of reasoning. However, this instance of their noticing similarity could indicate their attempts to construct similarities between the interview task and their prior experience in QMS.

5 Discussion and conclusions

This study explored physics students' transfer of learning of eigenvalues and eigenvectors from learning experiences to interviews. The results show that when students started the quantum mechanics (QM) courses, after initial linear algebra learning experiences, their conceptions of eigenvalues and eigenvectors included synthetic-arithmetic and analytic-arithmetic modes of thinking (Sect. 4.1.1). As students progressed through the QM courses, regardless of their differing initial learning experiences, their conceptions showed changes to incorporate other modes of thinking such as analytic-geometric and analytic-structural (Sects. 4.1.2 and 4.1.3). Analysis with the AOT revealed several indications of influence from QM learning experiences for each participant. Overall, each participant seemed to construct similarities between his experiences and Prof. Clay's explicit instructional moves, class interactions and activities. Taken together, these results illustrate that prerequisite knowledge of a student may not necessarily tell us much about a student's future process and progress, and that explicit instructional moves, in-class engagements and activities may help students build similarities and find ways to orient their generalization (transfer) process.

The results of students' struggles with modes of thinking after an initial linear algebra learning experience align with the existing linear algebra studies (e.g., Larson and Zandieh 2013; Stewart and Thomas 2010). The results contribute to this body of knowledge by exploring 'what happens' to students' modes of thinking after this initial experience. Students incorporated different modes of thinking in ways that seemed to be "equally useful," for students' conception, and "each in its own context, and for specific purposes, and especially when they are in interaction" (Sierpinska 2000, p. 233). I believe that these interactions were afforded by the various, mostly abstract, contexts of three quantum courses. This particular observation supports the claim that physics could 'bridge' various forms of thinking (Thompson et al. 2016) and align with Wawro et al.'s (2017) observation of a student exhibiting flexible ways of thinking in different notations, which was also afforded by a quantum physics course.

The AOT analysis added more nuance to the analysis of students' conceptions. When the focus of analysis moved from the product (i.e., students' conceptions) to the process of students' reasoning, indications of influences of learning experiences were observed. These observations provide opportunities for educators to enhance their course designs and in-class interactions. For example, both participants mentioned that discussion of a geometric interpretation of eigenvectors was the most interesting thing that they learned during LAW. As Milo mentioned, this interpretation provided him "something concrete" that he thought of as "at least the very basic concept of what an eigenvector is." It seems that the LAW experience provided opportunities for Milo to develop an additional mode of thinking for an eigenvector while retaining the existing ones.

Both students' observations and experiences with geometric representation of eigenvectors align with Prof. Clay's careful design of activities. She was purposeful in focusing students' attention on geometric representations on the second day of LAW and in revisiting this idea with algebraic representations on the fourth day. Furthermore, both Gus and Milo started to use the eigenvalue equation, $A\vec{v} = \lambda\vec{v}$ as they reasoned and even computed eigenvectors. Their process indicated the influence of Prof. Clay's comments on the use of the equation $A\vec{v} = \lambda\vec{v}$ instead of $(A - \lambda I)\vec{v} = \vec{0}$, her explicit discussion of the appropriateness of the former one in other contexts and her consistent usage of $A\vec{v} = \lambda\vec{v}$. It is possible that students found this equation more similar to others that they worked on in the subsequent quantum courses. In the QMS course, for example, students examined the Schrödinger equation, $i\hbar\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle$, noticing that $\frac{d}{dt}$ represented an operator (i.e., it was similar to matrix A) whereas $|\psi(t)\rangle$ represented the eigenfunction of the operator. Even though in previous studies students' progression from $A\vec{v} = \lambda\vec{v}$ to $(A - \lambda I)\vec{v} = \vec{0}$ is stated to be desired (Thomas and Stewart 2011), in abstract cases (e.g., QM courses) the latter equation may not be as meaningful.

This study adapted Lobato's (2003) actor-oriented transfer framework at a tertiary level by capturing course learning experiences, rather than short teaching interventions. This process did have some limitations. For example, participants' initial learning experiences of linear algebra were not accessible as they had taken a prerequisite course prior to the start of the study. Also, in-class small group interactions were not audio-recorded and these interactions could have provided more insights on students' authentic learning experiences. However, whole class discussions were recorded and analyzed. These interactions seemed to influence students' construction of similarities and resonate with the calls for active learning in tertiary level courses (CBMS 2016).

In closing, I echo Lobato and Siebert's (2002) emphasis on the importance of carefully designed courses. They stated that "simply trying to 'teach for understanding' in the hopes that students will transfer that understanding is far too general a guide to be useful in designing instruction" (p. 113). It is as equally important to be explicit during instruction as it is to design courses. As educators, we need to explicitly state reasons for our language use, example choices, processes of thinking, and solving problems. Specifically, we need to make the invisible work of experts visible to students, as in the case of Prof. Clay. Additionally, it is important to re-evaluate the meaning of prerequisite knowledge or course requirements for subsequent courses. Emphasizing and exploring process skills such as recognition of similarities, connection making and problem solving could open more doors for students' future learning and transfer.

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