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ORIGINAL ARTICLE

Exploring everyday examples to explain basis: insights into student understanding from students in Germany

Michelle Zandieh[1] · **Aditya Adiredja**[2] · **Jessica Knapp**[3]

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Abstract

There is relatively little research specifically about student understanding of basis. Our ongoing work addresses student understanding of basis from an anti-deficit perspective, which focuses on the resources that students have to make sense of basis using everyday ideas. Using data from a group of women of color in the United States, we previously developed an analytical framework to describe student understanding about basis, including codes related to characteristics of basis vectors and roles of basis vectors in the vector space. In this paper, we utilize the methods of the previous study to further enrich our findings about student understanding of basis. By analyzing interview data from students in Germany, we found that this group of students most often used ideas that we describe by the roles generating, structuring, and traveling, and the characteristics different and essential. Some of the themes that emerged from the data illustrate common pairings of these ideas, students' flexibility in interpreting multiple roles within one everyday example, and the ways that the roles and characteristics motivate students to create additional examples. We also discuss two ways that differences between the German and English languages were pointed out by students in the interviews.

Keywords Linear algebra · Basis · Everyday examples · Student thinking · Conceptual metaphor

1 Introduction

The current study aims to further enrich our previous findings about student understanding of basis in linear algebra (Adiredja and Zandieh 2017). Contributing to a need for more research on this central theoretical topic, that study accessed students' understanding of basis through their explanations of the concept using everyday examples. Contrary to existing findings in the literature, students in the study provided rich explanations about basis, from which we developed a series of analytic codes that delineated aspects of understanding basis. For our purposes in that paper, we invited undergraduate women of color (WOC)[1] as participants of the study, which was situated in the United States (US).

In the current study, we intend to enrich the findings from the previous study about students' understanding of basis by interviewing a different population of students. By collecting interview data from primarily master's level male students studying in Germany, the research team had the opportunity to explore the use of the codes developed in the first data set with a very different population. We do not intend to compare the knowledge or understanding of the two sets of students. Rather we wish to use a very different sample from the first study to determine in what ways the methods and codes developed in the first study can illuminate students' understanding in a different population, and how they can be further refined through an application to a broader set of data. So in this study we investigate the following questions:

Michelle Zandieh
zandieh@asu.edu

Aditya Adiredja
adiredja@math.arizona.edu

Jessica Knapp
jknapp1@harding.edu

1 Science and Mathematics Faculty, College of Letters and Sciences, Arizona State University, Polytechnic campus, 6073 S. Backus Mall MC 2580, Mesa, AZ, USA

2 Department of Mathematics, The University of Arizona, 617 N. Santa Rita Ave, P.O. Box 210089, Tucson, AZ 85721, USA

3 Harding University, 915 E. Market Ave., Searcy, AR 72149-5615, USA

[1] Women who self-identified as Black/African American, Asian or Asian American, or Hispanic or Latinx, or Native American and Indigenous students (Ong et al. 2011).

1. How did the current study participants explain the concept of basis using ideas from their everyday lives?
2. What affordances and limitations of the analytical codes from Adiredja and Zandieh (2017) will be highlighted when we use them to analyze the explanations of a different population of students? What further insights into student understanding of basis might we gain through the use of these codes?

2 Literature review

A common definition of a basis of a vector space is a linearly independent set that spans (i.e., generates) the vector space. This definition introduces the centrality of the concept for its connections to other concepts in linear algebra. Thus a study of student understanding of basis can also provide insight into studies related to span, linear independence, and vector space. Moreover, the connection of basis to eigen theory establishes the importance of the concept in its application to other areas within and outside of mathematics.

There is relatively little literature specifically on student understanding of basis. Stewart and Thomas (2010) had students develop concept maps for basis, whereas Bagley and Rabin (2016) and Schlarmann (2013), each asked students to create a basis for a given space. In all three cases students exhibited a variety approaches that at times relied more on the linear independence aspects of basis and at times the notion of a spanning set. Additionally, each of the papers discussed students' use of symbols and graphical terminology (e.g., line, plane). None of these papers drew on students' everyday contexts.

Other studies have explored the teaching and learning of span or linear independence without a specific focus on basis. Dogan-Dunlap (2010) and Çelik (2015) delineated more fine-grained aspects of linear independence referred to by students in symbols and graphical representations. Wawro (2014) and Selenski et al. (2014) analyzed respectively classroom discussions and individual interviews regarding the Invertible Matrix theorem. Their categorizations involved detailing aspects of both span and linear independence, overlapping many of the categories in Dogan-Dunlap and Çelik.

Recently more work has focused on curricular approaches based on different design frameworks (Wawro et al. 2013; Cárcamo et al. 2017; Trigueros and Possani 2013). For example, Wawro et al. (2013) and Cárcamo et al. (2017) developed learning trajectories based on Realistic Mathematics Education (Gravemeijer 1999) that include connections to extra mathematical settings. Studies based within these curricular approaches find results that overlap with the aforementioned studies but also sometimes see additional types of student thinking. For example, Plaxco and Wawro (2015) delineated classical categories for student understanding of span and linear independence such as geometric, vector algebraic and matrix algebraic, but they also found a fourth category of travel.

Travel is one of the many everyday contexts students used in Adiredja and Zandieh (2017). Although the study did not interview students who were part of a specific curriculum, a number of students spontaneously discussed basis in the context of driving or moving around a room. Students' explanations also involved creative ideas related to their backgrounds (religion), experiences (friendships), and other interests (planets and stars). The analysis revealed analytical codes capturing the roles and necessary characteristics of basis vectors, which we will elaborate in the "Methods" section. The roles codes correspond to intuitive ideas related to span, whereas the characteristic codes are related to linear independence.

A professor who was not part of the original study implemented tasks from the Adiredja and Zandieh (2017) study in her classroom, and was struck by her students' creativity and ownership of ideas in the everyday examples they generated (Adiredja et al. in press). This instructor reflected,

> [The task] involves higher order thinking in asking the students to go beyond remembering the definition, understanding it, and applying it as many typical questions would. It requires students to create a new example and to analyze it. The question is also conceptual in asking students to generalize and/or apply a concept to another context, which might not have a formal mathematics attached to it.

For this instructor, the examples revealed students' creativity and agency, and aspects of basis that was salient for them. In her assessment, the act of constructing everyday examples was also cognitively demanding activity that engaged students' conceptual understanding.

3 Theoretical framework

Examples and example generation play an important part in doing and learning mathematics (Bills et al. 2006). Research suggests that exploring student generated examples is key for students engaged in learning formal definitions (Edwards and Ward 2004; Knapp 2009; Zazkis and Leikin 2008). Several studies have investigated students' example generation in the context of linear algebra (e.g., Aydin 2014; Bogomolny 2007). Aydin (2014) looked at how students generated a mathematical example for the concept of linear (in)dependence. Likewise, Bogomolny (2007) noted that "example-generation tasks provide a view of an individual's schema of basic linear algebra concepts" (p. 2–71).

While these studies asked students to generate a mathematical example, Adiredja and Zandieh (2017) asked students to generate everyday examples for basis. In natural

Table 1 Students' educational backgrounds

Student	Major	Current semester	Previous education
Andreas	CSE	2nd sem. masters	Bachelors in CSE
Colin	Pure mathematics/ informatics	6th sem. bachelors	
Fritz	CSE	3rd sem. masters	Dental tech, bachelors in medical engineering
Johann	CSE	1st sem. masters	Diplom[a] in mathematics
Linus	Physics	1st sem. masters	Bachelors in physics
Matthias	CSE	4th sem. bachelors	
Simon	CSE	4th sem. bachelors	
Tobias	CSE	2nd sem. masters	Bachelors in mathematics/economics

[a]A Diplom is an previous degree program in Germany that is equivalent to a master's degree. Completing the curriculum can take up to 5 years, though it often takes longer. Johann completed his diplom in 7 years

language, it is common to use an everyday example as a conceptual metaphor for abstract ideas (Lakoff and Johnson 1980). Conceptual metaphors structure our thoughts and ideas and as such can give insight into student thinking about a mathematical topic (e.g., Lakoff and Núñez 2000; Zandieh and Knapp 2006; Barton 2008; Oehrtman 2009; Zandieh et al. 2017). In this paper, the everyday examples generated by students serve as conceptual metaphors that provide insight into their understandings. This theoretical idea further motivates the use of everyday examples along with their potential benefits in instruction as suggested in Adiredja et al. (in press).

An anti-deficit perspective on student mathematical thinking (Adiredja, in press) guides the design of this study and Adiredja and Zandieh (2017). An anti-deficit perspective begins with the assumption that students have resources that can support them to successfully reason about and with mathematics. The perspective acknowledges that learning requires time. Imperfect articulations of mathematical ideas and some inconsistencies in a student's current conception are a natural part of the learning process. An anti-deficit perspective focuses on optimizing the use of students' existing resources by investigating students' current understandings. Such a perspective is motivated by the ongoing influence of research that focuses on students' deficits in mathematics.[2] These studies tend to focus on the inherent problems in students' knowledge and attribute those problems to deficiencies in the students. The anti-deficit perspective broadly guides our analysis and supports the application of the analytical framework from Adiredja and Zandieh (2017). We discuss implications of our findings as related to the anti-deficit perspective at the end of the paper.

4 Methods

4.1 Data collection

4.1.1 Research context

Three months prior to conducting this study, the first author visited the university to explore the suitability of the context for research. The institution is a university[3] in southwest Germany. At that time, the first author interviewed one doctoral student and two applied mathematics faculty members from the department to discuss linear algebra instruction at this university. As part of this process, we obtained copies of the lecture notes most often used in teaching linear algebra and were able to confirm the consistency of the definitions for basis, span and linear independence with those used typically in the United States and more specifically in our earlier project (Adiredja and Zandieh 2017). From this experience, we concluded that students at this university would be suitable participants for the study. We recruited students with the help of the two applied mathematics faculty members at the university using email and by advertising in one of the faculty member's applied mathematics class.

The desired depth and detail of analysis favored the use of a small number of research subjects and videotaped individual interviews (diSessa et al. 2016). Participants were eight male students at the university, and all students were from the same state as the university or a neighboring state. At the beginning of the interview we asked students information about their background and past mathematics courses. This self-reported information is provided in Table 1. With the exception of Colin and Linus, a mathematics and a physics

[2] The prevalence of deficit narratives about students and their learning has been highlighted in research commentaries about equitable teaching (Bartell et al. 2017).

[3] The university visited for this study focused on both research and teaching with bachelors, masters, and PhD programs in engineering, mathematics, medicine and the natural sciences. This university has a high reputation within the German university system.

major, respectively, all the other students were majors in the bachelors or masters program for Computational Science and Engineering (CSE).[4] Pseudonyms were selected to reflect the origin of students' names.

4.1.2 Interview procedure

We used the same introductory tasks and interview protocol used in Adiredja and Zandieh (2017) (Appendix A). We conducted the interview in English, and participants who volunteered did so knowing the interview would be conducted in English. The main interviewer (the first author) had minimal knowledge of German. The camera person was a graduate student in mathematics education in Germany, who spoke English as a first language and was fluent in German. She translated for the inteviewer or student when needed. We found the students able to communicate a wide variety of understandings in English. We provide more details about the influence of the German language on the interview in the Results section.

After answering questions about their background, students started the interview by solving four tasks related to basis that might be asked in a beginning linear algebra course (Appendix B). This was followed by more general questions about student's understanding of basis, including their interpretation of a common definition for basis as a "a linear independent set that spans the vector space" (Q1(a) of Appendix A). The use of this common definition from the US was not an issue with the students in Germany. They were familiar with this common definition, and the lecture notes used at this university used a very similar definition. We did not analyze this part of the interview, but it provides additional context for the interview.

Following the introductory part of the interview we asked the questions that are the focus for this study:

1. Can you think of an example from your everyday life that describes the idea of a basis?
2. How does your example reflect your meaning of basis? What does it capture and what does it not?

Sometimes students were not able to come up with a response directly after being asked the first question. The interviewer would then use follow up questions, such as "How would you explain basis to a student who did not know anything about linear algebra?" or "How would you explain basis to someone who had just finished high school but hadn't studied basis yet?" These questions proved helpful for students to construct the examples.

At the end of the interview the students were given another opportunity to share an example from their everyday life. While we used the same protocol for this study as in Adiredja and Zandieh (2017), some differences in follow up questions naturally occurred because of the responses of the students and because the principal interviewer was different. In addition, these interviews were scheduled for 60 min total in contrast to the 90 min total available in the earlier study. We transcribed the interviews and organized the transcripts by turns, marked by changes in speaker.

4.2 Data analysis

4.2.1 Determining an everyday context

We focused on analyzing turns when students responded to the two questions listed above. We also analyzed turns when students introduced an example from their everyday life to explain basis, even if they were not in response to the two questions. The following is our method to identify an everyday context in students' explanations in each marked turn. An everyday context in an explanation has to include a clearly identified non-mathematical object. The explanation ideally includes objects for both the basis vectors and the vector space.

It is also important for students to make a connection to basis when discussing their everyday example. There were instances when students discussed what might appear as an everyday context without making a connection to basis. We did not report this as an everyday context for this study. While we use the term "everyday example," students' examples did not have to involve a physical experience that students have had in the real world. Students in Adiredja and Zandieh (2017) used contexts like planets in the universe without having been to space. Our focus is for students to generate an example that uses a familiar context (physical or theoretical), which ideally is distinct from R^n.

4.2.2 Analytical codes and coding

We coded each turn marked as consisting of an everyday example using the codes presented in Adiredja and Zandieh (2017). The first and third author coded the transcript. Any disagreements were resolved by the two authors. The second author then verified the coding by the other two authors, and disagreements were resolved by the first and second author. Characteristic codes describe important conditions that a basis vector, or a set of basis vectors have. Roles codes describe the roles that basis vectors have in relationship to the larger vector space. Table 2 includes a summary of the different roles and characteristic codes. Each code has a

[4] This course of study focuses on computer-aided mathematical modeling and simulation to study problems in the physical and life sciences as well as in engineering.

Table 2 Codes description, common indicators, and examples

Roles	Information	Characteristics	Information
Generating	Creating the space through combining vectors. E.g., **make** a recipe, **build** with blocks	Minimal	The set being the smallest size or having the least amount of vectors to fulfill the role in the space. E.g., **minimum** number of pieces of clothing, **least amount** of myself to cover the space
Covering	Filling or encompassing the space. E.g., **cover** the room, **fill** in the gaps	Non-redundant	Not wanting extraneous objects in the set. E.g., a skeleton would have **extra information you don't need**; you don't want to have the same pair of shoes
Structuring	Organizing the space by using crucial vectors from which the rest of the space can be delineated. E.g., shows you **where everything is, describes** everything you need	Different	The object being sufficiently distinct compared to other objects in the set. E.g., each of us had a **different** [chore] to do; each continent is physically **independent** from the others
Traveling	Moving through the space (physical or metaphorical). E.g., to **get anywhere** in the world, **to reach** a decision based on scriptures	Essential	The critical importance of each *object* in the set. E.g., **crucial** information to define your life, you **need all three of them**
Supporting	Contributing an important aspect to the space. E.g., **to get something** from each friend group; each continent **contributes** to the world market	Systematic	The purposeful choice of basis vectors to make the relationship between the vectors and the space to be orderly and efficient E.g., operations that **works every time**; axis about which you can rotate **easily**

This table is constructed and adapted from Adiredja and Zandieh (2017)

description and two examples that we drew from Adiredja and Zandieh (2017). In each of the examples, the bolded texts are common indicators of the code. With some, the whole phrasing of the example is needed to capture the idea and thus does not include indicator words. Some characteristic codes also focus on the set, some on the vectors, and some on both. They are highlighted accordingly using italics.

The code representative was included as one of the codes in Adiredja and Zandieh (2017). However, we did not end up finding it useful in our analysis. Further analysis with our data determined that the code representative was always embedded in essential. The definition of representative emphasized the way that the selected object exemplified the essential category. In Adiredja and Zandieh (2017) this code was operationalized by focusing on the act of choosing a representative for each category, for example, selecting a particular friend to represent the goofy friend and another for the serious friend. A characteristic code is generally meant to provide a specification for including or excluding a vector in a set of basis vectors. In this case, the selection of the vector (e.g., the particular friend) was predetermined by its membership in the essential category. The selection of the particular vector is less important but having one to represent the particular subspace is essential. For this reason, we removed representative as a characteristic code. Our analysis with the new set of data for this paper found a new code systematic that we defined in the table and discuss at length in the "Results" section.

After coding, we analyzed the data for themes in the coding including common contexts, relationships between codes, and relationships between codes and contexts. In our presentation of the transcript, we use bolded texts to indicate phrases that capture the codes. In cases where two codes are situated next to each other in the transcript, we differentiate them using ***bold italics*** and **black bolded texts**. We use regular *italics* to refer to codes in the text. We removed most hedges (e.g., like, kinda, um) from the presentation of transcripts and used ellipsis (/…/) to assist the reader in following the students' explanations.

5 Results

The first "Results" section about students' everyday examples includes answers to Research Question 1 about how participants of this study made sense of basis using everyday ideas. We provide summary information in the form of tables to capture the range and type of student explanations. The emergent themes and analysis of frequency of use of codes in the second section allow us to illustrate student explanations in ways that also address Research Question 2. In other words, the themes illustrate the ways in which students interrelate the different ways of understanding basis, and how our codes were able to reveal such nuances.

5.1 Students' everyday examples

The students utilized approximately nine distinct contexts to describe their everyday examples. In the description of the contexts we include the objects and actions taken within the context. Two more commonly used contexts were materials

Table 3 Everyday examples

Student	Everyday context
Andreas	The masts of a sail spanning the sail Legs of a starfish as directions to access all of the sea floor Bricks to build something bigger Elements as basis for all molecules
Colin	Pencils pointing in different directions Stones and people to build a house
Fritz	Two rules to get to any point (on the floor of) a room Economic factors that influence how much money is worth The conditions which are needed to describe the weather at a particular point
Johann	8 binary bits to describe a space of numbers The ingredients needed for a recipe
Linus	Using principal axes to describe moments of inertia Directions to go to find a point in the room
Matthias	North and east directions (or northwest and northeast directions) to make any vector on a map Bricks to build towers
Simon	Stones, glass and wood to build up a house or other item built with these materials
Tobias	Directions to reach every point in the room

Table 4 Students' everyday contexts and their associated roles and characteristic codes

Student	Examples	Roles	Characteristics
Andreas	Sailboat	Covering	Non-redundant, different
	Starfish	Traveling	Non-redundant, essential
	Bricks	Generating	Different
	Molecules	Generating, structuring	None
Colin	Pencils	Generating	Different
	Building a house	Generating	Essential
Fritz	Room	Traveling, structuring	Systematic
	Value of money	Supporting, generating	Essential
	Weather	Structuring	Essential
Johann	Bits for numbers	Structuring	Different
	Recipe	Generating	Different
Linus	Moment of inertia	Structuring	Essential, systematic
	Room	Traveling, structuring	None
Matthias	Map	Traveling	Essential
	Towers	Generating	None
Simon	Building a house	Generating	Minimal, different
Tobias	Room	Traveling	Minimal

to build a structure and set of directions to travel in a room or to travel generally. Some of the contexts were drawn from scientific settings (computer bits, molecules, moments of inertia). Others were explanations using more common every day ideas, such as the value of money, sails, and weather. These contexts were constructed during the interview, at times using objects available in the room (e.g., a pencil, the room). Table 3 provides all the contexts that each student discussed in the interview. In the next section, we elaborate on the details of some of these contexts to illustrate what students considered to be important characteristics and roles of the basis vectors.

5.2 Roles and characteristic codes in students' examples

In this section we investigate students' use of intuitive ideas as expressed in the roles and characteristic codes. The first part is a report the frequency of use of each code and patterns reflected in the number of use. The second part is a report on the emergent themes about the relationship between the codes that help explain the ways that students made sense of basis. Table 4 provides an overview of the different examples from each student and the codes that they involved.

Table 5 Roles and Characteristic Codes Count

Roles Code	Count	Characteristic code	Count
Generating	8	Minimal	2
Structuring	6	Different	5
Covering	1	Non-redundant	2
Traveling	5	Essential	6
Supporting	1	Systematic	2

5.2.1 Frequency of use of codes

From Table 4 we can begin to see patterns in student use of the roles and characteristics codes. A simple count in Table 5 illustrates that for roles codes, *generating* is the most common code followed by *structuring* and *traveling*. *Covering* and *supporting* are uncommon. For the characteristic codes in this set of data, *different* and *essential* were used fairly often, with *minimal, non-redundant* and *systematic* being less common (see Table 5).

There are also patterns we can see in Table 4 in terms of codes that occurred with particular contexts or codes that occurred together. Especially notable are the four instances when two roles codes were used by a student to discuss the same example. In what follows we present the four emergent themes about student reasoning about basis using everyday examples. Each theme illustrates a pattern that we saw in the data and may include quotes from one or more students. Cumulatively the following sections also provide illustrations of each of the roles and characteristics codes as well as most of the everyday examples discussed by these students.

5.2.2 Four emergent themes

5.2.2.1 Commonality of generating and different in the context of building a physical structure *Generating* was the most commonly used role code, which was often paired with a commonly used characteristic code *different*. We also note that of the eight times students used *generating*, half of the times involved an everyday example of building a structure. For example, Simon began talking about pieces to build a vector space and contextualized it in building a house. He later emphasized the *different* characteristic, but he started first by noting the *minimal* idea.

Simon: It's like, the **smallest amount** of pieces- I don't know. And I can build up my vector space from it. So. Yeah. I don't know

Interviewer: Is there anything from real life that kind of works like a basis that you can think of?

Simon: Maybe it's like you're building up a house or something like that. You have, for example, **you have wood you have stone you have glass and you can build up a house from these three things**.

Simon already had in mind the notion of smallest amount of pieces, a *minimal* amount, to build and was able to immediately elaborate that into using three materials– wood, stone and glass– to build a house. When asked to what aspects of basis fit well with or were hard to explain in his example, he brought up the characteristic code of *different*.

Simon: So maybe it's good to see that **you have three different, completely different-** you build it up like linearly independent, but maybe can't get the vector space of the house- is maybe a bit difficult to see that.

Interviewer: Oh yeah, what would be the vector space in this example?

Simon: I guess it would be everything you can **build up** from stone, glass, and wood. It could be a house; it could be something else.

Simon did not elaborate on the notion of different, but he seems to be referring to the three different building materials and he connected this to the phrase "linearly independent." He also realized that with this set of vectors the vector space would not just be a house. It could be anything built from these materials.

The idea of *generating* in the context of building resonated with other students in the study. Colin provided an example. In the rare case when a student was having difficulty coming up with an everyday example, we have shared an example from another student. With Colin, the interviewer mentioned that another student had talked about basic elements such as stone, wood and glass to build a house. Not only did Simon's example seem to have resonated with Colin, he also built upon it and introduced another commonly used characteristic code, *essential*.

Colin: So maybe you can explain it as the **basic ingredients of something**?

Interviewer: Yeah, maybe throw out an example for us now that you are thinking in that direction.

Colin: Yes, for example, house you **need stones for basis. And people who build the house and other things**, and you can multiply these things and add these things to another one.

In addition to generalizing Simon's example as "basic ingredients of something," Colin emphasized the *essential* nature of stones and people to build the house. He also highlighted the idea of having linear combinations of these different vectors. This nicely illustrates the generative nature of an everyday example, and the potential for students to build on one another's ideas related to everyday examples.

5.2.2.2 Flexibility of Examples as Illustrated by the Codes As mentioned above, one pattern we observed was students' use of two roles codes for the same everyday example. The four cases of this pattern showcase the flexibility of the examples these students constructed. They also illustrate the utility of the roles codes in capturing different nuances in students' explanations of basis. For example, as part of our protocol we asked students if they could see basis as a way to generate a space or to describe a space. In his response, Andreas initially came up with his brick example to build (*generate*) a structure, as in the first theme above. However, later he came up with the example about molecules from chemistry to be able to talk about both perspectives on basis.

Andreas: Maybe it's like in chemics [chemistry]. You have the elements and they are like a basis for the whole molecular system. And you can either generate. **You can synthesize one molecule from all of the elements**, and the other way around, ***you can have a molecule and then you can do analytics on it and then you find out, "Oh, there is one Helium*** and one [hand motioned another element]."

We can see in this example both elements *generating* molecules, and also elements *structuring* molecules. In this example, Andreas could think of basic entities *generating* more complex entities. With the same example he illustrated the possibility of breaking down the complex entity into its simpler components and therefore seeing elements as structuring the space of molecules. Note that there is a bit of directionality involved in comparing *generating* and *structuring*. With *generating*, one starts with the basis (e.g., building blocks, elements) and creates the space (e.g., physical structure, molecules). With *structuring*, one considers the space (e.g., molecules), and then imagines being able to describe each entity in the space (e.g., a molecule) in terms of the basis (e.g., the elements).

Unlike Andreas' example, which was prompted by the interview protocol, Linus' and Fritz' voluntarily explained their room example using both *structuring* and *traveling* ideas. Coincidentally, at the time of the interview, Linus had just explained basis to someone who was learning about basis. His explanation illustrates the pairing of the two roles codes.

Linus: What I kind of did was, I just picked a room like this [room]. And I take a coordinate system, and then you can take- how do you call it, **the corner**. I don't know, if you look at the room, **there are kind of axes, and if you take the walls, and now you want to describe points here**. But ***it doesn't matter if you say go 2 meters in this direction and 3 meters in this direction and then 2 meters up there, and then you find it***.

Linus initially *structured* the room using a corner and the walls to envision the axes. He then discussed *traveling* in different directions to find a point. Fritz came up with a similar explanation of basis for a student but in two dimensions.

Fritz: I think if I wanted to explain to him the plane, for example. I would say ok, it's like somebody comes to you and says, "***You can go zero degrees forward or backward, and you can go ninety degrees forward or backward***." And this is **how you can describe the whole**. And you can get- by these two rules ***you can get to any point in this room***.

Fritz was not as explicit as Linus about *structuring*, but we used his phrase "describe the whole" as an indicator of *structuring*. Fritz was more explicit about the directions for *traveling* and the fact that forward and backward are allowed. Both students used the context of the room as their example and explained basis as a way to *structure* and to *travel* through the space. Our codes were able to capture this nuance in their explanations.

5.2.2.3 *Introducing* systematic *with* structuring As described in the "Methods" section, our analysis of the data led us to remove the old characteristic code of *representative* and add a new code, *systematic*. This characteristic code occurred twice in our data and in both cases it was paired with the roles code *structuring*. In this section we provide more detail about the characteristic of *systematic* in general, and about its relationship with *structuring*.

The characteristic code *systematic* focuses on the purposeful choice of basis vectors to make the relationship between the vectors and the space to be orderly and efficient. Just like *minimal, systematic* specifies a desired condition on the basis vectors to achieve their intended role. In the case of *minimal*, the set of basis vectors needs to be of the smallest size to say, *generate* or *cover* the space. Similarly, in the case of systematic, the set of basis vectors should be the efficient for say *structuring* or *traveling* through the space. Phrases like "works every time" or one can do a particular action "easily," can be indicators for the code *systematic*. Below, Linus connects *systematic* with *structuring*.

Linus: If you rotate it about this axis you can describe it easily. For example the eigenvectors correspond then to **the axis about which you can rotate it easily** or can do the calculations easily if you rotate it about this axis. For example, if you have a cube. Then you get /…/ a moment of inertia for symmetry reasons should be exactly those axes. So if you rotate it about that, you can do all the calculations easily.

Linus used the context of moment of inertia from physics to explain basis. We provide a quick explanation of this concept for the reader. Eigenvectors of an inertia tensor matrix represents the principal axes of the 3D object for which it could rotate with the least angular resistance. The inertia tensor matrix provides information about the moments of inertia (and products of inertia) of the object. When diagonalized, the matrix removes all the extraneous information about moments of inertia, and it is left with moments of inertia with respect to the three principal axes (i.e., the three eigenvectors). Each diagonal entry, the three eigenvalues, are the moment of inertia of the object with respect to each of the principal axes. In this way, the principal axes served the role of *structuring* the space, i.e., rotations of the object.

The eigenvectors were purposefully selected as a basis because this basis describes the rotation in the most simplified way (*systematic*). As Linus might have been noting, this basis (the matrix of eigenvectors) also happens to assist in calculating other quantities in physics, e.g., torque or angular momentum. However, the focus of *systematic* is on the affordances for the relationship between the vectors and the space. Note that the phrase "for symmetry reasons should be exactly those axes" also indicates the use of *essential*. Linus was emphasizing that the use of those principal axes were not optional to achieve symmetry of rotation.

Fritz offered another example of *systematic* connected to the *traveling-structuring* role pair described in the previous section. After discussing rules to describe and move about the room (see quote in Sect. 5.2.2.2) Fritz explained why one would choose a basis.

Fritz: And you are doing this because you want to, you want to have some operations, some certain operations that **works every time**. And this is why, I mean a student would ask me, "Why can't I just ***go there***?" Because I want to **describe it in an ordered manner, in a structured way**.

For Fritz, the *systematic* rules of movement facilitate the role of the basis vectors providing means for accomplishing the role goals of *structuring* and *traveling* through the space. Fritz emphasized the ordered and structured way that his rules ("go zero degrees forward or backward" and "ninety degrees forward or backward") allowed him to "describe the whole" and "get anywhere in this room."

The examples of Linus and Fritz illustrate how *systematic* is intimately tied to the goals inherent in the roles codes.

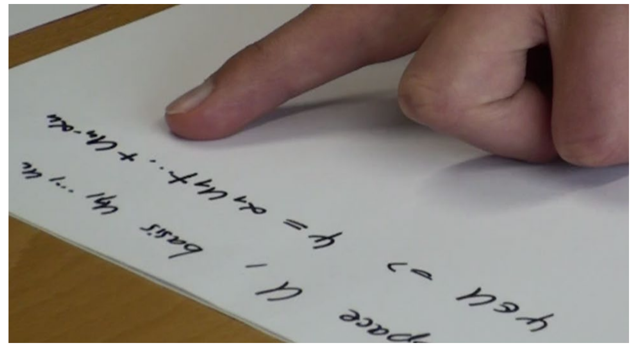

Fig. 1 Andreas pointing at the linear combination as "the description of the thing."

This may occur more often with *structuring* as the definition of *systematic* as ordered and efficient has some overlap with the idea of structure. However, as with the example of Fritz we can see that systematic may occur with other roles as well.

5.2.2.4 Transitions between examples and covering and supporting Six of our students gave more than one example during the interview. Most often this was in response to an interviewer's request. However, in three cases, the student transitioned to a new example while evaluating the first example. The first case was prompted by the interview protocol, whereas the other two cases happened because students wanted their examples to apply to higher dimension.

The first case is related to the molecule example from Andreas reported in Sect. 5.2.2.2. When Andreas was asked whether basis as a way to generate or describe something, he responded with *generating* using his bricks example:

Andreas: Either you already have something then you describe it, or you don't have it yet, then you generate it. And it's always um. Yes, generally it's like you have a lot of bricks. And **you can build something with it**. And each brick is ***unique***. And then you can build something bigger. And this is kind of a fundamental thing.

Note that Andreas discussed the bricks *generating* something bigger, but also bothered to mention that each brick is "unique" potentially pointing to the characteristic of *different* (*generating-different* pairing from the first emergent theme). Then Andreas explained this idea using symbolic notation. He wrote a *U* for a vector space, and a vector phi as a linear combination of the basis vectors (see Fig. 1).

Andreas: And the other way around you have the generation thing because all phi can be generated by

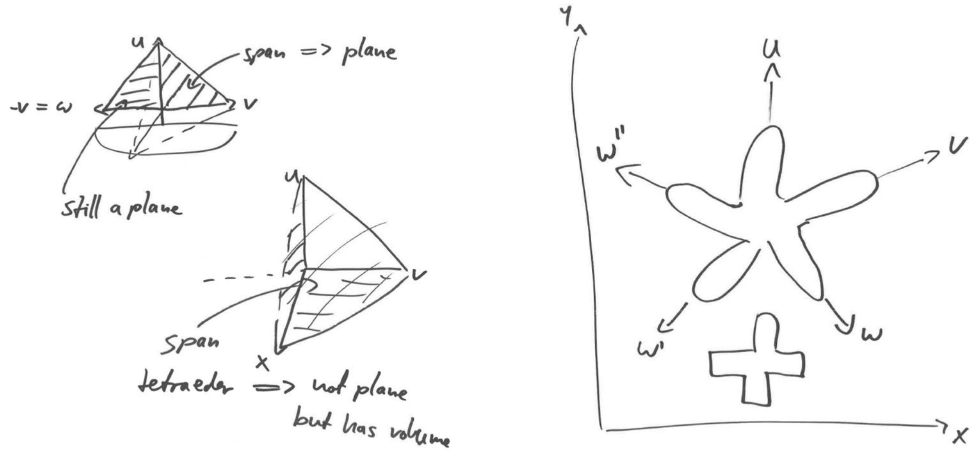

Fig. 2 Andreas' drawing of his sailboat and starfish examples

> these elements (see Fig. 1). So it's like um. And with the bricks I meant the u_1 until u_n cause they are like basic elementary things and then **you can build up complex things with those basic things**.

At this moment Andreas abruptly switched to the example of molecules in chemistry without any prompt from the interviewer. He may have done so because he did not see how to talk about describing in terms of the bricks. As we explained in Sect. 5.2.2.2, with the molecule example, Andreas was able to talk about both aspects of basis.

Another transition occurred when the student found the first example could not be scaled to consider a higher number of basis vectors. In addition to illustrating this transition, the next set of examples also illustrates *covering* and *supporting*, which were rarely used by the students. Andreas started with his sailboat example (Fig. 2). He explained, "You have a sailboat with that mast, of the boat. Then you can put that as vector *u* and there is a vector *v*, and then you **span the sail** in between." Andreas even generalized this example where the sail extends into a 3D space, instead of just a 2D plane. However, he quickly noted the limitation of his example: "It does only work for 3 dimensions. If you go in any higher dimension, there's no imagination there anymore."

He began to consider other examples at which point he noted what he was attending to in terms of the characteristics the new example needed. "The important thing is that … the student has to understand that vectors which are basically point in **the same direction** as all the other vectors before have already." He continued, "[if] you can assess the point already with one of the other vectors or a combination then ***it isn't actually a new vector and it can be crossed out or x'ed out from the basis***." Andreas was considering the new example in terms of the role of *traveling*, and attending to the characteristics of *non-redundancy* and *difference*. With these ideas in mind Andreas chose a starfish to describe basis (Fig. 2). "You have sea stars; … they just walk straight in one of those directions." He thought of the starfish legs as the vectors in the spanning set. "So they just declare one of their legs as front. And then they march on. And so they **can access any point on the sea floor** which is our plane again." Andreas also noticed the characteristic of *non-redundancy* for the sea stars, "the question is, is this ***even necessary to have five directions***, and then you can say no … there are only two vectors necessary."

A similar transition occurred during the interview with Fritz. As we included in the discussion about the *systematic* code, Fritz used rules to move around in a room as one of his examples. After discussing that example, Fritz mentioned that he had always thought of basis in a 3D space. His next example was his attempt to have an example involving higher dimensions and stepped away from a physical space.

Fritz: I could think that there is something in financial [finance, or "finanziell" in German] that you can- that basis maybe **create something** when you have more dimensions than 3. There's a lot of dimensions in financial I would say. I'm not sure, but I could imagine that there's a lot of ***influences*** in some equations.

Fritz came up with a specific example in the context of finance: the value of money.

Fritz: For example, when I want to know the real value of my money- I mean money is just a piece of paper and doesn't have any value. But there comes in- How many money have the other guys around me, how many money will there in the future or was there in the past. What do I want to buy, and what is what I want to buy worth.

Fritz is highlighting multiple dimensions that determine the value of money: how much money do others have, how much money will there be in the future and in the past, what does he want to buy, and how much do they cost. As we noted in Table 4, his further elaborations of this example include the *generating, supporting*, and *essential* codes. These transition points in both Andreas' and Fritz' examples highlighted the value students placed on examples which could be scaled to larger dimensions, more importantly, they provided insight into what characteristics the students held as important in their own evaluation of their examples.

5.3 Students' references to the German language

While English language learning is compulsory in most schools in Germany, translations between the two languages impacted the way that students explained basis. The first example is the relationship between the words "room" and "space" in German. During his interview, Linus mentioned that the word for space in German is "Raum," the same as the word for room. He argued that this would make it easier to describe the importance of basis to get to "a point in space" to his co-workers at the factory who might not know much about basis.

Some students actually used the word "room" in place of "space" in their explanations. Johann did this in his explanation of basis using bits to describe a number: "An option would be to use those eight (inaudible) to describe your-your room of numbers you can you can use." We posit that this fact, along with the fact that room is an everyday object in close proximity to the students during the interview, further increases the resonance and thus the commonality of using the room in students' explanations using everyday context.

A second example comes from the word "basis" (same spelling, pronounced "bah-sis") which in German also means "base" and "foundation." Colin mentioned this while discussing everyday use of the word basis independent from mathematics. He said, "Basis is the, if you are in politics you have a party. And all of your, want to ask all your, all of your colleagues in the party, all the people in the party, you say it's the basis, for example." He was referring to a base of a political party. After some discussion, the camera person clarified, "You don't [use the word 'base' in German]. You have 'bah-sis' for everything," and Colin nodded in agreement. We also posit that the use of the word basis to mean foundation in German also further contributed to the use of the building context. As we discussed in the first emergent theme, contexts related to building a structure were common for students in this study.

6 Discussion

This paper aims to gain further insights from and into the findings about student understanding of basis from Adiredja and Zandieh (2017). Our analysis shows the utility of the methods and codes from the previous study to uncover how a different population of students made sense of basis. Our analysis also illuminates the nature of students' understandings and how different ideas might operate together. We found that students used ideas related to particular roles codes (*generating, structuring*, and *traveling*) and characteristic codes (*different* and *essential*) more frequently. The meaning of particular words in German might have contributed to the frequent use of these ideas. Four additional themes about student understanding emerged from the data: (1) the common pairing of *generating* and *different* in the context of building, (2) students' flexibility in interpreting multiple roles within one everyday example, (3) the use of a new idea, *systematic*, with *structuring*, and (4) the ways that the roles and characteristics motivated students to create additional examples. Applying the analytical codes to a new set of data also exposed the limitation of the previous code *representative* and generated a new code *systematic*.

6.1 An anti-deficit perspective

In addition to the contributions of this paper in uncovering some patterns about student understanding of basis, and generalizing and refining previously used methods, the paper touches on two issues related to an anti-deficit perspective that came out of our previous work.

6.1.1 Students' own resources and language

The first issue is the importance of focusing on what students know and the resources they bring to the table. We recognize that there are limitations to the students' examples and explanations. However, instead of focusing on the deficits of their understanding, we met students where they were and recognized the mathematics that they were producing (Adiredja, in press).

We also took this approach when it came to the students' use of the English language to explain their reasoning. To the best of our ability and with the assistance of our collaborator, we accounted for the way the German language might have impacted students' encoding of their mathematical knowledge (see Sect. 5.3). In fact, what could have been interpreted as mistakes in using the English language, we interpreted as affordances of the German language in connecting with some productive intuitive ideas.

Nevertheless, mathematical ideas are encoded in some type of language, verbal or otherwise, and translations across languages can impact such encoding. Studies have demonstrated the utility of accessing knowledge using students' primary language (e.g., Spanish; Civil 2011; Turner and Celedón-Pattichis 2011). We posit that conducting the interview in German might reveal additional knowledge and other intuitive examples from this group of students beyond what we were able to uncover.

6.1.2 Research and social narratives

The second issue is the idea of challenging social narratives about particular groups of students. We explored the anti-deficit perspective in Adiredja and Zandieh (2017) wherein we specifically invited undergraduate women of color (WOC) to the study. This was in part due to dominant deficit narratives in the US about WOC as academically and intellectually less able than White and Asian male students (Ong et al. 2011; Solórzano and Yosso 2002; Teo 2008; Valencia 2010).[5] The findings of our previous study challenged these deficit narratives by showcasing the learning resources from a group of eight WOC to explain basis using everyday examples.

In the current study, we further support the effort to challenge deficit narratives by showing the utility and generalizability of the findings from the previous study for the current population of students. We were able to use a framework that was developed from studying a group of eight WOC to understand the sense-making of eight European men. This directionality is significant because studies of differences in academic achievement between groups of students in the US often position members of racial minority groups and women in a deficit way using White male students as the standard of practice and performance (Gutiérrez 2008, 2013).

6.2 Implications for practice

Curriculum designers may be able to use some of the everyday examples as experientially real starting points (Gravemeijer 1999) for task sequences. In fact, Wawro et al. (2012) describe a task sequence that uses the idea of *traveling* on a magic carpet and other modes of transportation to develop a curriculum on span and linear independence. Zandieh et al. (2018) describe an initial idea to create a computer game based on that magic carpet ride sequence.

Students' examples reveal the intricacy of students' understanding and the variety of nuances students bring to the concept of basis. The roles and characteristics codes highlight these nuances in ways that can be useful to teachers and curriculum developers. A teacher could use these codes and some of the patterns of their interactions as types of understandings to look for in their student work and to bring out in whole class discussion.

As we alluded to in the literature review, engaging students in constructing and analyzing everyday examples has potential benefits in the classroom (Adiredja et al., in press). In this paper, we also observed creativity in the students' constructions of examples that are distinct from R^n. We also saw an example of students' building on another's student's everyday example. Such interaction can also occur in the classroom and serve as an opportunity for students to reflect on their classmates' and their own understanding of basis. Altogether the task has the potential to provide additional insights for instructors about their students beyond traditional assessments.

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[5] Narratives includes stereotypes, but emphasize their role in providing scripts for interactions with the object of the narratives (Nasir et al. 2013).

Appendix A

Q1. (a) What does a basis mean to you?
Follow up:
(i) [*If students felt like their course did not cover basis formally*]: Some students have said that a basis of a vector space is a linear independent set that spans the vector space. What do you think that student means by that? Then go to (iii).

(ii) [*If they only mention one but not the other*]: Some students have said that a basis of a vector space is a linear independent set that spans the vector space. What do you think that student means by that? Then go to (iii).

(iii) [*If they mention both span or linear independence*]: What does each of those things mean to you?

Q1. (b) How would you explain it to a student who is about to take a Linear Algebra course?

Q2. (a) Can you think of an example from your every day life that describes the idea of a basis?
(b) How does your example reflect your meaning of basis? What does it capture and what does it not?

Q3. Could basis be relevant for any of the tasks you did? If so, how?

Q4. Can you see a basis as a way to describe something? If so, what is the something? How?

Q5. Can you see basis as a way to generate something? If so, what is the something? How?

Q6. Go through each task, and ask if they CAN possibly see basis in them.
Follow up: Can you express #3 in parametric form?

[If time permits] **Q7.** (a) Some students say that a basis is a minimal spanning set, what do you think the student means by that?
(b) Some students say that a basis is a maximal linear independent set, what do you think the student means by that?

Appendix B

Review Tasks

Student Code: ______________________________

1. Describe the span of the following set of vectors $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$

2. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$. What is the dimension of column space of A?

3. Let $B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 3 & 6 \end{bmatrix}$. Describe the solution or solutions of the equation $B\vec{x} = \vec{0}$.

4. Let $C = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$. Find the eigenvector or eigenvectors of the matrix C associated with the eigenvalue $\lambda = 3$.

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